School of Management Studies

# BCOC-134
# Business Mathematics and Statistics

## PART A : BUSINESS MATHEMATICS

### MATRICES



### DIFFERENTIALS CALCULUS



### BASIC MATHEMATICS OF FINANCE

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# COURSE INTRODUCTION

This is one of the core courses in B.Com programme under CBCS scheme. The main objective of this course is to familiarize the students with the application of Mathematics and Statistical techniques which will facilitate in business decision making. This course consists of two parts, viz., **PART- A: Business Mathematics** comprising of 11 units and **PART B: Business Statistics** comprising of 7 units (unit 12 to unit 18). The brief introduction of Part-A is as follows:

**PART A: BUSINESS MATHEMATICS**

This Part of the course, **Business Mathematics**, aims at introducing the learners to basic mathematical applications in the area of **matrices**, **differential calculus** and **financial mathematics** to solve simple business and economic problems. This part consists of 11 Units.

**MATRICES**

**Unit 1 : Introduction to matrices** discusses the concept, types of matrices, matrix algebra, Transpose of Matrix and its calculation.

**Unit 2: Determinants** explains computation of the value of determinants, its properties minors and cofactors and application of cramer's rule to solve system of linear equations.

**Unit 3: Inverse of Matrices** defines the inverse matrix, its properties and computation of methods of: i) determinant and Adjoint Route, ii) Elementary Operations Route, explains inverse and Rank of a Matrix and systems of equations.

**Unit 4: Application of Matrices in Business and Economics** discusses application and use of matrices for Business and Economic decision making.

**DIFFERENTIAL CALCULUS**

**Unit 5: Mathematical Functions and Types** defines functions and there types such as algebraic, transcendental, and inverse and composite functions. Its also deals with graph of some functions and functions relating to Business and Economics.

**Unit 6: Limit and continuity** deals with the limit of a function its properties, method of factorization and properties of continuity.

**Unit 7: Differentiation** covers the differentiation by first principle rule of differentiation, standard derivatives. It also discusses differentiation of implicit functions, using logarithms and parametric function.

**Unit 8: Maxima & Minima Functions** explains the higher order derivatives, increasing and decreasing functions and also the function of maxima and minima.

**Unit 9 Applications of differentials** deals with demand and supply functions, elasticity of demand and supply functions, average and marginal costs, revenue functions and profit maximization.

**BASIC MATHEMATICS OF FINANCE**

**Unit 10: Interest rates** discusses meaning and concept of interest, different types of interest and special cases of compound rate of interest.

**Unit 11: Compounding and Discounting** covers the calculation of normal and effective rates of interest, present value and types of discounts.

# UNIT 1 INTRODUCTION TO MATRICES

**Structure**



## 1.0 OBJECTIVES

After going through this unit, you will be able to understand:

i) Basic concept of matrix;

ii) Types of the matrices;

iii) Basic operations of matrix algebra; and

iv) Transpose of a matrix.

## 1.1 INTRODUCTION

Matrix (matrices in plural) is an arrangement of numbers into rows and columns. Because of its features of (i) compact notation for describing sets of data and (ii)efficient methods for manipulating data sets, it becomes a handy tool for finding solutions to problems which can be represented in linear equation system. Needless to say, matrix algebra finds wide applications covering the fields such as Engineering, Economics and Business, Sociology, Statistics, Physics, Medicine and Information Technology. For a better understanding of the applications, consider the following examples: sociologists use matrices to study the dominance within a group; demographers use these to study births and survivals, industries and businesses take the help of matrices for fast and accurate in decision making in the areas like evaluation of customers preferences to produce and sell. Some use linear programming techniques that is based on matrix formulations of data to maximise profit and thus plan production or availability of raw materials. Use also is made of the matrices to arrive at a decision on the location of business, marketing of the products or arranging financial resources. Economists use matrices to examine Inter-Industry flows, for studying game theory and to construct the system of social accounting.

Moreover, in medical studies, scientists use data in matrix form to determine a statistically valid rate of efficacy of a drug before prescribing it in hospitals and pharmacies. Many IT companies also use matrices as data structures to track user information, perform search queries, and manage databases.

**Check Your Progress 1**

1) What is a matrix?

2) Why would industries and businesses use matrices?

3) Matrix formulation of data that is used in linear programming is used for what purposes?

## 1.2 MATRIX

**Definition:** A matrix is defined as a rectangular array of numbers arranged in rows and columns enclosed by a pair of brackets viz., [ ] or ( ) . For example, the following array of numbers shows a matrix as

$$\begin{bmatrix} 11 & 42 & 22 & 84 \\ 10 & 15 & 60 & 25 \\ 41 & 28 & 45 & 51 \end{bmatrix}$$

On the basis of number of rows and columns that a matrix has, we decide its **dimension** or its **order**. By convention, rows are expressed first while columnssecond in a matrix. Since the above matrix has 3 rows and 4 columns, we say that its dimension (or order) is 3 x 4,

The numbers that appear in the rows and columns are called **elements** of the matrix. In the matrix above, the element in the first column of the first row is 11; the element in the second column of the first row is 42. Following the same logic, we can identify the other elements.

A matrix is usually denoted by a capital letter and it's elements by corresponding small letters with two subscripts which indicate row and column. For example, an element represented as $a_{23}$ in a matrix, is read as its to be position in $2^{nd}$ row and $3^{rd}$ column. Thus, a matrix having m rows and n columns can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & a_{3n} \\ . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & . & a_{mn} \end{bmatrix}$$

The above matrix can also be written as

A = [$a_{ij}$}$_{mxn}$ where I = 1,2,3…m

J=1,2,3…..n

Indicating a m x n order matrix.

### 1.2.1 Types of Matrices

We will discuss the most commonly used matrices to be able to use these in business related problems. Some other types will be taken up once we get familiar with transpose of matrix.

1) **Rows Matrix:** A matrix which has only one row ora matrix of order 1 x n is called row matrix.

**Example 1**:

$$\begin{bmatrix} -3 & 0 & 1 \end{bmatrix}$$

2) **Columns Matrix:** A matrix which has only one column ora matrix of order m x 1 is called column matrix.

**Example2**:

$$\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

3) **Rectangular Matrix:** A matrix is said to be rectangular if the number of rows is not equal to the number of columns.

**Example 3**:

$$\begin{bmatrix} 3 & 7 & 9 \\ 4 & 6 & 9 \end{bmatrix}$$

4) **Square Matrix:** A matrix in which the number of rows is equal to the number of columns is called square matrix i.e., the matrix of order m x n is a square matrix, if m = n.

**Example 4**:

$$\begin{bmatrix} 1 & -2 & 1 & -3 \\ -3 & 0 & 5 & 1 \\ 2 & 2 & 1 & -2 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

5) **Diagonal Matrix:** A square matrix in which all the elements except the diagonal elements are zero is called diagonal matrix.

Square Matrix A = [$a_{ij}$] is a diagonal matrix if $a_{ij}$ = 0 for all I≠j

**Example5:**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

6) **Scalar Matrix:** A diagonal matrix in which all the diagonal elements are the same is called scalar matrix.

**Example 6:**

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

7) **Identity Matrix (Unit matrix)**:A scalar matrix in which all the diagonal elements are one is called unit matrix or an identity matrix. An identity matrix is denoted by capital letter I.

**Example 7**:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8) **Triangular Matrix**: A square matrix is said to be triangular if all of its elements above the main diagonal are zero **(lower triangular matrix)** or all of its elements below the main diagonal are zero **(upper triangular matrix)**.

**Example 8**:

i) Lower Triangular Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 5 & -1 & -1 & 0 \\ -2 & 3 & 2 & 1 \end{bmatrix}$$

ii) Upper Triangular Matrix

$$\begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

9) **Null or Zero Matrix**: A square matrix in which all the elements are zero is called zero matrix or null matrix. It is denoted by capital letter O.

**Example 9**:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

10) **Symmetric Matrix:** Square Matrix A = $[a_{ij}]$ is a Symmetric Matrix if $a_{ij} = a_{ji}$ for all i&j. we will revisit this matrix after covering the transpose of matrix in this unit.

**Example10**: $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & -1 \end{bmatrix}$

11) **Sub Matrix:** A matrix obtained by deleting some rows or columns or both of a given matrix is called sub matrix of the given matrix.

**Example 11**:

Matrices $\begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 5 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 5 \\ 5 & -1 \end{bmatrix}$ are the Sub matrix of

$$\begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & -1 \end{bmatrix}$$

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**Check Your Progress 2**

1) What are diagonal elements of a matrix?

2) Find the elements $a_{21}$, $a_{34}$, $a_{24}$ and $a_{11}$ in the following matrix

$$\begin{bmatrix} -5 & 12 & 5 & 9 \\ 7 & 6 & 3 & 1 \\ 3 & 2 & 0 & 5 \\ 8 & 7 & -4 & 2 \end{bmatrix}$$

Also find diagonal elements.

3) Find x and y if

$\begin{bmatrix} x+y & 2 \\ 1 & x-y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}$

4) Classify the following matrices:

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & -1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} -5 \\ 7 \\ 3 \\ 8 \end{bmatrix}$

(iv) $\begin{bmatrix} 7 & 6 & 3 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$

(vii) $\begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix}$ (ix) $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

## 1.3 MATRIX ALGEBRA

In this section we will discuss the basic operations of matrices. We start with the idea of matrix equality before taking up the operations of addition and multiplication. In matrix algebra, the elements are ordered numbers and therefore operations on them have to be done in ordered manner. It may be useful to note that while we deal with the main operations such as addition and multiplication. Other operations viz., subtraction and division are derived out of those.

### 1.3.1 Equality of Matrices

Two matrices are equal if the following three conditions are met:

i) Each matrix has the same number of rows.

ii) Each matrix has the same number of columns.

iii) Corresponding elements within each matrix are equal.

The above conditions simply require that matrices under consideration are exactly the same.

**Example 12**:

Consider the two matrices given below:

$A = \begin{bmatrix} 2 & x \\ y & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$

If $A = B$, then x = 3 and y = 4, since corresponding elements of equal matrices have to be equal.

Further, suppose that we are given a matrix as follows:

$C = \begin{bmatrix} r & s & t \\ x & y & z \end{bmatrix}$. Then C is neither equal to A nor to B as C has three columns. Consequently, matrix C is not equal to either A or B.

### 1.3.2 Addition and Subtraction of two Matrices

- Matrices can be added or subtracted if and only if they are of the same order.
- The sum or difference of two (m x n) matrices is another (m x n) matrix whose elements are the sum or difference of the corresponding elements of the given matrices.

For two matrices A = $[a_{ij}]_{m \times n}$ and B = $[b_{ij}]_{m \times n}$

$$A \pm B = C$$

where C = $[c_{ij}]_{m \times n}$ and $c_{ij} = a_{ij} \pm b_{ij}$ for all i & j.

**Example 13:**

$$\text{For } A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 3 & 5 \\ 5 & -1 & -3 \end{bmatrix}$$

Here

$$A + B = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 7 & 3 & 5 \\ 5 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+7 & 2+3 & 3+5 \\ 2+5 & 1+(-1) & 4+(-3) \end{bmatrix} = \begin{bmatrix} 7 & 5 & 8 \\ 7 & 0 & 1 \end{bmatrix}$$

And

$$A - B = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 7 & 3 & 5 \\ 5 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0-7 & 2-3 & 3-5 \\ 2-5 & 1-(-1) & 4-(-3) \end{bmatrix} = \begin{bmatrix} -7 & -1 & -2 \\ -3 & 2 & 7 \end{bmatrix}$$

**Negation of a Matrix**: The negation of a Matrix A is denoted by –A which is obtained by replacing all the elements of A by their negation. For example, if

$A = \begin{bmatrix} 1 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix}$ then $-A = \begin{bmatrix} -1 & -3 & -5 \\ -5 & 1 & 1 \end{bmatrix}$

So, the subtraction of two matrices A and B can be expressed as the sum of A and the negation of matrix B.

$$A – B = A + (-B)$$

### 1.3.3 Multiplication of Matrix by a Scalar Quantity

If a Matrix is multiplied by a scalar quantity, then all the elements are multiplied by that quantity.

If a Matrix $A = [a_{ij}]_{m \times n}$ is multiplied by some scalar quantity λ then

$$\lambda A = \lambda[a_{ij}]_{m \times n}$$

$$= [\lambda a_{ij}\}_{m \times n}$$

For example, if $A = \begin{bmatrix} 7 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix}$, then $3A = 3\begin{bmatrix} 7 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 21 & 9 & 15 \\ 15 & -3 & -3 \end{bmatrix}$

**Properties of Addition of Matrices**

1) **Addition of Matrices is Commutative:** If A and B are two matrices of same order, then

$$A + B = B + A$$

2) **Addition of Matrices is Associative:** If A, B and C are three matrices of same order, then

$$(A + B) + C = A + (B + C)$$

3) **Existence of Additive Identity:** If A is a matrix and O is the null matrix of the same order as that of A, then

$$A + O = O + A = A$$

4) **Existence of Additive Inverse:** For any Matrix,

$$A + (-A) = (-A) + A = O$$

The following example illustrates these properties:

Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 & 3 \\ 6 & -1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 6 & 1 & -7 \\ -2 & 0 & 5 \end{bmatrix}$

and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then

$A + B = \begin{bmatrix} 1 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 1 & 3 \\ 6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 8 \\ 11 & -2 & 1 \end{bmatrix}$;

$B + A = \begin{bmatrix} 5 & 1 & 3 \\ 6 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 8 \\ 11 & -2 & 1 \end{bmatrix} = A + B$;

$(A + B) + C = \begin{bmatrix} 6 & 4 & 8 \\ 11 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 1 & -7 \\ -2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 5 & 1 \\ 9 & -2 & 6 \end{bmatrix}$;

$B + C = \begin{bmatrix} 5 & 1 & 3 \\ 6 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 1 & -7 \\ -2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 11 & 2 & -4 \\ 4 & -1 & 7 \end{bmatrix}$;

$A + (B+C) = \begin{bmatrix} 1 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 11 & 2 & -4 \\ 4 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 12 & 5 & 1 \\ 9 & -2 & 6 \end{bmatrix} = (A + B) + C$;

$A + O = \begin{bmatrix} 1 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix} = A$ and

$A + (-A) = \begin{bmatrix} 1 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & -3 & -5 \\ -5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$.

### 1.3.4 Multiplication of Two Matrices

Two matrices meet the requirement of multiplication if the number of columns of first matrix is equal to the number of rows of second matrix. If the matrix $A$ is of order $m\ x\ n$ i.e.,it has $m$ rows and $n$ columns, then matrix $B$ must be of order $n\ x\ p$ where $n$ is number of rows and $p$ is number of columns which is not necessarily equal to $m$. Then the product $AB$ is another matrix $C = A\ x\ B$ of the order $m\ x\ p$ (number of rows of $A$ and number of columns of $B$).

Let $A = [a_{ij}]_{mxn}$ and $B = [b_{ij}]_{nxp}$ be two matrices.Then the product AB is the Matrix C,

where $C = [c_{ij}]_{mxp}$, $c_{ij} = \sum_{k=1}^{k=n} a_{ik} b_{kj}$ for i =1,2,3….m & j=1,2,3,4……p

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}$$

where

$c_{11} = a_{11} b_{11} + a_{12} b_{21} + \ldots + a_{1m} b_{m1}$

$c_{12} = a_{11} b_{12} + a_{12} b_{22} + \ldots + a_{1m} b_{m2}$

$c_{1p} = a_{11} b_{1p} + a_{12} b_{2p} + \ldots + a_{1m} b_{mp}$

$c_{21} = a_{21} b_{11} + a_{22} b_{21} + \ldots + a_{2m} b_{m1}$

$c_{22} = a_{21} b_{12} + a_{22} b_{22} + \ldots + a_{2m} b_{m2}$

$c_{2p} = a_{21} b_{1p} + a_{22} b_{2p} + \ldots + a_{2m} b_{mp}$

$c_{n1} = a_{n1} b_{11} + a_{n2} b_{21} + \ldots + a_{nm} b_{m1}$

$c_{n2} = a_{n1} b_{12} + a_{n2} b_{22} + \ldots + a_{nm} b_{m2}$

$c_{np} = a_{n1} b_{1p} + a_{n2} b_{2p} + \ldots + a_{nm} b_{mp}$.

Remark: In the matrix product AB, the matrix A is called the pre-factor and matrix B is called post-factor.

**Example 14**:

Let A = $\begin{bmatrix} 1 & 3 & 5 \\ 5 & -1 & -1 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 1 & 2 \\ 5 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

Here the order of matrix A is 2 x 3 and the order of matrix B is 3 x 3, so the product AB is defined.

AB = $\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$

Where $c_{11}$ = (1x0) + (3x5) + (5x-1) = 0 + 15 – 5 = 10

$c_{12}$ = (1x1) + (3x2) + (5x2) = 1 + 6 + 10 = 17

$c_{13}$ = (1x2) + (3x1) + (5x1) = 2 + 3 + 5 = 10

$c_{21}$ = (5x0) + (-1x5) + (-1x-1) = 0 - 5 + 1 = -4

$c_{22}$ = (5x1) + (-1x2) + (-1x2) = 5 - 2 - 2 = 1

$c_{23}$ = (5x2) + (-1x1) + (-1x1) = 10 - 1 - 1 = 8

Thus, AB = $\begin{bmatrix} 10 & 17 & 10 \\ -4 & 1 & 8 \end{bmatrix}$

Consider these matrices A and B to see whether the product BA is defined. You will find that it is not. Why? Because the number of columns in B is not equal to the number of rows in A. This shows that *matrix multiplication is not* ***commutative.***

For two matrices A and B, if AB and BA both are defined, then it is not necessary that they are equal.

For example, if A = $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then

AB = $\begin{bmatrix} 10 & 14 \\ -1 & 0 \end{bmatrix}$ and BA = $\begin{bmatrix} 5 & 1 \\ 11 & 5 \end{bmatrix}$

Here AB ≠ BA

***For two matrices A and B if AB = O, then it is not necessary that either of A, B is a null matrix.***

**Example 15**:

Let A = $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Here AB = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ = O but neither A = O nor B = O.

**Properties of Matrix Multiplication**

1) **Associativity:** Matrix multiplication is associative. For three matrices A, B and C of order m x n,n x p and p x q respectively,

(AB) C = A (BC)

2) **Distributive over Addition:** Matrix Multiplication is distributive over matrix addition. For three matrices A, B and C of order m x n, n x p and p x q respectively,

A (B+C) = AB + AC

**3) Identity:** For any matrix A of order m x n, there is an identity matrix $I_n$ of order n x n and an identity matrix $I_m$ of order m x m such that $I_m$ A = A = A $I_n$.

For a square matrix A of order nxn, $I_n$ A = A $I_n$ = A

**Example 16**:

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, then show that

(I) (AB) C= A (BC)

(II) A (B+C) = AB + AC

(III) AI = IA = A

**Solution:**

i) $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 11 & -12 \end{bmatrix}$

$(AB)\,C = \begin{bmatrix} 5 & -6 \\ 11 & -12 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -11 \\ 11 & -23 \end{bmatrix}$

$BC = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -5 \end{bmatrix}$

$A\,(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 5 & -11 \\ 11 & -23 \end{bmatrix}$

Therefore, (AB) C= A (BC)

ii) $B + C = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$

$A\,(B+C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 14 & -11 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 11 & -12 \end{bmatrix}$

$AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$

$AB + AC = \begin{bmatrix} 5 & -6 \\ 11 & -12 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 14 & -11 \end{bmatrix}$

Therefore, A (B+C) = AB + AC

iii) $I\,A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$

$A\,I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$

Hence, AI = IA = A

**Check Your Progress 3**

1) Check the following two matrices. State if they are equal. Give reason to support your answer.

$$\begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 3 & 6 \\ 5 & 4 & 8 \end{bmatrix}$$

2) Suppose that the following two matrices are equal. What are the values of x and y?

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} x & 3 \\ 5 & y \end{bmatrix}$$

3) When do you say matrix multiplication is defined?

4) Explain with example the Properties of Matrix Multiplication.

5) When would you say a matrix operation is not commutative?

6) Why would you say that matrix addition is associative?

## 1.4 TRANSPOSE OF A MATRIX

The new matrix obtained by interchanging rows and columns of the original matrix is called its transpose. Suppose we have a matrix A= $[a_{ij}]$of order $m \times n.$ We interchanged its raw and column. The matrix thus derived is known as transpose of A and denoted by $A^T$ or $A'$. Thus, if

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 7 & -5 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 3 & 4 & 7 \\ 2 & 1 & -5 \end{bmatrix}.$$

For any Matrix A, AA' and A'A are always defined but need not to be equal. In the above matrices AA' and A'A are defined but not equal because the order of AA' is 3 x 3 while the order of A'A is 2 x 2.

**Properties of Transpose of a Matrix**

**i)** **(A')' = A**

**ii)** **(kA)' = k A'** where k is some scalar quantity.

**iii)** **(A + B)' = A' + B'**

**iv)** **I' = I**

**v)** **(AB)' = B' A'**

**Example17**:

Let $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A' = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ and $(A')' = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = A$

$3A = 3 \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 0 & 3 \end{bmatrix}$

$(3A)' = \begin{bmatrix} 6 & 0 \\ 9 & 3 \end{bmatrix}$ and $3A' = 3\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 9 & 3 \end{bmatrix} = (3A)'$

A + B = $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 2 & 2 \end{bmatrix}$

(A + B)' = $\begin{bmatrix} 5 & 2 \\ 7 & 2 \end{bmatrix}$

A' = $\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ and B' = $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ so, A' + B' = $\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 7 & 2 \end{bmatrix}$ = (A + B)'

AB = $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 11 \\ 2 & 1 \end{bmatrix}$ and (AB)' = $\begin{bmatrix} 12 & 2 \\ 11 & 1 \end{bmatrix}$

B'A' = $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ 11 & 1 \end{bmatrix}$ = (AB)'

### 1.4.1 Symmetric Matrix

Matrix A is called symmetric matrix if A' = A. For example, if

A = $\begin{bmatrix} 1 & 5 & 4 \\ 5 & 2 & -1 \\ 4 & -1 & 3 \end{bmatrix}$,

A' = $\begin{bmatrix} 1 & 5 & 4 \\ 5 & 2 & -1 \\ 4 & -1 & 3 \end{bmatrix}$ = A. So A is a Symmetric Matrix.

### 1.4.2 Skew Symmetric Matrix

Matrix A is called skew symmetric matrix if A' = -A. For example, if

A = $\begin{bmatrix} 0 & 5 & 4 \\ -5 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}$,

A' = $\begin{bmatrix} 0 & -5 & -4 \\ 5 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix}$ = -$\begin{bmatrix} 0 & 5 & 4 \\ -5 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}$ = -A. So A is a skew symmetric matrix.

### 1.4.3 Orthogonal Matrix

Matrix A is called orthogonal matrix if AA' = A'A = I. For example, if

A = $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$,

A' = $\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$.

Now, AA' = $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly, it also can be proved that A'A = I

**Check Your Progress 4**

1) What do you mean by is transpose of a matrix?

2) You are given a matrix A=$\begin{bmatrix} 2 & 3 & 4 \\ 1 & -5 & 9 \end{bmatrix}$. Verify that $[\boldsymbol{A}']' = A$.

3) How do you get an orthogonal matrix using transpose rule?

## 1.5 LET US SUM UP

In this unit we have discussed matrices which help find unique solution to problems when expressed in equations of linear forms. The basic operations viz., additions and multiplications have been taken up after introducing the concept of a matrix and types of matrices. We have seen that matrix defined as a rectangular array of numbers arranged in rows and columns. There are matrices such as row and column matrices, identity, diagonal, null, symmetric, rectangular, triangular and orthogonal. The unit closes with a brief discussion on transpose of matrix which means obtaining a new matrix by interchanging rows and columns of the original matrix. Some special matrices like orthogonal and skew symmetric have been discussed in this part.

## 1.6 KEY WORDS

**Diagonal Matrix**:Non-zero elements only in the diagonal running from the upper left to the lower right.

**Equality of Matrices**:Two matrices are equal if each matrix has the same number of rows, columns and corresponding elements within each are also equal.

**Identity matrix**:A matrix usually written as **I**, with 1 (ones) on the main diagonal and zeros elsewhere.

**Lower Triangular Matrix**: A special kind of square matrix with all its entries above the main diagonal aszero.

**Matrix Multiplication**: A feasible operation when the number of columns in a first matrix is equal to number of rows in a second matrix.

**Matrix**:A way of representing data in a rectangular array.

**Negation of a Matrix**: Elements of a matrix with their replacement by their negation.

**Orthogonal Matrix**:Matrix A is called orthogonal matrix if AA' = A'A = I

**Rectangular Matrix**:A matrix with the number of rows not equal to the number of columns.

**Scalar Matrix**: A diagonal matrix in which all the diagonal elements are the same.

**Scalar:** A single constant, variable, or expression.

**Skew Symmetric Matrix**:Matrix A is called Skew Symmetric matrix if A' = -A

**Square Matrix**:A matrix in which the number of rows is equal to the number of columns.

**Sub Matrix**:A matrix obtained by deleting some rows or columns or both of a given matrix is called sub matrix of a given matrix.

**Symmetric Matrix**:A matrix is symmetric if it equals its own transpose.

**Dimension(s) or Order**: The number of rows and the number of columns in a matrix.

**Transpose of Matrix**:New matrix obtained by interchanging the rows and columns of the original.

**Upper Triangular Matrix**:A special kind of square matrix with all its entries below the main diagonal as zero.

**Zero (or null) Matrix**:Matrix whose elements are all zeros.

## 1.7 SOME USEFUL BOOKS


- Allen, R.G.D., "Mathematical Analysis for Economists", London: English Language Book Society and Macmillan, 1974.
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- Chiang, A. and Kalvin Wainwright, Fundamental Methods of Mathematical Economics (Paperback), Mac Grow Hill, 2017.
- Dowling, Edward,T. "Schaum's Outline Series: Theory and Problems ofMathematics for Economists", New York: McGraw Hill Book Company, 1986.
- K. Sydsaeter and P. Hammond, Mathematics for Economic Analysis, PearsonEducational Asia, Delhi, 2002.
- Wegner, Trevor. (2016). *Applied Business Statistics: Methods and Excel-Based Applications,* Juta Academic. ISBN 9781485111931
- Yamane, Taro, "Mathematics for Economists: An Elementary Survey",New Delhi: Prentice Hall of India Private Limited, 1970.


## 1.8 ANSWER OR HINTS TO CHECK YOUR PROGRESS

**Check Your Progress 1**

1) Matrix is an arrangement of numbers into rows and columns.
2) Matrices are used by industries and businesses to arrive at a decision on the location of business, marketing of the products or arranging financial resources.
3) To plan production or availability of raw materials.

**Check Your Progress 2**

1) Diagonal Elements:All the elements $a_{ij}$are called diagonal elements if i=j. The elements $a_{11}$, $a_{22}$, $a_{33}$ …$a_{nn}$ are diagonal elements. In the above matrices the diagonal elements are 1, 4 in matrix A and 2, 0, -2 are in matrix B and 1, 0, 1, 2 are in matrix C.

2) $a_{21}$ = 7, $a_{34}$ = 5, $a_{24}$ = 1 and $a_{11}$ = -5. Diagonal Elements are -5, 6, 0, 2.

3) x = 5, y = -2.

4) (i) Identity Matrix (ii) Lower Triangular Matrix (iii) Column Matrix (iv) Row Matrix (v) Null Matrix (vi) Upper Triangular Matrix (vii) Scalar Matrix (viii) 2 x 3 Matrix (ix) 4 x 3 Matrix

**Check Your Progress 3**

1) These two matrices are not equal since they are not of the same dimensions.

2) $x$ =1 and $y$ =4

3) The matrix multiplication AB is defined only when the number of columns in A is equal to the number of rows in B.

4) Explain associativity, distributive and identity properties.

5) Consider matrices A and B to see whether the product BA is defined. If not, that could be due to the number of columns in B is not equal to the number of rows in A. Such a result indicates that matrix multiplication is not commutative.

6) Because (A+B) +C = A +(B+C)

**Check Your Progress 4**

1) The new matrix obtained by interchanging rows and columns of the original matrix is called its transpose.

2) Get $[A]'=\begin{bmatrix} 2 & 1 \\ 3 & -5 \\ 4 & 9 \end{bmatrix}$, then transpose it to get $[\boldsymbol{A}']' = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -5 & 9 \end{bmatrix}$

3) Take an example such that $AA' = A'A$=I

## 1.9 EXERCISES WITH ANSWER/HINTS

1) You are told that the following two matrices are equal. What are the values of x, y, and z?

$$A = \begin{bmatrix} 4 & 0 \\ 6 & -2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} x & 0 \\ 6 & y+4 \\ \frac{z}{3} & 1 \end{bmatrix}$$

**Ans.:** Given that $A = B$, we must have all the corresponding entries equal. So, we know that $a_{1,1} = b_{1,1}$, $a_{1,2} = b_{1,2}$, $a_{2,1} = b_{2,1}$, and so forth. Thus, $4 = x$, $-2 = y + 4$ and $3 = {}^{z}/_{3}$

Rewriting the matrices as $\begin{bmatrix} 4 & 0 \\ 6 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 6 & y+4 \\ \frac{z}{3} & 1 \end{bmatrix}$, then we solve for x= 4, y = –6, and z = 9.

2) Why is matrix multiplication not commutative?

**Ans.:** When we do not use square matrices, we cannot even try to commute multiplied matrices as the sizes wouldn't match. But even with square matrices, we don't have commutative feature always. For example, consider case of 2×2 matrices A and B.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$,

then, $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{21} + a_{22}b_{22} \end{bmatrix}$

$BA = \begin{bmatrix} a_{11}b_{11} + a_{21}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \end{bmatrix}$

It may be noted that these matrices are not be the same unless we make some very specific restrictions on the values for A and B. Since we take the rows from the first matrix and multiply by columns from the second, such a process switching the order changes the values.

3) If $A = \begin{bmatrix} 2 & -5 & 1 \\ -2 & -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 & 0 \\ 5 & -2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 7 & -6 & 2 \\ 1 & -4 & 11 \end{bmatrix}$, then evaluate

i) A + B

ii) B – C

iii) 2A + B – C

**Ans.:** (i) $\begin{bmatrix} 5 & -1 & 1 \\ 3 & -3 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} -4 & 10 & -2 \\ 4 & 2 & -8 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4) Find the matrices A and B from the following relations

$2A - B = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$ and $2B + A = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{bmatrix}$

**Ans.:** $A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

5) If $A = \begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$, find the matrix X such that 3A + 5B + 2X = O

**Ans.** $X = \begin{bmatrix} -16 & -14 \\ -47/2 & -69/2 \end{bmatrix}$

6) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ then show that

(i) A (B + C) = AB + AC (ii) (AB)C = A(BC)

7) If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, find (A – 2I) (A – 3I).

8) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, show that products AA' and A'A are symmetric but not equal.

# UNIT 2 DETERMINANTS

**Structure**



## 2.0 OBJECTIVES

After going through this unit, you will be able to understand:

1) Basic concept of Determinant
2) Difference between matrix and determinant
3) Cofactors and Minors of a Determinant
4) Application of determinant in solving a system of linear equations (Cramer's Rule)

## 2.1 INTRODUCTION

In the preceding unit, we have seen simple operations of matrices which can be used as a background material for obtaining solution in linear form problems. In the present unit, we extend the discussion to determinant which was discovered by Cramer during solving of system of linear equations.Note that determinant is a numerical value and can be computed from the elements of a square matrix.

The determinant of a matrix $A$ is denoted $\det(A)$, $\det A$, or $|A|$. Seen geometrically, it can be viewed as the volume scaling factor of the linear transformation described by the matrix. We get a determinant in positive or negative value form.

**Check Your Progress 1**

1) Distinguish between matrix and determinant
2) How Cramer discovered determinant?
3) Determinant can be computed only from which type of matrix?

## 2.2 COMPUTATION OF VALUE OF A DETERMINANT

The value of determinant can be calculated by the following procedure: Consider a square matrix. For each element of its first row or first column get its submatrix. Next multiply each of the selected elements with the corresponding determinant of the submatrix. Finally, add them with alternate signs.

**Determinant of a Matrix whose order is one(n=1)**

As a base case, the value of determinant of a 1*1 matrix is the single value itself.

That is to say that if the Matrix A has only one element, then the element itself is the determinant of matrix A, i.e.,

if A = [a], then |A| = a.

For Example, if A = [2] then |A| = 2.

**Determinant of a Matrix whose order is two (n=2)**

If the order of square matrix A is 2, then|A| = (multiplication of diagonal elements) – (multiplication of off diagonal elements), i.e.,

if A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then |A| = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ = ($a_{11}$ x $a_{22}$) – ($a_{21}$ x $a_{12}$).

For example, if A = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$then |A| = (2 x 2) – (1 x 3) = 4 – 3 = 1.

**Determinant of a Matrix whose order is three (n=3)**

**Let A =** $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$,

Then $$det\ (A) = det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

In other words, go across the first row of the matrix $A$,$(a\ b\ c)$ . Multiply each entry by the determinant of the$2 \times 2$ matrix obtained from $A$ by crossing out the row and column containing that entry.Then we add and subtract the resulting terms, alternating signs (add the $a$-term, subtract the $b$-term, add the $c$-term.)

For example, if $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{bmatrix}$, then

|A| = +2$\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}$ -1$\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}$+ (-1)$\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$

= 2(-3 + 2) -1(-3 - 1) – 1(-2 - 1) = -2 + 4 + 3 = 5.

We can use the same method to compute the determinant of a 4 × 4 matrix.In fact, the method seen above can be applied for any square matrix of any size. Note that you don't have to use the first row only; you can use any row or any column, as long as you know where to put the plus and minus signs. Thus,

if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & a_{3n} \\ . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & a_{nn} \end{bmatrix}$ then $|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}|M_{ij}|$, Where $M_{ij}$ is the sub matrix of A obtained by striking off $i^{th}$ row and $j^{tht}$ column.

**Check Your Progress 2**

1) What is the value of the determinant of a matrix with a single element?

2) You are given the following matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

   Find the value of its determinant.

3) What formula would you use to find the determinant of a square matrix of any size?

## 2.3 PROPERTIES OF DETERMINANTS

**i)** *If all the rows and columns of a determinant are interchanged then the value of determinant does not change, i.e.,* $|A| = |A^t|$

For example, $\begin{vmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 6 & 2 & 0 \\ 5 & 1 & -3 \end{vmatrix}$

**ii)** *If any two rows/columns of a determinant are interchanged, the value of determinant remains the same but the sign is reversed.*

For example, $\begin{vmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{vmatrix} = - \begin{vmatrix} -2 & 0 & -3 \\ 3 & 2 & 1 \\ 1 & 6 & 5 \end{vmatrix}$

**iii)** *If any two rows/columns of a determinant are identical, the value of determinant is zero.*

For example, $\begin{vmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0$

**iv)** *If all the elements of a row/column of determinant are multiplied by some number (say, k) then the value of the determinant is multiplied by that number.*

For example, $\begin{vmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{vmatrix} = 56$ and $\begin{vmatrix} 2x1 & 2x6 & 2x5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{vmatrix}$

$$=2x\begin{vmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{vmatrix} = 2x56 = 112.$$

**v)** *If all the elements of a row/column of a determinant are added/subtracted k-times the corresponding elements of another row/column, the value of determinant remains unchanged.*

For example, $\begin{vmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{vmatrix} = 56$ and $\begin{vmatrix} 1+(2x6) & 6 & 5 \\ 3+(2x2) & 2 & 1 \\ -2+(2x0) & 0 & -3 \end{vmatrix} = \begin{vmatrix} 13 & 6 & 5 \\ 7 & 2 & 1 \\ -2 & 0 & -3 \end{vmatrix} = 56$

**vi)** *If all the elements of a row/column of a determinant are sum/difference of two or more elements then determinant can be expressed as the sum/difference of two or more determinants.*

For example, $\begin{vmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 2+3 \\ 3 & 2 & 0+1 \\ -2 & 0 & 2+(-5) \end{vmatrix} = \begin{vmatrix} 1 & 6 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 6 & 3 \\ 3 & 2 & 1 \\ -2 & 0 & -5 \end{vmatrix} = -24 + 80 = 56$

**Check Your Progress 3**

1) You have interchanged two rows/columns of a determinant. What happens to its value?

2) John is doing some manipulation with the rows and columns of a determinant but getting the same value of it. What do you think he is doing?

3) Seeta has multiplied all the elements of the last row of the square matrix A with a constant $k$. What value of the corresponding determinant should she get because of her action?

4) Smith made changes in a determinant so as to obtain two identical columns. What value would he get of such a determinant?

## 2.4 MINORS AND COFACTORS

### Minor of an element of a determinant

Let |A| = $|a_{ij}|$ be a determinant of order n. The minor of the element $a_{ij}$ which exists in $i^{th}$ row and $j^{th}$ column of the determinant |A|, is the determinant left by striking off $i^{th}$ row and $j^{th}$ column of |A|.

For a determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ of order 3, the minor of $a_{11}$ is $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, which is obtained by deleting 1st row and 1st column of

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Similarly the minor of $a_{22}$ is $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ which is obtained by deleting 2nd row and 2nd column of

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

**Illustration:**

Given matrix = $\begin{bmatrix} 7 & 5 & 9 \\ 3 & 8 & 4 \\ 6 & 2 & 1 \end{bmatrix}$, find the minors of $a_{32}$ and $a_{23}$.

The minor of $a_{32}$ is $\begin{vmatrix} 7 & 9 \\ 3 & 4 \end{vmatrix}$ = (7x4) – (9x3) = 28 – 27 = 1.

The minor of $a_{23}$ is $\begin{vmatrix} 7 & 5 \\ 6 & 2 \end{vmatrix}$ = (7x2) – (5x6) = 14 – 30 = -16.

**Cofactor of an element of a determinant**

The cofactor of an element of a determinant is the signed minor of that element. The sign of minor is determined on the value of $i$ and $j$, i.e., the row and column in which the element exists. The cofactor of element $a_{ij}$ is denoted by $A_{ij}$.

The cofactor of $a_{ij} = (-1)^{i+j}$ minor of $a_{ij}$.

For a determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ of order 3,

the cofactor of $a_{11}$ is $A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (-1)^2\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Similarly, the cofactor of $a_{32}$ is $A_{32} = (-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = (-1)^5\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

$$= -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

The sign of minor is + if i + j is even and the sign of minor is – if i + j is odd.

**Illustration:**

For the matrix $A = \begin{bmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{bmatrix}$, let us find out the cofactors of all the elements of the corresponding determinant of the matrix.

$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix} = (-1)^{1+1}(2x\text{-}3 - 0x1) = -6 - 0 = -6.$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 1 \\ -2 & -3 \end{vmatrix} = (-1)^{1+2}(3x\text{-}3 - 1x\text{-}2) = -(-9 + 2) = 7.$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ -2 & 0 \end{vmatrix} = (-1)^{1+3}(3x0 - 2x\text{-}2) = (0+ 4)= 4.$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 6 & 5 \\ 0 & -3 \end{vmatrix} = (-1)^{2+1}(6x\text{-}3 - 5x0) = -(-18 -0) = 18.$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 5 \\ -2 & -3 \end{vmatrix} = (-1)^{2+2}(1x\text{-}3 - 5x\text{-}2) = (-3+10) = 7.$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 6 \\ -2 & 0 \end{vmatrix} = (-1)^{2+3}(1x0 - 6x\text{-}2) = -(0+12) = -12.$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix} = (-1)^{3+1}(6x1 - 2x5) = 6 - 10 = -4.$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix} = (-1)^{3+2}(1x1 - 5x3) = -(1 - 15)= 14.$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} = (-1)^{3+3}(1x2 - 6x3) = 2 - 18 = -16.$

**Miscellaneous Examples:**

**Example1:** If $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & -1 & 4 & 1 \\ -2 & 0 & -3 & 3 \\ 4 & 3 & 1 & 2 \end{bmatrix}$, find |A|.

**Solution:**

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & -1 & 4 & 1 \\ -2 & 0 & -3 & 3 \\ 4 & 3 & 1 & 2 \end{bmatrix}$$

$|A| = 1\begin{vmatrix} -1 & 4 & 1 \\ 0 & -3 & 3 \\ 3 & 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 & 1 \\ -2 & -3 & 3 \\ 4 & 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 3 & -1 & 1 \\ -2 & 0 & 3 \\ 4 & 3 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 & 4 \\ -2 & 0 & -3 \\ 4 & 3 & 1 \end{vmatrix}$

$|A| = 1M_{11} - 2M_{12} + 0M_{13} + 1M_{14}$

$M_{11} = \begin{vmatrix} -1 & 4 & 1 \\ 0 & -3 & 3 \\ 3 & 1 & 2 \end{vmatrix} = (-1)(-6 - 3) -4(0 - 9) + 1(0 +9) = 9 + 36 + 9 = 54$

$M_{12} = \begin{vmatrix} 3 & 4 & 1 \\ -2 & -3 & 3 \\ 4 & 1 & 2 \end{vmatrix}$ = 3(-6 – 3) - 4(-4 – 12) + 1(-2 + 12) = -27 +64 +10 =47

$M_{13} = \begin{vmatrix} 3 & -1 & 1 \\ -2 & 0 & 3 \\ 4 & 3 & 2 \end{vmatrix}$ = 3(0 – 9) +1(-4 – 12) + 1(-6 – 0) = -27 -16 - 6 = -49

$M_{14} = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 0 & -3 \\ 4 & 3 & 1 \end{vmatrix}$ = 3(0 + 9) +1(-2 + 12) + 4(-6 – 0) = 27 + 10 - 24 = 13.

$|A| = 1M_{11} - 2M_{12} + 0M_{13} + 1M_{14}$

|A| =1x54 – 2x47 + 0x(-49) + 1x13

= 54 – 94 + 0 + 13 = -27.

**Example2:** Evaluate $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix}$

**Solution:**

Take common a from first row, b from second row and c from third row.

$$\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = abc\begin{vmatrix} 0 & b^2 & c^2 \\ a^2 & 0 & c^2 \\ a^2 & b^2 & 0 \end{vmatrix}$$

Now, take common $a^2$ from first column, $b^2$ from second column and $c^2$ from third column. We obtain $a^3b^3c^3\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = a^3b^3c^3$ [0(0 – 1) – 1(0 – 1) + 1(1 -0)] = $a^3b^3c^3$[0+1+1) = 2 $a^3b^3c^3$.

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**Check Your Progress 4**

1) What do you mean by a minor of a square matrix?

2) What is a cofactor of a square matrix?

3) How would you get a cofactor in a square matrix?

4) You are given following determinant:

$\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$.

Find, for all its elements, minors and cofactors.

## 2.5 USE OFCRAMER'S RULEIN DETERMINANTS TO SOLVE SYSTEM OF LINEAR EQUATIONS

**Cramer's Rule:**

This method was given by Swiss mathematician Gabriel Cramer to solve a system of $n$ linear equations in $n$ variables using determinants.

Let the system of n linear equations with n variable $x_1, x_2, x_3, x_4, \ldots\ldots x_n$ be

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2$

.

.

.

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots\ldots\ldots + a_{nn}x_n = b$. The above system can be written in Matrix form

AX = B

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & a_{3n} \\ . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ b_n \end{bmatrix}$$

$$\text{Let } D = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & a_{3n} \\ . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & a_{nn} \end{vmatrix}$$

Apply the property of determinant $C_1 \rightarrow x_1 C_1$

$$x_1 D = \begin{vmatrix} x_1a_{11} & a_{12} & a_{13} & . & a_{1n} \\ x_1a_{21} & a_{22} & a_{23} & . & a_{2n} \\ x_1a_{31} & a_{32} & a_{33} & . & a_{3n} \\ . & . & . & . & . \\ x_1a_{n1} & a_{n2} & a_{n3} & . & a_{nn} \end{vmatrix}$$

Apply the property of determinant $C_1 \rightarrow C_1 + x_2C_2 + x_3C_3 + \ldots\ldots\ldots + x_nC_n$

We have

$$x_1 D = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \cdots .. a_{1n}x_n & a_{12} & a_{13} & . & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + \cdots .. a_{2n}x_n & a_{22} & a_{23} & . & a_{2n} \\ a_{31}x_1 + a_{32}x_2 + \cdots .. a_{3n}x_n & a_{32} & a_{33} & . & a_{3n} \\ . & . & . & . & . \\ a_{n1}x_1 + a_{n2}x_2 + \cdots .. a_{nn}x_n & a_{n2} & a_{n3} & . & a_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} b_1 & a_{12} & a_{13} & . & a_{1n} \\ b_2 & a_{22} & a_{23} & . & a_{2n} \\ b_3 & a_{32} & a_{33} & . & a_{3n} \\ . & . & . & . & . \\ b_n & a_{n2} & a_{n3} & . & a_{nn} \end{vmatrix} = D_1$$

Where $D_1$ is the determinant of A after replacing first column by $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ b_n \end{bmatrix}$

So $x_1 D = D_1$ which provides $x_1 = \frac{D_1}{D}$

Similarly, it can be shown that $x_2 D = D_2$

Where $D_2$ is the determinant of A after replacing $2^{nd}$ column by $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ b_n \end{bmatrix}$

Which provides $x_2 = \frac{D_2}{D}$ similarly $x_n = \frac{D_n}{D}$

where $D_n$ is the determinant of A after replacing $n^{th}$ column by $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ b_n \end{bmatrix}$

***Note: For any system of linear equations***

***If $D \neq 0$, the system has unique solution and system is consistent.***

***If $D = 0$ and all $D_1, D_2, D_3, D_4 \ldots\ldots D_n$ are also equal to zero, then the system has infinite solutions and system is consistent.***

***If $D = 0$ and at least one of $D_1, D_2, D_3, D_4 \ldots\ldots D_n$ is not equal to zero, then the system has no solution and system is inconsistent.***

**Illustrations:**

i) Solve the following system of equations by **Cramer's Rule**.

$x - 4y - z = 11$

$2x - 5y + 2z = 39$

$-3x + 2y + z = 1$

**Solution:**

We have $D = \begin{vmatrix} 1 & -4 & -1 \\ 2 & -5 & 2 \\ -3 & 2 & 1 \end{vmatrix} = 1(-5 -4) + 4(2 + 6) -1(4 - 15) = -9 +32 + 11 = 34$

Here $D \neq 0$, so solution is unique and can be obtained as

$x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D}$

$$D_1 = \begin{vmatrix} 11 & -4 & -1 \\ 39 & -5 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 11(-5 – 4) + 4(39 – 2) - 1(78 + 5) = -99 + 148 - 83 = -34$$

$$D_2 = \begin{vmatrix} 1 & 11 & -1 \\ 2 & 39 & 2 \\ -3 & 1 & 1 \end{vmatrix} = 1(39 –2) – 11(2 + 6) – 1(2 + 117) = 37 – 88 –119 = -170$$

$$D_3 = \begin{vmatrix} 1 & -4 & 11 \\ 2 & -5 & 39 \\ -3 & 2 & 1 \end{vmatrix} = 1(-5 –78) + 4(2 + 117) +11(4 –15) = -83 + 476 –121 = 272$$

So, $x = \frac{D_1}{D} = \frac{-34}{34} = -1$, $y = \frac{D_2}{D} = \frac{-170}{34} = -5$ and $z = \frac{D_3}{D} = \frac{272}{34} = 8$

ii) Solve the following system of equations by **Cramer's Rule**.

$x – 3y –8z = -10$

$3x + y - 4z = 0$

$2x + 5y + 6z = 13$

**Solution:**

We have $D = \begin{vmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{vmatrix} = 1(6 + 20) + 3(18 + 8) -8(15 – 2) = 26 +78 - 104 = 0$

$$D_1 = \begin{vmatrix} -10 & -3 & -8 \\ 0 & 1 & -4 \\ 13 & 5 & 6 \end{vmatrix} = -10(6 +20) +3(0 + 52) – 8(0 – 13) =-260 + 156 + 104 = 0$$

$$D_2 = \begin{vmatrix} 1 & -10 & -8 \\ 3 & 0 & -4 \\ 2 & 13 & 6 \end{vmatrix} = 1(0 + 52) +10(18 + 8) -8(39 –0) = 52 + 260 – 312 = 0$$

$$D_3 = \begin{vmatrix} 1 & -3 & -10 \\ 3 & 1 & 0 \\ 2 & 5 & 13 \end{vmatrix} = 1(13 –0) + 3(39 – 0) - 10(15 – 2) = 13 + 117 – 130 =0$$

Here $D = D_1 = D_2 = D_3 = 0$, the system is constent with infinite solutions.

II) Solve the following system of equations by **Cramer's Rule**.

$x – 3y + 4z = 3$

$2x -5y +7z = 6$

$3x -8y + 11z = 11$

**Solution:**

We have $D = \begin{vmatrix} 1 & -3 & 4 \\ 2 & -5 & 7 \\ 3 & -8 & 11 \end{vmatrix} = 1(-55 + 56) + 3(22-21) + 4(-16+15) = 1 +3 - 4 =0$

$$D_1 = \begin{vmatrix} 3 & -3 & 4 \\ 6 & -5 & 7 \\ 11 & -8 & 11 \end{vmatrix} =3(-55 + 56) +3(66–77) + 4(-48 + 55) = 3 – 33 + 28 = -2\neq 0$$

Since D = 0 and $D_1 \neq 0$, so the system has no solution. The system is inconsistent.

**Check Your Progress 5**

1) What is Cramer's Rule?

2) While using Cramer's rule, when would you say that the system of linear equations is inconsistent?

3) Give an example of system of linear equations.

4) What is the meaning of inconsistent system of linear equations?

## 2.6 LET US SUM UP

In this unit we have discussed the solution mechanism involved in a determinant. It is seen that determinant is a numerical value and can be computed from the elements of a square matrix.

We have learnt to compute the value of determinant which can be calculated by the following procedure: take a square matrix; for each element of its first row or first column we get its submatrix. Next, multiply each of the selected elements with the corresponding determinant of the submatrix. Finally, add them with alternate signs.

We started value of determinant of a 1*1 matrix is the single value itself. If the order of square matrix A is 2, then |A| = (multiplication of diagonal elements) – (multiplication of off diagonal elements). In case of a square matrix of order $3 \times 3$we need to go across the first row of the matrix. Multiply each entry by the determinant of the $2 \times 2$ matrix obtained by crossing out the row and column containing that entry. Then we add and subtract the resulting terms, alternating signs (add the $a$-term, subtract the $b$ -term, add the $c$-term.)The method seen in case of a $3 \times 3$can be extended to any square matrix of any size. We have also noted that there no need to confine ourselves to use the first row or column only while computing the value of a determinant. Any row or column can be used after ascertaining its position to include the plus and minus signs.

Discussing the properties of the determinant we have seen that interchanging of two rows/columns leaves the value unchanged with only its sign reversed; the value of determinant equals zero when two of its rows or columns are identical; the value of the determinant remains unchanged when all the rows and columns are interchanged. The same result is obtained by adding/subtracting all the elements of a row / column of a determinant $k - times$the corresponding elements of another row/column;the value of the determinant is $k$-time, if we multiply all elements of a row or column with a constant $k$.

We are introduced to the concept of minor and cofactor of a square matrix to compute the value of determinant. Whereas the value obtained from the determinant of a square matrixby deleting out a row and a column corresponding to the element of a matrix is called its minor, the cofactor is defined as the signed minor. Towards the last part of the unit we have been exposed to the use of determinants to solve a system of linear equations and learnt the Cramer's rule in that context.

## 2.7 KEY WORDS

**Cofactor**:The signed minor.

**Cramer's Rule**:Methodof solving a system of $n$ linear equations in $n$ variables using determinants.

**Determinant**:A numerical value computed from the elements of a square matrix

**Linear Equation System**:A collection of two or more linear equations involving the same number of variables.

**Minor**:Value obtained from the determinant of a square matrixby deleting out a row and a column corresponding to the element of a matrix.

## 2.8 SOME USEFUL BOOKS


- Allen, R.G.D., "Mathematical Analysis for Economists", London: English Language Book Society and Macmillan, 1974.
- Archibald, G.C., Richard G.Lipsey. "An Introduction to a Mathematical Treatment of Economics", Delhi: All India Traveller Bookseller, 1984
- Chiang, A. and Kalvin Wainwright, Fundamental Methods of Mathematical Economics (Paperback), Mac Grow Hill, 2017.
- Dowling, Edward,T. "Schaum's Outline Series: Theory and Problems of Mathematics for Economists", New York: McGraw Hill Book Company, 1986.
- K. Sydsaeter and P. Hammond, Mathematics for Economic Analysis, PearsonEducational Asia, Delhi, 2002.
- Yamane, Taro, "Mathematics for Economists: An Elementary Survey", New Delhi: Prentice Hall of India Private Limited, 1970.


## 2.9 ANSWER OR HINTS TO CHECK YOUR PROGRESS

**Check Your Progress 1**

1) A matrix is a simply an ordered arrangement of elements in a tabular form while determinant is a single numerical value which is associated to a square matrix only.

2) During solving of system of linear equations.

3) Square matrix.

**Check Your Progress 2**

1) The determinant of a 1×1 matrix is that number itself.

2) proceed along the first row, starting with the upper left component a. We multiply the component a by the determinant of the "submatrix" formed by ignoring a's row and column. In this case, this submatrix is

the 1×1 matrix consisting of d, and its determinant is just d. So the first term of the determinant is ad.

Next, proceed to the second component of the first row, which is the upper right component b. Multiply b by the determinant of the submatrix formed by ignoring b's row and column, which is c. So, the next term of the determinant is bc. The total determinant is simply the first term ad **minus** the second term bc.

3) Let the matrix be $A_{n\times n}$. Then $|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}|M_{ij}|$, Where $M_{ij}$is the sub matrix of A obtained by striking off $i^{th}$row and $j^{tht}$ column.

**Check Your Progress 3**

1) The value of determinant remains the same with its sign reversed.

2) He is interchanging (i). all the rows and columns of the determinant; and (ii) adding/subtracting all the elements of a row/column of a determinant $k - times$ to the corresponding elements of another row/column.

3) She should get the value of the determinant $k$-times of the original value.

4) He would get the value of determinant equal zero.

**Check Your Progress 4**

1) Value obtained from the determinant of a square matrixby deleting out a row and a column corresponding to the element of a matrix.

2) The cofactor is defined as the signed minor. Cofactor of an element $a_{ij}$, denoted by $A_{ij}$is defined by $Aij = (-1)^{i+j} Mij$, where $Mij$ is minor of $a_{ij}$.

3) A cofactor is the number one gets by removing the column and row of a designated element in a matrix**.** The cofactor is always preceded by a positive (+) or negative (-) sign, according as the element is in a $+$ or $-$ position.

4) Minor of the element $a_{ij}$ is $M_{ij}$.
Here $a_{11} = 1$. So $M_{11}$ = Minor of $a_{11} = 3$
$M_{12}$ = Minor of the element $a_{12} = 4$
$M_{21}$ = Minor of the element $a_{21} = -2$
$M_{22}$ = Minor of the element $a_{22} = 1$

Next go over to cofactor of $a_{ij}$ which is written as$A_{ij}$. So,
$A_{11} = (-1)^{1+1}\ M_{11} = (-1)^2 (3) = 3$
$A_{12} = (-1)^{1+2}\ M_{12} = (-1)^3 (4) = -4$
$A_{21} = (-1)^{2+1}\ M_{21} = (-1)^3 (-2) = 2$
$A_{22} = (-1)^{2+2}\ M_{22} = (-1)^4 (1) = 1.$

**Check Your Progress 5**

1) Methodof solving a system of $n$ linear equations in $n$ variables using determinants.

2) When determinant D = 0 and at least one of $D_1, D_2, D_3, D_4 \ldots\ldots D_n$ is not equal to zero.

3) Take $2x + y = 5$ and $-x + y = 2$. When working together, we have a system.

4) A system of linear equations is called inconsistent if it has no solutions.

## 2.10 EXERCISES WITH ANSWER/HINTS

Solve the following system of equations using Cramer's rule.

| | | | |
|---|---|---|---|
| i) | $5x - 7y + z = 11$; | $6x - 8y - z = 15$; | $3x + 2y - 6z = 7$; |
| ii) | $x + 2y - 2z = -7$; | $2x - y + z = 6$; | $x - y - 3z = -3$; |
| iii) | $6x + y - 3z = 5$; | $x + 3y - 2z = 5$; | $2x + y + 4z = 8$ |
| iv) | $2x - 3y - 4z = 29$; | $-2x + 5y - z = -15$; | $3x - y + 5z = -11$ |
| v) | $2x - y + z = 4$; | $x + 3y + 2z = 12$; | $3x + 2y + 3z = 10$; |

**Answers** (i) x = 1; y = -1; z = -1; (ii) x = 1; y = -2; z = 2; (iii) x = 1; y = 2; z = 1; (iv) x = 2; y = -3; z = -4; (v) No solution

# UNIT 3 INVERSE OF MATRICES

**Structure**



## 3.0 OBJECTIVES

After going through this unit, you will be able to understand:

1) Concept of the inverse of a matrix
2) Finding Inverse of a matrix using adjoint
3) Elementary operations
4) Finding Inverse of a matrix using elementary operations
5) Rank of a matrix

## 3.1 INTRODUCTION

In the first unit of this block under the title, 'Introduction to Matrices', we have seen matrix operations of addition, subtraction and multiplication. However, as it may be recalled, there was no discussion on division of a matrix. It is because of the underlying reason that a matrix cannot be divided. While we cannot do that, there is a related concept to work with for that purpose. It is called "inversion" of a matrix.

To get an intuitive idea of inverse, it is useful to recall that a simple equation like $4x = 8$ is solved if divided both the sides by 4. The result of such a move is the solution of x=2. Just note that instead of dividing by 4, we could have resorted to multiplication of ¼ in both the sides of the equation to solve the problem and the answer could have been 2. What is done is to take the help of reciprocal of 4/1 in the multiplication. Thus, reciprocal ¼ is the inversion of 4/1.We get 1 on multiplying 4/1 with its reciprocal 1/4. Matrix inversion can be thought of similar to such an operation with reciprocal numbers.

## 3.2 INVERSE MATRIX

Let $A$ be a matrix. Then inverse of $A$ is written as$A^{-1}$. Since we cannot divide a matrix, we do not write in reciprocal form of $1/A$. Our search for similarity of inverse matrix with scalar number system will show that just as multiplication of 4 and its reciprocal, ¼, yields 1, our multiplication of a matrix by its inverse will give the **Identity Matrix**. That is, $A \times A^{-1} =$ **I**which would be a square matrix with 1 in the diagonal.

### 3.2.1 Definition of Inverse Matrix

The **inverse** of a square $n \times n$matrix $A$, is another $n \times n$matrix denoted by $A^{-1}$ such that

$$AA^{-1} = A^{-1}A = \boldsymbol{I}$$

where $\boldsymbol{I}$is$n \times n$identity matrix. That is, multiplying a matrix by its inverse producesan identity matrix. Note, however, that not all square matrices have an inverse matrix. If the determinant of thematrix is zero, then it will not have an inverse. Since matrix whose determinant is zero is known as **singular**, we stipulate the condition that only a non-singular matrix will have an inverse. Such a matrix is also called invertible matrix.

### 3.2.2 Properties of Inverse Matrix

1) *A square matrix is invertible if and only if it is non-singular.*
2) *The inverse of the inverse of a matrix is matrix itself i.e.,* $(A^{-1})^{-1} = A$
3) *The inverse of the transpose of a matrix is same as the transpose of its inverse i.e.,* $(A')^{-1} = (A^{-1})'$
4) *If A and B are two invertible matrices of the same order then AB is also invertible which holds true* $(AB)^{-1} = B^{-1}.A^{-1}$.

### 3.2.3 Inverting a 2 × 2 Matrix

With a view to familiarize ourselves with the idea of computation of inverse, we discuss below, as an example, a $2 \times 2$ matrix.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

where *a*, *b*, *c* and *d* are numbers. Its inverse is written as

$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where$(ad - bc)$is the determinant value. It needs to be not equal to zero for moving forward with process of inversion. Thus,wecould do the computation only when we have a square matrix and a non-zero determinant.

**Example 1**: Invert the matrix $A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$

**Solution:**

$A^{-1} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{4\times3-5\times2}\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1.5 & -2.5 \\ -1 & 2 \end{bmatrix}$.

Similar to solving single equation like 4x=8, we can solve for unknown value of a matrix X, using the inversion technique. For example, if we are given that $XA = B$where matrices $A$ and$B$ are known, we can use inversion to find the matrix $X$.To do that write

$XAA^{-1} = BA^{-1}$. Since $AA^{-1} = I$, we get

$XI = BA^{-1}$ or, $X = BA^{-1}$ (due to multiplication $X$ with identity matrix). Remember that all the conditions of inversion like $A$ is a square matrix and invertible and order of $B$ is amenable to multiplication rule are satisfied to get the required solution.

## 3.2.4 Computing Inverse of Bigger Matrices

We have seen above the inverse of a 2x2matrix. Compared to larger matrices such as a 3x3, 4x4, …, it is easy to compute the inverse. Forlarger matrices, however, we will have to use two main methods while working out the inverse. Such methods are

i) inverse of a matrix using Minors, Cofactors and Adjugate (or, Adjoint) and

ii) inverse of a matrix using Elementary Row Operations.

We have not included the use a computer, such as the Matrix Calculator, in the present unit which can be used for inverting a matrix.

Note: Inversion of a diagonal matrix is obtained by replacing each element in the **diagonal** with its reciprocal. For example, take a diagonal matrix $C = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$. Then $C^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$. Verify that $CC^{-1} = I$ where $I$ is identity matrix.

---

**Check Your Progress 1**

1) How would you explain the idea of inversion of a matrix?

2) What is the result of multiplying a square matrix with its inverse?

3) You can apply the operation of inverse to which type of matrix?

4) Write the definition of inverse of a matrix.

5) What is a singular matrix?

6) How would you identify an invertible matrix?

7) How do you verify that you have correctly calculated an inverse matrix?

8) Can you perform division operation on matrix?

## 3.3 MATRIX INVERSE METHOD: DETERMINANT AND ADJOINT ROUTE

In the following, we discuss one of the two methods mentioned above

on inverting a matrix. As a background for that the concept of adjoint of a matrix which is transpose of a cofactor in a square matrix is covered first. The inverse which is obtained by dividing the adjoint with determinant is given as a second step under a separate subsection.

### 3.3.1 Adjoint of a Matrix

To invert a matrix using first of the two methods mentioned above, it is needed to understandthe concept adjoint of a matrix.

If$A$ = $[a_{ij}]$isa square matrix of order n, then adjoint of $A$ is defined to be transpose of matrix $[A_{ij}]$ of order n, where $A_{ij}$ is cofactor of $a_{ij}$in |A|. In other words, let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & a_{3n} \\ . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & a_{nn} \end{bmatrix}$$

$$Adj(A) = \text{ transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} & . & A_{1n} \\ A_{21} & A_{22} & A_{23} & . & A_{2n} \\ A_{31} & A_{32} & A_{33} & . & A_{3n} \\ . & . & . & . & . \\ A_{n1} & A_{n2} & A_{n3} & . & A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{21} & A_{31} & . & A_{n1} \\ A_{12} & A_{22} & A_{32} & . & A_{n2} \\ A_{13} & A_{32} & A_{33} & . & A_{n3} \\ . & . & . & . & . \\ A_{1n} & A_{2n} & A_{3n} & . & A_{nn} \end{bmatrix}$$

Here $A_{11}$ is the cofactor of $a_{11}$ in |A|

$A_{12}$ is the cofactor of $a_{12}$ in |A| and so on.

**Remarks**

1) ***If A be a square matrix of order n, then*** $A.Adj(A) = Adj(A).A = |A|.I_n$***, where*** $I_n$ ***is an identity matrix of order n.***

2) $adj(AB) = adj(A).adj(B)$

3) $adj(A') = (adj(A))'$

**Example 2**:Find the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{bmatrix}$$

and verify the theorem $A.\,adj(A) = adj(A).\,A = |A|.\,I_n$

**Solution:**

$Adj(A)$ = transpose of $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Given that $A = \begin{bmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{bmatrix}$, let us find out the cofactors of all the elements of the matrix.

$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix} = (-1)^{1+1}(2x\text{-}3 - 0x1) = -6 - 0 = -6.$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 1 \\ -2 & -3 \end{vmatrix} = (-1)^{1+2}(3x\text{-}3 - 1x\text{-}2) = -(-9 + 2) = 7.$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ -2 & 0 \end{vmatrix} = (-1)^{1+3}(3x0 - 2x\text{-}2) = (0 + 4) = 4.$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 6 & 5 \\ 0 & -3 \end{vmatrix} = (-1)^{2+1}(6x\text{-}3 - 5x0) = -(-18 - 0) = 18.$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 5 \\ -2 & -3 \end{vmatrix} = (-1)^{2+2}(1x\text{-}3 - 5x\text{-}2) = -3 + 10 = 7.$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 6 \\ -2 & 0 \end{vmatrix} = (-1)^{2+3}(1x0 - 6x\text{-}2) = -(0 + 12) = -12.$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix} = (-1)^{3+1}(6x1 - 2x5) = 6 - 10 = -4.$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix} = (-1)^{3+2}(1x1 - 5x3) = -(1 - 15) = 14.$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} = (-1)^{3+3}(1x2 - 6x3) = 2 - 18 = -16.$

So $adj(A)$ = transpose of $\begin{bmatrix} -6 & 7 & 4 \\ 18 & 7 & -12 \\ -4 & 14 & -16 \end{bmatrix}$

$$adj(A) = \begin{bmatrix} -6 & 18 & -4 \\ 7 & 7 & 14 \\ 4 & -12 & -16 \end{bmatrix}$$

Also $|A| = 1(-6 - 0) - 6(-9 + 2) + 5(0 + 4)$

$= -6 + 42 + 20 = 56$

Now,

$$A.\,adj(A) = \begin{bmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{bmatrix}\begin{bmatrix} -6 & 18 & -4 \\ 7 & 7 & 14 \\ 4 & -12 & -16 \end{bmatrix}$$

$$= \begin{bmatrix} -6+42+20 & 18+42-60 & -4+84-80 \\ -18+14+4 & 54+14-12 & -12+28-16 \\ 12+0-12 & -36+0+36 & 8+0+48 \end{bmatrix}$$

$$= \begin{bmatrix} 56 & 0 & 0 \\ 0 & 56 & 0 \\ 0 & 0 & 56 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(I)}$$

Similarly,

$$adj(A).A = \begin{bmatrix} -6 & 18 & -4 \\ 7 & 7 & 14 \\ 4 & -12 & -16 \end{bmatrix} \begin{bmatrix} 1 & 6 & 5 \\ 3 & 2 & 1 \\ -2 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -6+54+8 & -36+36-0 & -30+18+12 \\ 7+21-28 & 42+14+0 & -12+28-16 \\ 12+0-12 & -36+0+36 & 8+0+48 \end{bmatrix}$$

$$= \begin{bmatrix} 56 & 0 & 0 \\ 0 & 56 & 0 \\ 0 & 0 & 56 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(II)}$$

$$\begin{bmatrix} 56 & 0 & 0 \\ 0 & 56 & 0 \\ 0 & 0 & 56 \end{bmatrix} = 56 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|.I_3 \ldots\ldots\ldots\ldots \quad \text{(III)}$$

From (I), (II) and (III),

$Adj(A).A = A.adj(A) = |A|.I_3$

### 3.3.2 Computation of Inverse using Adjoint of a Matrix

Let $A$ be a square matrix of order n. The square matrix $B$ of order $n$ is called the inverse matrix of $A$ if

$$A \times B = B \times A = I_n$$

The inverse of $A$ is denoted by $A^{-1}$, i.e.,$B = A^{-1}$ so that

$$A \times A^{-1} = A^{-1} \times A = I_n$$

In the preceding subsection we have seen that $A.adj(A) = |A|.I_n$,

which gives $A.\frac{adj(A)}{|A|.} = I_n$.Thus $A$ is invertibleand $A^{-1} = \frac{adj(A)}{|A|.}$ where $|A| \neq 0$

So,only **non-singular matrices are invertible**.

**Example3**: Find the inverse of matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}$ and hence show that $A \times A^{-1} = A^{-1} \times A = I_3$

**Solution:**

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}$$

|A|= 1(2 – 4) +2(-6 -8) +3(3 +2) = -2 – 28 +15 = -15

The cofactors of elements of matrix $A$ are

$A_{11}$ = (2 – 4) =-2

$A_{12}$ = -(-6 – 8) =14

$A_{13}$ = (3+2) =5

$A_{21}$ = -(4 – 3) = -1

$A_{22}$= (-2 – 6) =-8

$A_{23}$ = -(1 + 4) =-5

$A_{31}$ = (-8 + 3) =-5

$A_{32}$ = -(4 – 9) = 5

$A_{33}$ = (-1 + 6) = 5

$Adj(A)$ = transpose of $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

= transpose of $\begin{bmatrix} -2 & 14 & 5 \\ -1 & -8 & -5 \\ -5 & 5 & 5 \end{bmatrix}$

$=\begin{bmatrix} -2 & -1 & -5 \\ 14 & -8 & 5 \\ 5 & -5 & 5 \end{bmatrix}$

Since $A^{-1} = \frac{\boldsymbol{adj(A)}}{|A|.}$,

$A^{-1}= -\frac{\mathbf{1}}{\mathbf{15}}\begin{bmatrix} -2 & -1 & -5 \\ 14 & -8 & 5 \\ 5 & -5 & 5 \end{bmatrix}$

So, $A \times A^{-1} = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} . (-\frac{\mathbf{1}}{\mathbf{15}}) \begin{bmatrix} -2 & -1 & -5 \\ 14 & -8 & 5 \\ 5 & -5 & 5 \end{bmatrix}$

$=-\frac{\mathbf{1}}{\mathbf{15}}\begin{bmatrix} -2-28+15 & -1+16-15 & -5-10+15 \\ -6-14+20 & -3+8-20 & -15-5+20 \\ -4+14-10 & -2-8+10 & -10+5-10 \end{bmatrix}$

$=-\frac{\mathbf{1}}{\mathbf{15}}\begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

Similarly, it also can be proved that $A^{-1} \times A = I_3$

**Example 4**: Find the matrix $X$ such that $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} X = \begin{bmatrix} 10 & 4 \\ -5 & 9 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 10 & 4 \\ -5 & 9 \end{bmatrix}$.

So, the above relation can be rewritten as $AX = B$

Consequently, $X = A^{-1}B$

Here $|A|$= (2x4 – (-1x3) = 8+3 = 11.

$Adj(A)$ = transpose of $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

= transpose of $\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$

= $\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$

$A^{-1} = \frac{\boldsymbol{adj(A)}}{|A|.}$, so, $A^{-1} = \frac{\mathbf{1}}{\mathbf{11}}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$

$X = A^{-1}B$

Hence,

$= \frac{\mathbf{1}}{\mathbf{11}}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 10 & 4 \\ -5 & 9 \end{bmatrix}$

$= \frac{\mathbf{1}}{\mathbf{11}}\begin{bmatrix} 40+15 & 16-27 \\ 10-10 & 4+18 \end{bmatrix}$

$= \frac{\mathbf{1}}{\mathbf{11}}\begin{bmatrix} 55 & -11 \\ 0 & 22 \end{bmatrix}$

$=\begin{bmatrix} 5 & -1 \\ 0 & 2 \end{bmatrix}$.

**Example 5**: If $A$is a square matrix of order 3 and $detA = 5$ then what is $det\ [(2A)^{-1}]$?

Solution: If $A$ is of order 3 then, $A^{-1}$ is also of order 3. Again, $det\ (cA) = c^n$(det A) where $n$ is the order of the matrix. A determinant being a scalar number, $det\ A = \frac{1}{\det A^{-1}}$. Thus, $det\ [(2A)^{-1}] = 2^3 det[A]$ and $det[A^{-1}] = \frac{1}{8.5} = \frac{1}{40}$.

**Check Your Progress 2**

1) If $A$is a square matrix of order $n$, then what is $A$.adj($A$)?

2) You have two adjoint matrices $adj\ (A)$ and $adj\ (B)$. State how would you find out $adj(AB)$.

3) How can you show that $adj(A') = (adj(A))'$?

## 3.4 MATRIX INVERSE METHOD: ELEMENTARY OPERATIONS ROUTE

Elementary operations play a crucial role in finding the inverse. Take a matrix$A$ such that its inverse ($A^{-1}$) exists. To find the inverse of a matrix using elementary row or column operations, write $A = IA$and apply a sequence of row or column operations on $A = IA$ till we get, $I = BA$. The matrix $B$ will be the inverse matrix of $A$.

### 3.4.1 Elementary Matrix Operations

An elementary matrix is a square matrix that has been obtained by performing an elementary row or column operation on an identity matrix. In other words, when elementary operations are carried out on identity matrices, they give rise to elementary matrices.

**Example 6**: If we take the identity matrix and multiply its first row by 3, we obtain the elementary matrix.

$$E = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the following two matrices are also considered elementary matrices:

$(i) E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $(iii) E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

To see the method of obtaining these matrices you need to

i) interchange two rows of matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and get $E_1$ and

ii) take the second row of the identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and add 2 times of first row to it for getting $E_2$.

There are three kinds of elementary matrix operations.

1) Interchange two rows (or columns).
2) Multiply each element in a row (or column) by a non-zero number.
3) Multiply a row (or column) by a non-zero number and add the result to another row (or column).

When these operations are performed on rows, they are called **elementary row operations**; and when they are performed on columns, they are called **elementary column operations**.

The transformations on a given matrix into basic operations are written as,

1) Interchange of any two rows (columns)

   $R_i \leftrightarrow R_j$ / $C_i \leftrightarrow C_j$

2) Multiplication of any row (column) by a non-zero scalar

   $R_i \to kR_i$ / $C_i \to kC_i$

3) Addition to one row (column), of another row (column) multiplied by any non-zero scalar

   $R_i \to R_i + kR_j$ / $C_i \to C_i + kC_j$

If square matrix $B$ is obtained from matrix $A$ by applying one or more elementary operations, then matrices $A$ and $B$ are called equivalent matrices and denoted as $A \sim B$.

### 3.4.2 Computation of Inverse using Elementary Row Operations

Let $A$ be a non-singular square matrix. So, we have

$$A = I \times A$$

If elementary row operations are applied on $A$ from left hand side and on $I$ from right hand side and we obtained a new form like

$I = B \times A,$

then matrix $B$ is the inverse of matrix $A$.

**Example 7:** Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ using elementary operations.

Solution:

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

So, $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Apply $R_2 \rightarrow R_2 + R_1$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Apply $R_2 \rightarrow \frac{1}{5} R_2$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -\frac{2}{5} \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Apply $R_1 \rightarrow R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -\frac{6}{5} \\ 0 & 1 & -\frac{2}{5} \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_3 \rightarrow R_3 + R_1$

$$\begin{bmatrix} 1 & 0 & -\frac{6}{5} \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{2}{5} & 1 \end{bmatrix} A$$

Apply $R_3 \rightarrow 5R_3$

$$\begin{bmatrix} 1 & 0 & -\frac{6}{5} \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ 2 & 2 & 5 \end{bmatrix} A$$

Apply $R_1 \rightarrow R_1 + \frac{6}{5} R_3$ and

$R_2 \rightarrow R_2 + \frac{2}{5} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A$$

So, $A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

**Check Your Progress 3**

1) What is an elementary matrix?

2) When would you get elementary matrices?

3) List the three kinds of elementary matrix operations

4) James obtained a square matrix from another matrix by applying elementary operations. Then he said of deriving a rectangular matrix due to operations. Would you say James is correct? Give reasons to support your answer.

5) State the meaning of elementary operators.

## 3.5 INVERSE AND RANK OF A MATRIX

There are two ways to determine whether the inverse of a square matrix exists, viz.,

**i) Compute its determinant** and find that it is not equal to zero, and

**ii) Determine its rank** and see if it is invertible.

We have already seen the first of the two ways above. Let us look into the other way, i.e., rank of a matrix now. Note that the rank of a matrix is a unique number associated with a square matrix. If the rank of an $n$ x $n$ matrix is less than $n$, the matrix does not have an inverse.

### 3.5.1 Rank of a Matrix

The maximum number of linearly independent rows in a matrix is called the **row rank.** Similarly, the maximum number of linearly independent columns in a matrix is called the **column rank.**

If $A$ is an $m \times n$ matrix, that is, if $A$ has $m$ rows and $n$ columns, then its

Row rank of $A \leq m$ and column rank of $A \leq n$ ----------------(1)

It is not so obvious, however, that for any matrix $A$,

the row rank of $A$ = the column rank of $A$ ---------------------(2)

Because of this, there is no point in distinguishing between row rank and column rank.

Therefore, if $A$ is order $m$ x $n$, it follows from the inequalities in (1) that

$rank(A_{m\times n}) \leq \min(m, n)$ where $min(m, n)$ denotes the smaller of the two numbers $m$ and $n$ (or their common value if $m = n$). For example, the rank of a 3 x 5 matrix can be no more than 3, and the rank of a 4 x 2 matrix can be no more than 2. Note that the rank of matrix A is denoted by ρ(A).

**Remarks**

- A zero matrix or Null matrix is said to have rank zero.
- The elementary transformations do not alter the rank of a matrix.

As linear independence is associated with the identification of the rank of a matrix, it seems pertinent to understand the concept clearly.

## 3.5.2 Linear Independence

One **vector**(i.e., variable representing both a magnitude and a direction) is dependent on other vectors if it is a **linear combination** of the other vectors. That is, if we take scalar multiples of some vectors and add these up to get a new vector, then it is said to be a **linear combination** of the other vectors.

**Example 8**: Let $\boldsymbol{x} = 2\boldsymbol{y} + 3\boldsymbol{z}$ be given with

$\boldsymbol{x} = \begin{bmatrix} 37 \\ 26 \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ and $\boldsymbol{z} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$.

Operate on y and z to get

$$2\begin{bmatrix} 5 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 2*5+3*9 \\ 2*7+3*4 \end{bmatrix}.$$

It may be noted that $2\boldsymbol{y}$ **and** $3\boldsymbol{z}$ are scalar multiples. Therefore, $\boldsymbol{x}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$.

Thus, a set of vectors is **linearly independent** if no vector in the set under consideration is

i) a scalar multiple of another vector in the set,

or,

ii) a linear combination of other vectors in the set;

Conversely, a set of vectors is linearly dependent if any vector in the set is

i) a scalar multiple of another vector in the set, or,

ii) a linear combination of other vectors in the set.

Consider the row vectors

$\boldsymbol{a} = [1 \quad 2 \quad 3]$, $\quad \boldsymbol{b} = [5 \quad 6 \quad 7]$, $\quad \boldsymbol{c} = [6 \quad 8 \quad 10]$,

$\boldsymbol{x} = [2 \quad 4 \quad 6]$, $\quad \boldsymbol{y} = [0 \quad 0 \quad 1]$, $\quad \boldsymbol{z} = [0 \quad 1 \quad 0]$

We can make the following observations:

| | | |
|---|---|---|
| i) | $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent | Neither is a scalar multiple of the other |
| ii) | $\boldsymbol{a}$ and $\boldsymbol{x}$ are linearly dependent | 2**a**=**x** |
| iii. | $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ arelinearly dependent | **c** is a linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$ as $\boldsymbol{a}+\boldsymbol{b}=\mathbf{c}$ |
| iv. | $\boldsymbol{x}$ ,$\boldsymbol{y}$and $\boldsymbol{z}$ are independent | Neither is a scalar multiple nor a linear combination of other |

### 3.5.3 Invertibility and Rank of a Matrix

Once we determine the linearly independent rows or columns, the rank of a matrix is known. Since one of our concerns in this present unit is to draw inference on invertibility of a matrix on the basis of its rank, we deal with linearly independent rows or columns of a square matrix and ascertain its rank. Such a result will indicate the invertibility of the matrix under consideration.

For example, take the matrix

$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

It may be seen that neither the columns nor the rows of this 2x2 matrix could be amenable to conditions linear dependency. So, the rank of the matrix is equal to 2 and is full ranked. On the basis of this result, we can say that it is an invertible matrix.

To check that the matrix indeed has an inverse apply the elementary row operation.

To use elementary row operations, we may write A = IA, i.e.,

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} A$$

Apply $R_2 \to R_2 - 2R_1$ such that

$$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A.$$

Again, apply $R_2 \to -\frac{1}{5} R_2$ to get

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} A$$

Further, taking

$R_1 \to R_1 - 2R_2$ we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix} A$$

giving $A^{-1} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$.

Also we can assess that less than full rank square matrix is not invertible.

**Example 9**: Let $A$ is the 4 x 4 matrix of

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{bmatrix}$$ A is the 4x4 matrix

Its four row vectors are,

$$r_1 = (1, -2, 0, 4)$$
$$r_2 = (3\ \ 1, 1, 0)$$

$r_3 = (-1, -5, -1, 8)$ and

$r_4 = (3, 8, 2, -12)$.

It may be seen that

$$r_3 = 2r_1 - r_2$$

and

$$r_4 = 3r_1 + 2r_2$$

Thus, vectors $r_3$ and $r_4$ can be written as linear combinations of $r_1$ and $r_2$. That means the maximum number of independent rows is 2 and the rank of this matrix is 2. Hence the matrix is not invertible.

**Example10**: Determine the rank of the following 4 by 4 matrix and see if it has an inverse:

$$C = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Since $r_2 = r_4 = -r_1$ and $r_3 = r_1$, all rows but the first will vanish. As only 1 nonzero row remains, rank $C = 1$. So, the matrix is not invertible.

**Check Your Progress 4**

1) How would you determine the rank of a matrix on the basis of its minors?
2) Stuart has a 3x3 matrix whose rank is 2. Can he invert the matrix?
3) What is a vector?
4) State the meaning of linear combination of vectors.
5) When would you say that a set of vectors is linearly dependent?

## 3.6 SOLVING SYSTEM OF LINEAR EQUATIONS BY INVERSE MATRIX

This section will explain the method used to solve a system of linear equations by using inverse matrices. To help understand this method, we will largely rely on examples.

### 3.6.1 Systems of Equations

Suppose your family is going to National Zoological Park, New Delhi. The family members consisted of one adult and two children for which the total cost of entry fee is Rs.50. Your friend's family joins you in this trip. They have three adults and five children and their total cost of entry fee is Rs.140. Your task is to find out the cost of ticket per each child and adult.

To solve the problem,you need to have two equations to be able to find two unknowns. So, you have a **system of equations**. For that purpose, you have to set up the system of equations, using $x$ as the cost of one adult ticket, and $y$ as the cost of one child ticket.

System of equations:

$$x + 2y = 50$$

$$3x + 5y = 140$$

Although there are many ways to solve systems of linear equations like the ones at our disposal, we solve systems of equations by using inverse matrices.

Note that the present method of solution can be obtained by going through two steps, viz.,

i) find the coefficient matrix and its inverse matrix and

ii) use the inverse matrix to solve the equations.

The **coefficient matrix** is consisted of coefficients, or, numbers in front of each variable in the system of equations.

Seen in terms of the above example, in the first equation the coefficients are 1 of $x$ and 2 of $y$. In the second equation, we have 3 of $x$ and 5 of $y$. Their arrangement in matrix form will generate a 2x2 matrix. Let us call it matrix $A$. Thus, we have the coefficient matrix of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

In the coefficient matrix, the elements of first row are the coefficients of the first equation whereas the second row is composed of the coefficients from the second equation. Again, the first column of the matrix is formed from the coefficients of the $x$'s and the second column are the coefficients of the $y$'s.

Since we have two variables, we will have a 2x1 matrix called variable matrix. So, the variable matrix is

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

A third matrix, called the **constant matrix,** contains the constants of the system of equations. Let us call this matrix $B$,whichis a 2x1 matrix because it has two rows and one column.

Write it as

$$B = \begin{bmatrix} 50 \\ 140 \end{bmatrix}$$

With these arrangements, we have a system of equations inmatrix form which can be used to solve the system and find the answer.

Now that we know the required matrices, putting them together, a matrix equation is obtained. Such an equation contains a coefficient matrix, a variable matrix and a constant matrix, making use of which the system be solved. Essentially, we know that if we multiply matrix $A$ times matrix $X$, it will equal matrix $B$.

So, we write the matrix equation as $A(X) = B$

i.e., $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 50 \\ 140 \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 50 \\ 140 \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 50 \\ 140 \end{bmatrix}$ (applying adjoint method of inversion) or, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$.

Hence the solution is the cost of ticket per child is Rs. 10 while it is Rs. 30 per adult.

Keeping the above discussion in mind, we generalise the method to solve the system of linear equations for $n$ variables in the following.

Let the system of $n$ linear equations with $n$ variable $x_1, x_2, x_3, x_4, \ldots\ldots x_n$ be

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2$

.

.

.

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots\ldots\ldots + a_{nn}x_n = b_n$

The above system can be written in Matrix form

$$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & a_{3n} \\ . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ b_n \end{bmatrix}$

From $AX = B$

$$X = A^{-1}B$$

So, the multiplication of matrix $A^{-1}$and $B$ gives the solution of the system.

$A^{-1}$ can be obtained by any method (either adjoint or elementary row operations method).

**Example 11:**

Solve the following equations

x + 2y - 2z = -7

2x - y + z = 6

x - y - 3z = -3

**Solution:** Here $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & 1 \\ 1 & -1 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} -7 \\ 6 \\ -3 \end{bmatrix}$

Now, we find out $A^{-1}$by taking into account

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & 1 \\ 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

From $R_2 \rightarrow R_2 - 2R_1$ and

$R_3 \rightarrow R_3 - R_1$,

$$\begin{bmatrix}1 & 2 & -2\\0 & -5 & 5\\0 & -3 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\-1 & 0 & 1\end{bmatrix} A$$

From $R_2 \rightarrow \frac{-1}{5} R_2$

$$\begin{bmatrix}1 & 2 & -2\\0 & 1 & -1\\0 & -3 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\\frac{2}{5} & -\frac{1}{5} & 0\\-1 & 0 & 1\end{bmatrix} A$$

From $R_1 \rightarrow R_1 - 2R_2$ and

$R_3 \rightarrow R_3 + 3R_2$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & -1\\0 & 0 & -4\end{bmatrix} = \begin{bmatrix}\frac{1}{5} & \frac{2}{5} & 0\\\frac{2}{5} & -\frac{1}{5} & 0\\\frac{1}{5} & -\frac{3}{5} & 1\end{bmatrix} A$$

From $R_3 \rightarrow \frac{-1}{4} R_3$ ,

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{5} & \frac{2}{5} & 0\\\frac{2}{5} & -\frac{1}{5} & 0\\-\frac{1}{20} & \frac{3}{20} & -\frac{1}{4}\end{bmatrix} A$$

From $R_2 \rightarrow R_2 + R_3$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{5} & \frac{2}{5} & 0\\\frac{7}{20} & -\frac{1}{20} & -\frac{1}{4}\\-\frac{1}{20} & \frac{3}{20} & -\frac{1}{4}\end{bmatrix} A$$

So, $A^{-1} = \begin{bmatrix}\frac{1}{5} & \frac{2}{5} & 0\\\frac{7}{20} & -\frac{1}{20} & -\frac{1}{4}\\-\frac{1}{20} & \frac{3}{20} & -\frac{1}{4}\end{bmatrix}$

Taking $X = A^{-1}B$, we get

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\frac{1}{5} & \frac{2}{5} & 0\\\frac{7}{20} & -\frac{1}{20} & -\frac{1}{4}\\-\frac{1}{20} & \frac{3}{20} & -\frac{1}{4}\end{bmatrix}\begin{bmatrix}-7\\6\\-3\end{bmatrix}$$

$$= \begin{bmatrix}\frac{-7}{5} + \frac{12}{5} + 0\\-\frac{49}{20} - \frac{6}{20} + \frac{3}{4}\\\frac{7}{20} + \frac{18}{20} + \frac{3}{4}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{5}{5}\\-\frac{40}{20}\\\frac{40}{20}\end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

So, x=1; y=-2; and z=2.

**Check Your Progress 5**

1) What do you mean by system of equations?

2) What is a coefficient matrix?

3) State the meaning of constant matrix.

4) How do you solve system of equations?

## 3.7 LET US SUM UP

In this unit we have discussed inverse of a matrix. It is said that the intuitive idea behind the concept is related to that ofdivision operation in scalar number system, where division of a number $a$ yields the same result as multiplication of $1/a$. A matrix, which does not have the operation of division, can be thought of such a multiplication of reciprocals in number system and written for a matrix $A$ as$A^{-1}$ instead of $\frac{1}{A}$.

We have seen that inverse of a matrix A, written as $A^{-1}$, is derived from another matrix $A$ such that $AA^{-1} = I$, where $I$ is the identity matrix. The inversion however, is confined to square matrices. As some square matrices do not have non-zero determinants, only a non-singular matrix will have an inverse and such a matrix is called invertible matrix.

Two methods of matrix inverse have been presented. They are inverse of matrix using (i). Minors, Cofactors and Adjoint and (ii)Elementary Row Operations. While the first of these two requires computation of an adjoint matrix which is transpose of a cofactor matrix, the second takes the help of transformation to a square matrix that has been obtained by performing an elementary row or column operation on an identity matrix.

We have been exposed to find rank of a matrix. Using the rank, we attempt to determine inverse of a square matrix. In the process,it is explained that the rank of matrix which refers to a number according the maximum number of linearly independent rows/columns in a matrix. The rank of a square matrix, if less than full, indicates that the matrix is not invertible.

In the last part of the unit, we learnt to find the solution of a system of equations using an inverse matrix. In the process, we discussed the conversion of a system of linear equations into matrix format and worked out the values of unknown variables.

## 3.8 KEY WORDS

**Adjoint (*Adjugate*) *Matrix*:**A square *matrix* that is the transpose of the cofactor ***matrix*** *of the given matrix*.

**Coefficient Matrix**:Coefficients of the variables in a set of linear equations.

**Constant Matrix**:Constants of the system of equations.

**Elementary Matrix**: A square matrix that has been obtained by performing an elementary row or column operation on an identity matrix.

**Elementary Operators**: Each type of elementary operation performed by matrix multiplication, using square matrices.

**Inverse Matrix**: a square matrix $A^{-1}$ derived from another square matrix $A$ such that $AA^{-1} = I$, where I is the identity matrix.

**Linear Combination**: Scalar multiples of some vectors added to get a new vector.

**Linearly Dependent**: A set of vectors such that at least one of them can be written as a linear combination of the others.

**Linearly Independent**: A set vectors in which no vector cannot be written as a linear combination of the others.

**Non-Singular Matrix**: A square matrix with its determinant non-zero.

**Rank of a Matrix**:Maximum number of linearly independent rows/columns in a matrix.

**Singular Matrix**: A square matrix whose determinant is zero.

**Systems of Equations**: Collection of equations that are dealt with together.

## 3.9 SOME USEFUL BOOKS


- Allen, R.G.D., "Mathematical Analysis for Economists", London: English Language Book Society and Macmillan, 1974.
- Archibald, G.C., Richard G.Lipsey. "An Introduction to a Mathematical Treatment of Economics", Delhi: All India Traveller Bookseller, 1984.
- Chiang, A. and Kalvin Wainwright, Fundamental Methods of Mathematical Economics (Paperback), Mac Grow Hill, 2017.
- Dowling, Edward,T. "Schaum's Outline Series: Theory and Problems of Mathematics for Economists", New York: McGraw Hill Book Company, 1986.
- K. Sydsaeter and P. Hammond, Mathematics for Economic Analysis, PearsonEducational Asia, Delhi, 2002.
- Yamane, Taro, "Mathematics for Economists: An Elementary Survey", New Delhi: Prentice Hall of India Private Limited, 1970.


## 3.10 ANSWER OR HINTS TO CHECK YOUR PROGRESS

### Check Your Progress 1

1) There is no division operation of a matrix as it is done in case of scalars. To meet the requirement of division it can be thought of multiplication of reciprocals in number system which produces the same result. So, if there is a matrix $A$, one can write $A^{-1}$ instead of $\frac{1}{A}$.

2) An identity matrix.

3) Square matrix.

4) The inverseof a square matrix $A_{n\times n}$, is another matrix denoted by $A^{-1}$ such that$AA^{-1} = A^{-1}A = \boldsymbol{I}$,where $\boldsymbol{I}$isa$n \times n$identity matrix.

5) Matrix whose determinant is zero

6) A square matrix that has a non-zero determinant (or, a non-singular matrix).

7) Multiply it by the original matrix to get the identity matrix.

8) No. There is no matrix operation equivalent of division. The closest we can get to division by a matrix is multiplying by its inverse.

**Check Your Progress 2**

1) $|A|.I_n$, where $I_n$ is an identity matrix of order $n$.

2) By multiplying $adj(A)$ with $adj(B)$.

3) Take a matrix $A$ and find adjoint of $A$ and $A$'. Use your results to establish the stated equality.

**Check Your Progress 3**

1) A matrix which differs from the identity matrix by one single elementary row operation.

2) When elementary operations are carried out on identity matrices.

3) i.Interchange two rows (or columns); ii. multiply each element in a row (or column) by a non-zero number; and iii. multiply a row (or column) by a non-zero number and add the result to another row (or column).

4) James is wrong. He would get equivalent matrices.

5) Each type of elementary operation may be performed by matrix multiplication, using square matrices called **elementary operators**.

**Check Your Progress 4**

1) Rank of a non-zero matrix is defined as r if at least one of its r-rowed minors is not equal to zero while every (r+1)-rowed minor, if any, is zero.

2) No. A square matrix not full rank is singular.

3) Variable representing both a magnitude and a direction

4) Take scalar multiples of some vectors and add these up to get a new vector.

5) If any vector in the set is

   i) a scalar multiple of another vector in the set, or,

   ii) a linear combination of other vectors in the set.

**Check Your Progress 5**

1) A collection of two or more equations with a same number of unknown as the number of equations.

2) The matrix formed by the coefficients in a linear system of equations.

3) A matrix that contains the constants of the system of equations.

4) Find values for each of the unknowns that will satisfy every equation in the system.

## 3.11 EXERCISES WITH ANSWER/HINTS

1) if $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, find adj(A).

2) if $A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & -3 \end{bmatrix}$, verify A.adj(A) = adj(A).A = |A|.I

3) $A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$, Find matrix B, if AB equals (I) $\begin{bmatrix} 22 & 6 \\ 11 & 3 \end{bmatrix}$ (II) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (III) $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

4) $B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Find X, if BX = C

5) If $A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}$, find $A^{-1}$

6) Solve the following equations using inverse of a matrix

$5x - 6y + 4z = 15$

$7x + 4y - 3z = 19$

$2x + y + 6z = 46$

7) Solve the following equations

$2x + 6y + 11 = 0$

$6x + 20y - 6z - 3 = 0$

$6y - 18z + 1 = 0$

8) What do you understand by the term rank of a matrix? Find out the rank of the following matrix:

$A = \begin{bmatrix} 7 & -1 & 0 \\ 1 & 1 & 4 \\ 13 & -3 & -4 \end{bmatrix}$

9) Find the value of x such that the rank of the following matrix is less than 3.

$$\begin{bmatrix} 3 & 5 & 0 \\ 3 & x & 2 \\ 9 & -1 & 8 \end{bmatrix}$$

10) Taking the following matrix, what would you comment on its invertibility? Give reasons in support of your answer.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

11) Given the following matrix, examine its rank and find its inverse using the method of adjoint. Comment on its invertibility.

$$\begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -3 \end{bmatrix}$$

**Answers**

1) $\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$

3) (I) $\begin{bmatrix} 32 & 9 \\ -110 & -30 \end{bmatrix}$ (II) $\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$ (III) $\begin{bmatrix} -4 & 2 \\ 14 & -8 \end{bmatrix}$

4) $\begin{bmatrix} \frac{5}{3} \\ \frac{7}{3} \end{bmatrix}$

5) If $A = \begin{bmatrix} \frac{1}{15} & \frac{1}{15} & \frac{1}{3} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

6) $x = 3; y = 4; z = 6$

7) Equations have no solution.

8) 2

9) $x = 1$

10) Since A is a square matrix of order 3 so $\rho(A) \leq 3$

Now,

$|A| = 1(32 - 30) - 2(24 - 20) + 3(18-16) = 2 - 8 + 6 = 0$

|A| is the only minor of order 3 which is zero however a minor of order 2 is not equal to zero, viz.

$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0.$

So, a square matrix with less than full rank is not invertible.

11) The matrix is linearly independent with a rank 3. Determinant is -24 and inverse of the matrix is

$$\begin{bmatrix} 0.5 & 0 & -0.17 \\ -0.5 & 0.25 & 0.33 \\ -1 & 0.5 & 0.33 \end{bmatrix}.$$

# UNIT 4 APPLICATION OF MATRICES IN BUSINESS AND ECONOMICS

**Structure**



## 4.0 OBJECTIVES

After going through this unit, you will be able to understand application of matrices to

- Matrix Representation of Data;
- Market demand and supply Equilibrium;
- National Income Model; and
- Input-Output Analysis.

## 4.1 INTRODUCTION

In the preceding Unit 3, we have learnt Solving System of Linear Equations by matrix algebra. The present unit discusses application of the tool to some of the themes in business and economics. It may be useful to note that matrix algebra is widely used many areas including those of demand-supply equilibrium, national income determination and input-output analysis. We see such applications in the following.

## 4.2 MATRIX REPRESENTATION OF DATA

Matrices are the most convenient way to compactly represent the data. To see this feature let us take the following examples:

**Example 1:** A firm has three factories (F1, F2 and F3) and four warehouses (W1, W2, W3, W4). It wants to find the minimum per unit transport cost from each factory to each warehouse. Given the cost data in matrix form in the following, find the minimum as well as maximum unit cost for each factory to transport its product to warehouses.

***Warehouses***

| | $K$ | $W1$ | $W2$ | $W3$ | $W4$ |
|---|---|---|---|---|---|
| | $F1$ | 15 | 17 | 20 | 12 |
| **Factories** | $F2$ | 21 | 10 | 17 | 19 |
| | $F3$ | 10 | 27 | 18 | 16 |

**Answer**:

i) Per unit cost is minimum from F1, F2 and F3 to W4, W2 and W1 respectively.

ii) Per unit maximum cost from F1, F2 and F3 to W3, W1 and W2 respectively.

**Example 2:** In an elocution contest, a participant can speak either of the five languages, viz., Hindi, English, Punjabi, Gujarati and Tamil. A college (say, No.1) sent 30 students of which 10 offered to speak in Hindi, 9 in English, 6 in Punjabi, 3 in Gujarati and rest in Tamil. Another college (say, No.2) sent 25 students of which 7 spoke in Hindi, 8 in English, 10 in Punjabi. Out of 22 students from third college (say, No.3), 12 offered to speak in Hindi, 5 in English and 5 in Gujarati.

Write the information given above in matrix form.

**Answer:**

| | **Hindi** | **English** | **Punjabi** | **Gujarati** | **Tamil** |
|---|---|---|---|---|---|
| **College 1** | 10 | 9 | 6 | 3 | 2 |
| **College 2** | 7 | 8 | 10 | 0 | 0 |
| **College 3** | 12 | 5 | 0 | 5 | 0 |

**Example 3:** A finance company has offices located in every division, every district and every taluka in a certain state in India. Assume that there are five divisions, 30 districts and 200 talukas in the state. Each office has 1 head clerk, 1 cashier, 1 clerk and 1 peon. A divisional office has, in addition, an office superintendent, 2 clerks, 1 typist, and 1 peon. A district office has, in addition 1 clerk, 1 peon, The basic monthly salaries ( in Rs.) are as follows:

| | |
|---|---|
| Office superintendent | 15000 |
| Head clerk | 12000 |
| Cashier | 11750 |
| Clerks and typists | 11500 |
| Peons | 7500 |

Using matrix notation, find

i) The total number of posts of each kind in all the offices taken together.

ii) The total basic monthly salary bill of each kind of office and,

iii) The total basic monthly salary bill of all the offices taken together.

**Answer:**

The number of offices can be arranged in a row matrix:

| | Division | District | Taluka |
|---|---|---|---|
| $A =$ | [ 5 | 30 | 200 ] |

Staff compostion can be arranged in 3 x 6 matrix

O H C T Cl P

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

where O = Office Superintendent

H = Head Clerk

C = Cashier

T = Typist

Cl = Clerk

P = Peon

The column matrix D will have the elements that correspond to basic monthly salary

$$D = \begin{matrix} O \\ H \\ C \\ T \\ Cl \\ P \end{matrix} \begin{bmatrix} 15000 \\ 12000 \\ 11750 \\ 11500 \\ 11500 \\ 7500 \end{bmatrix}$$

i) Total Number of posts of each kind in all the offices are the columns of the matrix $AB$

$$A.B = [5 \quad 30 \quad 200] \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} = [3 \quad 235 \quad 235 \quad 5 \quad 275 \quad 270]$$

ii) Total basic monthly salary bill of each kind of offices is the elements of matrix $BD$

$$B.D = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 15000 \\ 12000 \\ 11750 \\ 11500 \\ 11500 \\ 7500 \end{bmatrix} = \begin{bmatrix} 99750 \\ 61750 \\ 42750 \end{bmatrix}$$

iii) Total bill of all the offices is the element of the matrix

$$\begin{bmatrix} 5 & 30 & 200 \end{bmatrix} \begin{bmatrix} 99750 \\ 61750 \\ 42750 \end{bmatrix} = [10901250]$$

**Check Your Progress 1**

1) Why would you prefer to store data of a business operation in matrix form?

2) You use two modes of travel, bus and train, while commuting between home and your study centre. The cost of travel for the last two days is recorded as follows:

   Day one: Rs. 20 (in bus)+Rs. 30(in train)=Rs.50 (total)

   Day two: Re.0 (in bus)+ Rs. 60 (in train)=Rs. 60 (total)

   Write the above data in matrix format.

3) The total cost of manufacturing three types of motor car is given by the following table:

| Type of motor Car | Labour (hrs) | Materials (units) | Subcontracted Work (units) |
|---|---|---|---|
| Car A | 40 | 100 | 50 |
| Car B | 80 | 150 | 80 |
| Car C | 100 | 250 | 100 |

   Labour cost Rs 2 per hour, units of material cost Rs 1 each and unit of sub-contracted work cost Rs 3 per unit. Find the total cost of manufacturing 3000, 2000 and 1000 vehicles of type A, type B and type C respectively using matrices.

## 4.3 MARKET DEMAND AND SUPPLY EQUILIBRIUM

It is seen in microeconomics that the demand for a commodity and its supply are expressed as functions of its own price only. The resultant equilibrium of such a relation is given by the equality between demand and supply equations. To see the process of determination of equilibrium price and quantity, let us consider a hypothetical example of car industry.

**Example** 4: If demand and supply curves for car are:

$D = 100 - 6P, S = 28 + 3P$ , where $P$ is the price of cars, what is the quantity of car bought and sold at equilibrium?

**Answer**: We know that the equilibrium quantity will be where supply equals demand. So first we'll set supply equal to demand:

100 - 6P = 28 + 3P. Re-arranging we get:

$$72 = 9P \quad \Rightarrow P = 8.$$

Since we have worked out the equilibrium price, we can solve for the equilibrium quantity by simply substituting $P = 8$ into the supply or the demand equation. That is, if $P$ is substituted into the supply equation to get:

S = 28 + 3x8 = 28 + 24 = 52.

Thus, the equilibrium price = 8, and the equilibrium quantity = 52.

However, this simple market formulation presupposes that the demand for and the supply of a commodity are not influenced by other factors such as prices of substitutes and complementary goods. In reality such types of assumptions do not hold good. To accommodate such scenarios a better option could be an inclusion of prices of other commodities also into demand and supply equations.

It may be useful to point out that the above-mentioned approach to demand-supply equilibrium can be extended to a larger number of interrelated markets of related commodities. In such a framework, the equilibrium prices and quantities of the included commodities comprising a large number of supply and demand equations can be obtained using a system of simultaneous equations. We do not undertake such an exercise at present. But to understand the underlying process, let us consider a two-commodity market model and solve for equilibrium price-quantity combinations with the help of techniques of matrix algebra.

**Example 5:** Consider a two-commodity (*x* and *y*) market model given by

**Demand Equations:**

$D_x = 11 - 2P_x + P_y$;

$D_y = 8 + P_x - P_y$.

**Supply Equations:**

$S_x = 1 + 3P_x$;

$S_y = 6 + P_y$.

where,$D_x$and $D_y$ →quantity demanded for *xandy;*

$S_x and\ S_y$ → quantity supplied of *xandy;and*

$P_x$ and $P_y$ → price of *x* and *y.*

Find the equilibrium prices and quantities.

Equilibrium condition for *x:*

$D_x = S_x = Q_x$.

Therefore, in terms of the given equations we have

$$11 - 2P_x + P_y = 1 + 3P_x$$

$$\text{or}, 3P_x + 2P_x - P_y = 11 - 1$$

or,$5P_x - P_y = 10$ --- (1)

Again, set the equilibrium condition for *y, i.e.,*

$D_y = S_y = Q_y$

or, $8 + P_x - P_y = 6 + P_y$

or, $P_x - 2P_y = 6 - 8$

or, $-P_x + 2P_y = 2$ --- (2)

Writing equations (1) and (2) together in the matrix form, we have

$$\begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

Next, solve the above equation system for the two prices by the matrix inverse method.

Hence, $\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 2 \end{bmatrix}$.

As Determinant of $\begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} = 9$ and adjoint of $\begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$.

$\begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.22 & 0.11 \\ 0.11 & 0.56 \end{bmatrix}$.

Thus, $\begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.42 \\ 2.22 \end{bmatrix}$

So, $\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} 2.42 \\ 2.22 \end{bmatrix}$,

yielding equilibrium prices, $P_x = 2.42$ and $P_y = 2.22$.

Plugging these prices into demand equations we get the equilibrium quantities of x and y.

Thus, $Q_x = 8.38$ and $Q_y = 8.20$.

Besides the matrix inversion method used above in solving for equilibrium price-quantity, we may also be used Cramer's rule to obtain the results.

**Check Your Progress 2**

1) You are asked to compute the equilibrium demand and supply price-quantity combination using matrix inversion. How would you like to formulate the demand supply equations?

2) While computing demand-supply equilibrium using matrix algebra, what other method can be used besides matrix inversion?

3) In demand-supply analysis, what condition must be met for establishing equilibrium?

## 4.4 NATIONAL INCOME MODEL

The national income model is another area of application of matrix algebra.We will consider a simple two-equation national income model for an economy. In order to keep the exposition simple, we have not included government and trade relationship with external countries in the present analysis.

Basically, our objective is geared towards the computation of equilibrium national income as well as consumption of an economy. For that purpose, we start with two equations given below:

$$Y = C + I_0 \quad (1)$$

$$C = a + bY \quad (2)$$

where, $Y$ is national income, $C$ is consumption, $I_0$ is autonomous investment, $a$ and $b$ are constants.

Out of these two equations, first one shows that the total value of national income is seen from expenditure approach. That is, the total expenditure of the economy incurred on goods and services is equal to consumption expenditure and investment expenditure. The other equation depicts a linear relationship between income and consumption. In the present context, $'a'$, the intercept term of the linear line is assumed to be greater than zero as it is generally postulated that consumption remains positive even when there is no income; the slope coefficient $'b'$ is taken to be greater than zero but less than one as general psychological tendency of the human beings is to consume less than the total income.

To compute the values of income and consumption at equilibrium we rearrange (1) and (2) as follows:

$$Y - C = I_0 \quad (3)$$

$$-bY + C = a \quad (4)$$

Writing equations (3) and (4) together in the matrix form we get

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 \\ a \end{bmatrix}$$

or, $$\begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}^{-1} \begin{bmatrix} I_0 \\ a \end{bmatrix} \quad (5)$$

Now, we can obtain equilibrium Y as well as C from (5) either by Cramer's rule or matrix inversion method.

**Example 6:**

Given the following national income model

$Y = C + I$

$C=5 + \frac{3}{4}Y$

$I= 10$

Find Y and C.

Putting the value of I from the third equation into the first equation

$Y = C + 10$

or, $Y - C = 10$ (6)

Again, the second equation $C= 5 + \frac{3}{4}Y$ can be rearranged as

$-\frac{3}{4}Y + C=5$ (7)

Equations (6) and (7) can be written in the matrix equation as

$$\begin{bmatrix} 1 & -1 \\ -\frac{3}{4} & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \qquad (8)$$

Using Creamer's rule to we get

$$Y = \frac{\begin{vmatrix} 10 & -1 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -\frac{3}{4} & 1 \end{vmatrix}} = \frac{15}{\frac{1}{4}} = 60$$

and $C = \frac{\begin{vmatrix} 1 & 10 \\ -\frac{3}{4} & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -\frac{3}{4} & 1 \end{vmatrix}} = \frac{12.5}{\frac{1}{4}} = 50$, i.e., the required equilibrium values.

**Check Your Progress 3**

1) What conditions are imposed on consumption equation $C = a + bY$ in the national income model?

2) In the national income model you have studied, which variables have been computed while determining equilibrium?

3) Take the Example 6 of Section 4.4 given above and solve for equilibrium values of income and consumption using inverse matrix method.

## 4.5 INPUT-OUTPUT ANALYSIS

Input-Output(I-O) analysis is another area where matrix algebra becomes handy in the derivation of results. We discuss the process involved in it to start with. I-O analysis is also known as the inter-industry analysis as it explains the interdependence and interrelationship among various industries. For example, in the two-industry model, coal is an input for steel industry and steel is an input for coal industry, though both are the output of respective industries.

### 4.5.1 Assumptions

The economy is divided into finite number of sectors (industries) on the basis of the following assumptions:

i) Each industry produces only one homogeneous output.

ii) Production of each sector is subject to constant returns to scale, i.e., two-fold change in every input will result in an exactly two-fold change in the output.

iii) Input requirement per unit of output in each sector remains fixed and constant. The level of output in each sector (industry) uniquely determines the quantity of each input, which is purchased. Moreover, if 5 men per Rs. 100000 of investment are required at any level of operation, it is assumed that the same ratio will be required no matter how much the size of the firm expands or contracts.

iv) The final demand for the commodities is given from outside the system. The total amount of the primary factor (e.g., labour) is also given. Presence of these two assumptions makes the system open ended and for this, the model is called '*open model*'. In contrast to this, in the '*closed model*', all the variables are determined within the system.

### 4.5.2 Input-Output Table

I-O table shows the disposition of the total products and total inputs among the different industries. Let us assume that an economy consists of 2 producing sectors only. In addition, the production of each sector is being used as the input in all the sectors and used for final consumption. Suppose,

i) $X_1$ and $X_2$ are total outputs of 2 sectors;

ii) $F_1$ and $F_2$ are the amounts of final demand, consumption for output of these sectors.

iii) $X_{ij}$ be the amount of output of the $i^{th}$ Industry used up as an intermediary input by the $j^{th}$ industry ( i,j = 1,2)

iv) L represents the given amount of primary factor (here, labor) and $L_i$ is the amount of primary factor used in the $i^{th}$ Industry.

Then the following table represents the I-O table for the simplified economy.

| Producing sector number | Total output of the sector | Input requirements of producing sectors | | Requirements for final uses |
|---|---|---|---|---|
| | | $X_1$ | $X_2$ | |
| 1<br>2 | $X_1$<br>$X_2$ | $X_{11}$<br>$X_{21}$ | $X_{12}$<br>$X_{22}$ | $F_1$<br>$F_2$ |
| Primary input (labor) | Total primary input = L | $L_1$ | $L_2$ | |

It is important to note that items in the above table are *physical units per year*. Since the entries in any row are all measured in the same physical units, there is no problem in undertaking addition across the rows. The 'total output' column gives output of each commodity and the overall input of labor. However, items in the same column are not measured in the same units, so that it would not be correct to add down the columns.

It may be seen from the table that requirements of the first industry are recorded in the third column: $X_1$ units of output of the first industry was produced with the use of $X_{11}$ units of first good, $X_{21}$ units of second good and $L_1$ units of labor. Similar meanings will follow for other column, i.e., Columns 2 and 4. The 'final demand' i.e., Column (5), shows the commodity available for consumption. It is assumed that labor is not consumed directly.

If we suppose that each unit of output of each industry has a price of Re.1 and each unit of labor receives a wage rate of Re.1, then each entry of the above table can be expressed in terms of money value (rather than being a physical unit). It is then possible to add down the columns. The sum of each column gives the total cost of the corresponding industry. Thus, revenue of industry 1 accrues from the sale of $X_1$ units (= $X_{11}$ + $X_{12}$+ $F_1$) and cost of that industry is ($X_{11}$ + $X_{21}$ + $L_1$) units. It is true for the other industry also.

Taking into account the producing sectors we can write the total output in each as

$$X_1 = X_{11} + X_{12} + F_1 \quad (1)$$

$$X_2 = X_{21} + X_{22} + F_2 \quad (2)$$

and

$$L = L_1 + L_2$$

That is, $X_i = \sum_{j=1}^{2} X_{ij} + F_i$ and $L = \sum_{i=1}^{2} X_i$

Here, $X_i$ = total output of the i[th] sector, $X_{ij}$ = output of i[th] sector used as input in j[th] sector, and $F_i$ = final demand for i[th] sector.

The above identity states that all the output of a particular sector could be utilised as an input in one of the producing sectors of the economy and/or as a final demand. Basically, therefore, I-O analysis is nothing more than finding the solution of these simultaneous equations.

## 4.5.3 Technological Coefficient Matrix

From the assumption of fixed input requirements, we see that in order to produce 1 unit of j[th] commodity, the input used of i[th] commodity must be a fixed amount, which we denote by $a_{ij}$; thus $a_{ij} = \frac{X_{ij}}{X_j}$. If $X_j$ represents the total output of the j[th] commodity (or j[th] producing sector) the input requirements of the i[th] commodity will be equal to $a_{ij}X_j$ or, $X_{ij} = a_{ij}X_j$. We call $a_{ij}$an*input coefficient.*

Let us consider $X_{ij}$ and $X_j$ in terms of value. So, $a_{ij}$ gives the value of input per unit of output. Thus, $a_{ij}$, = 0.02 indicates that to produce one-rupee worth of the *j*th commodity 20 paise worth of the *i*th commodity would be required.

It may be noted that the sum of elements in each column gives the requirement of secondary input to produce a rupee worth of output in that producing sector. Therefore, the sum of the elements in each column of the input coefficient matrix should be less than 1. This is because it does not include the cost of the primary inputs. In the two-sector example we have considered,$\sum_{i=1}^{2} a_{ij}$with $j = 1,2$.

If we take the value of output at Re. 1, it must be fully absorbed by the payment to all factors of production. Without an inclusion of the amount of payment made to primary input (labour here), the column sum must be less than Re. 1. Thus, the value of the primary input needed in producing a unit of the jth commodity should be

$$1 - \sum_{i=1}^{2} a_{ij}.$$

For The economy considered above, if the cost of primary inputs per rupee worth of output for the two producing sectors are $l_1$ and $l_2$, we write

$$l_1 = 1 - (a_{11} + a_{21})$$

and

$l_2 = 1 - (a_{12} + a_{22})$.

The total output for each producing sector can be expressed in terms of the input coefficients, by replacing $X_{ij} = a_{ij}X_j$ in equations (1) and (2). Thus, for out two producing sector economy,

$X_1 = a_{11}X_1 + a_{12}X_2 + F_1$

or, $X_1 - a_{11}X_1 - a_{12}X_2 = F_1$

or, $(1 - a_{11})X_1 - a_{12}X_2 = F_1$ (3)

Similarly,

$X_2 = a_{21}X_1 + a_{22}X_2 + F_2$

or, $-a_{21}X_1 + X_2 - a_{22}X_2 = F_2$

or, $-a_{21}X_1 + (1 - a_{22})X_2 = F_2$ (4)

Writing equations (3) and (4) in the matrix form

$$\begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad (5)$$

See that matrix $\begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix}$ can be obtained by subtracting $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix}$, an input coefficient matrix from $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,an identity matrix, i.e., $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix}$.

So, equation (5) can be written as

$(I - A)\, X = F$ (6)

where

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

and

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

Note that order of matrices A, X and F is according to number of sectors we have considered.

The matrix ($I$-$A$) is called the *technology matrix.*

If ($I$ -$A$) is non-singular (that is, $|I - A| \neq 0$, equation (6) can be solved for $X$. Thus,

$$X = (I - A)^{-1} F \quad (7)$$

If D = | I-A | = determinant of matrix [I-A] and $D \neq 0$, then the inverse matrix can be written as $[I - A]^{-1} = \begin{bmatrix} \frac{1-a_{22}}{D} & \frac{a_{12}}{D} \\ \frac{a_{21}}{D} & \frac{1-a_{11}}{D} \end{bmatrix} = \frac{1}{D}\begin{bmatrix} 1 - a_{22} & a_{12} \\ a_{21} & 1 - a_{11} \end{bmatrix}$

Therefore, $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{D}\begin{bmatrix} 1 - a_{22} & a_{12} \\ a_{21} & 1 - a_{11} \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$

or, $X_1 = \frac{(1-a_{22})F_1 + a_{12F_2}}{D}$, $X_2 = \frac{a_{21F_1} + (1-a_{11})F_2}{D}$ where, D $= \begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix} = (1-a_{11})\,(1-a_{22}) - a_{12}a_{21}$

### 4.5.4 Hawkins-Simon Conditions

If the I-O solution gives negative outputs, then it means that more than one unit of a product is used up in the production of every one unit of that product; it is not realistic and such a system is not a viable. Hawkins-Simon condition guards against such eventualities. For our basic equation $X = [I-A]^{-1}F$ not to give negative numbers as a solution, we need to have the matrix $[I-A]$ such that

i) the determinant of the matrix must always be positive, and

ii) the diagonal elements: $(1-a_{11})$, $(1-a_{22})$ should all be positive. In other words, elements, $a_{11}$, $a_{22}$ should all be less than one. That means one unit of output of any sector should use not more than one unit of its own output. These are called **Hawkins-Simon conditions**. Moreover, the first condition, $D > 0$, implies that (for 2 industry case) $\begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix} > 0$, or, $(1-a_{11})\,(1-a_{22}) - a_{12}a_{21} > 0$

This condition stipulates that the direct and indirect requirement of any commodity to produce one unit of that commodity must also be less than one.

**Example 7:** Suppose [I-A] $= \begin{bmatrix} 0.1 & -0.1 \\ -0.8 & 0.2 \end{bmatrix}$

Then from [I-A] = $\begin{bmatrix} 0.1 & -0.1 \\ -0.8 & 0.2 \end{bmatrix}$ we can get the value of the determinant [I-A] = (-) 0.06, which is less than zero. As the Hawkins-Simon conditions are not satisfied, no solution will be possible in this case.

### 4.5.5 Closed and Open Input-Output Models

In the above example, besides two industries, our model contains exogenous sector of final demand, which supplies primary input factors (labor services, which are not produced by these two industries) and consumes the output of two producing industries (not as input). Such an I-O model is known as '*open model*'. It includes exogenous sectors in terms of 'final demand bill' along with the endogenous sectors in terms of $n$ producing sectors. I-O model, which has endogenous final demand sector, is known as '*closed model*'.

**Check Your Progress 4**

1) What is an input coefficient matrix?

2) What is an open input-output model?

3) The input coefficient matrix is given as $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{pmatrix}$. Get the technology matrix.

4) Discuss the importance of Hawkins-Simon conditions in an input-output model.

## 4.6 LET US SUM UP

In this unit we discussed some of the applications of matrix algebra in business and economics. We learnt the presentation of data in matrix form taking the help of examples. Computation of demand-supply equilibrium from the simultaneous equationsusing matrix was shown for our understanding. The technique for solving national income model by matrix methods was explained.

In the last part of the unit we were exposed to input-output analysis and were shown the use of technique of matrix algebra in deriving the solution. In the process we learnt the formulation of input-out transaction matrix and technology matrix before the steps required for solving an open input-output model was demonstrated.

## 4.7 KEY WORDS

**Closed Input-Output Model:** Exogenous sector of the open **input-output model** is absorbed into the system as just another industry such that the entire output of each producing sector is absorbed by other producing sectors as secondary inputs or intermediate products. Essentially, the household sector is treated as one of the industries and no portion of the output is sold in the market as final product.

**Endogenous Variable**: Dependent ***variable*** generated within a model and, therefore, its value is changed (determined) by one of the functional relationships in that model.

**Exogenous Variable**: A variable whose value is determined from outside a given model.

**Hawkins-Simon Condition**: More than one unit of a product cannot be used up in the production of every unit of that product. Condition requires that all principal minors of the technology matrix must be positive.

**Input Coefficient Matrix**: A matrix of different secondary inputs required by producing sectors per unit of output.

**Input-Output Model:** An economic model that represents the interdependencies between different sectors (industries) of a national economy.

**Input-Output Transaction Matrix**: A matrix showing the distribution of total output of one industry to all other industries as inputs and for final demand.

**Model**: A set of equations, functional relationships and identities that seek to explain some economic phenomenon.

**Open Input-Output Model**: A model in which the producing sectors interact with household sector of an economy through their purchase of primary inputs and sales of final products.

**Primary Inputs**: Basic inputs of a production process such as labour.

**Technology Matrix**: A matrix obtained by subtracting a given input coefficient matrix from an identity matrix.

## 4.8 SOME USEFUL BOOKS


- Abraham, W.I., *National Income and Economic Accounting:* Prentice-Hall Inc., New Jersey, Chapter 7 (including the Appendix, 1969.
- Chiang, A. and Kalvin Wainwright, Fundamental Methods of Mathematical Economics (Paperback), Mac Grow Hill, 2017.
- Stafford, L.W.T., *Mathematics for Economists:* The English Language Book society and Macdonald & Evans Ltd., London, Chapter 17, 1977.
- Wisniewski, M., *Introductory Mathematical Methods in Economics:* McGraw-Hill Book Company, London, Chapter 7, 1991.


## 4.9 ANSWER OR HINTS TO CHECK YOUR PROGRESS

**Check Your Progress 1**

1) Because of its compact and convenient feature.

2)

| Day | Bus | Train | Total |
|---|---|---|---|
| 1 | 20 | 30 | 50 |
| 2 | 0 | 60 | 60 |

3) Labour hours, material used and sub-contracted work for three types of cars A, B, and C matrix:

$X = \begin{bmatrix} 40 & 100 & 50 \\ 80 & 150 & 80 \\ 100 & 250 & 100 \end{bmatrix}$, labour cost per unit, material cost and cost of sub-contracted work $Y = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, cost of each car A, B, C = $XY = \begin{bmatrix} 330 \\ 550 \\ 750 \end{bmatrix}$, total cost of manufacturing three cars A, B, C is given by the matrix

$XYZ = [3000 \quad 2000 \quad 1000] \begin{bmatrix} 330 \\ 550 \\ 750 \end{bmatrix}$ =[990000 + 1100000 + 750000] = [2840000]

**Check Your Progress 2**

1) Consider demand and supply equations which include factor(s) influencing quantity of supply and demand besides own price of the commodity.

2) Use Cramer’s rule.

3) Demand equals supply.

**Check Your Progress 3**

1) $a > 0$ and $0 < b < 1$.

2) Income and consumption.

3) Do yourself after reading the method given in Section 4.3.

**Check Your Progress 4**

1) A matrix of different secondary inputs required by producing sectors per unit of output.

2) Model contains exogenous sector of final demand, which supplies primary input factors (labor services-which are not produced by these two industries) and consumes the output of two producing industries (not as input).

3) $$A = \begin{pmatrix} (1-a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (1-a_{22}) & -a_{23} \\ -a_{31} & -a_{32} & (1-a_{33}) \end{pmatrix} = \begin{pmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{pmatrix}$$

4) More than one unit of a product cannot be used up in the production of every unit of that product.

## 4.10 EXERCISES WITH ANSWER/HINTS

1) A large energy company produces electricity, natural gas, and oil.

The production of a rupee's worth of electricity requires inputs of Rs. 0.30 from electricity, Rs. 0.10 from natural gas and Rs. 0.20 from oil. The production of a rupee's worth of natural gas requires inputs of Rs. 0.30 from electricity, Rs.0.10 from natural gas and Rs.0.20 from oil. Production of a rupee's worth of oil requires inputs of Rs. 0.10 from each sector. Find the output for each sector that is needed to satisfy a final demand of Rs. 25 crore for electricity, Rs. 15 crore for natural gas and Rs. 20 crore for oil.

2) The daily cost of operating a hospital C is a linear function of the number of in-patients I and out-patients P plus a fixed cost $a$, i.e.,

$$C = a + bP + dI$$

Given the following data of 3 days, find the values of a, b and d by setting up a system of linear equations and using the matrix inverse.

| Day | Cost (Rs) | No. of Inpatients | No.of outpatients |
|---|---|---|---|
| 1 | 6950 | 40 | 10 |
| 2 | 6725 | 35 | 9 |
| 3 | 7100 | 40 | 12 |

3) A trust fund has Rs 10000 that is to be invested into two types of bonds. The first bond pays 5% interest per year and the second bond pays 6% interest per year. Using matrix algebra, determine how to divide Rs 10000 among the two types of bonds so as to obtain an annual interest Rs 550.

4) A man invested Rs 30000 into three different investments at the annual rate of interest 2%, 3% and 4% respectively. The total annum income is Rs 1000. If the income from the first and second investment together is

Rs. 50 more than the income from third, find the amount of each investment by using matrix algebra.

5) Given the national income model

$Y = C + I + G$ ($G$: Government expenditure)

$C = 5 + \frac{3}{4}Y$

$I = 10$

$G = 10$

Using matrix inverse method, find Y and C.

**Answers**

1) Let $x_1$= the total output of the electric company ($E$)

$x_2$= the total output of the natural gas company ($G$)

$x_3$= the total output of the oil company ($O$)

**Total amount of electricity** needed is the sum of amounts of electricity needed to produce electricity, natural gas and oil (internal demand) plus the final (external) demand of electricity.

$x_1 = .30x_1 + .30x_2 + .10x_3 + 25$

**Total amount of natural gas** needed is the sum of amounts of natural gas needed to produce electricity, natural gas and oil (internal demand) plus the final (external) demand of natural gas.

$x_2 = .10x_1 + .10x_2 + .10x_3 + 15$

**Total amount of oil** needed is the sum of amounts of oil needed to produce electricity, natural gas and oil (internal demand) plus the final (external) demand of oil.

$x_3 = .20x_1 + .20x_2 + .10x_3 + 20$

Using the technology matrix $M$, the final demand matrix $D$and total demand matrix $X$, we get

$X = MX + D$, with solution$X = [I - M]^{-1}$D

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3 & .3 & .1 \\ .1 & .1 & .1 \\ .2 & .2 & .1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 25 \\ 15 \\ 20 \end{bmatrix}$$

$$I - M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .3 & .3 & .1 \\ .1 & .1 & .1 \\ .2 & .2 & .1 \end{bmatrix} = \begin{bmatrix} .7 & -.3 & -.1 \\ -.1 & .9 & -.1 \\ -.2 & -.2 & -.9 \end{bmatrix}$$

$$[I - M]^{-1} = \begin{bmatrix} 1.58 & 0.58 & 0.24 \\ 0.22 & 1.22 & 1,16 \\ 0.4 & 0.4 & 1.2 \end{bmatrix}$$

$$[I-M]^{-1}D = \begin{bmatrix} 1.58 & 0.58 & 0.24 \\ 0.22 & 1.22 & 1,16 \\ 0.4 & 0.4 & 1.2 \end{bmatrix}\begin{bmatrix} 25 \\ 15 \\ 20 \end{bmatrix} = \begin{bmatrix} 53 \\ 27 \\ 40 \end{bmatrix}$$

That is, the total output of electricity is Rs. 53 crore, the total output of natural gas is Rs. 27 crore, the total output of oil is Rs. 40 crore.

2) Substituting the tabulated values in $C = a + bP + dI$, we obtain the following set of linear equations

$$a + 10b + 40d = 6950$$

$$a + 9b + 35d = 6725$$

$$a + 12b + 40d = 7100$$

The above system of equations in the matrix form is

$$\begin{bmatrix} 1 & 10 & 40 \\ 1 & 9 & 35 \\ 1 & 12 & 40 \end{bmatrix} \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} 6950 \\ 6725 \\ 7100 \end{bmatrix} \quad \text{……………………(I)}$$

Now $A^{-1} = \frac{adj(A)}{|A|.}$ Where $|A| = \begin{vmatrix} 1 & 10 & 40 \\ 1 & 9 & 35 \\ 1 & 12 & 40 \end{vmatrix} = -10$

andadj (A) = $\begin{bmatrix} 60 & -80 & 10 \\ 5 & 0 & -5 \\ -3 & 2 & 1 \end{bmatrix}$

So, from the relation (I), we get

$$\begin{bmatrix} a \\ b \\ d \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} 60 & -80 & 10 \\ 5 & 0 & -5 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 6950 \\ 6725 \\ 7100 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -50000 \\ -750 \\ -300 \end{bmatrix} = \begin{bmatrix} 5000 \\ 75 \\ 30 \end{bmatrix}$$

Hence a = 5000, b = 75 and d = 30.

3) Suppose Rs.$a$ and (10000 – $a$) be invested in the first and second types of bonds respectively. So the value of these bonds can be written in the row matrix form

A = [$a$ (10000- $a$)]

and the amount received from two bonds as interest can be written in the form of column matrix

$$B = \begin{bmatrix} 5\% \\ 6\% \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.06 \end{bmatrix}$$

Hence the total interest = AB

$$= [\, a \;\; (10000 - a)] \begin{bmatrix} 0.05 \\ 0.06 \end{bmatrix} = [600 - 0.01a]$$

But total annual interest = Rs 550 (given)

Thus, 600 – 0.01 $a$ = 550

0.01 $a$ = 50

a = 5000

Investment in first bond = Rs. 5000

Investment in second bond = 10000 – 5000 = Rs 5000

4) Let x, y and z be the amount of three investments. The given data can be expressed in the following system of linear equations

$x + y + z = 30000$ ……………………(i)

$.02x + .03y + .04z = 1000$ ……….(ii)

$.02x + .03y - .04z = 50$ …………….(iii)

The equation (ii) and (iii) may be rewritten as

$2x + 3y + 4z = 100000$ ……………(ii)

$2x + 3y – 4z = 5000$ …(iii)

These equations are expressed in matrix form AX = B

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 3 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 30000 \\ 100000 \\ 5000 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 3 & -4 \end{vmatrix} = -8$

$\text{Adj}(A) = \begin{bmatrix} -24 & 7 & 1 \\ 16 & -6 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

Now $A^{-1} = \frac{adj(A)}{|A|.} = -\frac{1}{8}\begin{bmatrix} -24 & 7 & 1 \\ 16 & -6 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

$X = A^{-1}B$

$= -\frac{1}{8}\begin{bmatrix} -24 & 7 & 1 \\ 16 & -6 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 30000 \\ 100000 \\ 5000 \end{bmatrix}$

$= \begin{bmatrix} 1875 \\ 16250 \\ 11875 \end{bmatrix}$

Hence the investments at 2%, 3% and 4% are Rs 1875, Rs 16250, Rs 11875 respectively.

5. Y=100; C=80.

# UNIT 5 MATHEMATICAL FUNCTIONS

**Structure**

## 5.0 OBJECTIVES

After going through this unit, you will be able to understand:

- meaning of a function;
- different types of functions; and
- use of functions in Business and Economics.

## 5.1 INTRODUCTION

A function relates an input (or, argument) to an output. Notice the three main components, viz., input, relation and output present in describing a function. For example, in producing an output, you have used an input. It may so happen that you have got the output whose value is doubled the value of the input. That means the relationship between input and output is a simple function of multiplying 2, viz., $input \times 2 = output$.

We use "$f$" or any other letter like $g$ ... to name a function. To read a function, for example, we say "$f\ of\ x\ equals\ x\ squared$" and write it as$f(x) = x^2$. Here input $x$, if takes a value 3, output becomes 9 and we can write $f(3) = 9$.

The format of the function as output depends on inputs is useful represent many relations in the form of graphs. In this unit we discuss graphic forms of some mathematical function.

## 5.2 FUNCTIONS AND THEIR TYPES

On the basis of general idea of a function seen above, it can be defined formally.Some important functions find application in economics and business. We take these up in this section.

### 5.2.1 Definition of Function

A function relates each element of a set with exactly one element of another set. In defining the function as above, we need to note following:

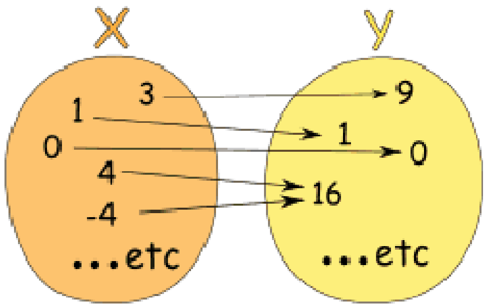
i) "each element" implies every element in a set, say X, is related to some element of another set say, Y;

ii) "exactly one" implies that a function is single valued. That is, $f(3) =$ 8 or 9 is not right.

If a relationship does not follow those two rules, then it is not a function although it is still a relationship. In that event, the elements of both the sets are simply ordered pairs. They are called **ordered** pairs because the input always comes first, and the output second.

**Example1**: Let the relationship be $x \rightarrow x^2$ as given in the figure below:

**Figure 5.1: Relation between X and Y**




**Source: Math is Fun (taken from Internet)**

In the relationship given above, set X contains the elements of $x$ and set Y has the element of $x^2$. It is a function as

- every element in X is related to Y
- no element of X has two or more relationship.

### 5.2.1.1 Domain, Codomain and Range

In the example given above

the set "X" is called the Domain,

- the set "Y" is called the Codomain, and
- the set of elements that get pointed in Y (the actual values produced by the function) is called the Range.

A function is uniquely represented by its graph which is the set of order pairs (x, f(x)). When the domain and the codomain are sets of numbers, each such pair may be considered as the Cartesian coordinates of a point in the plane. In general, these points form a curve, which is also called the graph of the function. This is a useful representation of the function, which is commonly used everywhere. For example, graphs of functions are commonly used in newspapers for representing the evolution of price indexes and stock market indexes.

Sometimes a relation or a function does not follow directly the stated pattern discussed above. The case of implicit function falls in this category. We see it before going further.

A function is **explicit** is when it shows us how to go directly from independent input $x$ to dependent output $y$.

For example, take$y = x^3 - 5$. See that if you know $x$, then you can find $y$.

That is how we write $y = f(x)$.

**Implicit** function is one where the independent input and its dependent output are not given directly as is done with $y = f(x)$.

**Example 2**: $x^2 - 5xy + y^3 = 0$

As you may notein case of the above equation, it is difficult to go directly from $x$ to $y$.

Out of many types, we will discuss the following functions which are commonly used.

- *Functions*
  - *Algebraic*
    - *Polynomial*
      - *Constant*
      - *Linear*
      - *Quadratic*
    - *Rational*
    - *Piecewise*
    - *Power*
  - *Transcendental*
    - *Exponential*
    - *Logarithmic*

### 5.2.2 Function Types: Algebraic

The following discussion introduces the important functions often used to solve problems. As summarised above, we have two broad groups of functions, viz., **algebraic and transcendental.** Let us attempt to understand these accordingly.

There is a group of functions called **algebraic functions** where a function $f(x)$ satisfies $p(x, f(x)) = 0$ as a polynomial in $x$ and $y$ with integer coefficients. These functions can be constructed using only a finite number of elementary operations, such as plus, minus, multiplication and division.

#### 5.2.2.1 Polynomial Function

A polynomial in the variable $x$ is a function that can be written in the form,

$F(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$

where $a_n, a_{n-1}, \ldots a_2, a_1, a_0$ are constants. We call the term containing the highest power of $x$ (i.e., $a_n x^n$) the *leading term*, and $a_n$ the *leading coefficient*. The degree of the polynomial is the power of $x$ in the leading term i.e., degree $n$. Again, its degree 0, 1, and 2 are polynomials which are constant, linear and quadratic functions while degree 3, 4, and 5 are polynomials with special names: cubic, quartic, and quintic functions. Polynomials with degree $n > 5$ are just called $n^{th}$ degree polynomials. See examples of polynomials in the following:

$p_1(x) = 2x + 5,$

$p_2(x) = 2x^3 - x + 5,$

$p_3(x) = 2x^7 - 5x^3 + 5x^2.$

**Constant Function**: A **constant function** is a linear function for which the range does not change no matter which member of the domain is used. That is, $f(x_1) = f(x_2)$ for any $x_1$ and $x_2$ in the domain.

**Linear Function**: *Linear functions* are those whose graph is a straight line. A ***linear function*** has the following form. y = f(x) = a + bx.

**Quadratic Function**: A quadratic function is one of the forms: $f(x) = ax^2 + bx + c$, where $a$, $b$, and $c$ are numbers with a $\neq 0$.

**Power Function:** A *power function* is a function that can be represented in the form:

$f(x) = kx^p$,

*where k and p are real numbers, and k is known as the coefficient. See that this function is constituted of a variable base raised to a fixed power just like you find in a single term of a polynomial function.*

*All the functions given below are power functions:*

The constant and only x term functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The reciprocal and reciprocal squared functions are power functions with negative integral powers because they can be written as $f(x) = x^{-1}$ *and* $f(x) = x^{-2}$
The square and cube root functions are power functions with fractional powers because they can be written as $f(x) = x^{\frac{1}{2}}$ and $f(x) = x^{\frac{1}{3}}$.

### 5.2.2.2 Rational Function

A rational function is defined by a rational fraction, *i.e.,* an algebraic fraction such that both the numerator and the denominator are polynomials.

**Example 3:** The **function** $R(x) = (-2x^5 + 4x^2 - 1) / x^9$ is a

**rational function** since the numerator and the denominator are polynomials. Note that the value of denominator should not be equal to zero.

### 5.2.2.3 Piecewise Function

We can have functions that behave differently depending on the input value. That is, it is defined on a sequence of intervals. The absolute value function as given below is an example of a piecewise function.

**Example 4:**

$$|x| = \begin{cases} -x & for\ x < 0 \\ 0 & for\ x = 0 \\ x & for\ x > 0 \end{cases}$$

We may consider the rate structure of Indian income tax to appreciate such a function. It may be seen from the income tax rates for 2019-20 is given below.

| **Income tax Slabs** | **Tax rate** |
|---|---|
| Income up to Rs.2.5 lakh | No tax |
| Income above Rs. Rs.2.5 to 5 lakh | 5% |
| Income more than Rs.5 to 10 lakh | Rs.12500 plus 20% of total income exceeding Rs. 5 lakh |
| Income above Rs. Rs. 10 lakh | Rs.1,12,500 + 30% of total income exceeding Rs.10,00,000 |

Taking rate of tax in the table above as a function of income level, we can construct the piecewise function.

### 5.2.3 Function Types: Transcendental

This class of functions is not expressible as a finite combination of the algebraic operations of addition, subtraction, multiplication, division, raising to a power, and extracting a root. Examples of these types include the logarithmic and exponential functions.

#### 5.2.3.1 Exponential Function

If $a$ is any number such that $a > 0$ and $a \neq 1$, then we have an exponential function in the form,

$f(x) = a^x$, where $a$ is called the base and $x$ can be any real number.

Notice that in the function above,$x$ is in the exponent and the base is a fixed number. This is exactly the opposite from what we've seen in algebraic functions. There the base has been the variable, $x$ in most cases, and the exponent was a fixed number. We will see some examples of exponential functions shortly.

Before we proceed further, we should address the restrictions on $a$. We avoid taking one and zero as the function would be,

$f(x) = 0^x = 0$ and $f(x) = 1^x$=1.

Such constant functions won't have many of the properties that general exponential functions have.

Next, we do not take negative numbers. Such numbers would result in getting complex values out of the function. For example, taking $a = -4$, the function would be,

$$f(x) = (-4)^x \Rightarrow f(1/2) = (-4)^{\frac{1}{2}} = \sqrt{-4},$$

a complex number. As we only want real numbers to arise from function evaluation, we require that $a$not be a negative number.

Note that taking a function

$f(x) = e^x$, where $e$ is the " Euler's Number " = **2.718281828459...**

we get the natural version of exponential function.

#### 5.2.3.2 Logarithmic Function

The function $y = log_b x$, where $x, b > 0$ and $b \neq 1$ gives the basic logarithm function, which is read " $y$ equals the log of $x$, base $b$" or " $y$ equals the log, base $b$, of $x$."

The above function is equivalent to $x = b^y$. Note that when no base is shown, the base is understood to be 10. Also note that the domain of the logarithmic function is the set of all positive real numbers and the range is the set of all real numbers.

While evaluating logarithmic functions, the bases used most often are base 10 and base $e$. $Log$ base 10,i.e $log_{10}$ is known as the **common logarithm** and is written as log.

The logarithm of a number is the exponent to which we must raise the base to get the number. e.g.,

$\log_2 8 = 3$ because $2^3 = 8$.

$\log_3 27 = 3$ because $3^3 = 27$

$\log_{10} 100 = 2$ because $10^2 = 100$.

If $log$ base $e$, i.e., $log_e$, is used, it is known as the **natural logarithm** and is written as l$n$.

Having seen both exponential and logarithm function above, it would be useful to note the relationship between the two. That is, a logarithmic function is inverse of an exponential function, and vice versa. Thus, $a^x$ (an exponential function) is the inverse function of $log_a(x)$ (a logarithm function).

If $m$, $n$ and $a$ be positive numbers, then important properties of logarithmic functions are:

1) log (m.n) = log m + log n; logarithm of product is the sum of logarithms.
2) log ( m/n) = log m - log n; logarithm of quotient is the difference of logarithms.
3) $\log (m^n) = n \log m$; logarithm of power of a number is the exponent times the logarithm of that number.
4) $\log_e (e^x) = x$
5) $\log_e e = 1$
6) $\log_a 1 = 0$
7) $\log_a a = 1$.

## 5.2.4 Function Types: Inverse and Composite

In addition to algebraic and transcendental functions, it would be useful to know two more types of functions, viz., inverse and composite. These two functions figure in algebraic and transcendental functions.

### 5.2.4.1 Inverse Function

Inverse functions are defined asf(x) $=$ y $if\ and\ only\ if$ g(y) $=$ x.In other words, a function $f$ has an inverse function only if for every $y$ in its range there is only one value of $x$ in its domain for which $f(x) = y$. This inverse function is unique and is frequently denoted by $f^{-1}$ and called "$f$ inverse."

**Example 5**: Let the $f$ be a function withdomain $a, b$ and $c$ and with codomain 1, 2 and 3 (see figure below). See that $f^{-1}$ is inverse of $f$. The reason is element $a$ maps to 3 in $f$ while 3 maps to $a$ in $f^{-1}$. The same is true for other elements.

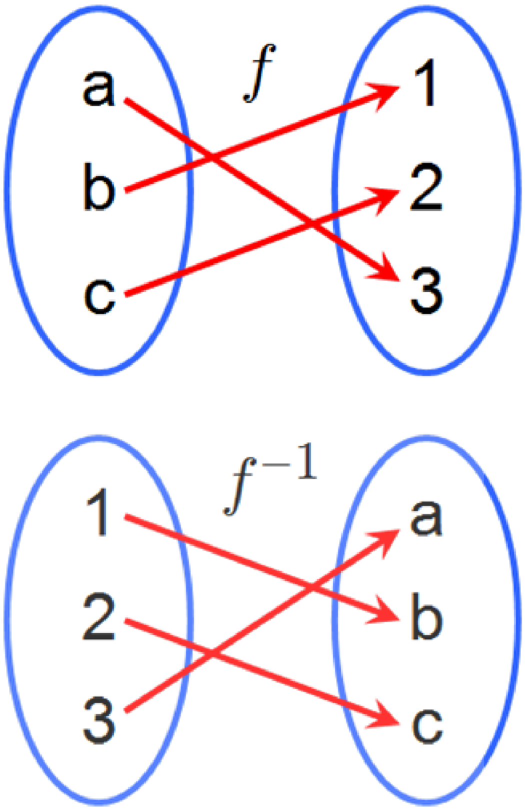



**Fig. 5. 2: Inverse Function**

**Source:** Wikipedia

**Example 6**: Consider function $f(x) = 5x - 7$. To reverse it means we need to get $x$ back from some output value, say $y$. This is given by the function,

$g(y) = \frac{y+7}{5}$.{\displaystyle g(y)={\frac {y+7}{5}}.}

**Example 7:** Find the inverse function for f(x) = 3x − 7.

Answer: Write y = 3x − 7 and solve for x as a function of y.

That is, $x = \frac{y+7}{3}$.

**Example 8**: Find the inverse function for

$$f(x) = \frac{x}{x+1}$$

Answer: Let $y = x/(x+1)$, and solve for $x$in terms of $y$.

We have from above $yx + y = x$,so that $y = x(1 - y)$

or, $x = \frac{y}{1-y}$

Thus, $f^{-1}(x) = \frac{x}{1-x}$.

**Example 9**: Find the inverse of function $f(x) = e^{x-3}$

**Answer:**

Given $y = e^{x-3}$

Taking $ln$ of both sides we obtain
$x-3 = \ln y$

or, $x = ln\, y + 3$

Change $x$ into $y$ and $y$ into $x$ to obtain the inverse function.

$f^{-1}(x) = y = ln\, x + 3$

**Example10**: Find the inverseof the function given by

$f(x) = 3\, ln(4\,x - 6) - 2$

**Answer**:

- Write $f$ as an equation, change from logarithmic to exponential form.
  $y = 3\, ln(4\,x - 6) - 2$,
  which gives $ln(4\,x - 6) = (y + 2)/3$
- Changing from logarithmic to exponential form we get
  $4x - 6 = e^{\left(\frac{y+2}{3}\right)}$.
- Solving for $x$ yields
  $4x = e^{\left(\frac{y+2}{3}\right)} + 6$
  and finally,$x = (1/4)e^{\left(\frac{y+2}{3}\right)} + 3/2$
- Changing $x$ into $y$ and $y$ into $x$ to obtain the inverse function.
  $f^{-1}(x) = y = (1/4)e^{\left(\frac{x+2}{3}\right)} + 3/2$

### 5.2.4.2 Composite Function

Composite function implies applying one function to the results of another. That is, the result of f( ) is sent through g( ) and written as (g º f)(x), which means g(f(x)).Thus,if we have two functions $f(x)$ and $g(x)$, then we can define a composite function $h(x) \equiv f(g(x))$.

For example,if $f(x) = x^3$ and $g(x) = 2x - 1$, then

$h(x) = (2x-1)^3 = 8x^3 - 12x^2 + 6x - 1$.

On the other hand, if we define the composite function $k(x) \equiv g(f(x))$, then we write $k(x) = 2(x^3) - 1$. Notice that $h(x)$ and $k(x)$ are different functions.

Taking into account natural exponential and logarithm functions, we evaluate their relationship with composite function the following:

Given $f(x) = e^x$ and $g(x) = \ln x$, then

$f(g(x)) = e^{\ln x} = x$

$g(f(x)) = \ln (e^x) = x$.

When composing functions, the order matters. For example, consider the function

$f(x) = 2x + 3$ and $g(x) = x^2$. Let

$(g \circ f)(x) = g(f(x))$.Thus, $(g \circ f)(x) = (2x+3)^2$. Now reverse the order $f$ and $g$. So,

$(f \circ g)(x) = f(g(x)) = f(x^2)$

Since $f(x) = 2x + 3, f(x^2) = 2x^2 + 3$.

We get a different result due to change of order of $f$ and $g$.

**Example 11**: Given $f(x) = 2x + 3$ and $g(x) = -x^2 + 5$, find $(f \circ g)(x)$.

Answer: Plugging the formula for $g(x)$ into the formula for $f(x)$, we get

$(f \circ g)(x) = f(g(x))$
$= f(-x^2 + 5)$
$= 2(\qquad) + 3$

$= 2(-x^2 + 5) + 3$
$= -2x^2 + 10 + 3$
$= -2x^2 + 13.$

**Check Your Progress 1**

1) What are functions?
2) Explain the difference between domain and range.
3) State the meaning of an inverse function.
4) What does a composite function imply?
5) What do you understand by linear functions?
6) What do you understand by Quadratic functions?
7) Explain the difference between Exponential and Logarithm functions.
8) What is the difference between common logarithm and natural logarithm functions?

## 5.3 FUNCTIONS AND CARTESIAN COORDINATES

We have already seen that a function is a technical term used to define relations between variables. A variable $y$ is called a function of $x$ if for every value of $x$ there is a definite value of $y$.

For example, $y = 2x + 3$,

where $x$ is independent variable and $y$dependent variable as its value depends upon the value of $x$.

For a better understanding of the functional relationship between the variables let us return to cartesian coordinate system that is composed of a horizontal line and a vertical line perpendicular to each other (see Figure 3). Remember that these lines are called coordinate axes. The point where they intersect each other is called the origin (0). Distance of a point from horizontal axis or $x$-axis is called the coordinate, coordinate and the distance of the point from vertical axis or $y$-axis is called the x-coordinate(abscissa). To the right of $y$-axis,$x$ coordinates are positive and to the left of it, x-coordinates are negative. Above the $x$-axis $y$ coordinates are positive and below it they are negative.

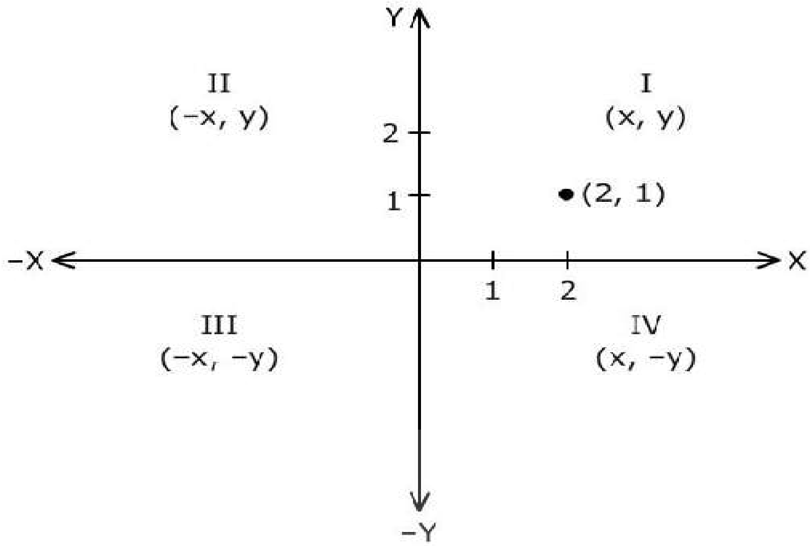



**Fig. 5.3: Cartesian Coordinate System**

The sign of the coordinate in each quadrant is shown in the figure. Note that quadrants are numbered anticlockwise.

Each point in the coordinate system is associated with ordered pair of numbers known as coordinates, showing the location of point in relation to origin. For example, the point (2, 1) is 2 units on$x$-axis and 1 unit on-axis measured from the origin.

**Check Your Progress 2**

1) What is a cartesian coordinate system?

2) How would you describe each point in the coordinate system?

3) Taking a point on horizontal axis, what would you can measure?

## 5.4 GRAPH OF SOME FUNCTIONS

Writing a function as $y = f(x)$, we take the elements of the domain $(x_i)$ as independent variable and that of range $y_i$ as dependent variable. It helps record the values of $x$ and $y$ while plotting the graphs.

### 5.4.1 Graphing Functions of Straight-Line Types

#### 5.4.1.1 Linear Function

To see the graph of a linear function consider depicting the relationship between price$(x)$ and market demand $(y)$ for a commodity which is often depicted by a straight line.To plot the graph for evaluating the relationship, it is convenient to prepare a table called, t chart, comprising the values of $x$ and $y$as a first step. If a function takes the form of$\boldsymbol{y = 7 - 5x}$**,** then the$t -$ chart is as under:

**Table**: t-chart of $\boldsymbol{y = 7 - 5x}$

| $x$ | $y = 7 - 5x$ |
|---|---|
| -1 | 12 |
| 0 | 7 |

| | |
|---|---|
| 1 | 2 |
| 2 | -3 |
| 3 | -8 |

Plotting the values of the chart we get the following graph.

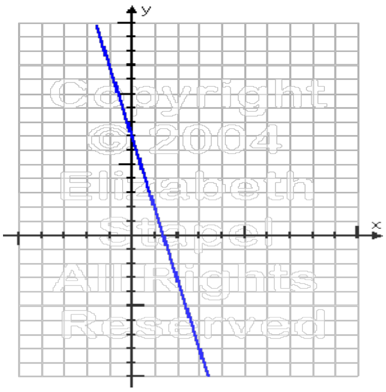



**Fig.5.4: Graph of Function $y = 7 - 5x$**

**Source: Internet**

The graph has x-intercept on a point where *y* is zero and y-intercept on a point where x is zero. For example, to find the *x* and *y* intercepts of the equation

$3x + 4y = 12$, proceed as follows:

To find the *x*-intercept, set *y* = 0 and solve for *x*. That is,

$3x + 4(0) = 12$ *or*, $x = 12/3 = 4$. Similarly, to find the *y*-intercept, set *x* = 0 and solve for *y*.

That is, $3(0) + 4y = 12$ *or*, $y = 12/4 = 3$. Thus,the *x*-intercept is (4, 0) and the *y*-interceptis (0, 3).

### 5.4.1.2 Absolute Value Function

Absolute Value Function is given as f(x) = |x| indicating that we need to consider modulus |x| of a real number keeping it as the non-negative value without regard to its sign. That is, |x| = x for a positive x, |x| = −x for a negative x (in which case −x is positive), and |0| = 0. Thus, the absolute value of 2 is 2, and the absolute value of −2 is also taken as 2. The graph of such as function is as follows:

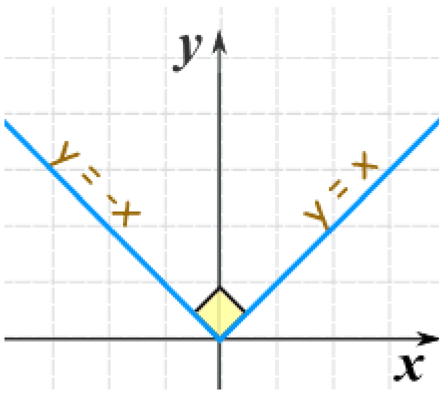



**Fig.5.5: Graph of Absolute Value Function**

**Source:** Internet

For example, $let\ f$ is a function given by $(x)\ =\ |x-2|$ . The $y$ intercept is given by

$\big(0\,,f(0)\big) =\ (0\,,|-2|) =\ (0\,,2);\,x$ intercept is at the point $(2\,,0)$ since we solve for$|x\ -\ 2|\ =\ 0$. Since $|x\ -\ 2|$ is either positive or zero for x = 2, the domain of $f$ is the set of all real numbers andthe range of $f$ is given by the interval $[0\,,+\infty)$.

#### 5.4.1.3 Step Function

A step function (or, staircase function) is a piecewise function containing all constant "pieces". The constant pieces are observed across the adjacent intervals of the function, as they change value from one interval to the next. A step function is discontinuous (not continuous). You cannot draw a step function without removing your pencil from your paper.

Features of step functions are: open circles and/or closed circles on the graph (open = point not on graph; closed = point is on graph);

- horizontal "pieces"
- discontinuous (cannot be drawn without removing your pencil from the paper)
- may or may be a function.
- Domain: all reals; Range: all integers; y-intercept = 0; x-intercept: [0,1).

Consider the function $f(x) = \begin{cases} -3; & x < 2 \\ 0; & -2 \le x \le 1 \\ 3; & x > 1 \end{cases}$ . Its graph is given below:

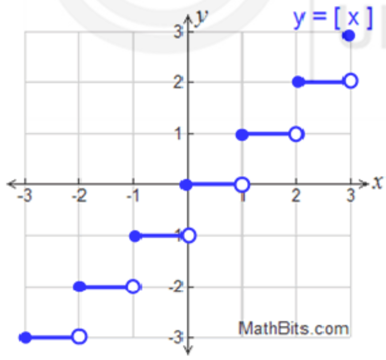



**Fig.5.6: Graph of Step Function**

**Source:** Internet



### 5.4.2 Graphing Functions: Curvy Types

Determining the nature of the function that produces curves requires additional considerations compared to linear types. When you graphed straight lines, you only needed two points to graph the line, though you generally plot three or more points just to be on the safe side. However, three points will almost certainly *not* be enough points for graphing curves of quadratic or cubic functions.

### 5.4.2.1 Quadratic Functions

Quadratic functions help describing demand, cost, revenue and profit which you will find while covering microeconomic analysis. The general technique for graphing quadratics is the same as for graphing linear equations. However, since quadratics graph are curvy lines (called "parabolas"), rather than the straight lines generated by linear equations, there will have to be some additional considerations.

The most basic quadratic function is $y = x^2$. We will use the following F chart to draw the graph.

**Table: T-chart of $y = x^2$**

| $x$ | $y = x^2$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |

Based only on this experiment the plotted graph will produce a straight line. Thus, the graph is not correct presentation of the function. More points need to be considered. By extending the above table with more points we construct the following table.

**Table: t-chart of $y = x^2$**

| X | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| Y | 9 | 4 | 1 | 0 | 1 | 4 | 9 |

With this table, we draw the following graph:

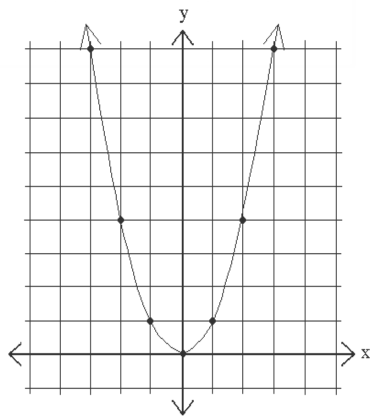



**Fig.5.7: Graph of Equation $y = x^2$**

**Source:** Internet

See that the graph is a parabola.

Note that the parabola does not have a constant slope. In fact, as $x$increases by 1, starting with $x = 0$, $y$increases by 1, 3, 5, 7,… . As $x$decreases by 1, starting with $x = 0$, $y$again increases by 1, 3, 5, 7,… .

Consider a quadratic function of the form: $f(x) = ax^2 + bx + c$ with $a \neq 0$. The graph of such a function takes one of the two general forms shown in the following figure, depending on the sign of $a$.

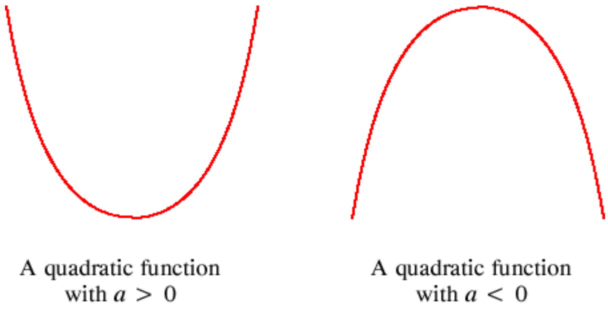



**Fig.5.8: Graph of Quadratic Function**

**Source:** Internet

For example, let us sketch the graph of function $y = 2x^2 - 8x + 6$. The coefficient of $x^2$ is positive, so the graph is $U - shaped$. The function is sketched in the following figure.

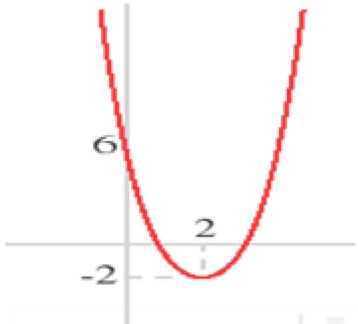



**Fig.5.9: Graph of Function $y = 2x^2 - 8x + 6$**

**Source:** Internet

### Computation of Vertex

Consider a generic quadratic expression:$y = ax^2 + bx + c$.Start with completing the square on the equation to have $y = a[x^2 + bx/a + c/a]$ or, $y = a[(x + b/2a)^2 - (b/2a)^2 + c/a]$

The expression $-(b/2a)^2 + c/a$ is a constant and it does not depend on $x$. So, we can replace it with $k$. Thus, we write $y = a[(x + b/2a)^2 + k]$.

Now, depending on whether $a$ is positive or negative, the parabola given by $y$ will either have a maximum or minimum. Since $a$ and $k$ are fixed, this must occur when $(x + b/2a)^2 = 0$. Hence, $x = -b/2a$ which implies that the function y is at a minimum or a maximum when this is true.

Since parabolas are symmetric over a vertical line, let us call that line $x = k$. This means that if the graph crosses the x-axis, then, $ax^2 + bx + c = 0$ to have real solutions they must be equidistant from $x = k$.So $(k, 0)$ must be the midpoint of the segment with endpoints at the zeros of the quadratic or, $k$ is the average of the zeros. From the quadratic formula, the two zeros of the quadratic are,$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ so their sum is $\frac{-b}{a}$ and their average is $k = -\frac{b}{2a}$. That means, the $x - coordinate$ of the vertex must be $-\frac{b}{2a}$ .

**Example 12**: Find the vertex of $y = 3x^2 + x - 2$ and graph the parabola.

**Ans:** To find the vertex, look at the coefficients$a$,$b$, and $c$. The formula for the vertex gives: $h = \frac{-b}{2a} = \frac{-(1)}{2x3} = \frac{-1}{6}$.

Then $k$ is obtained by evaluating $y$at $h = -1/6$:

$$k = 3(-1/6)^2 + (-1/6) - 2 = 3/36 - 1/6 - 2$$
$$= 1/12 - 2/12 - 24/12 = -25/12$$

Since the vertex is at $(-\frac{1}{6}, -\frac{25}{12})$T-chart can be prepared as follows:

| $x$ | $y = 3x^2 + x - 2$ |
|---|---|
| -2 | 8 |
| -1 | 0 |
| 0 | -2 |
| 1 | 2 |
| 2 | 12 |

The graph can be drawn with the specification vertex as follows:

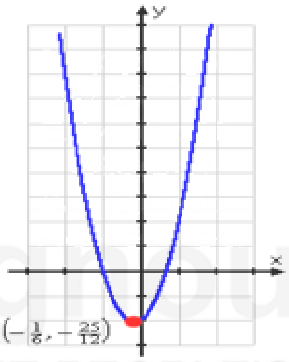



**Fig.5.10: Graph of $y = 3x^2 + x - 2$**



**Source:** Internet

### 5.4.2.2 Cubic Function

Cubic functions are encountered in the discussion of total cost in microeconomic analysis. Consider a cubic function of the form$f(x) = ax^3 + bx^2 + cx + d$. Its graph is as follows:

The "basic" cubic function is$f(x) = x^3$. The graph is:

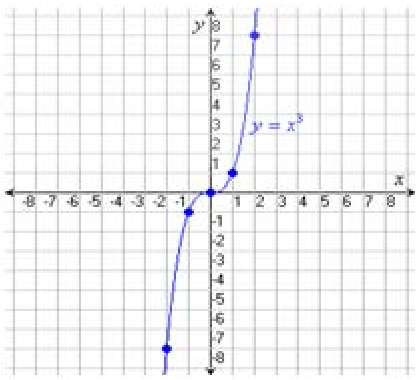



**Fig.5.11: Graph of Cubic Function**

**Source:** Internet

Let f is a cubic function given by $f(x) = x^3$

a) Find the $x$ and $y$ intercepts of the graph of $f$.

b) Find the domain and range of $f$.

c) Sketch the graph of $f$.

a) The y intercept is given by $(0,\ f(0)) = (0\,,0)$ . The $x$ coordinates of the $x$ intercepts are the solutions to $x^3 = 0$. The x intercept is at the point $(0\,,0)$.

b) The domain of $f\ (x)$ is the set of all real numbers. Since the leading coefficient of $x^3$ is positive, the graph of $f$ is up on the right and down on the left and hence the range of $f$ is the set of all real numbers.

c) Table of values are:

| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x) = x^3$ | -8 | -1 | 0 | 1 | 8 |

Now, the graph can be drawn with this values.

## 5.4.3 Graphing Functions: Asymptotic Types

We are familiar with exponential growth models such as population growth and compound interest. The functions representing such behaviour will be used to see the graphs.

### 5.4.3.1 Square Root Function

As the name indicates we take a variable x to write the function in the form of $f(x) = \sqrt{x}$.

Because the domain of f is the set of all positive real numbers and zero, we might construct a table of values as follows:

| | | | | | |
|---|---|---|---|---|---|
| X | 0 | 1 | 4 | 9 | 16 |
| $\sqrt{x}$ | 0 | 1 | 2 | 3 | 4 |

Plotting its graph yields the following figure:

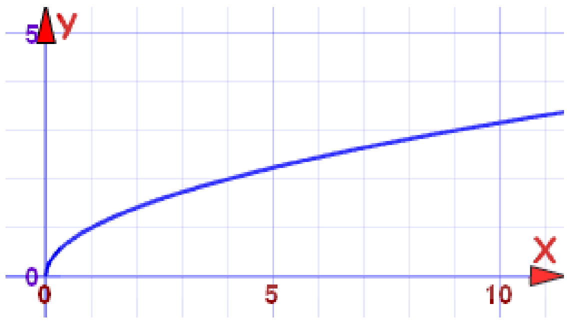



**Fig.5.12: Graph of Square Root Function**

**Source: Internet**

### 5.4.3.2 Exponential Function

Consider natural exponential of the form $x \to e^x$ where $e$ is Euler's number, a transcendental number approximately 2.718281828. Since x is an exponent and the exponent can be any real number, the domain is all real numbers. The range is the y-values. Since the graph never intersects or goes below the x-axis, the y-values are not zero or negative. Its graph is given below.

Draw a graph so that at any point (x, y) on the graph the slope is equal to y, the vertical coordinate of the point. Notice that the higher up the point on the graph is (the larger the y value), the steeper the slope. So, we make this graph, starting at the point (x, y) =(0, 1) (where the graph has slope equal to 1). So, the graph starts out going up as we move to the right. You will see that as you draw the graph from left to right it keeps going up faster and faster.

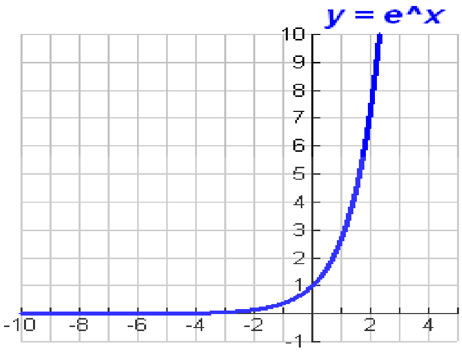



**Fig.5.13: Graph of Exponential Function**

**Source: Internet**

### 5.4.3.3 Logarithmic Function

The logarithmic function slowly goes to positive infinity as the variable increases and slowly goes to negative infinity as itapproaches 0. Afunction given by $y = log_b x$, where $b$ is any **number such that $b > 0, b \neq 1, and\ x > 0$ is called a logarithmic function the function is read "log base *b* of *x*".** Its graph crosses the *x*-axis at (1, 0). When *b*> 1, the graph increases; 0 <*b*< 1, the graph decreases.

In thelogarithmic function, the domain is all positive real numbers (never zero) while the range is all real numbers. The graph is asymptotic to the *y*-axis - gets very, very close to the *y*-axis but does not touch it or cross it.

The graph of $y = log_b x$, *b*> 1 with

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- x-intercept: (1, 0)
- Increasing

may be seen below:

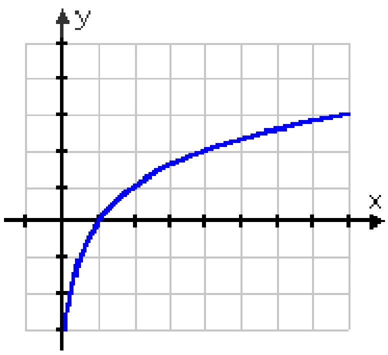



**Fig.5.14: Graph of Logarithmic Function**

**Source:** Internet

**Check Your Progress 3**

1) Draw the graph of $y = (-5/3)x - 2$.

2) List two graphs that would be of curvy-type.

3) How do you describe the shape taken by a logarithmic function?

4) What is a vertex?

## 5.5 FUNCTIONS RELATED TO BUSINESS AND ECONOMICS

In recent years, economic decision making has become more and more mathematically oriented. Faced with huge masses of statistical data, depending on hundreds or even thousands of different variables, business analysts and economists have increasingly turned to mathematical methods to help them describe what is happening, predict the effects of various policy alternatives, and choose reasonable courses of action from the large of possibilities. In this section some functions which are useful in business and economics are discussed.

### 5.5.1 Demand Function

Demand is a function of price. For each price level of a good, there is a corresponding quantity demanded that consumer will demand. If p is the price per unit and $x$ is the demand for a good, then demand function can be written as:

$$x= f(p)$$

where $x$ is dependent variable and p is the independent variable. Here, $x$ and p are positive because negative prices and quantity are meaningless. Normally, there is a negative and linear relation between price and demand.

### 5.5.2 Supply Function

Supply is a function of price. For each price level of a good, there is a corresponding quantity supplied by a producer. If p is the price per unit and $x$ is the supply for a good, then supply function can be written as:

$$x= g(p)$$

where $x$ is dependent variable and p is the independent variable. Here, $x$ and p are positive because negative prices and quantity are meaningless. Normally, there is a positive and linear relation between price and supply.

### 5.5.3 Cost Function

Total cost is the sum of fixed cost(FC) and variable cost(VC). Fixed cost is the sum of all cost that is independent of the level of output, e.g., rent, salaries of permanent workers etc. Variable cost is dependent on output. It increases with an increase in output and vice versa.e.g., raw material, salaries of temporary workers etc.

Let C be the total cost, then total cost function is defined as:

C= C(x),

where $x$ is the total units of output.

### 5.5.4 Revenue Function

Revenue function shows the relation between price of the commodity and output sold. If $x$units of an output are sold at a price p per unit, then the total revenue R(x) is given by

$$R(x) = x \cdot p$$

### 5.5.5 Profit Function

Profit function is derived by considering profit earned and prices of inputs and output. If R(x) and C(X) be the total revenue and total cost functions respectively, then profitP(x) = R(x) – C(X), where P(x) = profit, and $x$ is units of output.

The point where revenue from sales is equal to the cost of production is called as break-even point i.e., no profit, no loss point. In other words, profit is equal to zero (i.e., total revenue is equal to total cost).

### 5.5.6 Consumption Function

It is proposed that consumption is the function of national income. If C denotes total consumption in the economy and $Y$ denotes the total national income, then

$$C = f(Y)$$

is called as the consumption function.

**Example 13**: A business man sells 2000 items per month at a price of Rs.10 each. It is estimated that monthly sales will increase by 250 items for each Re. 0.25 reduction in price. Find the demand function corresponding to this estimate.

**Answer** From the estimate, increases 250 units each time drops Re. 0.25 from the original cost of Rs.10. This is described by the equation

$$x = 2000 + \left(\frac{10-p}{0.25}\right)$$

$$= 1200 - 1000p$$

$$\text{or, } p = 12 - \frac{x}{1000}$$

**Check Your Progress 4**

1) What is a demand function?

2) How would you describe a revenue function?

3) How would you formulate a consumption function?

## 5.6 LET US SUM UP

In this unit we have discussed the mathematical functions used to study economic and business themes. For that purpose, we started with a broad understanding of function used to define relations between variables. A variable $y$ is called a function of $x$ if for every value of $x$ there is a definite value of $y$. Coming to types of functions, we have tried to categorized these into algebraic (containing ordinary numbers, variables and operators like addition, subtraction, multiplication and division) and transcendental (not expressible as a finite combination of the algebraic operations). Thereafter we have been exposed to the graphs of some important functions. There is a brief preliminary discussion on application of functions in the themes of demand, supply, cost, revenue, profit and consumption in the last section of the unit.

## 5.7 KEY WORDS

**Algebraic**: Expression or equation in which a finite number of symbols are combined using only the operations of addition, subtraction, multiplication, division, and exponentiation with constant rational exponents.

**Asymptotic**: A curve and a line that get closer but do not intersect.

**Codomain**:Set of all possible output values of a function.

**Domain**:Set of values of the independent variable(s) for which a function or relation is defined.

**Ordered Pair**: Two elements, say, input and output$(a, b)$, must follow the ordering such that the input always comes first, and the output second in a relation.

**Parabola**:A curve where any point is at an equal distance from a fixed point; and a fixed straight line.

**Range**: All the output values of a function.

**Rational Number**: Number that can be represented as the ratio of two integers.

**Real Number**: A value that represents a quantity along a *number* line.

**Transcendental**: A function which "transcends," i.e., cannot be expressed in terms of algebra.

**Vertex**: A corner or a point where lines meet.

## 5.8 SOME USEFUL BOOKS


- Allen, R.G.D., “Mathematical Analysis for Economists”, London: English Language Book Society and Macmillan, 1974.
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- Chiang, A. and Kalvin Wainwright, Fundamental Methods of Mathematical Economics (Paperback), Mac Grow Hill, 2017.
- Archibald, G.C., Richard G.Lipsey. “An Introduction to A mathematical Treatment of Economics”, Delhi: All India Traveller Bookseller, 1984
- Yamane, Taro, “Mathematics for Economists: An Elementary Survey”,New Delhi: Prentice Hall of India Private Limited, 1970.
- K. Sydsaeter and P. Hammond, Mathematics for Economic Analysis, PearsonEducational Asia, Delhi, 2002.

## 5.9 ANSWER/HINTS TO CHECK YOUR PROGRESS

**Check Your Progress 1**

1) A function is a technical term used to define relations between variables.

2) The set of all possible value of independent variable in a function is called the domain of function and the set of all values of dependent variable is called range of a function.

3) A function $f$ has an inverse function only if for every $y$ in its range there is only one value of $x$ in its domain for which $f(x) = y$.

4) Composite function implies applying one function to the results of another.

5) Linear functions are functions with highest power 1.

6) Having the highest degree '2'.

7) Logarithmic and exponential functions are inverse of each other

8) Logarithms to the base 10 are called common logarithms and logarithms to the natural base are called natural logarithms.

**Check Your Progress 2**

1) System that is composed of a horizontal line and a vertical line perpendicular to each other.

2) By ordered pair of numbers known as coordinates.

3) Distance from vertical axis or $y$-axis (abscissa).

**Check Your Progress 3**

1) Do yourself after reading Sub-Section 5.4.1.1

2) Quadratic and cubic functions.

3) The logarithmic function goes slowly to positive infinity as the variable increases and slowly goes to negative infinity as itapproaches 0.

**Check Your Progress 4**

1) Relation between quantity demanded and price of a commodity. It is postulated that quantity demanded depends on price.

2) Revenue earned depends on price of the commodity sold.

3) By taking consumption as the dependent variable while income as the independent variable.

4) Parabola has a vertical axis of symmetry and a turning point called vertex.

## 5.10 EXERCISES WITH ANSWERS/HINTS

1) A business man sells 2000 items per month at a price of rupees10 each. It is estimated that monthly sales will increase by 250 items for each

Rs. 0.25 reduction in price. Find the demand function corresponding to this estimate.

2) Suppose the cost to produce some commodity is known to be linear. Find cost as a function of output if costs are Rs. 4000 for 250 units and Rs. 5000 for 350 units.

3) A garment manufacturer is planning production of new variety of t-shirts. It involves initially a fixed cost of Rs. 1.5 lacs and a variable cost of Rs. 150 for producing each shirt. If each shirt can be sold at Rs. 350, then find: i) the cost function ii) the revenue function iii) the profit function and iv) the break-even point

4) Suppose there is demand of 60 units of a product when its price is Rs. 18 per unit and 40 units when its price is Rs. 28 each. Find the demand function, assuming that it is linear.

5) When the unit price of an item is Rs. 5, the daily supply will be 100. When the price is increased to Rs. 10, the daily supply is found to be 200. Find the supply function, assuming that it is linear.

6) A company decides to set up a small production plant for manufacturing electronic clocks. The total cost for initial set-up (fixed cost) is Rs. 9 lakhs. The additional cost (i.e., variable cost) for producing each clock is Rs. 300. Each clock is sold at Rs. 750. During the first month, 1,500 clocks are produced and sold:

   i) Determine the cost function C(x) for the total cost of producing x clocks.

   ii) Determine the revenue function R(x) for the total revenue from the sale of x clocks.

   iii) Determine the profit function P(x) for the profit from the sale $x$ clocks.

   iv) What profit or loss the company incurs during the first month when all the 1,500 clocks are sold?

   v) Determine the break-even point.

7) A manufacturer earns Rs. 5500 in the first month and Rs. 7000 in the second month. On plotting these points, the manufacturer observes a linear function may fit the data.

   i) Find the linear function that fits the data.

   ii) Using your model make a prediction of the earning for the fourth month.

8) A salesman earns Rs. 380 in the first week, Rs. 660 in the second week and Rs. 860 in the third week. On plotting the points (1, 380), (2, 660) and (3, 860), the salesman feels that a quadratic function may fit the data.

   i) Find the quadratic function that fits the data.

   ii) Using the model make a prediction of the earning for the fourth week.

9) A firm wants to launch a new product. It observes that the fixed cost of the new product is Rs. 35000 and the variable cost per unit is Rs. 500. The revenue function of the new product is $5000x - 100x^2$. Find i) Profit ii) Break-even values iii) the values of $x$ results in loss.

**Answers**

1) p= 12- x/1000;  x>2000

2) C (x)= 10x +1500

3) i) C (x)= 150000 +150x;

   ii) R(x) = 350x

   iii) P(x) =200x – 150000

   iv) 750

4) p = 48 -1/2 x.

5) p = 1/20 x.

6) i) C(x) = 9,00,000 + 300x

   ii) R(x) = 750x

   iii) 450x – 9,00,000

   iv) P(1500) = - 2,25,000

   v) x = 2,000

7) i) Take y = mx + c, where $y$ denotes earnings, $x$ denotes the months and m and c are constants. From given data obtain,

$$5500 = m + c \qquad \ldots 1$$

$$7000 = 2m + c \qquad \ldots 2$$

Solve these to get, 1500 = m and c = 4000 and linear equation is

$$y = 1500x + 4000.$$

   ii) Earning for the fourth month is

$$y = 1500 \times 4 + 4000$$

$$= 6000 + 4000 = 10000$$

8) i) Let the quadratic function is

$y= ax^2 + bx + c$

where y stand for earnings and $x$ for weeks.

From above data to obtain

$380 = a.\ 1^2 + b.\ 1 + c$

$660 = a.\ 2^2 + b.\ 2 + c$

$860 = a.\ 3^2 + b.\ 3 + c.$

Solve these to get a= -40, b = 400 and c= 20.

Therefore, the required function is:

$y = -40\ x^2 + 400\ x + 20$

ii) The predicated earning for the fourth week is:

$y = -40 \text{ x } 16 + 400 \text{ x } 4 + 20$

= Rs. 980.

9) i) Given $R(x) = 5000x - 100x^2$ (revenue function)

$C(x) = FC + VC$ (Cost Function)

$= 35000 + 500x$

$P(x) = R(x) - C(x)$ (Profit Function)

$= 5000x - 100x^2 - (35000 + 500x)$

$= -100x^2 + 4500x - 35000$

ii) For break-even values, $P(x) = 0$

$P(x) = -100x^2 + 4500x - 35000 = 0$

$x^2 - 45x + 350 = 0$

$(x-10)(x-35) = 0$

$x = 10, 35$

iii) For loss making values; $P(x) < 0$

$-100x^2 + 4500x - 35000 < 0$

i.e., $(x-10)(x-35) > 0$

This is possible if $x<10$ and $x>35$.

# UNIT 6 LIMIT AND CONTINUITY

**Structure**



## 6.0 OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of limit of a function and
- Explain the concept of continuous function.

## 6.1 INTRODUCTION

The present unit discusses two basic concepts, viz., limit and continuity, which are adopted widely in differential calculus. We consider these two themes together as the concept of a limit is closely connected to that of continuity. We will see later that a function is continuous at a point, if the limit exists at that point and is equal to the corresponding value of the function.

**Idea of Limits of Functions**

When we discussed about functions in the preceding unit, we tried to see the values of functions at specific points. For example, our concern was to discern the value of f(x) if, say, x=1.

The idea behind limit is to analyze the value that function is "approaching" when its input "approaches" a specific value. To appreciate underlying idea, see the following Graph 1:

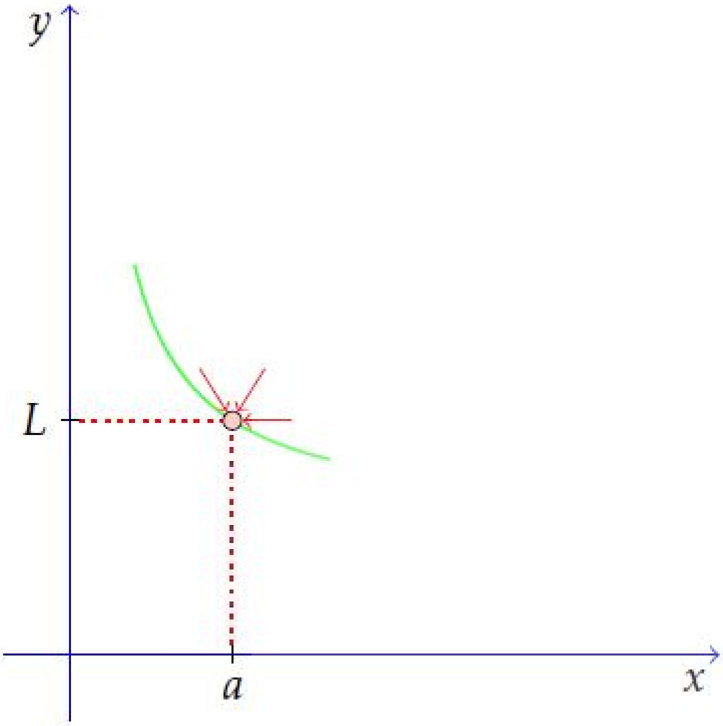



**Fig. 6.1: Approximation of Limit**

It may be seen that when $x$ approaches the value "$a$" in the $x$ axis, the function $f(x)$ approaches "$L$" in the $y$ axis. Let us pay attention to the hole at the point $(a, L)$ given in the form of a small circle. Around that area, we don't necessarily know the value of function $f$ at $x = a$. That is, we cannot work out the exact value of the function but can see what it should be as we get closer and closer. Such a situation is expressed as $\lim_{x \to a} f(x) = a$ .

**Example 1:** Let us take a function$f(x) = \frac{x^2-1}{x-1}$ and try to solve it for $x = 1$. Plugging the value $x = 1$ into $\frac{x^2-1}{x-1}$ we get $\frac{0}{0}$. There is a difficulty in the solution obtained. We do not know the value as $\frac{0}{0}$ is indeterminate. Therefore, we need to find some other way toderive an answer.

Instead of working out for $x = 1$, we try approaching 1 closer and closer in the following way:

| $x$ | $\frac{x^2-1}{x-1}$ |
|---|---|
| 0.5 | 1.50000 |
| 0.9 | 1.90000 |
| 0.99 | 1.99000 |
| 0.999 | 1.99900 |
| 0.9999 | 1.99990 |
| 0.99999 | 1.99999 |

See that as $x$ gets closer to 1, the value of $\frac{x^2-1}{x-1}$ gets closer to 2. With such a result, an interesting situation is obtained. For, when x=1, we don't know the answer as it is indeterminate. But in another approach, we could see that it is going to be 2. That is to say, we want to give the answer "2" but can't do so. Instead, we say, the limit of $\frac{(x^2-1)}{x-1} = 2$ as $x$ approaches 1. Essentially, we are saying, ignoring what happens when we get there, weknow of getting closer and closer to 2 as $x$ goes closer and closer to 1.

**The Idea of Continuous Functions**

Basically, we say a function is continuous when we can graph it without lifting a pencil from the paper. Here's an example of what a continuous function looks like Graph 2:

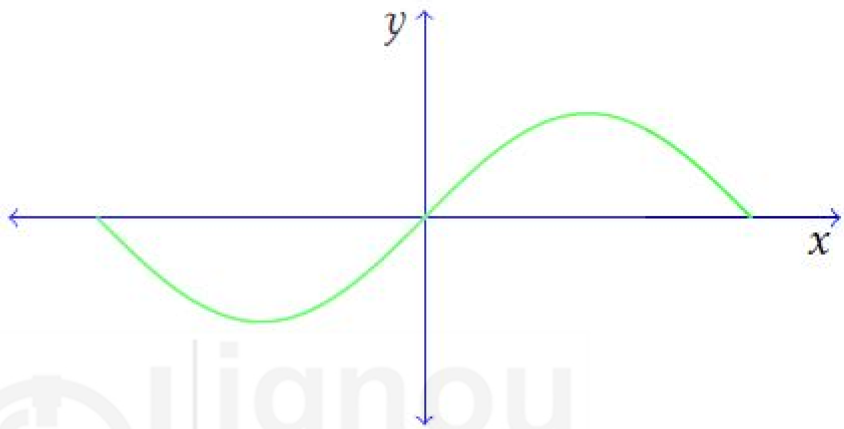



**Fig. 6.2: Continuous Curve**

If there is a break in the curve shown above in the graph, then the function is not continuous. Intuitively, a continuous function indicates that small changes in the input of the function would result in small changes in the output.

We will return to see the precise definitions of limit and continuity concepts introduced above.

**Check Your Progress 1**

1) What is the intuitive idea behind limit of a function?

2) List the commonly understood idea behind continuity of a function.

3) How would you like to explain the relation between limit and continuity of function?

## 6.2 LIMIT OF A FUNCTION

We have seen above in the introduction section that a function may have a limiting value as the independent variable approaches a particular real number. This limiting value is known as a limit, provided it exists. Symbolically,

$$\lim_{x \to c} f(x) = L$$

Above equation implies that as $x$ approaches $c$, the limit of $f(x)$ equals $L$. Here $x$ approximately equals to $c$ (i.e., $x$ can be less than $c$ or greater than $c$ but x$\neq c$).

To find out the limit, we need to work on the value of function for two different values of $x$ - one for $x$ greater than c (right-hand limit) and other for $x$ less than c (left-hand limit) and verify whether both of these are equal or not. The limit of a function will exist if and only if both left-hand and right-hand limits are equal, i.e.,

If, $\lim_{x\to c+} f(x) = L$ and $\lim_{x\to c-} f(x) = L$, then $\lim_{x\to c} f(x) = L$

**Example 2**: Find out the limit of $x^2$ when $x \to 2$.

**Solution:** To find out the limit, following tables are created with assumed values of $x$ and the corresponding values of $f(x)$:

When $x < 2$,

| $X$ | 1 | 1.4 | 1.7 | 1.8 | 1.9 | 1.99 | 1.995 | 1.999 |
|---|---|---|---|---|---|---|---|---|
| **f($x$)** | 1 | 1.96 | 2.89 | 3.24 | 3.61 | 3.9601 | 3.98 | 3.996 |

When $x > 2$,

| $X$ | 3 | 2.6 | 2.4 | 2.2 | 2.1 | 2.01 | 2.005 | 2.001 |
|---|---|---|---|---|---|---|---|---|
| **f($x$)** | 9 | 6.76 | 5.76 | 4.84 | 4.41 | 4.0401 | 4.02 | 4.004 |

It is clear from above tables, when $x$ is approximately equal to 2, the f($x$) is approximately equal to 4, i.e.,

$\lim_{x\to 2-} x^2 = 4$ and $\lim_{x\to 2+} x^2 = 4$

Therefore,

$$\lim_{x\to 2} x^2 = 4$$

**Example 3:** Find out the limit of f($x$)= $\frac{x^2-4}{x-2}$when $x$→2.

**Solution**: when $x$<2 then $f(x)$ equals

| $x$ | 1 | 1.4 | 1.7 | 1.8 | 1.9 | 1.99 | 1.995 | 1.999 |
|---|---|---|---|---|---|---|---|---|
| **f($x$)** | 1 | 3.4 | 3.7 | 3.8 | 3.9 | 3.99 | 3.995 | 3.999 |

When $x$>2, the f($x$) equals

| $x$ | 3 | 2.6 | 2.4 | 2.2 | 2.1 | 2.01 | 2.005 | 2.001 |
|---|---|---|---|---|---|---|---|---|
| **f($x$)** | 5 | 4.6 | 4.4 | 4.2 | 2.1 | 4.01 | 4.005 | 4.001 |

It is clear from above tables, when $x$ is approximately equals 2, the $f(x)$ is approximately equals 4.

$\lim_{x\to 2-} f(x) = 4$ and $\lim_{x\to 2+} f(x) = 4$

Hence,

$$\lim_{x\to 2} f(x) = 4$$

The method of finding limiting values of a function at a given point by putting the values of the variable very close to that point may not always be convenient.

We, therefore, need other methods for calculating the limits of a function as $x$(independent variable) tends to a finite quantity, say,$a$.

**Example 4**: Find $\lim_{x\to 3} f(x), where\ f(x) = \frac{x^2-9}{x-3}$

**Solution:**

We can solve it by the method of substitution,steps of which are as follows:

1) We consider a value of $x$close to,say,$x = a + h$,where$h$isaverysmallpositive number. Clearly, as $x \to a,\ h \to 0$

   For $f(x) = \frac{x^2-9}{x-3}$, we write $x = 3 + h$, when $x \to 3$ then $h \to 0$.

2) Simplify $f(x) = f(a + h)$

   Now, $f(3 + h) = \frac{(3+h)^2-9}{(3+h)-3}$

   $= \frac{9+h^2+6h-9}{h}$

   $= \frac{h^2+6h}{h}$

   $= h + 6$

3) Put$h = 0$

   $f(3 + 0) = f(3) = 6$

i.e., $\lim_{x\to 3} f(x) = \lim_{h\to 0}(h + 6) = 6$.

In the above while defining limit we have seen a condition on existence of a limit. When we say the limit does not exist, it means that the limit is either infinity or not defined. In case of the limit of a function 'tends to infinity', its value gets arbitrarily larger. If it doesn't get closer to any value, the limit does not exist.

If the variable tends to a finite value, then the function must get closer to a number as the variable gets closer to the finite value. Again, if it doesn't get closer to any value, then the limit does not exist. It could be because the left- and right-hand limits are not equal, or because they're equal to infinity.

### 6.2.1 Properties of Limit

If $c$ be any constant and $\lim_{x\to a} f(x)$ then the following are true:

1) The limit of a constant is constant, i.e., $\lim_{x\to a} k = k$, where k is constant.

2) $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$

3) The limit of a sum or difference is the sum or difference of the limits. i.e., $\left[\lim_{x\to a}[\ f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

4) The limit of a product is the product of limits. i.e., $\lim_{x\to a}[f(x).\, g(x)] = \lim_{x\to a} f(x).\, \lim_{x\to a} g(x)$.

5) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ given, $\lim_{x\to a} g(x) \neq 0$

6) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ for all positive integer values of n

7) $\lim_{x\to a} x^n = a^n$ for all positive integer values of n

8) $\lim_{x\to a} \log[f(x)] = \log[\lim_{x\to a} f(x)]$

9) $\lim_{x\to a} \exp\,[f(x)] = \exp[\lim_{x\to a} f(x)]$

### 6.2.2 Some Standard Limits

1) $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}, n$ be positive integer

2) $\lim_{x\to 0}(1+x)^{1/x} = e$

3) $\lim_{x\to 0} \frac{a^x-1}{x} = \log_e a$ (where a>0)

4) $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

5) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

**Examples 5:** Evaluate the following limits

1) $\lim_{x\to 2} 9$

2) $\lim_{x\to 2} 3x^2$

3) $\lim_{x\to 2}( 5x^2 + 3x + 7)$

4) $\lim_{x\to 1}[(x^2+4)(3x-2)]$

5) $\lim_{x\to 1} \frac{2x^2+x-3}{x^3+9}$

**Solutions:**

1) $\lim_{x\to 2} 9 = 9$

2) $\lim_{x\to 2} 3x^2 = 3.\lim_{x\to 2} x^2 = 3.\ (2)^2 = 3 \times 4 = 12$

3) $\lim_{x\to 2}( 5x^2 + 3x + 7) = \lim_{x\to 2} 5x^2 + \lim_{x\to 2} 3x + \lim_{x\to 2} 7 = 5(2)^2 + 3(2) + 7 = 33$

4) $\lim_{x\to 1}[(x^2+4)(3x-2)] = \lim_{x\to 1}(x^2+4).\lim_{x\to 1}(3x-2)$

$= [\lim_{x\to 1} x^2 + \lim_{x\to 1} 4].[\lim_{x\to 1} 3x - \lim_{x\to 1} 2] = (1 + 4)\,(3 - 2) = 5$

5) $\lim_{x\to 1} \frac{2x^2+x-3}{x^3+9} = \frac{\lim_{x\to 1}(2x^2+x-3)}{\lim_{x\to 1}(x^3+9)} = \frac{2+1-3}{1+9} = \frac{0}{10} = 0$

---

**Check Your Progress 2**

1) Explain the left-hand and right-hand limit.

2) Is the limiting value of a function exist when left-hand limit is not equal to the right-handlimit?

3) What is $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$?

4) What is value of $\lim_{x\to 0} \frac{e^x-1}{x}$?

### 6.2.3 Method of Factorization

If f($x$) and g($x$) are two functions such that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and we have to find $\lim_{x\to a} \frac{f(x)}{g(x)}$, then we obtain a $\frac{0}{0}$ form, which is meaningless. Note that $\frac{0}{0}$is called as indeterminate form. Other indeterminate form is ±∞/±∞. Such limits are solved by method of factorization.

Consider the example $\lim_{x\to 5} \frac{x^2-3x-10}{x^2-6x+5}$, as $x$→5, both numerator and denominator approach zero. This is an indeterminate form. In such case, we follow the following steps:

1) Factorise$f(x)$ and $g(x)$to get,

$$\lim_{x\to 5} \frac{x^2-3x-10}{x^2-6x+5} = \lim_{x\to 5} \frac{(x+2)(x-5)}{(x-1)(x-5)}$$

2) Simplify the equation

$$\lim_{x\to 5} \frac{(x+2)(x-5)}{(x-1)(x-5)} = \lim_{x\to 5} \frac{(x+2)}{(x-1)}$$

3) Putting the values of $X$, we get

$$\lim_{x\to 5} \frac{(x+2)}{(x-1)} = \frac{7}{4}$$

**Example 6:** Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Here, we do the following steps:

**Step 1.** Rationalise the factor containing square root.

**Step 2.** Simplify

**Step 3.** Put the value of $x$ and get the required result.

**Solution**: given, $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Rationalising,$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \text{x} \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

$$= \lim_{x\to 0} \frac{\sqrt{1+x}^2-\sqrt{1-x}^2}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$=\lim_{x\to 0} \frac{1+x-1+x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$=\lim_{x\to 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$=\lim_{x\to 0} \frac{2}{(\sqrt{1+x}+\sqrt{1-x})}$$

Putting the value $x = 0$, we get

$$\lim_{x\to 0}\frac{2}{\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{2} = 1$$

**Example 7**: Evaluate $\lim_{x\to\infty}\frac{2x^2+5x+1}{3x^3-2x+4}$

**Solution**: As $x$ approaches ∞, both the numerator and denominator approach ∞.Therefore, the given function takes the indeterminate form $\frac{\infty}{\infty}$. However, we can change the form of the quotient so that a conclusion can be drawn as to whether or not it has a limit. This is done by dividing both numerator and denominator by the highest power of $x$ that occurs in the denominator. Thus, dividing both the numerator and denominator by $x^3$, we get

$$\lim_{x\to\infty}\frac{2x^2+5x+1}{3x^3-2x+4} = \lim_{x\to\infty}\frac{\frac{2}{x}+\frac{5}{x^2}+\frac{1}{x^3}}{3-\frac{2}{x^2}+\frac{4}{x^3}}$$

$$= \frac{\lim_{x\to\infty}\frac{2}{x}+\lim_{x\to\infty}\frac{5}{x^2}+\lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty}3-\lim_{x\to\infty}\frac{2}{x^2}+\lim_{x\to\infty}\frac{4}{x^3}}$$

$$= \frac{0+0+0}{3-0+0} = \frac{0}{3} = 0$$

**Example 8:** For the demand function $p = \frac{a}{x+b}$ (where $p$ is price, $x$ is quantity demanded and $a$ and $b$ are constants), show, by using the concept of limit, that the demand increases to infinitely large amount as the price falls. Also, show that total revenue reaches a limiting value as the quantity demand increases.

**Solution:** Given, $p = \frac{a}{x+b}$

Rearranging equation,

$$x = \frac{a}{p} - b$$

Putting the limit $p \to 0$,

$\lim_{p\to 0} x = \lim_{p\to 0}[\frac{a}{p} - b] = \infty$

Therefore, demand approaches ∞, when price approaches 0

Now, $TR = px = x[\frac{a}{x+b}] = \frac{ax}{x+b}$

$\lim_{x\to\infty} TR = \lim_{x\to\infty}\frac{ax}{x+b} = \lim_{x\to\infty}\frac{a}{1+\frac{b}{x}} = a$

Thus, $TR$ reaches a limiting value '$a$' as quantity demanded increases.

---

**Check Your Progress 3**

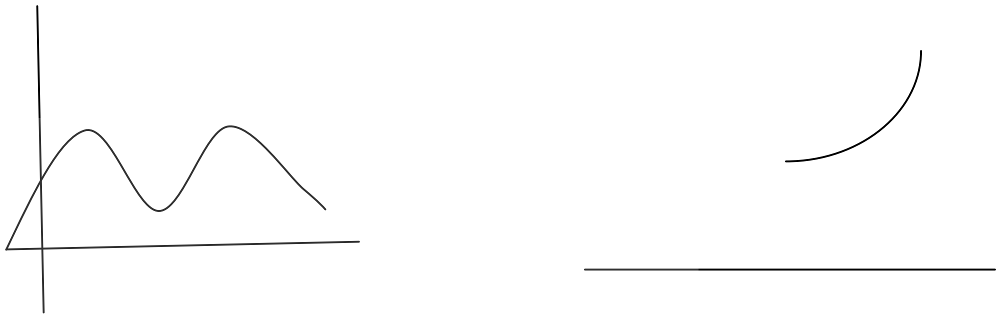
1) Explain the indeterminate forms of limit.

2) What is the limiting value of constant when $x$ approaches $n$?

3) When would you like to use factorization method of finding the limit of a function?

## 6.3 CONTINUITY

We have seen above in the introduction section that a function is said to be continuous if the graph of the function has no breaks, i.e., if its graph can be drawn without lifting pencil off the paper. The graph of a continuous function and discontinuous function are given below:



a: Continuous function b:Discontinuous function

**Fig. 6.3: Continuous and Discontinuous Function**

In other words, a function is continuous if minor changes in the independent variable generate minor changes in the function values. Mathematically, a function $f$ is said to be continuous at $x = c$ if

i) the function is defined at $x = c$, and

ii) $\lim_{x \to c} f(x) = f(c), i.e, \lim_{x \to c-} f(x) = \lim_{x \to c+} f(x)$ = f(c)

If $f(x)$ is **continuous** for every value of $x$ in its **domain**, it is said to be **continuous** throughout the interval.

A function, which is not continuous at a point, is said to have a **discontinuity** at that point.

A function $f$ is a continuous over an interval $(a, b)$ if it is continuous at every point within the interval.

**Example 9**: Examine the continuity of the function $f(x) = x - a$ at $x = a$

**Solution**: Put $x = a + h$,so that when $x \to a$ then $h \to 0$. Now,

$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h)$and at $x = a$,

$\lim_{h \to 0} f(a + h) = \lim_{h \to 0} ((a + h) - a)$

$= 0$

Also, $f(a) = a - a = 0$

Hence, $f(a) = \lim_{h \to 0} f(a + h) = 0$

Thus $f(x)$ is continuous at $x = a$.

### 6.3.1 Properties of Continuity

1) $f(x) = a$ (constant) is continuous for all real number$x$.

2) $f(x) = x^n$,$n$ is natural number, is continuous for all real number $x$.

3) polynomial functions are continuous for all real numbers.

4) If $f(x)$ and $g(x)$ are two continuous function then, $f(x) \pm g(x)$ are also continuous. i.e., sum and difference of two continuous functions are also continuous.

5) If $f(x)$ and $g(x)$ are two continuous function, then $f(x)$x $g(x)$ is also continuous. i.e., product of two continuous functions is also continuous.

6) If $f(x)$ and $g(x)$ are two continuous function, then $f(x)/g(x)$ is also continuous (provided $g(x) \neq 0$), i.e., quotient of two continuous functions is also continuous.

**Example 10**: A function $f(x)$ is defined as

$$f(x) = \begin{cases} x+1, if -1 \leq x < 0 \\ x, if\ 0 \leq x \leq 1 \\ 2-x\ if\ 1 \leq x \leq 2 \end{cases}$$

Show that it is discontinuous at $x = 0$ but is continuous at $x = 1$.

**Solution**: When $x = 0$,we get

$$f(0) = 0$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x = 0$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (x+1) = 1$$

Since, $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$,$f$ is not continuous at $x = 0$.

When $x = 1$,

$f(1) = 1$. Now,

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+}(2-x) = 2-1 = 1;$

$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x = 1.$

Since $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = 1 = f(1),$

$f$ is continuous at $x = 1$.

**Example 11:** Find the points of discontinuity of the function

$$\frac{2x^2 + 6x - 5}{12x^2 + x - 20}$$

**Solution**: The given function will be discontinuous at the point where denominator is equal to zero. i.e.,

$12x^2 + x – 20 = 0$

or, $12x^2 + 16x - 15x – 20 = 0$

or, $4x (3x + 4) - 5 (3x + 4) = 0$

or, $(4x – 5) (3x + 4) = 0$. So

$x = \frac{5}{4}$ or, $-\frac{4}{3}$

which are the points of discontinuity.

**Example 12:** Examine the continuity at $x = 0$ of the function $f(x)$ defined as under:

$f(x) = \frac{x}{1+e^{\frac{1}{x}}}$ at x≠0

$= 0$ at x = 0

**Solution:** R.H.L. $\lim_{h\to 0+} f(x) = \lim_{h\to 0} \frac{0+h}{1+e^{\frac{1}{h}}} = \lim_{h\to 0} \frac{h}{1+e^{\frac{1}{h}}} = 0$

L.H.L. $\lim_{h\to 0-} f(x) = \lim_{h\to 0} \frac{0-h}{1+e^{-\frac{1}{h}}} = \lim_{h\to 0} \frac{-h}{1+e^{-\frac{1}{h}}} = 0$

Also, f(0) =0

Since R.H.L. = L.H.L = 0 = f(0), the given function is continuous at $x = 0$.

**Example 13:** A function is defined as under

$$y = f(x) = \frac{x^2 - x - 12}{x-4}$$

i) Evaluate the limit of y when $x \to 4$

ii) Is the function continuous at $x = 4$? why?

**Solution:** Given the function,

$$y = f(x) = \frac{x^2 - x - 12}{x - 4}$$

i) Applying the limit at $x = 4$, we get

$$\lim_{x\to 4} y = \lim_{x\to 4} f(x) = \lim_{x\to 4} \frac{x^2 - x - 12}{x - 4} = \lim_{x\to 4} \frac{(x-4)(x+3)}{x-4}$$

$$= \frac{\lim(x+3)}{x \to 4} = 4 + 3 = 7$$

ii) At $x = 4$, function is evaluated as

$$y = f(x) = \frac{x^2 - x - 12}{x-4} = \frac{4^2 - 4 - 12}{4-4} = \frac{0}{0} \text{ (Not defined)}$$

Since, the function y is not defined at $x = 4$, the function is not continuous at $x = 4$.

**Example14:** A shopkeeper charges Rs. 25 per item for buying 20 or less items. He gives some rebate if items bought are more. If the items bought are 50 or less, then a rebate of Re. 1 per item and for purchase of more than 50 items, rebate of Rs. 2 per item. Find the cost function. What are the points at which this is not continuous?

**Solution:** let 'x' denotes the number of items then the cost function C(x) is shown as

$$C(x) = \begin{cases} 25x & 1 \le x \le 20 \\ 24x & 20 < x \le 50 \\ 23\,x & x > 50 \end{cases}$$

At x = 20,C(20) = 25 x 20 = 500

L.H.L = $\lim_{x\to 20^-} C(x) = \lim_{x\to 20^-} C(20) = \lim_{x\to 20^-} 25x = 25X20 = 500$ (left hand limit)

R.H.L = $\lim_{x\to 20^+} C(x) = \lim_{x\to 20^+} 24x = 24\,X20 = 480$ (right hand limit)

Since, LHL ≠ RHL,the function C(x) is not continuous at x = 20-0

At x = 50, C(50) = 24 x 50 = 1200

L.H.L = $\lim_{x\to 50^-} C(x) = \lim_{x\to 50^-} C(50) = \lim_{x\to 50^-} 24x = 24\,X 50 = 1200$ (left hand limit)

R.H.L = $\lim_{x\to 50+} C(x) = \lim_{x\to 50^+} 23x = 23\,X50 = 1150$ (right hand limit)

Since, LHL ≠ RHL, the function C(x) is not continuous at x = 50.

Therefore, the function is discontinuous at x = 20 and x = 50.

**Example 15**: Anelectric company charges from its customers the following amount for services: Rs. 5 for first 20 kilowatt hours or less, 25 paise per kilowatt hour for the next 80-kilowatt hours and 10 paise per kilowatt for any hours above 100 kilowatt hours. If x is the number of kilowatt hours, express the total cost C as a function of x. Also test the continuity of $C$at x = 20 and x = 100.

**Solution**: Given,

$$C(x) = \begin{cases} 5, & if\ x \le 20 \\ 5 + \frac{1}{4}(x-20), if\ 20 < x \le 100 \\ 25 + \frac{1\,X(x-100)}{10}, if\ x > 100 \end{cases}$$

At x = 20, C(20) = 5

L.H.L = $\lim_{x\to 20-} C(x) = \lim_{x\to 20-} 5 = 5$

R.H.L = $\lim_{x\to 20+} C(x) = \lim_{x\to 20+} 5 + \frac{1}{4}(x-20) = 5$

Hence, $\lim_{x\to 20} C(x) = 5 = C(20)$

Therefore, the $C(x)$ is continuous at x =20.

Similarly, we can show that $C$ is continuous at x = 100.

**Check Your Progress 4**

1) Explain continuous and discontinuous functions.

2) Are polynomial functions continuous for all real numbers?

3) Construct a function that is discontinuous at $x = 0$ but is continuous at $x = 1$.

## 6.4 LET US SUM UP

In this unit we have been exposed to the concepts of limit and continuity of a function. We have learnt that the intuitive idea of a limit is one of analyzing the value which a function is "approaching" when its input "approaches" a specific value. On the other hand, we say a function is continuous when we graph it without lifting the pencil from the paper. That is, to be qualified as a continuous function, it should not have breaks. With such a feature in place, a continuous function indicates that small changes in the input of the function would result in small changes in the output. Along with these, we have been told the properties of limit of a function.

Discussing the formal definition of limit of a function, we learnt that the limiting value is known as a limit, provided it exists. To understand such a concept, we need to find out the limit of a function from two different sides called right-hand limit and left-hand limit. Once computing these, we have to proceed for verifying whether both of them are equal. For, the limit of a function would exist if and only if both left-side and right-side limits are equal. We also learnt the method of finding limit through factorization when the value of a function attains indeterminate form.

Coming to the concept of continuity we have seen that a function $f$ is said to be continuous at $x = c$ if conditions

i) the function is defined at $x = c$, and

ii) $\lim_{x\to c} f(x) = f(c), i.e, \lim_{x\to c-} f(x) = \lim_{x\to c+} f(x) = \text{f}(x)$ are satisfied. We have been also showed the properties of a continuous function.

## 6.5 KEY WORDS

**Continuity**:A function *f*(x) is **continuous** provided its graph is continuous. More formally, a function *f*(x) is said to be **continuous** at x = a, provided $lim_{x\to a} f(x)$ exists, finite and is equal to f $(a)$.

**Indeterminate Forms**:whose limit cannot be determined solely from the limits of the individual functions.

**Limit**: Value that a *function* (or sequence) "approaches" as the input "approaches" some value.

**Method of Factorization**:It is a technique to finding limits that works by cancelling out common factors. It is normally used to transform an indeterminate form into one that allows for direct evaluation.

**Method of Substitution**:is a method of determining limits where the approaching value is substituted into the function and the result is evaluated.

**Natural Number**: Those used for counting.

**Real Number**: A continuous quantity that can represent a distance along a line. Thus, the real numbers include all the rational numbers, such as the integer −5 and the fraction 4/3, and all the irrational numbers, such as √2.

**Real Valued Function**: whose values are real numbers.

## 6.6 SOME USEFUL BOOKS


- Allen, R.G.D., "Mathematical Analysis for Economists", London, English Language Book Society and Macmillan, 1974.
- Chiang, A. and Kalvin Wainwright, Fundamental Methods of Mathematical Economics (Paperback), Mac Grow Hill, 2017.
- K. Sydsaeter and P. Hammond, Mathematics for Economic Analysis, PearsonEducational Asia, Delhi, 2002.

## 6.7 ANSWER/HINTS TO CHECK YOUR PROGRESS

**Check Your Progress 1**

1) Value that a *function* approaches as the input approaches some value.

2) When we can graph a function without lifting a pencil from the paper.

3) A function $f(x)$ is continuous at a point $x_0$ if the limit exists at $x_0$ and is equal to $f(x_0)$.

**Check Your Progress 2**

1) For RHS limit, value of a function is just greater than the limit whereasfor LHS limit, value of a function is just less than the limit.

2) no

3) It is a property of limit of a function.

4) 1

**Check Your Progress 3**

1) the value of the equation is of the form of $\frac{0}{0}$ $and$ $\frac{\infty}{\infty}$.

2) Constant

3) In cases of functions with two indeterminate forms viz., $0/0$ and $\frac{\infty}{\infty}$.

**Check Your Progress 4**

1) A function f is a continuous over an interval $(a, b)$ if it is continuous at every point within the interval, otherwise discontinues.

2) Yes

3) See the discussion given in Sub-Section 6.3.1 and answer.

## 6.8 EXERCISES WITH ANSWER/HINTS

1) Find the limit of the following functions:

a) $\lim_{x\to 0} 7$

b) $\lim_{x\to 0} x$

c) $\lim_{x\to 3}(3x + 4)$

d) $\lim_{x\to 1}(3x^4 - 2x^3 + 4x^2 + 5x - 7)$

e) $\lim_{x\to 2}[(x^2 + 5)/(x - 1)]$

f) $\lim_{x\to\infty}[\frac{2x^2 - x+1}{3x^2 + 2x+5}]$

g) $\lim_{x\to\infty}[\frac{2x^3 + 3x^2 + 5}{-5x^3 + 8x-17}]$

2) Discuss whether the following functions are continuous or not:

a) $\lim_{x \to 3} (2x^2 - 3x + 5)$

b) $\lim_{x \to 5} [(x^2 - 7)/(x + 3)]$

3) Examine the continuity at x = 0 for the following function

$$f(x) = \begin{cases} \frac{xe^{1/x}}{1+e^{1/x}}, if\ x \neq 0 \\ 0, \quad if\ x = 0 \end{cases}$$

4) Given, y = f(x) = $\frac{x^2 - x - 20}{x - 5}$

a) Evaluate the limit of $y$ as $x \to 5$.

b) Is this function continuous at $x = 5$? Why?

5) Discuss the continuity of the function $f$, where

$$f(x) = \begin{cases} x + 2, for\ x < 1 \\ 4x - 1, for\ 1 \leq x \leq 3 \\ x^2 + 5, \ for\ x > 3 \end{cases}$$

**Answers:**

1) a) 7 b) 0 c) 13 d)3 e) 9 f) 2/3 g) -2/5

2) a)continuous b)continuous

3) continuous

4) a) 4 b) No, since it is not defined at $x = 5$.

5) Continuous at $x = 1$, discontinuous at $x = 3$.

# UNIT 7 CONCEPT OF DIFFERENTIATION

**Structure**



## 7.0 OBJECTIVES

After studying this unit, you will be able to:

- understand the concept of derivative, its meaning, computation and interpretation;
- comprehend several rules for calculating derivatives; and
- use logarithm in differentiation.

## 7.1 INTRODUCTION

The rate of change of one variable of function with respect to another on which it depends is called the derivative of the function. Differentiation is the process of the finding out the derivative of a continuous function. A derivative is the limit of the ratio of the small increment in one variable of the function corresponding to a small increment in the argument (other variable) as the later tends to zero.

## 7.2 DIFFERENTIATION BY FIRST PRINCIPLE

Differentiating a function from "first principles" means we start from scratch and use algebra to find a general expression for the slope of a curve, at any value $x$. To appreciate the underlying idea, we start with some general idea of the concept.

Let $y = f(x)$ be a function defined in an interval $(a, b)$. Let $x = c$ be any point of the interval, so that $f(c)$ is the corresponding value of the function. Let( $c + h$) be any other point of this interval which lies to the right or left of $c$ according as $h$ is positive or negative. The corresponding value of the function is $f(c + h)$. Then $f(c + h) - f(c)$ is the change in the dependent variable $y$ corresponding to the change $h$ in the independent variable $x$. Consider the ratio: $\frac{f(c+h)-f(c)}{h}$ of these two changes whichis a function of $h$ and is not defined for $h = 0$, $c$ being a fixed point.

Definition: A function $y = f(x)$ is said to be differentiable at $x = c$ if

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists and the limit is called the derivative of the function $f(x)$ at $x = c$. it is denoted by $f'(c)$ or $y_1(c)$.

**Steps**

1) Put the given $f(x)$ equal to $y$ i.e.,

$$y = f(x) \qquad (1)$$

2) Increase $x$ by a small quantity $\Delta x$ and corresponding increase in $\Delta y$

So $y + \Delta y = f(x + \Delta x)$ (2)

3) Subtracting (1) from (2)

$$y + \Delta y - y = f(x + \Delta x) - f(x)$$

or, $\Delta y = f(x + \Delta x) - f(x)$

4) Dividing both sides by $\Delta x$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

5) Put $\Delta x \xrightarrow{\lim} 0$ both sides such that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Expression on LHS represents the derivative which is the instantaneous rate of change in the dependent variable corresponding to a change in the independent variable. The function is differentiable only if the limit exists and the function must be continuous at that point.

**Check Your Progress 1**

1) What is derivative of a function?

2) When would you say that a function is differentiable?

3) What do you mean by differentiation from first principle?

## 7.3 RULES OF DIFFERENTIATION

In this section, we will discuss the rules and properties for finding derivatives for different kinds of functions. Importantly, we will learn the Product Rule and the Power Rule that offer shortcuts to differentiation. See the accompanied figure for computing important derivatives.

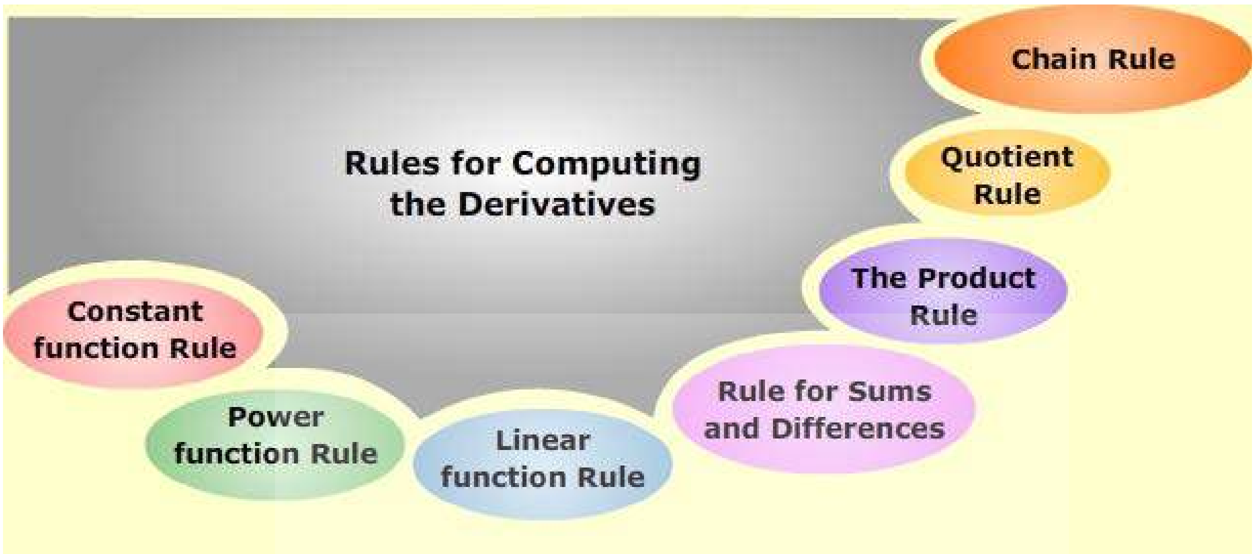



**Fig. 7.1: Rules for Computing Derivatives**

**Rule 1:**Constant function rule

Derivative of a constant is zero.

That is, if $y = f(x) = c$, then

$\frac{dy}{dx} = 0$

**If Y= constant, Y will not change for any value of X**

**Example1**:Find the derivative of the following functions:

i) y = 8

ii) y = 6000c

iii) y = 5/6

**Solution:**i) y = 8

$$\frac{dy}{dx} = 0$$

(ii) y = 6000c

$$\frac{dy}{dx} = 0$$

(iii) y = 5/6

$$\frac{dy}{dx} = 0$$

**Rule 2:** Power Function Rule

If$n$ is any real number, and $y = f(x) = x^n$, then $\frac{d}{dx}x^n = n\,x^{n-1}$

**Derivative of Power function= power$^x$ coefficient$^x$ Variable$^{power-1}$**

**Example 2**:

i) y= $x^{3/2}$

**Solution:** $\frac{dy}{dx} = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}$ $x^{1/2}$

ii) y = x

**Solution:** $\frac{dy}{dx} = 1.x^{1-1}$= 1

iii) y = $\frac{1}{\sqrt{x}}$

**Solution:** y = $x^{-1/2}$

$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}$

iv) y = 2 $x^6$

**Solution:** $\frac{dy}{dx} = 2.6$ $x^{6-1}$

=12 $x^5$

v) y = $\frac{8}{3}$.$x^{2/3}$

**Solution:** $\frac{dy}{dx} = \frac{8}{3}.\frac{2}{3}x^{\frac{2}{3}-1} = \frac{16}{9}$. $x^{-1/3}$

vi) y = $\frac{100}{\sqrt[3]{x}}$

**Solution:** y= $100.x^{-1/3}$

$\frac{dy}{dx}$= 100. - $\frac{1}{3}x^{-\frac{1}{3}-1}$

= - $\frac{100}{3}$ $x^{-4/3}$

**Rule 3:** The Linear Function Rule

The derivative of a linear function, $y = ax + b$,is equal to $a$, the coefficient of $x$.

If $y = ax + b$, then $\frac{dy}{dx} = a\,x^0 + 0$

That is, $\frac{dy}{dx} = a$

**Example 3**:

If $y = 3x + 7$, then $dy/dx = 3 + 0$

or,dy/dx = 3

**The derivative of a linear function, $y = ax + b = coefficient\ of\ x.$**

**Rule 4:** Rule of Sums and Differences

If f and g are differential functions, and

$F(x) = f(x) \pm g(x)$ then, we have:

$F'(x) = f'(x) \pm g'(x)$

**The derivative of a sum or a difference of two functions is the same as the sum or difference of their individual derivatives.**

**Example 4**:

i) $y = x^8 + x^{2/3}$

**Solution:**

$$\frac{dy}{dx} = \frac{d}{dx}.x^8 + \frac{d}{dx}.x^{\frac{2}{3}}$$

or, $\frac{dy}{dx} = 8.x^{8-1} + \frac{2}{3}.x^{\frac{2}{3}-1} = 8.x^7 + \frac{2}{3}.x^{-1/3}$

ii) $y = 3x^3 + 4x^2 - 5x + 5$

**Solution:**

$\frac{dy}{dx} = \frac{d}{dx}.3x^3 + \frac{d}{dx}.4x^2 - \frac{d}{dx}.5x + \frac{d}{dx}.5$

$= 3.3x^{3-1} + 4.2x^{2-1} - 5x^{1-1} + 0$

$= 9x^2 + 8x - 5$

iii) $y = \frac{x^3+1}{x}$

**Solution:**

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3+1}{x}\right)$$

$= \frac{d}{dx}.\frac{x^3}{x} + \frac{d}{dx}.\frac{1}{x}$

$= \frac{d}{dx}x^2 + \frac{d}{dx}.\frac{1}{x}$

$= \frac{d}{dx}x^2 + \frac{d}{dx}x^{-1}$

$= 2x - x^{-2}$

$= 2x - \frac{1}{x^2}$

**Rule 5:** The Product Rule

The derivative of a product, $y = f(x).g(x)$ is equal to the first function multiplied by the derivative of the second function plus the second function multiplied by the derivative of the first function, i.e.,

$$\frac{d}{dx}[f(x).g(x)] = f(x).\frac{d}{dx}[g(x)] + g(x).\frac{d}{dx}[f(x)]$$

or, $\frac{d}{dx}[f(x).g(x)] = f(x).\ [g'(x)] + g(x).\ [f'(x)]$

**Example 5**:

i) $y = 5x^4(3x - 7)$

**Solution:**

$$\frac{d}{dx}[5x^4(3x-7)] = 5x^4.\frac{d}{dx}[3x-7] + (3x-7).\frac{d}{dx}[5x^4]$$

or, $\frac{dy}{dx} = 5x^4.3 + (3x-7).20x^3$

or, $\frac{dy}{dx} = 15x^4 + 60x^4 - 140x^3$

or, $\frac{dy}{dx} = 75x^4 - 140x^3$

ii) $y = (4x^3 - 5x + 7)(3x^4 - 2x^3 + 2)$

**Solution:**

$$\frac{d}{dx}[(4x^3 - 5x + 7).(3x^4 - 2x^3 + 2)]$$

$$= (4x^3 - 5x + 7).\frac{d}{dx}[(3x^4 - 2x^3 + 2)]$$

$$+ \quad (3x^4 - 2x^3 + 2).\frac{d}{dx}(4x^3 - 5x + 7$$

$$\frac{dy}{dx} = (4x^3 - 5x + 7)(12x^3 - 6x^2) + (3x^4 - 2x^3 + 2).(12x^2 - 5)$$

$$= 84x^6 - 48x^5 - 75x^4 + 120x^3 - 18x^2 - 10$$

**Rule 6:** The Quotient Rule

The derivative of the given expressionis $y = \frac{f(x)}{g(x)}$ equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator (where $g(x) \neq 0$). i.e.,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

i.e., $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).\ f'(x) - f(x)\,g'(x)}{[g(x)]^2}$

**Example 6**:

Find dy/dx from

i) $y = \frac{2x+1}{x-1}$

**Solution:**

$$\frac{dy}{dx} = \frac{(x-1)\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)(2) - (2x+1)(1)}{(x-1)^2} = \frac{-3}{(x-1)^2}$$

ii) $y = \frac{x^3-x^2+1}{x^2+1}$

**Solution:**

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d}{dx}(x^3-x^2+1) - (x^3-x^2+1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(3x^2-2x) - (x^3-x^2+1)(2x)}{(x^2+1)^2} = \frac{x^4+3x^2-4x}{(x^2+1)^2}$$

**Rule 7:**Chain Rule

The derivative $(dy/dx)$ of a functionof a function, $y = f(u)$, where $u = g(x)$, is equal to the derivative of the first function with respect to $u$ times the derivative of the second function with respect to $x$. i.e.,

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

**Example 7**: Find $\frac{dy}{dx}$ of the following

i) if $y = u^2$ and $u = 2x^3 + 5x + 1$

**Solution:**

$$\frac{dy}{du} = \frac{d}{du}u^2$$

or, $\frac{dy}{du} = 2u$

*Also,* $\frac{du}{dx} = \frac{d}{dx}.(2x^3 + 5x + 1)$

or, $\frac{du}{dx} = 6x^2 + 5$

or, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 2u.(6x^2 + 5)^2$

Putting the value of u in $\frac{dy}{dx}$,

$$\frac{dy}{dx} = 2(2x^3 + 5x + 1).(6x^2 + 5)$$

ii) $y = 4u^3$ and $u = 12x^2 + 5$

**Solution:**

$dy/du = 12u^2$

$du/dx = 24x$

So, $dy/dx = 12u^2(24x)$

or, dy/dx = $288xu^2$

putting the value of u in dy/dx,

dy/dx= $288x(12x^2+5)^2$

iii) y= $\sqrt{u}$ and u= $5 + 7x + x^3$

**Solution:**

$$\frac{dy}{du} = \frac{d}{du}.u^{\frac{1}{2}}$$

or, $\frac{dy}{du} = \frac{1}{2}.u^{-\frac{1}{2}}$

$and\ \frac{du}{dx} = \frac{d}{dx}.(5 + 7x + x^3)$ or, $\frac{du}{dx} = 0 + 7 + 3x^2 = 3x^2 + 7$

Thus, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ or, $\frac{dy}{dx} = \left(\frac{1}{2}.u^{-\frac{1}{2}}\right).(3x^2 + 7)$

$putting\ the\ value\ of\ u\ in \frac{dy}{dx}$,

$$\frac{dy}{dx} = \left(\frac{1}{2}.(5 + 7x + x^3)^{-\frac{1}{2}}\right).(3x^2 + 7)$$

$or, \frac{dy}{dx} = \frac{(3x^2 + 7)}{2\sqrt{(5+7x+x^3)}}$

**Note:** The chain rule is particularly helpful in finding the derivative of quantities raised to powers, as the derivative of $[f(x)]^n$. Let $y = u^n$ and $u = f(x)$. Applying the chain rule, we obtain,

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = nu^{n-1}.f'(x)$

Therefore, $\frac{d}{dx}.[f(x)]^n = n\,[f(x)]^{n-1}.f(x)$

iv) $\frac{(2x+5)^4}{(5x-7)^4}$

**Solution:**

Let y = $\frac{(2x+5)^4}{(5x-7)^4}$. Then $y$ can be written as $y = u^4$, where $u = \frac{2x+5}{5x-7}$.

Thus, by chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Now, $\frac{dy}{du} = 4u^3$

Also, $\frac{du}{dx} = \frac{d}{dx}.\frac{2x+5}{5x-7}$.

Applying quotient rule,

$$\frac{du}{dx} = \frac{(5x-7)(2) - (2x+5)(5)}{(5x-7)^2}$$

$$\Rightarrow \frac{du}{dx} = \frac{(10x-14) - (10x+25)}{(5x-7)^2} = \frac{-39}{(5x-7)^2}$$

Thus, $\frac{dy}{dx} = 4u^3 . \frac{-39}{(5x-7)^2}$

Putting the value of u,

$$\frac{dy}{dx} = 4\left(\frac{2x+5}{5x-7}\right)^3 . \frac{-39}{(5x-7)^2} = -156\,\frac{(2x+5)^3}{(5x-7)^5}$$

**Check Your Progress 2**

1) Find $\frac{d}{dx}(3x+5)$

2) Compute is the derivative of $x^n$

3) What is the Product Rule?

4) Using chain rule differentiate $y = (3x+1)^2$.

5) What is quotient rule of differentiation?

## 7.4 STANDARD DERIVATIVES

Logarithmic and exponential functions are the most commonly used functions after algebraic functions. The standard derivatives of logarithmic and exponential functions are given below:

**i) Derivative of logarithm function**

If y = log x

then, $\frac{dy}{dx} = \frac{1}{x}$

(When base of the log is not mentioned, it is taken as e)

**ii) Derivative of exponential function**

If y = $e^x$

then, dy/dx = $e^x$

iii) **Derivative of $a^x$**, where a is constant

If y = $a^x$

then, dy/dx = $a^x . \log_e a$

**Note:** the chain rule is also useful in finding the derivative of y = log u and y = $e^u$ where u is an appropriate function of x. then we can find differentiation by applying chain rule. For example, if y = log u and u= f(x), then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{u}.\frac{du}{dx}$$

and if y = $e^u$, u = f(x)

$$\frac{dy}{dx} = e^u\frac{du}{dx}$$

**Example 8**:

Find $\frac{dy}{dx}$ of the following

i) y = log ($5x^2$ + 7)

**Solution:** This function is of the form y = log u, where u = $5x^2$ + 7. Thus,

$$\frac{dy}{dx} = \frac{1}{5x^2 + 7}.\frac{d}{dx}(5x^2 + 7) = \frac{1}{5x^2 + 7}.(10x)$$

$$= \frac{10x}{5x^2 + 7}$$

ii) y = $x^2$ log (3x + 7)

**Solution:** applying product rule

$$\frac{dy}{dx} = x^2\frac{d}{dx}log(3x+7) + \log(3x+7)\frac{d}{dx}x^2$$

or, $\frac{dy}{dx} = x^2\left(\frac{1}{3x+7}\right)(3) + \log(3x+7).(2x) = \frac{3x^2}{3x+7} + 2x\log(3x+7)$

iii) y = $e^{5x^2+4x+8}$

**Solution:**

$$\frac{dy}{dx} = e^{5x^2+4x+8}\frac{d}{dx}.(5x^2 + 4x + 8) = e^{5x^2+4x+8}(10x+4)$$

iv) y = $\frac{\log x}{x^2}$

**Solution:** Applying the quotient rule,

$$\frac{dy}{dx} = \frac{x^2.\frac{d}{dx}.(\log x) - \log x.\frac{d}{dx}(x^2)}{x^4} = \frac{x^2.\left(\frac{1}{x}\right) - \log x.2x}{x^4} = \frac{x - 2x\log x}{x^4} = \frac{1-2\log x}{x^3}$$

v) y = $x^3$ + $8^x$ + log x

**Solution:** $\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}8^x + \frac{d}{dx}\log x = 3x^2 + 8^x \ \log e^8 + \frac{1}{x}$

**Check Your Progress 3**

1) Find the derivative of $g(t) = 5^t$

2) What is the derivative of $y = 3x^2 + 2e^x$?

3) Differentiate $f(t) = \frac{1+5t}{\ln(t)}$

## 7.5 DIFFERENTIATION OF IMPLICIT FUNCTIONS

Given a relation $f(x, y) = 0$, if it is not possible to express $y$ in terms of $x$ or vice-versa, then that relation is called an implicit function. To determine $dy/dx$ in such an equation, we differentiate both sides of the equation and obtain an expression for $dy/dx$ from resulting equation. The method is best illustrated with the help of following examples:

**Example 9**: Find $dy/dx$ of the following:

i) $x3 + y3 = xy$

**Solution:** By treating $y$ as a function of $x$ and differentiating both sides with respect to x, we get

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(xy)$$

$$or, 3x^2 + 3y^2.\frac{dy}{dx} = x.\frac{dy}{dx} + y$$

Rearranging, $3y^2\frac{dy}{dx} - x\frac{dy}{dx} = y - 3x^2$

or, $(3y^2 - x)\frac{dy}{dx} = y - 3x^2$.

$$Thus, \frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}$$

ii) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

**Solution:** differentiating both sides with respect to (w.r.t.) x, we get,

$$a.\frac{d}{dx}(x^2) + 2h\frac{d}{dx}(xy) + b\frac{d}{dx}(y^2) + 2g\frac{d}{dx}(x) + 2f\frac{dy}{dx} + \frac{d}{dx}c = \frac{d}{dx}0$$

or, $2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} + 0 = 0$

or, $\frac{dy}{dx}(2hx + 2by + 2f) + (2ax + 2hy + 2g) = 0$

or, $\frac{dy}{dx}(2hx + 2by + 2f) = -(2ax + 2hy + 2g)$

or, $\frac{dy}{dx} = \frac{-(2ax+2hy+2g)}{2hx+2by+2f} = -\frac{ax+hy+g}{hx+by+f}$

iii) $x\sqrt{1+y} + y\sqrt{1+x} = 0$

**Solution:** Rearrange the equation

$x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, we get

$x^2 (1 + y) = y^2 (1+ x)$

or, $x^2 + x^2y = y^2 + y^2x$

or, $x^2 - y^2 + x^2y - y^2x = 0$

or, $(x-y)(x+y) + xy(x-y) = 0$ (as $x^2 - y^2 = (x-y)(x+y)$)

or, $(x - y)(x + y + xy) = 0$

or $(x + y + xy) = 0$

(because $x \neq y$ $Thus\ x - y \neq 0$

Now, differentiating both sides,

$$\frac{dy}{dx} = \frac{d}{dx}\left[-\left(\frac{x}{1+x}\right)\right]$$

or, $\frac{dy}{dx} = -\frac{(1+x).1-x.1}{(1+x)^2} = -\frac{1}{(1+x)^2}$

**Check Your Progress 4**

1) How to differentiate an implicit function?

2) Find $\frac{dy}{dx}$, if you have 3x+2y=4

3) Explain the method of finding $\frac{dy}{dx}$, if given $\frac{x}{y^3} = 1$

## 7.6 DIFFERENTIATION USING LOGARITHMS

The method of logarithm differentiation is used to differentiate functions of the form $y = f(x)^{g(x)}$. With the help of this method, take natural log on both sides of the equation $y = f(x)^{g(x)}$ to obtain $log\ y = log\ [f(x)^{g(x)}]$. By using the properties of logarithm, we simplify the equation $og\ y = log\ [f(x)^{g(x)}]$ and then differentiate both sides with respect to $x$ and then solve for $dy/dx$. This method can also be used to differentiate functions which are the product of several functions.

**Note:Important properties of logarithm**

log (m.n) = log m + log n

log ( m/n) = log m - log n

log ($m^n$) = n log m

log ($e^x$) = x

log e = 1

**Example 10**: Find dy/dx:

i) $y = x^x$

**Solution:**

Taking log on both sides of the equation $y = x^x$, we get

$\log y = \log [x^x]$

$\log y = x \log x$

Differentiating both sides of the equation,

$$\frac{1}{y}\frac{dy}{dx} = x\ \frac{d}{dx}\ (\log x) + \log x\ \frac{d}{dx}x$$

or, $\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \log x$

or, $\frac{1}{y}\frac{dy}{dx} = 1 + \log x$

or, $\frac{dy}{dx} = y\ ( 1 + \log x)$.

Putting the value of y, $\frac{dy}{dx} = x^x\ ( 1 + \log x)$

ii) $y = \frac{(x+1)^{10}\ (x+5)(x+6)}{(x+7)^2}$

**Solution:** Taking log on both sides of the given equation,

$\log y = \log\left[\frac{(x+1)^{10}\ (x+5)(x+6)}{(x+7)^2}\right]$

$\log y = \log (x+1)^{10} + \log (x+5) + \log (x+6) – \log (x+7)^2$

$\log y = 10 \log (x+1) + \log (x+5) + \log (x+6) – 2 \log (x+7)$

Differentiating both sides w.r.t.$x$, we get,

$$\frac{1}{y}.\frac{dy}{dx} = \frac{10}{x+1} + \frac{1}{x+5} + \frac{1}{x+6} - \frac{2}{x+7}$$

or, $\frac{dy}{dx} = y.\left[\frac{10}{x+1} + \frac{1}{x+5} + \frac{1}{x+6} - \frac{2}{x+7}\right]$

$=\left[\frac{(x+1)^{10}\ (x+5)(x+6)}{(x+7)^2}\right].\left[\frac{10}{x+1} + \frac{1}{x+5} + \frac{1}{x+6} - \frac{2}{x+7}\right]$

iii) $x^y = y^x$

**Solution**: Taking log on both sides

$\log x^y = \log y^x$

$y \log x = x \log y$

Differentiating w.r.t.$x$, we get

$$y.\frac{d}{dx}\log x + \log x\,\frac{dy}{dx} = x.\frac{d}{dx}\,logy + logy\,\frac{d}{dx}\,x$$

or, $y.\frac{1}{x} + \log x\frac{dy}{dx} = x.\frac{1}{y}\frac{dy}{dx} + \log y$

or, $\log x\frac{dy}{dx} - x.\frac{1}{y}\frac{dy}{dx} = \log y - \frac{y}{x}$

or, $\left[\log x - \frac{x}{y}\right]\frac{dy}{dx} = \log y - \frac{y}{x}$

or, $\frac{[y\log x - x]}{y}\frac{dy}{dx} = \frac{x\log y - y}{x}$

or, $\frac{dy}{dx} = \frac{x\log y - y}{ylog\,x - x}.\frac{y}{x}$

iv) $y = (3x^2+5)^{1/x}$

**Solution:**Take log on both sides to get

$\log y = \log (3x^2+5)^{1/x}$

$\log y = \frac{1}{x}\log (3x^2+5)$

Differentiating both sides w.r.t.$x$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}.\frac{d}{dx}\log(3x^2 + 5) + \log(3x^2 + 5).\frac{d}{dx}.\frac{1}{x}$$

or, $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}.(6x) + \log(3x^2 + 5).(-x^{-2})$

or, $\frac{1}{y}\frac{dy}{dx} = 6 - \frac{\log(3x^2 + 5)}{x^2}$

or, $\frac{dy}{dx} = y.\left[6 - \frac{\log(3x^2 + 5)}{x^2}\right]$

or, $\frac{dy}{dx} = (3x^2+5)^{1/x}\left[6 - \frac{\log(3x^2 + 5)}{x^2}\right]$

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**Check Your Progress 5**

1) When would you prefer differentiation using logarithms?

2) Explain the steps you would follow to differentiate the function $y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$.

3) Simplify the natural logarithm of $10^x$**?**

## 7.7 DERIVATIVE OF INVERSE FUNCTION

Let $y = f(x)$ be a function of $x$ and suppose that we can solve this equation for $x$ in terms of $y$.So, we may write $x$ as a function of $y$, i.e., $x = g(y)$. Then $g(y)$ is called the inverse of $f(x)$. If $y = f(x)$ is a differentiable function at $x$ such that $f'(x) \neq 0$, then $x = g(y)$ is also differentiable at the corresponding value of $y$ and $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$.

**Example 11**:Find $dx/dy$ from the following

i) $y = \frac{3x+2}{x+1}$,

**Solution:** Differentiating $y$ w.r t. $x$

$$\frac{dy}{dx} = \frac{(x+1)(3) - (3x+2).1}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(x+1)^2}$$

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{\frac{1}{(x+1)^2}} = (x+1)^2$$

ii) $y = (x + 2)^{1/2}$

**Solution:** Differentiating $y$ w.r t. $x$

$$\frac{dy}{dx} = \frac{d}{dx}(x+2)^{\frac{1}{2}}$$

or, $\frac{dy}{dx} = \frac{1}{2}.(x+2)^{-\frac{1}{2}}.1$

or, $\frac{dy}{dx} = \frac{1}{2(x+2)^{\frac{1}{2}}}$

or, $\frac{dy}{dx} = \frac{1}{2\sqrt{(x+2)}}$

or, $\frac{dx}{dy} = \frac{1}{dy/dx} = 2\sqrt{(x+2)}$

**Check Your Progress 6**

1) What is an inverse function?

2) Prove that derivative of $y = f^{-1}(x)$ with respect to $x$ is $\frac{1}{f'(y)}$

3) If $y = f(x)$ is a differentiable function at $x$ such that $f'(x) \neq 0$, and $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, you are dealing with what kind of function?

## 7.8 DIFFERENTIATION OF PARAMETRIC FUNCTION

In parametric functions we use to express curves that can't be written in single-equation form. In the process, rather than defining $x$ and $y$ in terms of one another, we define these in terms of another variable *t* (known as the parameter). So, we have $x = f(t)$ and $y = g(t)$. These functions are then related to one another through the parameter. We can differentiate parametric equations using the chain rule.

Let $x = f(t)$ and $y = g(t)$ be two derivable functions where $t$ is a parameter.See that $dx/dt$ and $dy/dt$ both exist. So, $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$ ,provided $dx/dt \neq 0$.

**Example 12**: Find $\frac{dy}{dx}$ for the following:

i) y= $t^3$ and x = $t^2$

**Solution:**Since y= $t^3$, $\frac{dy}{dt} = 3t^2$.

Again from $x = t^2$ we get $dx/dt = 2t$

Hence, $dy/dx = (dy/dt)/(dx/dt) = (3t^2)/2t = \frac{3}{2}\, t$

ii) $x = (1-t)/(1+t)$ and $y = 2t^3 + 4^t$

**Solution:** $x = (1-t)/(1+t)$

$$dx/dt = ((1+t)(-1) - (1-t).1)/(1+t)^2$$

$$dx/dt = (-2)/(1+t)^2$$

$$y = 2t^3 + 4^t$$

$$dy/dt = 6t^2 + 4^t \; log4$$

Therefore,

$$dy/dx = (dy/dt)/(dx/dt)$$

$$dy/dx = (6t^2 + 4^t \; log4)/((-2)/(1+t)^2)$$

$$\frac{dy}{dx} = \frac{(6t^2 + 4^t \, log4).(1+t)^2}{-2}$$

---

**Check Your Progress 7**

1) What is a parametric function?

2) How parametric functions are differentiated?

3) Differentiate the following parametric function:

   $x = 2at^2$ and $y = 4at$.

## 7.9 LET US SUM UP

In this unit we have discussed the techniques of differentiation. Starting with the meaning of derivative of a function as the rate of change in the values of a function, we went on to learn that derivative can be expressed as $dy/dx$ or $f'(x)$ or $f_x$ when $y$ is differentiated with respect to $x$ for the function, $y = f(x)$.

There are certain basic rules that can be used to find or compute the derivative of several types of functions viz., The Constant Function rule, The Power Function Rule, The Linear Function Rule, The Rule for Sums and

Differences, The Product Rule, The Quotient Rule,Rule for a Function of a Function. Further, we are exposed to Derivative of a Logarithmic Function (i.e., if $y = logx$, then $dy/dx = 1/x$)as well as derivative of an exponential function (viz., $y = e^x \Rightarrow dy / dx = e^x$). Towards the concluding part of the unit we have covered the differentiation of inverse function and parametric function. In the process it is seen that in case of an inverse function, if y = f(x) is a differentiable function at x such that f' (x) ≠ 0, then x = g(y) is differentiable at the corresponding value of y and $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$. Considering differentiation of the parametric function, we have learnt that if $x = f(t)$ and $y = g(t)$ are two derivable functions with $t$ is a parameter, $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$ ,provided $dx/dt \neq 0$.

## 7.10 KEY WORDS

**Chain Rule:** A method for finding the derivative of a composition of functions.

**Derivative**: Measuring rate of change of function.

**Exponential Function**:Afunction of the following form: $f(x) = a^x$ where *x* is a variable and *a* is a constant called the base of the function.

**Implicit Differentiation:** A method for finding the derivative of an implicitly defined function or relation.

**Implicit Function:** Defined implicitly by an implicit equation, i.e., by associating one of the variables with the others.

**Inverse Function:** A function that "reverses" another function.

**Limit of a Function**: Analysis of the behavior of a function near a particular point.

**Logarithmic Differentiation:** A method for finding the derivative of functions such as $y = x^{\sin x}$ .

**Logarithmic Function:** A function such as $y = \log_a x$ or $y = \ln x$. It is inverse of an exponential function such as $y = a^x$ or $y = e^x$.

**Parametric Function**: A set of equations that expresses a set of quantities as explicit functions of a number of independent variables, known as "parameters."

**Power Function**: A function written as $y = f(x) = kx^n$ where ***k*** and ***n*** are real numbers.

**Power Rule:** The formula for finding the derivative of a power of a variable.

**Product Rule:** A formula for the derivative of the product of two functions.

**Quotient Rule:** A formula for the derivative of the quotient of two functions.

## 7.11 SOME USEFUL BOOKS

• Allen, R.G.D., "Mathematical Analysis for Economists", London: English Language Book Society and Macmillan, 1974.

- Archibald, G.C., Richard G.Lipsey. "An Introduction to a Mathematical Treatment of Economics", Delhi: All India Traveller Bookseller, 1984
- Chiang, A. and Kalvin Wainwright, Fundamental Methods of Mathematical Economics (Paperback), Mac Grow Hill, 2017.
- Dowling, Edward,T. "Schaum's Outline Series: Theory and Problems of Mathematics for Economists", New York: McGraw Hill Book Company, 1986.
- K. Sydsaeter and P. Hammond, Mathematics for Economic Analysis, PearsonEducational Asia, Delhi, 2002.
- Yamane, Taro, "Mathematics for Economists: An Elementary Survey", New Delhi: Prentice Hall of India Private Limited, 1970.


## 7.12 ANSWER OR HINTS TO CHECK YOUR PROGRESS

**Check Your Progress 1**

1) Rate of change of functions of one variable with respect to another on which it depends.

2) Function is differentiable only if the limit exists and the function must be continuous at that point.

3) Use of algebra to find a general expression for the slope of a curve.

**Check Your Progress 2**

1) 3

2) $nx^{n-1}$

3) Take $y = f(x).g(x)$ and find the derivative equals to the first function is multiplied by the derivative of the second function plus the second function multiplied by the derivative of the first function.

4) Differentiate ``the square" first, leaving (3*x*+1) unchanged. Then differentiate (3*x*+1). You will get $\frac{dy}{dx} = 2(3x+1)^{2-1}.\frac{d}{dx}(3x+1) = 2(3x+1).(3) = 6(3x+1)$

5) The quotient rule is a formal rule for differentiating functions where one function is divided by another.

   Thus, $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$

**Check Your Progress 3**

1) $(ln5)5^t$

2) $6x + 2e^x$

3) $\frac{5\ln(t) - (1+5t)\left(\frac{1}{t}\right)}{[\ln(t)]^2}$

**Check Your Progress 4**

1) Differentiate both sides of the equation and obtain an expression for dy/dx from resulting equation.

2) -3/2

3) First, solve the equation for y to get $y^3 = x$. So $y = x^{\frac{1}{3}}$. Then get $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

**Check Your Progress 5**

1) When required to find the *derivatives* of some complicated functions, we can simplify these by *using logarithms*.

2) Take logarithms of both sides and simplify to get

$\ln y = ln(x^5) - ln(1 - 10x) - ln(\sqrt{x^2 + 2})$. Then proceed just as you do for differentiation of implicit function to get $\frac{dy/dx}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}$.

3) $x$

**Check Your Progress 6**

1) Inverse function is a function that reverses another function. That is, $f(x) = y$ if and only if $g(y) = x$.

2) Take $x = f(y)$. Differentiate both sides with respect to $y$ such that

$\frac{dx}{dy} = \frac{d}{dy}(f(y))$, i.e., $\frac{dx}{dy} = f'(y)$ or, $\frac{dy}{dx} = \frac{1}{f'(y)}$.

3) Inverse function.

**Check Your Progress 7**

1) A class of functions exists which defined in terms of another variable *t* (known as the parameter).

2) Through chain rule.

3) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Since $\frac{dx}{dt} = 4at$, we get $\frac{dx}{dt} = \frac{1}{4at}$.

Again, $\frac{dy}{dt} = 4a$. Hence $\frac{dy}{dx} = 4a \times \frac{1}{4at} = \frac{1}{t}$

## 7.13 EXERCISES WITH ANSWER/HINTS

1) Differentiate with respect to x

   i) $y = 3x^5$

   ii) $y = x^3 + 3x$

   iii) $f(x) = 7x^2 - 8x + 5$

   iv) $f(x) = 8(10 - x^4)$

2) Find the derivative of the following functions:

   i) $y = \frac{3x+2}{4x-5}$

ii) $y = \frac{x^5 - x^3 + 20}{x^2 + 10}$

iii) $y = \frac{1 + \sqrt{x}}{\sqrt{x} - 1}$

3) Use the chain rule to find dy/dx of the following functions

i) $y = 2u^2 - 7u$ and $u = 5x - x^3$

ii) $y = u^2$ and $u = \frac{x+1}{x-1}$

iii) $y = \log u$ and $u = \sqrt{x} + 1/\sqrt{x}$

iv) $y = e^u$ and $u = x^6 - x^2 + 1$

4) Find dy/dx of the following

i) log (logx)

ii) $\log x^3$

iii) $(\log x)^3$

iv) $\text{Log}\,(3x^2 + 2x - 5)$

v) $\text{Log}\,(xe^x)$

vi) $(e^{3x} + 1)^4$

vii) $4^{2x^3 + 5x}$

viii) $e^{x^2} \log x$

5) if $y = \log\left[x + \sqrt{1 + x^2}\right]$, prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$

6) if $y = \left[\sqrt{x+1} + \sqrt{x-1}\right]$, prove that $\sqrt{x^2 - 1}\,\frac{dy}{dx} = \frac{1}{2}y$

**Answers to Exercises**

1) i) $15x^4$ ii) $3x^2 + 3$ iii) $14x - 8$ iv) $32x^3$

2) i) $23/(4x-5)^2$ ii) $\frac{3x^6 + 49x^4 - 30x^2 + 40x}{(x^2+10)^2}$ iii) $\frac{-\frac{1}{\sqrt{x}}}{(\sqrt{x}-1)^2}$

3) i) $(20x - 4x^3 - 7)(5 - 3x^2)$ ii) $\frac{-4\,(x+1)}{(x-1)^2}$ iii) $\frac{(x-1)}{2x\,(x+1)}$ iv) $(6x^5 - 2x)\,e^{x^6 - x^2 + 1}$

4) i) 1/ (xlogx) ii) 3/x iii) $\frac{3(logx)^2}{x}$ iv) $\frac{6x+2}{3x^2 + 2x - 5}$ v) (x+1)/x vi) $12e^{3x}(e^{3x+1})^3$

vii) $\log 4\,(6x^2+5)\,4^{2x^3+5x}$ viii) $e^{x^2}\left[\frac{1}{x} + 2x\,logx\right]$

# UNIT 8 MAXIMA AND MINIMA OF FUNCTIONS

**Structure**



## 8.0 OBJECTIVES

After going through this unit, you will be able to:

- Understand higher order derivatives;
- Find the intervals on which a given function is increasing or decreasing; and
- Find the maximum and minimum values of the function.

## 8.1 INTRODUCTION

In the preceding unit, we have seen the derivatives of functions. Going further from that level, an attempt is made in the present unit to use derivatives for finding the extreme points of the graph of a function. For example, we will show how the sign of the derivative is used to settle questions about intervals over which the graph of a function is rising or falling. The principal application of this is in locating high or low points on graphs and, in turn, these points are used to determine the maximum and minimum values attained by the function.

## 8.2 HIGHER ORDER DERIVATIVES

If $f(x)$is a differentiable function of '$x$', then $f'(x)$ or $\frac{dy}{dx}$ is the first derivative or first order derivative of $y = f(x)$with respect to (w.r.t) '$x$'.

Since the derivative of function is also a function, another derivative can also be found. The second order derivative, or second derivative, is the derivative of the first derivative of the function $f(x)$. Other notations are:

$\frac{d\,(^{dy}/_{dx})}{dx}$ or, $\frac{d\,(f'(x))}{dx}$ or, $\frac{d^2 y}{dx^2}$ or, $y_2$ or $f''(x)$.

Since $f''(x)$ is also a function, its derivative can also be found which is denoted as $f'''\,(x)$. For higher order derivatives, superscripts can be used i.e., $f^4$ = fourth derivative etc.

**Example 1**: $f(x) = 5x^4 + 6x^3 + 2x + 1$

$f'(x) = 20x^3 + 18x^2 + 2$

$f''(x) = 60x^2 + 36x$

$f'''(x) = 120x + 36$

**Example 2**: $y = 9x^4 + 7x^3 + 2x^2 + 5x + 7$

Find all possible order of derivatives.

**Solution**:

$Y_1 = dy/dx = 36x^3 + 21x^2 + 4x + 5$

$Y_2 = d^2y/dx^2 = 108x^2 + 42x + 4$

$Y_3 = d^3y / dx^3 = 216x + 42$

$y_4 = d^4 y / dx^4 = 216$

$Y_5 = d^5 y / dx^5 = 0$

$y_6 = d^6 y / dx^6 = 0$

Derivatives of fifth and higher orders will be zero.

**Example 3**: If $y = \left(\frac{1}{x}\right)^x$ find out second order derivative

**Solution:** Given $y = \left(\frac{1}{x}\right)^x$, take log on both sides to get

$\log y = x \log\frac{1}{x}$

$= x(\log 1 - \log x)$

$= -x\log x$ (as $\log 1 = 0$)

Differentiating both sides w.r.t. $x$

$$\frac{1}{y}.\frac{dy}{dx} = -\left[x\left(\frac{1}{x}\right) + \log x\,.1\right]$$

$= -\,(1 + \log x)$

or, $\frac{dy}{dx} = -y.(1 + logx)$

Again, differentiating w.r.t. $x$,

$$\frac{d^2y}{dx^2} = -\left[y\left(\frac{1}{x}\right) + (1 + \log x)\,.\frac{dy}{dx}\right]$$

Putting the value of $dy/dx$,

$$\frac{d^2y}{dx^2} = -\left[y\left(\frac{1}{x}\right) + (1 + \log x)\,.(-y.(1 + logx))\right]$$

or, $\frac{d^2y}{dx^2} = -\left[\frac{y}{x} - (1 + logx)^2\, y\right]$

or, $\frac{d^2y}{dx^2} = -y\left[\frac{1}{x} - (1 + logx)^2\right]$

Putting the value of $y$

$$\frac{d^2y}{dx^2} = -\left(\frac{1}{x}\right)^x\left[\frac{1}{x} - (1 + logx)^2\right]$$

**Interpretation of Derivatives of Different Orders**

The first derivative of the function $y = f(x)$ i.e., $dy/dx$, measures the rate of change of $y$ due to change in $x$. It gives the slope of the curve at a point.

1) If first order derivative is positive, i.e., if dy/dx > 0 at a point, then it implies that $y$ increases for a small increase in the value of $x$.

2) If first order derivative is zero, i.e., if dy/dx = 0 at a point, then it implies $y$ does not change for a small increase in the value of $x$.

3) If first order derivative is negative, i.e., dy/dx< 0 at a point, then $y$ decreases for a small increase in the value of $x$.

   The second derivative, $d^2y/dx^2$ gives the rate of change of dy/dx. Thus, if:

   a) $d^2y/dx^2>0$, the function has increasing slope.

   b) $d^2y/dx^2<0$, the function has decreasing slope.

---

**Check Your Progress 1**

1) What do you look in a derivative to draw your inference on rising or falling graph of a function?

2) If the value of a function is decreasing when its input is increasing, then the graph is showing what curve?

3) The slope of a curve at a point can be known from which order derivative of a function?

4) You are given the second order derivative of a function. What can you comment on the slope of a curve on the basis of such a piece of information?

5) You are given the profit function of a business activity and asked to offer your suggestion on the rate of change of profit. What would you do?

6) Make a list of signs of derivatives to draw your inference on increasing and decreasing curves of a function.

## 8.3 INCREASING AND DECREASING FUNCTIONS

Broadly speaking, a function, say, $f(x)$, is increasing when y increases as $x$ gets larger (i.e., looking left to right) and $f(x)$is decreasing when y decreases as $x$ gets larger. Such functions are of interest to us for determining rate of change of a variable. For example, we would be interested to know the speed of a car, or, rate of growing and declining population of a country.

### 8.3.1 Increasing Function

A function $y = f(x)$ is said to be increasing in an interval $[a, b]$, if $y$ increases as $x$ increases from $a$ to $b$.

If $x_1$ and $x_2 \in [a, b]$ and $x_2 \geq x_1$, then, $f(x_2) \geq f(x_1)$

Thus, a function is said to be an increasing function in $[a, b]$ if first order derivative is greater than zero (i.e.,$f'(x) > 0$) for all values of $x$ in an interval $[a, b]$. If $f'(q) > 0$, then the curve $y = f(x)$ increasesfrom left to right at the point $x = q$.

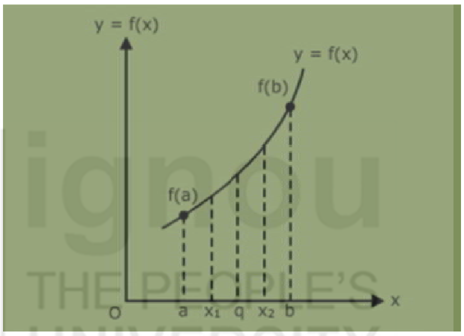



**Fig.8.1: Increasing Function**

### 8.3.2 Decreasing Function

A function $y = f(x)$ is said to be decreasing in the interval $[a, b]$ if $y$ decreases as value of $x$ increases in $[a, b]$ or vice versa. That is,

$$f(x_2) \leq f(x_1), \text{ when } x_2 \geq x_1 \text{ and } x_1 \text{and } x_2 \in [a, b]$$

If first order derivative is less than zero i.e., $f'(x) < 0$ for all values of $x$ in an interval $[a, b]$, then the function $y = f(x)$ is a decreasing function in $[a, b]$ or if $f'(q) < 0$, then the curve $y = f(x)$ falls from left to right at the point $x = q$.

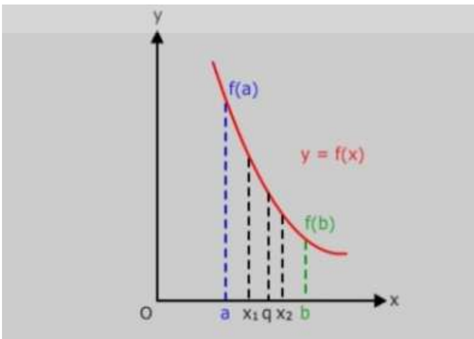



**Fig. 8.2: Downward Sloping Curve**

**Example 4**: Show that the function $y = f(x) = 3x^3 + 3x^2 + x - 1$ is increasing for allvalues of $x$.

**Solution**:$dy/dx = f'(x) = 9x^2 + 6x + 1$

$= (3x + 1)^2$

So, $dy/dx > 0$for all values of $x$.

Thus, the given function is increasing for all values of $x$.

**Example 5**: Find the interval in which $f(x)$ is (i)increasing (ii) decreasing

$f(x) = 2x^3 + 9x^2 + 12x - 1$.

**Solution**:$dy/dx = f'(x) = 6x^2 + 18x + 12$

$= 6(x^2+3x+2) = 6(x+1)(x+2)$

i) For $f(x)$ to be increasing, $f'(x)>0$

i.e., $6(x + 1)(x + 2) > 0$.

If $x < -2$, then $f'(x)$= 6 (-) (-)>0. That is, $f(x)$ is increasing

and if $x > -1$, then $f'(x)$= 6 (+) (+)>0. That is, $f(x)$ is increasing.

Hence, $f(x)$ is increasing when $x \in (-\infty, -2) U (-1, \infty)$

ii) If$f(x)$is decreasing, then $f'(x)<0$

i.e., $6(x+1)(x+2)<0$.

If $-2<x<-1$, then $f'(x)$= 6 (-)(+)<0. $f(x)$ is decreasing when $x \in (-2,-1)$.

Therefore, $f(x)$ is increasing when $x \in (-\infty,-2) U (-1,\infty)$ and $f(x)$ is decreasing when $x \in (-2,-1)$.

**Example 6**: Find the value of $x$ for which the given function

$f(x) = 2x^2 - 8x + 80$

is decreasing.

**Solution**: For the function to be decreasing, its first derivative, $f'(x)$ should be less than 0. That is

$f'(x) = 4x - 8 < 0$ for the function to be decreasing. So,

$4x - 8 < 0$,
or,

$4x < 8$

$x < 2$

Thus, the function is decreasing for $x$ less than 2 and increasing for more than 2 and has a stationary value at $x = 2$.

**Example 7**: Find the interval in which $f(x)$

$f(x) = -2x^3 + 3x^2 + 12x - 1$

is (i)increasing (ii) decreasing.

**Solution**: $f'(x) = -6x^2 + 6x + 12$

$= -6(x^2 - x - 2) = -6(x+1)(x-2)$.

i) For $f(x)$ to be increasing, $f'(x) > 0$

i.e., $-6(x+1)(x-2) > 0$.

If $-1 < x < 2$, then $f'(x) = (-)(+)(-) > 0$. That is, $f(x)$ is increasing.

Hence, $f(x)$ is increasing when $x \in (-1,2)$

ii) For $f(x)$ to be decreasing, $f'(x) < 0$

i.e., $-6(x+1)(x-2) < 0$.

If $x < -1$, then $f'(x) = (-)(-)(-) < 0$. So, $f(x)$ is decreasing

and if $x > 2$, then $f'(x) = (-)(+)(+) < 0$. That is, $f(x)$ is decreasing when $x \in (-\infty,-1) \cup (2,\infty)$.

Therefore, $f(x)$ is increasing when $x \in (-1,2)$ and $f(x)$ is decreasing when $x \in (-\infty,-1) \cup (2,\infty)$.

**Example 8**: Find the interval in which the function

$f(x) = x^4 - 2x^2$ is

i) increasing

ii) decreasing

**Solution**: $f(x) = x^4 - 2x^2$

$\Rightarrow f'(x) = 4x^3 - 4x$

$= 4x(x^2 - 1)$

$= 4x(x-1)(x+1)$

i) For $f(x)$ to be increasing, $f'(x)$ should be greater than zero.

$f'(x) = 4x(x-1)(x+1) > 0$

If $-1 < x < 0$ then $f'(x) = 4(-)(-)(+) > 0$. So, $f(x)$ is increasingand if $x > 1$, then $f'(x) = 4(+)(+)(+) > 0 \Rightarrow f(x)$ is increasing.

Hence, $f(x)$ is increasing when $x \in (-1,0) \cup (1,\infty)$.

ii) For $f(x)$ to be decreasing, $f'(x) < 0$

i.e., $4x(x-1)(x+1) < 0$.

If $-\infty < x < -1$, then $f'(x) = 4(-)(-)(-) < 0$. and if $0 < x < 1$, then $f'(x) = 4(+)(-)(+) < 0$. Therefore, $f(x)$ is decreasing when $x \in (-\infty,-1) \cup (0,1)$.

Thus, $f(x)$ is increasing when $x \in (-1,0) \cup (1,\infty)$ and $f(x)$ is decreasing when $x \in (-\infty, -1) \cup (0,1)$.

**Example 9**: The Revenue function of a firm is given by

$$R = [8{,}00{,}000 + (x - 300)^2]$$

Determine the values of $x$ for which total Revenue function is increasing and decreasing.

**Solution:** R'(x) = +2 ($x$ – 300)

For revenue function to be increasing R'($x$) > 0

i.e., $+2(x-300)>0$

Hence revenue function is increasing for more than/greater than 300 i.e., $x > 300$. For the Revenue function to be decreasing

R'($x$) < 0

i.e., $+2(x-300)<0$

when $x$=0 then R'($x$) < 0

and $x-300<0$

$x<300$.

Hence, the revenue function is decreasing for $x = 0$ *and* lying between 0 and 300 i.e., $0 \leq x< 300$.and it is stationary value at $x = 300$.

**Check Your Progress 2**

1) If a function is said to be an increasing function in interval $[a, b]$,what would you do to verify the claim?

2) Which order of derivative explains the slope of the function?

3) Explain a decreasing function on the basis of results obtained on first and second order derivatives.

## 8.4 MAXIMA AND MINIMA

The maxima or minima of a function is the largest orthe smallest value of the function, definedeither within a given range (the local or relativeextrema) or on the entire domain of a function. Maximum means upper bound or the largest possible value. The absolute maximum of a function is the largest number contained in the range of the function. That is, if $f(a)$ is greater than or equal to $f(x)$, for all $x$ in the domain of the function, then $f(a)$ is the absolute maximum. On the other hand, minimum means lower bound or least possible value. The absolute minimum of a function is the smallest number in its range and corresponds to the value of the function at the lowest point of its graph. If $f(a)$ is less than or equal to $f(x)$, for all $x$ in the domain of the function, then $f(a)$ is an absolute minimum.

### 8.4.1 First Derivative Test for Local Extreme Values

i) Find $f'(x)$

ii) Find all critical values of the function $f(x)$,i.e., all the values of $x$ where $f'(x)$=0.

iii) Each critical value of $x$, say $x = a$, determine the sign of the first derivative. If sign of the first derivative, $f'(x)$, changes from positive to negative as $x$ increases through $a$, then the function attains a local maximum at $x = a$. If $f'(x)$ changes from negative to positive as

$x$increases through $a$, then the function attains a local minimum at $x = a$.

iv) If $f'(x)$ does not change its sign as $x$ increases through $a$, then there is no local extremum at $x = a$. Such a point is called as point of inflection.

### 8.4.2 Second Derivative Test forLocal Extreme Values

A function, $f(x)$, is said to be

• maximum, if $f\,'(x) = 0$ and $f\,''(x) < 0$, at a critical value of $x = a$

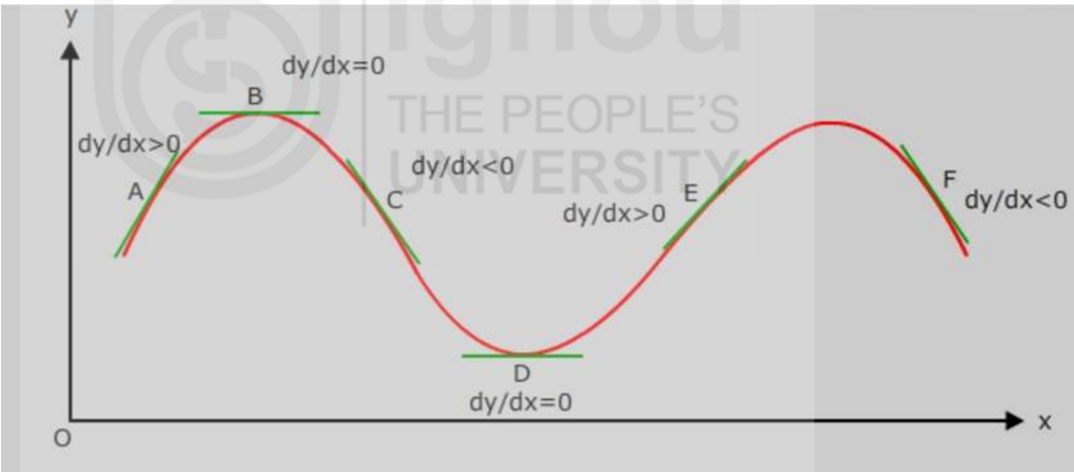
• minimum, if $f\,'(x) = 0$ and $f\,''(x) > 0$, at a critical value of $x = a$

A point where $f\,''(a) = 0$ and $f\,'''(a) \neq 0$ is called a point of inflection.

Thus,

| For a Relative Maximum | | For a Relative Minimum | |
|---|---|---|---|
| $dy/dx = 0$ | (First-order Condition) | $dy/dx = 0$ | (First-order Condition) |
| $d^2y/dx^2 < 0$ | (Second-order Condition) | $d^2y/dx^2 > 0$ | (Second-order Condition) |

Points of maxima and minima are shown in diagram below




A function is increasing at those points, where the first derivative is positive, i.e., at the point A & E in the diagram. A function is decreasing at those points where the first derivative is negative, i.e., at the point C & F in the diagram. At those points B & D, where the function is at a relative maximum or minimum, the first derivative of the function or slope of the function is equal to zero. This is essential condition for both a relative maximum and a relative minimum.

The second derivative measures the rate of change in the marginal function. If the first derivative is zero, indicating a zero slope and hence a plateau in the function, while second derivative is negative, it means the function is moving down from the plateau and should have been at a relative maximum. If the first derivative is zero and the second derivative is positive, it means that the function is moving upward from a valley and the function has a relative minimum. A point at which the first derivative equals zero is called a critical value or, a stationary value or an extreme value point. If the second derivative is equal to zero, but the third derivative does not equal zero, then the critical value is neither a maximum nor a minimum. There will be an inflexion point at which the function alters its rate of change. Point A in the figure is an inflexion point.

### 8.4.3 Steps for Maxima/Minima Using Second Order Derivative

i) Find the first derivative of the function $f(x)$. i.e., Calculate $f'(x)$.

ii) Set the first derivative equal to zero and find the real roots of the equation $f'(x)= 0$ to find the critical points of the independent variable, $x = a$.

iii) Find the second derivative $f''(x)$ of the function.

iv) Substitute each critical value in the second derivative. After the substitution of critical values, second derivative can be positive, negative or zero.

v) If $f''(a) = 0$, find $f'''(x)$, the third-order derivative of the function. Third Derivative can either be zero or non-zero. Information regarding maximization or minimization of a function can only be gathered at an even-ordered derivative whereas information regarding point of inflexion can be collected at an odd-ordered derivative.

**Example 10:** Find all the points of local maxima and minima of the function $f(x) = x^4 -8x^3 -22x^2 -24x+ 1$ by using first derivative rule.

**Solution:** Given, $f(x) = x^4 -8x^3 -22x^2 -24x+ 1$,

$f'(x) = 4x^3 -24x^2 - 44x -24$

$= 4(x-1)(x-2)(x-3)$.

Putting $f'(x)=0$,

$4(x-1)(x-2)(x-3) = 0$. So,

either x=1, x=2 or x=3.

If $x<1$, then $f'(x) = 4 (-)(-)(-)<0$;

If $1<x<2$, then $f'(x) = 4 (+)(-)(-)>0$;

If $2<x<3$, then $f'(x) = 4 (+)(+)(-)<0$;

If $x>3$, then $f'(x) = 4 (+)(+)(+)>0$.

When $x = 1$, $f'(x)$ changes sign from negative to positive as $x$ increases through

1. Hence, local minimum is at $x = 1$. The corresponding minimum value of the function $f(x)$ is $f(1) = 1-8+22-24+1 = -8$.

When $x = 2$, $f'(x)$ changes sign from positive to negative as $x$ increases through

2. Hence, local maximum is at $x = 2$. The corresponding maximum value of the function $f(x)$ is $f(2) = 16-64+88-48+1 = -7$.

When $x = 3, f'(x)$ changes sign from negative to positive as x increases through

3. Hence, local minimum is at $x = 3$. The corresponding minimum value of the function is $f(3)$ = 81-216+198-72+1 = -8.

**Example 11**: Find all the points of local maxima and minima of the function $f(x) = 2x^3 - 15x^2 - 36x + 60$ by using first derivative rule.

**Solution:** given, $f(x) = 2x^3 - 15x^2 + 36x + 60$,

$f'(x) = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6)$

$= 6(x-2)(x-3)$.

Put $f'(x)=0$

i.e., $6(x-2)(x-3)=0$.

Hence, $f'(x)=0$ when x=2 or x=3.

If x<2, then $f'(x) = 6(-)(-)>0$;

if 2<x<3, then $f'(x) = 6(+)(-)<0$ and

if x>3, then $f'(x) = 6(+)(+)>0$.

Since $f'(x)$ changes sign from + to - as x increases through 2, $x = 2$ is a point of local maximum. Similarly, $f'(x)$ changes sign from – to + as $x$ increases through 3.Therefore, $x = 3$, is a point of local minimum.

**Example 12**:Use second order derivative test to find all the points of maxima and minima of the function $f(x) = x^3 - 3x^2 - 9x + 60$.

**Solution:** Given $f(x) = x^3 - 3x^2 - 9x + 60$,

$f'(x) = 3x^2 - 6x - 9$

$= 3(x^2 - 2x - 3)$

$= 3(x+1)(x-3)$.

Put $f'(x) = 0$

i.e., $3(x+1)(x-3) = 0$. Hence,

either x= -1 or x=3.

Therefore, x= -1 and x=3 are the critical values of the function $f(x)$. Take $f''(x) = 6x-6$

when x= -1, $f''(x) = f''(-1) = 6(-1) - 6 = -6 - 6 = -12$.

Since, $f''(x)<0$ at x= -1, the function has maxima at point x=-1.

When x=3, $f''(x) = f''(3) = 6(3) - 6 = 18 - 6 = 12$

Since, $f''(x)>0$ at x= 3, the function has minima at x=3.

**Example 13**:Use second order derivative test to find all the points of maxima and minima of the function $f(x) = x^5 - 5x^4 + 5x^3 - 18$.

**Solution:** Given $f(x) = x^5 - 5x^4 + 5x^3 - 18$,

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$
$$= 5x^2 (x^2 - 4x + 3)$$
$$= 5x^2 (x-1)(x-3)$$

Putting $f'(x) = 0$

$$= 5x^2 (x-1)(x-3).$$

Solving the above equation gives x=0, x=1 and x=3.

We have $f''(x) = 20x^3 - 60x^2 + 30x$.

If x=1, then $f''(x) = -10 < 0$, therefore, there is a local maximum at x=1;

if x=3, then $f''(x) = 90 > 0$, therefore, there is a local minimum at x=3;

if x= 0, $f''(x) = 0$. Here second derivative test fails.

Now, finding third order derivative,

$f'''(x) = 60x^2 - 120x + 30$. So,

$f'''(0) = 30 > 0$

Since third order derivative is an odd number, function has neither a maximum value nor a minimum at x=0. Hence, function has point of inflexion at x=0.

**Example 14**:Find all the points of local maxima and minima of the function $f(x) = (1-x)^2 e^x$. Also find the corresponding maximum and minimum values.

**Solution:** $f(x) = (1-x)^2.e^x = (1 - 2x + x^2) e^x$

$$f'(x) = (1 - 2x + x^2) e^x + e^x (-2 + 2x)$$
$$= (1 - 2x + x^2 - 2 + 2x) e^x$$
$$= (x^2 - 1) e^x$$

Putting $f'(x) = 0$,

$(x^2 - 1) e^x = 0$

$x = \pm 1$ (because $e^x \neq 0$)

$f''(x) = e^x 2x + (x^2 - 1) e^x$

$= e^x (x^2 + 2x - 1)$

If x = +1, then $f''(x) = e(1 + 2 - 1) = 2e > 0$. Therefore, there is a local minimum at x =1.

Putting x=1 in $f(x)$ to find out corresponding value, we get

$$f(1) = (1-1)^2 e^1 = 0$$

If x = -1, then $f''(x) = e^{-1}(1 - 2 - 1) = -2e^{-1} < 0$. Therefore, there is a local maximum at x =-1.

Putting x=-1 in $f(x)$ to find out corresponding value,

$$f(-1) = (1+1)^2 e^{-1} = 4/e.$$

**Check Your Progress 3**

1) What do you mean by maxima or minima of a function?
2) State the meaning of absolute minimum of a function.
3) Which are the steps would you follow to do a first derivative test for local extreme values?
4) State the meaning of an inflexion point.
5) What is a stationary point of a function?
6) How would you propose to determine the extreme values, Maxima/Minima, of a function using second order derivative?
7) Explain the steps for finding Maxima and Minima of a function.
8) How critical value is found while working on maxima and minima of function?

## 8.5 LET US SUM UP

In this unit we have discussed the application of derivative of a function to find out the maximum and minimum points of a curve. In the process we have learnt the presentation of higher order derivatives and interpretation of some of these. For example, it is seen that first order derivative of a function indicates the rate of change of a variable giving thereby the slope of a point on a curve. The second derivative, on the other hand, provides information on the rate of change of a slope. Importantly, the increasing or decreasing slope can be known on the basis of the positive or negative sign of the first as well as second order derivative.

Signs of derivatives can also be used to know the increasing and decreasing curves. While a function is said to be an increasing function in interval $[a, b]$ if first order derivative is greater than zero (i.e.,$f'(x)> 0$) for all values of $x$ in an interval $[a, b]$, it is decreasing when first order derivative is less than zero i.e., $f'(x) < 0$ for all values of $x$ in same interval. If the first derivative is equal to zero, then we have a stationary point indicating neither increase nor decrease of the curve.

We have covered maxima and minima of a function as the largest and the smallest value of the function, defined either within a given range or on the entire domain of a function. When considered the values within a given range, we get the local or relative maxima or minima, while we reach to global or absolute extrema by taking into account entire domain of a function.

In order to compute the relative maxima and minima values, we learnt the first and second derivative tests where assessments are made on the basis of signs of first and second order derivatives. In addition, we are introduced to the identification of inflexion points where the sign of the curvature changes considering the signs of first and second order derivatives. Towards the last part of the unit, steps required to compute maxima/minima with the help of second order derivatives have been discussed.

## 8.6 KEY WORDS

**Absolute (Global) Minimum or Maximum**: The smallest or the largest value of the function on the entire domain of a function.

**Local (Relative) Minimum or Maximum**: The smallest or the largest value of the function within a given range.

**Point of Inflection**:A point on a curve at which the sign of the curvature changes.

**Revenue**:The income which a producer earns from its normalactivities, usually from the sale of goods and services.

**Slope**: The slope of a curve is a number that describes steepness of the curve.

**Stationary Point**:A point where the function stops increasing or decreasing.

## 8.7 SOME USEFUL BOOKS


- Allen, R.G.D., "Mathematical Analysis for Economists", London: English Language Book Society and Macmillan, 1974.
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- Archibald, G.C., Richard G.Lipsey. "An Introduction to A mathematical Treatment of Economics", Delhi: All India Traveller Bookseller, 1984
- Yamane, Taro, "Mathematics for Economists: An Elementary Survey", New Delhi: Prentice Hall of India Private Limited, 1970.
- K. Sydsaeter and P. Hammond, Mathematics for Economic Analysis, PearsonEducational Asia, Delhi, 2002.


## 8.8 ANSWER OR HINTS TO CHECK YOUR PROGRESS

**Check Your Progress 1**

1) Sign of the derivative

2) A decreasing curve.

3) First order derivative

4) Rate of change of a slope.

5) Get the first derivative of the function

6) Read Section 8.2 of the present unit and answer.

**Check Your Progress 2**

1) Find that the first order derivative is greater than zero (i.e.,$f'(x)$> 0) for all values of $x$ in an interval $[a, b]$.

2) Second order derivative

3) Check Sub-Section 8.3.2.

**Check Your Progress 3**

1) The largest and the smallest value of the function, defined either within a given range or on the entire domain of a function.

2) The smallest number in defined range and corresponds to the value of the function at the lowest point of its graph.

3) Read Sub-Section 8.4.1 and answer.

4) A point at which the function alters its rate of change.

5) In a differentiable function of one variable a point on the graph of the function where the function's derivative is zero. Informally, it is a point where the function stops increasing or decreasing.

6) Read Sub-Section 8.4.2 and answer.

7) Check Sub-Section 8.4.3

8) A critical point is an interior point in the domain of a function at which $f'(x) = 0$ or $f'$ does not exist.

## 8.9 EXERCISES WITH ANSWER/HINTS

1) Find the second order derivative of the following

   i) $y = ax^3 + bx^2 + cx + d$

   ii) $y = AL^\alpha$

2) If $y = e^x + \log x$, then prove that $y_3 = e^x + 2/x^3$

3) If $y = \frac{ax+b}{cx+d}$, prove that $2 y_1 y_3 = 3{y_2}^2$

4) If $y = 5x^4 + 9x^3 - 8x - 100$ then show that $y_5 = 0$

5) Find the intervals on which the following functions are increasing or decreasing

   i) $f(x) = x^3 - 6x^2 - 36x + 2$

   ii) $f(x) = 2x^3 - 15x^2 + 36x + 1$

   iii) $f(x) = (x+1)^3 (x-3)^3$

   iv) $f(x) = x^2 e^x$

6) Examine for the maximum and minimum values of the following functions:

   i) $f(x) = 3x^4 - 4x^3 + 1$

   ii) $f(x) = 2x^3 + 3x^2 - 36x + 10$

7) Examine the function $f(x)$ = $x^3$ -$3x^2$ +3x +7 is i) maximum ii) minimum iii) neither maximum nor minimum.

8) Show that the function $f(x)$ = $4x^3$ – 18 $x^2$ +27x -7 has neither maximum nor a minimum by using second derivative test.

9) Show that the function $f(x)$ = $x^3$ – 9 $x^2$ +30x +5 has neither maximum nor a minimum.

10) Show that the function $f(x)$ = $x^3$ – $6x^2$ +12x -5 has neither maximum nor a minimum.

11) Show that the function $f(x) = \frac{log x}{x}$ has a maximum value at x=e.

**Answers**

1) i) $30x^2$ -30x +7 ii) $A\alpha(\alpha-1)L^{\alpha-2}$

2) i) Increasing on (-∞, -2) and (6,∞), decreasing on (-2,6)

ii) Increasing on (-∞,2) and (3,∞), decreasing on (2,3)

iii) Increasing on (1,∞), decreasing on (-∞,1)

iv) Increasing on (-∞, -2) and (0, ∞), decreasing on (-2,0)

3) i) Local maximum at x=1, value 0, point of inflection at x=0

ii) Local maximum at x= -1/2, value 81/16, local minimum at x=2, value -10,point of inflection at x=0.

7) iii) Neither maximum nor minimum.

# UNIT 9 APPLICATION OF DERIVATIVES

**Structure**



## 9.0 OBJECTIVES

After going through this unit, you will be able to understand

- application of derivatives to elasticity of demand as well as supply; cost and revenue functions; profit maximization; and
- the importance of derivatives in maximum-minimum problems.

## 9.1 INTRODUCTION

In the preceding two units, we have discussed the techniques of differentiation and extreme values of a function. In the present unit, we extend differentiation techniques for applying to select business and economic themes. Basically, an attempt is made here to learn the use of derivatives to understand the rate of change as well as change in rate of change, which is the corner stone of marginal analyses in economics as well as business studies.

## 9.2 DEMAND FUNCTION AND SUPPLY FUNCTION

In microeconomic analysis, it is postulated that demand and supply are functions of price. While the quantity demanded is related to price inversely, in case of supply it has a direct relationship with price. Because of such a proposition, we have a downward sloping demand curve and upward sloping supply curve. For the sake of simplicity, we often assume the demand curve and supply curve for a good or service to be a linear line, although other non-linear types are also considered. The consequent slopes of demand and supply curves are, respectively, negative and positive. Let us use derivatives to see the process of deriving these results.

We know that a linear equation is of the form $y = mx + c$, where $m$ is the slope of the line and $c$ is the $y$ intercept. We can model a demand function as $q = -ap + b$(or, the inverse demand function as $p = -aq + b$),where $p$ and $q$ represent price and quantity respectively. In such a format, we move the price to the horizontal $(x)$ axis and quantity to the vertical $(y)$ axis as we think of quantity demanded or supplied as a function of price. In the event of presenting inverse demand function, we consider quantity in horizontal axis and price in vertical axis. You will see the use of this form when we come to derivation of marginal revenue.

The slope is negative in case of a normal demand curve, which slopes downward. Similarly, a supply curve is modelled by another linear equation of the form $q = cp + d$ where $c$ is the slope of supply curve with a positive sign.

### 9.2.1 Slope of Demand Curve

In case of the equation of demand curve,$q = -ap + b,$ differentiating $q$ with respect to (w.r.t.) $p$, we get $\frac{dq}{dp} = -a$. The slope is $-a$ and it gives the rate of change in quantity demanded in response to change in price. See that the downward sloping demand curve is represented by the negative slope. Reasons for this feature could be threefold, viz.,

Substitution effect: Consumer will substitute other goods to buy more or less according as price falls or rises.

Income effect: Rise or fall in real income as prices falls or rise

Law of diminishing utility: At a lower price, consumers are willing to consume more as they compare the price with the satisfaction that is derived from the consumption of that good.

### 9.2.2 Slope of Supply Curve

Considering the equation $q = cp + d$ above and differentiating $q$ w.r.t. $p$, we get the slope of the supply curve. So, $\frac{dq}{dp} = c$ is the slope of supply curve and it is positive, indicating a positive relationship between quantity supplied and price.

Reason for this may be as follows:

1) Profit motive: With rise in prices following an increase in demand, it becomes more profitable for businesses to increase their output;

2) Production and costs: Firm's production cost tends to rise as output expands. Consequently, a higher price is needed to cover these extra costs of production.

3) Incentive for new entrants: Higher prices may create an incentive for other businesses to enter the market leading to an increase in total supply.

**Check Your Progress 1**

1) Why the slope of the supply curve is positive for normal goods?

2) What reasons would you list for a negative slope of demand curve?

3) If you intend to consider an inverse demand function, then in what way you would have to write the demand function?

## 9.3 ELASTICITY: DEMAND AND SUPPLY FUNCTIONS

Elasticity refers to the responsiveness of the dependent variable of a function to changes in the independent variable. It is measured in terms of percentage changes instead of absolute changes. For example, we consider first the percent change in a variable *X* and write,

$$Percent\ change\ in\ X = \frac{Change\ in\ the\ variable\ X}{Original\ value\ of\ X}$$

Thus, we can say that the change in *X* is $\Delta X$. If *X* changes from *X* to $X + \Delta X$, then proportion change in $X = \frac{\Delta X}{X}$.

Seen in terms of numerical value, we would say that the value of *X* changes from 20 to 30. So, the proportion change is

$X = \frac{30-20}{20} = 0.5$. This is a percentage change in *X* of 50%. If concerned with elasticity, then we say the elasticity of *Y* with respect to *X. And w*e need to find the ratio of the percentage change in *Y* to the percentage change in *X*. That is,

Elasticity of Y with respect to $X = \frac{Percent\ change\ in\ Y}{percentage\ change\ in\ X}$.

Sometimes the elasticity is defined with a negative sign while at other in absolute values. For convenience, let us stick to the sign to emphasize the nature of the percent changes. When an elasticity negative and say, greaten than -1, it indicates that a percent increase in *X* corresponds to a greater percent decrease in *Y*. In such a case, we would say that the variable *Y* is elastic with respect to variable *X*.

An elasticity between -1 and 0 indicates that a percent increase in *X* corresponds to a smaller percent decrease in *Y*. So, the variable *Y* is inelastic with respect to the variable *X*. If the elasticity is of -1, then a percent increase in *X* corresponds to an identical percent decrease in the variable *Y*. In that case, the variable *Y* is said to be unit-elastic with respect to the variable *X*.

### 9.3.1 Price Elasticity of Demand

The most common use of elasticity in economics and business studies is price elasticity of demand (PED) or elasticity of demand with respect to price. Such a concept helps us explore the responsiveness of the consumer demand for some product to changes in the price of that product. If the price of a cup of coffee were to increase, the quantity sold will be influenced. To see the point, suppose that in 2016, Nestle increased the price of coffee from Rs. 4.95 to Rs. 5. 00. Due to such a change, the demand for coffees dropped from 440 units per day to 438 units per day. The percent change in price $P$ is

$\frac{\Delta P}{P} = \frac{5.00 - 4.95}{4.95} = 0.01$

or about 1%. The percent change in the demand $Q$ is

$\frac{\Delta Q}{Q} = \frac{438 - 440}{440} = -0.0045$ or about -0.45%. That is, with 1% increase in price there is 0.45% decrease in the quantity demanded of coffee. Thus,

the price elasticity of demand is

$$E = \frac{\text{Percent change in Q}}{\text{Percent change in P}} = \frac{-0.0045}{0.01} = -0.45$$

The result shows that 1% increase in price corresponds to a 0.45% drop in demand for coffee. On the other hand, we may also say that 1% drop in price corresponds to a 0.45% increase in demand.

Taking into account the relationship between demand and price in a functional form, we can use derivative of the demand function to calculate the price elasticity of demand. If we write that the change in price as being from $P$to $\Delta$Pand the corresponding change in demand as being from $Q$ to $\Delta Q$, then corresponding percent changes in price and demand are $\frac{\Delta P}{P}$and $\frac{\Delta Q}{Q}$. The definition of elasticity gives

$$E_d = \frac{\text{Percent change in Q}}{\text{Percent change in P}}$$

$$= \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}}$$

$$= \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} = \frac{\Delta Q}{Q} \cdot \frac{P}{\Delta P} = \frac{P}{Q} \frac{\Delta Q}{\Delta P}.$$

It may be noted that $\frac{\Delta Q}{\Delta P}$ represents the average rate of change of demand $Q$ with respect to price $P$. If we assume the change in price is small, we can replace the average rate of change with the instantaneous rate of change, viz., $\frac{dQ}{dP}$.

Let the demand function be $Q = f(P)$. Then, the price elasticity $E_d$ of demand is

$$E_d = \frac{P}{Q} \frac{dQ}{dP}$$

Specifically, we may also define point price elasticity of demand at a point $(p_0, q_0)$ on the demand curve as $PED = (dq/dp)\,(p_0/q_0)$.

**Example 1**: Let the demand curve be $q = -5\,p + 30$. Then $\frac{dq}{dP} = -5$ . If a point ($P_0$, $q_0$) = (1,25) is on the demand curve, how do we compute the PED at (1,25).

Using the formula, we get

PED = (-5)(1/25) = -1/5 = -0.20.
Further, if a point p=3 and q =15 is on the demand curve, then the PED at

(3,15) is equal to (-5)(3/15) = -1;

if a point $p = 5$ and $q = 5$ is on the demand curve, then the PED at

(5,5) is (-5)(5/5) = -5.

Thus, with the help of calculus, we see that the PED changes for every point on a straight-line demand curve.

***Note the simple mathematical relationship:***

$$\frac{\frac{dQ}{Q}}{\frac{dP}{P}} = \frac{dQ}{dP} \times \frac{P}{Q} = (1/slope) \times \frac{P}{Q}$$

**Example 2**: To find the point price elasticity of demand,let us consider the following example of a demand function:

$Q = 15{,}000 - 50P$

Given this demand curve, we need to find out price elasticity of demand at two different prices, P = 100 and P = 10.

So, take the derivative of the demand function when it's expressed with Q as a function of P. Since quantity (Q) goes down by 50 each time price (P) goes up by 1, we get, (ΔQ/ΔP)= -50

To find the quantity demanded at each price level we consider the price quantity combinations of : (100; 10,000), (10; 14,500)

Then point elasticity of demand yields,

$E_d$ = -50(100/10,000) = -.5
$E_d$ = -50(10/14,500) = -.034

Results obtained show that both elasticities are negative. That imply a downward sloping demand relationship. Moreover, they are relatively more elastic.

***Deriving Elasticity using Log Functions***

Here is another expression for elasticity that is sometimes useful. It turns out that elasticity can also be expressed as

$\frac{d\,lnQ}{dlnP}$.

The proof involves repeated application of the chain rule. We start by noting that

$$\frac{d\,lnQ}{dlnP} = \frac{d\,lnQ}{dQ}.\frac{dQ}{dlnP}$$

$$= \frac{1}{Q}\frac{dQ}{dlnP} \qquad (1)$$

We also note that $\frac{d\,Q}{dP} = \frac{dQ}{dlnP}\frac{d\,lnP}{dP} = \frac{dQ}{dlnP}\frac{1}{P}$

$$\Rightarrow \frac{d\,Q}{dlnP} = P\frac{dQ}{dP}.$$

Substituting this into equation (1), we have

$$\frac{d\,lnQ}{dlnP} = \frac{1}{Q}\frac{d\,Q}{dP}P = E_d.$$

## 9.3.2 Income Elasticity of Demand

The income elasticity of demand (YED) measures how effect of the change in a consumer's income on the demand for a specific product. It is defined as the ratio of proportionate change in quantity demanded due to proportionate change in income ($Y$). There is a positive relation between income and demand in case of normal goods; hence, sign of income elasticity of demand is positive. Thus, we define

$$E_y = \frac{proportionate\ change\ in\ demand}{proportionate\ change\ in\ income}$$

$$= \frac{\Delta Q/Q}{\Delta Y/Y}$$

$$= \frac{Y}{Q}\frac{\Delta Q}{\Delta Y}$$

In case of instantaneous change income, $E_y = \frac{Y}{Q}\frac{dQ}{dY}$

**Example 3**: If $x=4y^2$, where x is quantity demanded of a good and y is the consumer's income. Find the income elasticity of demand.

**Solution**: Given the demand function

$$x = 4y^2;$$

$$E_y = \frac{dx}{dy}.\frac{y}{x}$$

$$\frac{dx}{dy} = 4.2\,y = 8y$$

Therefore,

$$E_y = 8y.\frac{y}{4y^2} = 2$$

The higher the income elasticity of demand for a specific product, the more responsive it becomes to the change in consumers' income.

Now, we can measure the income elasticity of demand for different products by categorizing them as inferior goods and normal goods.

**Normal Goods**

The YED for a product can be elastic or inelastic based on its category—whether it is an inferior good or a normal good. When YED is more than zero, the product is income-elastic. Normal goods have a positive YED. That is, when the consumers' income increases, the demand for these goods also increases.

**Inferior Goods**

Inferior goods are called so because these have superior alternatives. Such goods have a negative income elasticity, that is, YED is less than 0. If the consumers' income increases, they demand less of these goods.

***Note: If sign of income elasticity of demand is negative, then the commodity is inferior.***

### 9.3.3 Elasticity of Supply ($E_s$)

It is defined as the ratio of proportionate change inquantity supplied to a proportionate change in its price.

$$E_s = \frac{proportionate\ change\ in\ supply}{proportionate\ change\ in\ price}$$

$$= (-)\ (\Delta Q_s/Q_s)/(\Delta P/P)$$

$$= (-)\ \frac{P}{Q_s}\frac{\Delta Q_s}{\Delta P} \text{ or } = \frac{P}{Qs}\frac{dQ_s}{dP}$$

If a supply curve is given as $q = 2p + 20$, then $dq/dp = 2$. If a point $p = 1$ and $q = 22$ is on the supply curve then the PES at (1,22) is (2)(1/22) = 1/11. If a point $p = 5$ and $q = 30$ is on the supply curve, then the PES at (5,30) is (2)(5/30) = 1/3.

**Example 5**:Forthe supply function $x=5 + 2p^2$, find the elasticity of supply at p=2.

**Solution:**Given the supply curve

$x = 5 + 2p^2$, differentiating w.r.t.$p$

$\frac{dx}{dp} = 4p$. Thus,

$$E_s = \frac{p}{x}.\frac{dx}{dp} = \frac{p}{5+2p^2}.4p$$

$$= \frac{4p^2}{5+2p^2}$$

When p=2; $E_s$ is equal to

$$E_s = \frac{4\times4}{5+8} = \frac{16}{13}$$

## 9.3.4 Cross Elasticity of Demand

Cross elasticity of demand is the responsiveness of demand for a product in relation to the change in the price of another related product. Mark word "related" product. Note that unrelated products have zero elasticity of demand. For example, an increase in the price of black gram will have no effect on the demand for ice cream.

We can measure the cross elasticity of demand by dividing the percentage of change in the demand for one product by the percentage of change in the price of another product. Thus,

$$Cross\ Elasticity\ of\ Demand = \frac{\%\ of\ change\ in\ the\ demand\ for\ Product\ A}{\%of\ change\ in\ the\ price\ of\ product\ B}$$

or, $(dQ\ /\ dP') * (P'/Q)$, where Q= quantity demanded, $P'$=price of a related product of Q.

It is useful to note that the cross elasticity of demand depends on whether the related product is a substitute or a complement (see below).

Substitute and Complementary Products

As mentioned earlier, cross elasticity measures the responsiveness of demand in relation to related products. And these related products can be either substitutes or complementary products. Let us understand the difference between the two**.**

**Substitute Products**

In case of substitute products an increase in the price of one will lead to an increase in demand for the competing product. For instance, an increase in the price of petrol will force consumers to go for diesel and increase the demand for diesel. Thus, the cross-price elasticity value for two substitute goods is always positive.

**Complementary Products**

In case of complementary goods, demand for two or more goods move together. For example, see the demand for coffee beans and coffee paper filters. If the price of coffee increases, then the demand for filters would reduce because the demand is reduced for coffee. The cross elasticity of demand for two complementary products is always negative.

**Example 4**:

We have a demand equation of $Q = 2000 - 4P + 5ln(P')$, find the cross-price elasticity of demand.

Differentiate with respect to $P'$ and get:

$dQ/dP' = 5/P'$.

Substituting $dQ/dP' = 5/P'$ and $Q = 2000 - 4P + 5ln(P')$ into cross-price elasticity of demand equation, we get
Cross-price elasticity of demand = $\left(\frac{5}{P'}\right) \times (P'/(2000 - 4P + 5ln(P')))$

To find the cross-price elasticity of demand at $P = 5$ and $P' = 10$, substitute these into cross-price elasticity of demand equation:

Cross-price elasticity of demand

= (5/P') × (P'/(2000 -4P + 5ln(P')))
= (5/10) × (10/(2000 - 20 + 5ln(10)))

= 0.5 × (10 / 2000 - 20 + 11.51) (as $ln(10) = 2.302585$)

= 0.5 × (10 / 1991.51)
= 0.5 × 0.002008
= 0.001004

Thus, cross-price elasticity of demand is 0.000502. Since it is greater than 0, goods are substitutes.

**Check Your Progress 2**

1) Explain the idea behind elasticity of demand or supply.

2) Why would you say that the cross elasticity of demand for two complementary products is always negative?

3) How do you identify an inferior good on the basis of income elasticity of demand?

4) What is cross elasticity of demand?

5) How would you interpret the elastic, inelastic and unit-elastic character of $y = f(x)$?

6) What is price elasticity of supply?

## 9.4 AVERAGE AND MARGINAL COST:

In this section we will discuss the cost of output production and see the process of applying derivatives for finding solutions to different cost related concepts.Let us start with the following cost concepts:

1) **Total Cost**: Total cost is the sum of two components, viz., fixed cost and variable cost. Symbolically, TC= FC+ VC, where TC= total cost, FC= fixed cost and VC= variable cost.

2) **Fixed Cost**: The cost remains constant for all levels of output (i.e., independent of quantity produced). For example, cost of machinery or rent of factory building fall in this category.

3) **Variable Cost**: The cost varies with level of output. For example, cost of raw materials forms a part of variable cost.

Given the total cost, we can derive the

**Average Cost**, which is defined as per unit cost of production and written as

AC = $\frac{TC}{q}$, where $q$ is units of output;and

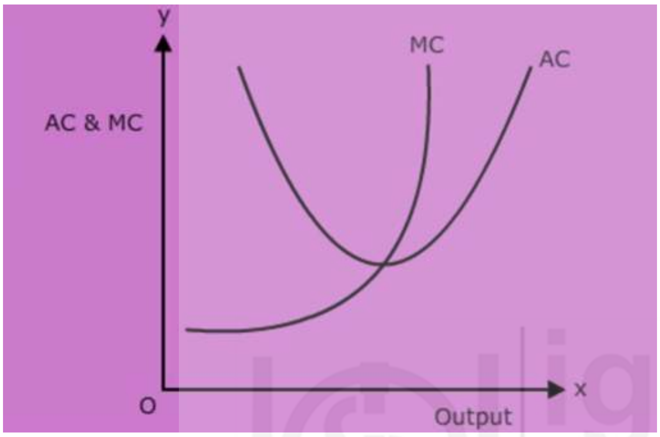
**Marginal Cost**, which is the rate of change in total cost when $q$ units are produced and defined as

MC= $\frac{dTC}{dq}$.

**Relation between AC and MC**

Relation between AC and MC can be explained with the help of diagram below:

| | | |
|---|---|---|
| $\frac{d(AC)}{dx} < 0$ | $\frac{d}{dx}(AC) = 0$ | $\frac{d}{dx}(AC) > 0$ |
| $\Rightarrow \frac{1}{x}(MC - AC) < 0$ | $\Rightarrow \frac{1}{x}(MC - AC) = 0$ | $\frac{1}{x}(MC - AC) > 0$ |
| or $(MC - AC) < 0$ | $\Rightarrow MC - AC = 0$ | or $MC > AC$ |
| or $MC < AC$ | $\Rightarrow MC = AC$ | |
| (When AC is sloping downwards, MC is below AC) | (When AC achieves its minimum, MC intersects AC) | (When AC curve is sloping upwards MC is above AC) |




**Example 6**:If TC = $2q^2 + 8q + 60$,

calculate AC and MC at q=3 and interpret the result.

**Solution**: Given TC = $2q^2 + 8q + 60$.

$$\text{Therefore, } AC = \frac{TC}{q}$$

$$= \frac{2q^2 + 8q + 60}{q}$$

$$= 2q + 8 + \frac{60}{q}.$$

$$MC = \frac{dTC}{dq}$$

$$= \frac{d(2q^2 + 8q + 60)}{dq}$$

$$= 4q + 8.$$

When q= 3, MC = 4x3 + 8 = 20.

**Interpretation:**It means when production increases by 1 unit from 3rd unit to 4th unit, then the total cost increases by Rs.20,approximately.

**Example 7**: Total Cost of the firm is

$C(x) = 0.005x^3 - 0.002\,x^2 - 30x + 5000$, where x is the output

Find out

i) Fixed Cost

ii) Variable Cost

iii) Average Cost

iv) Average Variable Cost

v) Marginal Cost

vi) Marginal Cost when 50 units are produced, interpret the results.

vii) Actual Cost of producing $51^{st}$ unit.

viii) Rate of change in Marginal Cost w.r.t. x

ix) Marginal Average Cost

**Solution**: Given $C(x) = 0.005x^3 – 0.002\,x^2 -30x +5000$, we have to find out

i) Fixed cost by putting x=0 (i.e., output = 0)and get

FC = 5000

ii) VC= TC – FC

$= 0.005x^3 – 0.002\,x^2 -30x +5000 -5000$

$= 0.005x^3 – 0.002\,x^2 -30x$

iii) AC= TC/x $= \frac{0.005x^3 - 0.002\,x^2 - 30x + 5000}{x} = 0.005x^2 - 0.002\,x - 30 + \frac{5000}{x}$

iv) Average Variable Cost = AVC= VC/x

$= \frac{0.005x^3 - 0.002\,x^2 - 30x}{x}$

$= 0.005x^2 - 0.002\,x - 30$

v) MC $= \frac{dTC}{dx} = \frac{d}{dx}\left(0.005x^3 - 0.002\,x^2 - 30x + 5000\right) = 0.015\,x^2 – 0.004x -30.$

vi) When x= 50, MC= $0.015\,(50)^2 – 0.004(50) -30 = 7.3$

**Interpretation:** It means if production increases by 1 unit from $50^{th}$ unit to $51^{th}$ unit, then the total cost increases by Rs.7.3, approximately.

vii) $C(51) = 0.005(51)^3 – 0.002\,(51)^2 -30(51) +5000 = 4128.05$

$C(50) = 0.005(50)^3 – 0.002\,(50)^2 -30(50) +5000 = 4120$

Actual cost of producing 51th unit = C(51) – C(50) = 4128.05 – 4120

= 8.05

viii) Rate of change in MC $= \frac{dMC}{dx} = \frac{d}{dx}\left(0.015x^2 - 0.004\,x - 30\right)$

$= 0.030x – 0.004$

ix) Marginal Average Cost = $\frac{dAC}{dx} = \frac{d}{dx}\left(0.005x^2 - 0.002\,x - 30 + \frac{5000}{x^2}\right)$

$$= 0.010x - 0.002 - \frac{5000}{x}$$

**Example 8**:Prove that slope of average cost curve is $\frac{1}{x}(MC - AC)$ for the TC= $ax^3 + bx^2 + cx + d$.

**Solution:** GivenTC= $ax^3 + bx^2 + cx + d$,

$$AC = \frac{ax^3+bx^2+cx+d}{x} = ax^2 + bx + c + d/x$$

$$\text{Slope of AC} = \frac{dAC}{dx} = \frac{d}{dx}\left(ax^2 + bx + c + \frac{d}{x}\right)$$

$$= 2ax + b - d/x^2$$

$$MC = \frac{dTC}{dx} = \frac{d}{dx}(ax^3 + bx^2 + cx + d)$$

$$= 3ax^2 + 2bx + c.$$

$$\text{Now, } \frac{1}{x}(MC - AC) = \frac{1}{x}\left(3ax^2 + 2bx + c - ax^2 - bx - c - \frac{d}{x}\right)$$

$$= \frac{1}{x}\left(2ax^2 + bx - \frac{d}{x}\right)$$

$$= 2ax + b - d/x^2$$

$$= \text{slope of AC}$$

Hence proved.

**Check Your Progress 3:**

1) When AC is minimum then value of MC is equal to what?

2) Name the cost which is incurred by the firm when firm produces zero output.

3) What are the components of total cost of production?

## 9.5 REVENUE FUNCTION

A revenue function depicts the relation between revenue and output. Note that revenue is the product of price($p$) and output ($q$), i.e., the price of the product and its quantity sold.Needless to say, if the price of a good increases, then the quantity sold decreases. So, revenuemay increase or decrease. To see this, let us write the total revenue $(TR) = pq = pf(q)$, where $p$ is the price per unit of the good and $q$ is the number of units of the commodity sold.

If we want to the see the nature of relation between revenue and output, we have to examine the price elasticity of demand. Indeed, there is a very useful relationship between price elasticity andrevenue change. To examine such an insight, let the total revenue be expressed as

$TR = pq$.

If we let the price change to be $p + \Delta p$and the quantity change to be$q + \Delta q$, then revenue obtained becomes

$$R = (p + \Delta p)(q + \Delta q)$$
$$= pq + q\Delta p + p\Delta q + \Delta p\Delta q.$$

Subtracting $TR$ from $R$ we have

$\Delta TR = q\Delta p + p\Delta q + \Delta p\Delta q.$

For small values of $\Delta p$ and $\Delta q$, we can neglect the last term and get the revenue as $\Delta TR = q\Delta p + p\Delta q$.

The rate of change of revenue per unit change in price is

$\frac{\Delta TR}{\Delta p} = q + p\frac{\Delta q}{\Delta p}.$

Thus, revenue increases when price increases if the elasticity of demand is less than 1 in absolute value. Similarly, revenue decreases when price increases if the elasticity of demand is greater than 1 in absolute value.

To see this is to return tothe formula for $\Delta R/\Delta p$ and rearrange it as follows:

$$\frac{\Delta TR}{\Delta p} = q + p\frac{\Delta q}{\Delta p}$$
$$= q\left[1 + \frac{p}{q}\frac{\Delta q}{\Delta p}\right]$$
$$= q[1 + E_d]$$

Since demand elasticity is stated be negative, we can also write this expression

as $q[1 - [|E_d|]]$

In this formulation, it is easy to see how revenue responds to a change in price:

if the absolute value of elasticity is greater than 1, then $\frac{\Delta TR}{\Delta p}$ must be negative and vice versa.

Intuitively, if demand is very responsive to price—that is, it is very elastic—then an increase in price will reduce demand so much that revenue will fall.

If demand is very unresponsive to price—it is very inelastic—then an increase in price will not change demand very much, and overall revenue will increase. Taking the dividing line of an elasticity of $-1$, we can say that in such a point, if the price increases by 1 percent, then the quantity will decrease by 1 percent,so overall revenue doesn't change at all. We have unit elasticity.

### 9.5.1 Relationship between AR, MR and Price Elasticity of Demand

We may extend the above result on revenue and price elasticity further to find its relationship with marginal revenue and average revenue. To work in that direction, let us define these terms first.

Average Revenue (AR) is the per unit revenue and can be computed as

$$AR = \frac{TR}{q} = \frac{pq}{q} = p$$

Marginal Revenue (MR)is the change in total revenue resulting from an additional unit of output sold. Thus,

$MR = \frac{dTR}{dq}$.

***Elasticity and Marginal Revenue:***

We have seen above that for small changes in price and quantity, the change in revenue is given by

$$\frac{\Delta TR}{\Delta p} = q + p\frac{\Delta q}{\Delta p}$$

or, $\Delta TR = q\Delta p + p\,\Delta q$

Dividing both the sides by $\Delta q$, we get

$\frac{\Delta TR}{\Delta q} = q\frac{\Delta p}{\Delta q} + p\frac{\Delta q}{\Delta q}$, or

MR=$\frac{\Delta TR}{\Delta q} = p + q\frac{\Delta p}{\Delta q}$, or

$$MR = p\left[1 + \frac{q}{p}\frac{\Delta p}{\Delta q}\right]$$

Remembering we are trying to derive the relationship between MR and $E_d$, let usdo undertake some refinement on the second term in $MR = p\left[1 + \frac{q}{p}\frac{\Delta p}{\Delta q}\right]$. Broadly, it looks like inverse of $E_d$. So, we can write $MR = p\left[1 + \frac{1}{E_d}\right]$. Since we have used to write $E_d$ attaching a negative sign, we do so here by writing $MR = p\left[1 - \frac{1}{|E_d|}\right]$, i.e., writting the $E_d$ in absolute term.

In interpreting such a result, we can say,

if elasticity of demand is $-1$, then marginal revenue is zero, i.e., revenue doesn't change when we increase output.

If demand is inelastic, then $|E_d|$ is less than 1, which means $1/|E_d|$ is greater than 1. Thus, $1-1/|E_d|$ is negative. So, revenue would decrease when we increase output.

***Elasticity, Average Revenue and Marginal Revenue:*** To see the relation between elasticity, average revenue and marginal revenue we simply work with $MR = p\left[1 - \frac{1}{|E_d|}\right]$. Just multiply $(\frac{q}{q})$ in the right-hand side to get

$$MR = \left(\frac{q}{q}\right)p\left[1 - \frac{1}{|E_d|}\right] = \frac{pq}{q}\left[1 - \frac{1}{|E_d|}\right]$$

$$= \frac{TR}{q}\left[1 - \frac{1}{|E_d|}\right]$$

$$= AR\left[1 - \frac{1}{|E_d|}\right]$$

$$or, MR - AR = \left[-\frac{AR}{|E_d|}\right]$$

$$or, -MR + AR = \left[\frac{AR}{|E_d|}\right]$$

$$or, \frac{AR - MR}{AR} = \frac{1}{|E_d|}$$

$$or, \frac{AR}{AR-MR} = |E_d|.$$

### 9.5.2 Maximizing Revenue

It is seen above that total revenue is influenced by price variation. In a market structure with monopoly, the producer controls the price. This is in contrast to pure competition, where price remains constant and it's independent from output. In that case, price, AR and MR are exactly the same. Let us see the revenue maximization issue under a monopoly market.

**Example 9**: Find the total revenue and marginal revenue of the perfect competitive firm with current price of Rs. 10 per unit.

**Solution**: Given, P= 10.

If q units are sold in the market then,

$TR = p.q = 10q$

$AR = \frac{TR}{q} = 10$

$MR = \frac{dTR}{dq} = 10$

Therefore, $P = AR = MR = 10$ which is constant.

Under monopoly, price is determined by the firm and not given to it by the market. The total revenue will be maximum at a level of output where first order differentiation of total revenue is zero and second order derivative is less than zero, i.e.,$TR'(q) = 0$ ( $i.e., MR = 0$) and $TR''(q) < 0$.

**Example10**: If demand function of the firm is given as $p = 5000 - 20x - x^2$, find TR,MR and comment on the nature of the firm.

**Solution**: Given, $p = AR = 5000 - 20x - x^2$

$$TR = p.x$$
$$= (5000 - 20x - x^2)\, x$$
$$= 5000x - 20x^2 - x^3$$

$$MR = \frac{dTR}{dx} = \frac{d}{dx}(5000x - 20x^2 - x^3)$$
$$= 5000 - 40x - 3x^2$$

Since, $AR \neq MR$, hence, firm is monopolist firm.

**Example 11**: Show that $E_d = \frac{AR}{AR-MR}$ at p = 5, where the demand function is given by $p = 50 - 3x$.

**Solution:** $p = 50 - 3x$

Differentiating both side w.r.t.x

$\frac{dp}{dx} = -3$

Taking reciprocal

$\frac{dx}{dp} = -\frac{1}{3}$

$E_d = -\frac{p}{x}\frac{dx}{dp} = -\frac{50-3x}{x} . -\frac{1}{3} = \frac{50-3x}{3x}$

If p= 5, then

5 = 50 – 3x

x= 45/3= 15

Putting the value of x in $E_d$

$E_d = \frac{50-3\times15}{3\times15} = \frac{50-45}{45} = \frac{1}{9}$.

TR = p.x = (50 – 3x)x

$= 50x - 3x^2$

AR = TR/x = p.x/x= p =50 – 3x

When x= 15,

AR= 50 – 3×15 = 50- 45= 5.

$MR = \frac{dTR}{dx} = 50 - 6x$

When x=15,

MR = 50- 6×15 = 50 – 90 = -40

$\frac{AR}{AR-MR} = \frac{5}{5-(-40)} = \frac{5}{45} = \frac{1}{9} = E_d$

Hence proved.

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**Check Your Progress 4**

1) What is a revenue function?

2) State the relationship between AR, MR and Price Elasticity of Demand.

3) How is price elasticity related with rate of revenue change?

4) Revenue maximization is not important in a firm under which market structure? Why?

## 9.6 PROFIT MAXIMIZATION

Economic theory normally uses the profit maximization assumption in studying the behaviour of firm just as it uses the utility maximization assumption for the individual consumer. This approach is taken to satisfy the need for a simple rational objective for the firm.

The profit-maximizing firm chooses both inputs and outputs so as to maximize the difference between total revenue and total cost, i.e., $\pi = R(q) - C(q)$

The firm will adjust variables under its control such that it can no longer increase the profit further. Thus, the firm looks at each additional unit of input or output with respect to its effect on profit.

Let us define revenue as $R(q) = p(q) \times q$, where $R$ and $p$ are functions of $q$. Note that we are taking inverse demand function $p(q)$, which depicts price as a function of quantity for getting the revenue function. As we will see below it helps calculate MR quickly, which is required for getting the profit-maximizing condition for firms regardless of market structure. Profit function is the function of total revenue and total cost and it can be expressed as:

$\pi = \pi(q) = TR(q) - TC(q)$, i.e.,

$\pi = p(q) \times q - C(q)$.

To maximise profit take

$$\frac{d\pi}{dq} = \frac{dR}{dq} - \frac{dC}{dq} = 0$$

$$\Rightarrow \frac{dR}{dq} = \frac{dC}{dq} = 0$$

$$\Rightarrow MR = MC.$$

Such an equality specifies the **first order condition** and we must add the **second order condition** of profit maximisation. So, we need to have

$$\frac{d^2\pi}{dq^2} < 0.$$

That is, at the optimal quantity (q*), marginal profitmust be declining. In terms of $MR$ and $MC$, weget $\frac{d^2\pi}{dq^2} < 0\ or,$

$TR''(q) - TC''(q) < 0$, or,

$TR''(q) < TC''(q)$, or,

$(MR)' < (MC)'$,

which means that the slope of marginal revenue must be less than the slope of MC curve at the profit maximizing output.

**Example 12**: The total cost of a firm is C= $1/3x^3 - 6x^2 + 40x + 15$. Find the equilibrium output if price is fixed at Rs.20 per unit.

**Solution:** Given

price (p) = 20,

Revenue (TR) = px

Thus, Revenue = R(x) = 20x

Profit = $\pi$(x) = TR(x)- TC(x)

$= 20x - (1/3x^3 - 6x^2 + 40x + 15)$

$= 20x - 1/3x^3 + 6x^2 - 40x - 15$

$= -1/3x^3 + 6x^2 - 20x - 15$

Differentiating with respect to x

$\pi'(x) = - x^2 + 12x - 20$

For maximization $\pi'(x) = 0$

That is, $- x^2 + 12x - 20 = 0$

or, $x(x-10) - 2(x-10) = 0$

or, $(x-10)(x-2) = 0$

Therefore, either x= 10 or x = 2

Calculating second-order derivative of the profit function with respect to x, we get,

$\pi''(x) = -2x + 12$

When x=10

$\pi''(10) = -2(10) + 12$

$= -20 + 12 = -8 < 0$

When x=2

$\pi''(2) = -2(2) + 12 = 8 > 0$

Since, second order derivative is less than zero at x = 10, profit is maximum when firm produces 10 units of output.

**Example 13**: A monopolist has the following demand and cost functions:

$C = 20 + 2x + 3x^2$ and $p = 50 - x$

Find the output level and price at which the profit is maximized.

**Solution:** Revenue (TR) = $px = x(50-x) = 50x - x^2$

Profit ($\pi$) = TR – TC

$= 50x - x^2 - 20 - 2x - 3x^2$

$= 48x - 4x^2 - 20$

Differentiating w.r.t. $x$

$d\pi/dx = \pi' = 48 - 8x = 0$

$x = 48/8 = 6$

Calculating second order derivative

$\pi'' = -8 < 0$

Hence, profit is maximum at x = 6, and equilibrium price is:

$P = 50 - x = 50 - 6 = 44$

Hence, equilibrium output is 6 units and equilibrium price is 44.

**Example 14**: A stereo manufacturer determines that in order to sell x units of a new stereo, the price per unit, in rupees, must be $p(x)= 1000 - x$. The manufacturer also determines that the total cost of producing x units is given by $C(x) = 3000 + 20x$.

a) Find the total revenue R(x).

b) Find the total profit P(x).

c) How many units must the manufacturer produce and sell in order to maximize profit?

d) What is the maximum profit?

e) What price per unit must be charged in order to make this maximum profit?

**Solution**:

a) Revenue = quantity × price

$$R(x) = x \times p$$

$$R(x) = x(1000 - x)$$

$$R(x) = 1000x - x^2$$

b) Profit = Total Revenue - Total Cost

$P(x) = R(x)- C(x)$

$P(x) = 1000x- x^2- (3000 + 20x)$

$P(x) = - x^2+ 980x - 3000$

c) $P'(x) = -2x + 980 = 0$

$-2x = -980$

$x = 490$

Since there is only one critical value, we can use the second derivative to determine whether or not it yields a maximum or minimum.

$P''(x) = -2$

Since $P''(x)$ is negative, $x = 490$ yields a maximum.

Thus, profit is maximized when 490 units are produced and sold.

d) The maximum profit is given by

$P(490) = -(490)^2+ 980(490) - 3000$

$P(490) = 237,100.$

Thus, the stereo manufacturer makes a maximum profit of 237,100 when 490 units are produced and sold.

e) The price per unit to achieve this maximum profit is

$p(490) = 1000 - 490$

$p(490) = 510.$

**Check Your Progress 5**

1) How do you get profit of a firm?

2) What are the two conditions to be satisfied for profit maximization?

3) Interpret the second order condition of profit maximization.

## 9.7 LET US SUM UP

This unit deals with application of derivatives to economic and business studies. For that purpose, it covers some select themes that find frequent use. Atthe outset, the negative slope of downward sloping demand curve and positive slope of a supply curve are shown by using the first order derivative.

We have learnt the derivation of elasticity of demand as well as supply function. Introducing the concept, it is said that elasticity refers to the responsiveness of the dependent variable of a function to change in the independent variable. We have seen the derivation of price elasticity of demand as $E_d = \frac{P}{Q}\frac{dQ}{dP}$ along with small changes in price and quantity. Another expression for elasticity in terms of log functions has been explained.

Stating that income elasticity of demand measures the effect of the change in a consumer's income on the demand for a specific product, we are also told that this category of elasticity helps identify normal goods (positive elasticity) and inferior goods (negative elasticity).

We are exposed to the concept of cross elasticity of demand which is defined as the responsiveness of demand for a product in relation to the change in the price of another related product. Use of this idea helps in knowing the elasticity of substitute goods (positive cross elasticity) and that of complementary goods (negative cross elasticity). At the last part of the section, the supply elasticity has been discussed. The ratio of proportionate change inquantity supplied to a proportionate change in its price is demonstrated using the derivative.

The process of applying derivatives for finding solutions different categories of cost has been covered. Relation between average and marginal costs has been derived using differentiation. An attempt is made to deal with revenue function, which depicts the relation between revenue and output. The rate of change of revenue per unit change in price is derived through technique of derivative and relation between marginal revenue, average revenue and elasticity of demand specified.

In the last part of the unit, the application of derivative to find first and second order conditions of profit maximization is discussed. We have learnt to derive the equality of marginal revenue and marginal cost in the first order condition. The derivation of rate of change in the slope of marginal revenue, which has to be less than the rate of change in the slope of marginal cost to satisfy the second order condition is explained.

## 9.8 KEY WORDS

**Complement Goods**: A good for which the demand increases as the price of an associated good decreases.

**Cost of Output**: Increase in cost as production increase.

**Elasticity of Demand**: Sensitiveness the demand for a good to changes in other economic variables, such as prices and income. It is calculated as the percent change in the quantity demanded divided by a percent change in another economic variable.

**Equilibrium**: A situation in which supply and demand are matched and price remains stable.

**Fixed Cost**:Cost that remains constant for all levels of output.

**Inverse Demand Function**: Demand function viewing price as a function of quantity demanded.

**Monopoly:** Market, with only one seller selling unique product and earn super normal profit.

**Perfect Competition**: A market with a large number of buyers and seller. Sellers sell homogenous product and earn normal profit. There is free entry and exit in the market.

**Price Elasticity of Supply:** Responsiveness to the supply of a good or service after a change in its market price.

**Profit**: Excess of total revenue over total costs.

**Substitute Goods**: As the price of one good increases, the demand for an alternative good meeting the same consumer needs, increases.

**Total Cost**: Combination of fixed cost and variable cost.

**Total Revenue**: Product of price/demand function and output.

**Variable Cost**: Cost that varies with the level of output.

## 9.9 SOME USEFUL BOOKS


- K. Sydsaeter and P. Hammond, Mathematics for Economic Analysis, PearsonEducational Asia, Delhi, 2002.
- Dowling, Edward,T. "Schaum's Outline Series: Theory and Problems of Mathematics for Economists", New York: McGraw Hill Book Company, 1986.
- Chiang, A. and Kalvin Wainwright, Fundamental Methods of Mathematical Economics (Paperback), Mac Grow Hill, 2017.
- Varian , Hal R., Intermediate Microeconomics: A Modern Approach, Springer (India) Pvt. Ltd. India, 2010.


## 9.10 ANSWER OR HINTS TO CHECK YOUR PROGRESS

**Check Your Progress 1**

1) Read Sub-Section 9.2.2 and answer.
2) Law of diminishing utility, income and substitution effects.
3) Ans.: $p = f(q)$

**Check Your Progress 2**

1) Elasticity refers to the responsiveness of the dependent variable in a function to change in the independent variable.

2) Due to requirement of complimentary goods moving together to satisfy demand.

3) Income elasticity of demand will be less than zero.

4) Cross elasticity of demand is the responsiveness of demand for a product in relation to the change in the price of another related product.

5) $Elastic\ > -1$;$Inelastic$=in between -1 and 0; $Unit\ elastic = -1$.

6) Proportionate change inquantity supplied to a proportionate change in its price.

**Check Your Progress 3**

1) AC = MC

2) Fixed cost

3) Fixed cost and variable cost.

**Check Your Progress 4**

1) A function that depicts the relation between revenue and output.

2) $E_d = \frac{\text{AR}}{\text{AR}-\text{MR}}$.

3) $\frac{\Delta TR}{\Delta q} = p\left[1 - \frac{1}{|E_d|}\right]$.

4) Firm operating under perfect competition. There is the optimum situation of P=AR=MR.

**Check Your Progress 5**

1) Taking the difference between the total revenue and total cost.

2) First order condition of $MR = MC$ and second order condition of $(MR)' < (MC)'$,

3) Slope of marginal revenue must be less than the slope of MC curve at the profit maximizing output.

## 9.11 EXERCISES WITH ANSWERS/HINTS

1) The demand function for a commodity is given by p= 20 - 2x. Find total revenue. Compute average revenue and marginal revenue at x=3 and interpret the results. Find the output level at which total revenue is maximum and the maximum revenue. Is marginal revenue rising at x=3?

2) A manufacturer finds her yearly uniform demand for her product to be 40,000 units. The cost of setting up a production run is Rs. 200 and the cost of carrying one unit in inventory is Re.1 per annum. Find the economic lot size that should yield a minimum total cost.

3) A firm produces x number of units at a total cost

C=Rs. ($x^3/10 - 5x^2 + 10x + 5$)

At what level of output will the marginal cost and the average variable cost attain their respective minimum?

4) The total revenue(R) and total cost(C) functions of a firm are given by: R= $30x - x^2$, C=20 + 4x, where x is the output. Find the output which maximizes profit and the maximum profit.

5) If $f(x) = ax^2 + bx + c$, show that supply relation is linear. Show that p must exceed $b + 2\sqrt{ac}$ if total costs are to be covered but that, if only variable costs are to be covered, p need exceed only b.

6) The demand q as a function of income y is given by 30q = 10 +3y. Obtain the expression for the income elasticity of demand and its value at y = Rs.250.

7) If MR= 100 and elasticity of demand with respect to output is 3, find AR.

8) For what value of q, the elasticity of demand for the given demand function $p = 4 - 3q^2$ becomes unitary.

9) What is the marginal revenue for a demand curve which has infinite elasticity?

   Find out the elasticity of demand whenthe demand curve is p = 50/(x +3).

10) Given the total cost function $C = x^3 - 6x^2 + 15x + 10$; find the marginal cost function. At what output will MC be an increasing function?

11) If the total manufacturing cost 'y' of making x units of a product

    is: y = 20x + 5000,

    a) What is the variable cost per unit?

    b) What is the fixed cost?

    c) What is the total cost of manufacturing 4000 units?

    d) What is the marginal cost of producing 2000 units?

12) The profit function of a company can be represented by $P = f(x) = x - 0.00001x^2$, where x is units sold. Find the optimal sales volume and the amount of profit to be expected at that volume.

13) What does Price Elasticity mean? Elaborate your answer.

14) The corn flakes industry decides to reduce the price of its product, from Rs.100 to Rs. 75. The company expects that the sales of corn flakes will increase from 10,000 units a month to 20,000 units a month. Calculate and comment on the price elasticity of demand.

15) A demand function is given as $x = 50 - 5p$. Compute the price elasticity of demand at p=5.

16) If the demand curve is $x = \frac{9}{\sqrt{p}}$; then show that the price elasticity of demand is constant and equals to -1/2.

**Answer of questions**

1) x=5, TR=50, yes
2) 4000
3) 50/3; 25
4) x= 13, Rs. 149
5) Hint: p=MC; p≥ minimum AC; p≥ minimum AVC
6) 75/76
7) 150

8) 2/3

9) MR=AR

10) –x/(x+3)

11) We have the cost-output equation: y = 20x + 5000. We know that, if the production increases, only total variable cost will increase in direct proportion but the fixed cost will remain unchanged in total. So, the derivative of y with respect to the increase in x by 1 unit will give the variable cost per unit.

a) Variable cost per unit =$\frac{d}{dx}$(cost-output equation)=$\frac{d}{dx}$ (20x + 5000) = 20. Variable cost per unit is 20.

b) Total fixed cost will remain unchanged even if we don't produce any unit. If we don't produce any unit, there will be no variable cost and only fixed cost will be the total cost. So, if we put x=0 in the cost-output equation, we will get the fixed cost. Fixed Cost = y = [20.(0) + 5,000] = 5000

c) If we put x=4000 in the cost-output equation, we will get the total cost of producing 4,000 units. Total cost of producing 4000 units = y = 20 (4,000) + 5,000 = 85,000.

d) We know that the marginal cost of 'n'th unit = TCn–TCn–1 Marginal cost of 2000th unit

= TC of 2000 units – TC of (2000–1) units

= [20 (2000) + 5000] – [20 (1999) + 5000] = [45,000 – 44,980] = 20.

12) The necessary condition for the optimal sales volume is that the first derivative of the profit function is equal to zero and the second derivative must be negative. Where the profit function is:

P = x – 0.00001$x^2$

$$\frac{dP}{dx} = \frac{d}{dx}(x - 0.00001x^2)$$

Marginal Profit = 1 – 0.00002x

To get maximum profit now we put marginal profit = 0

So, 1 – 0.00002x = 0

or, 0.00002x = 1

So, x = $\frac{1}{0.00002}$ = 50,000 units.

The second derivative of profit function, i.e.,

$$\frac{d^2p}{dx^2} = \frac{d}{dx}(1 - 0.00002x)$$

Now by putting the value of x in profit function we get maximum profit.

P = x – 0.00001$x^2$

= 50,000 – 0.00001$(50,000)^2$ = 50,000 – 0.00001(2500000000)

MR is defined as the change in the total revenue for the sale of an extra unit.

= 50,000 – 25,000 = 25,000.

The optimum output for the company will be 50,000 units of x and maximum profit at that volume will be 25,000.

13) Price elasticity of demand is defined as the negative of the ratio of proportionate change in quantity demanded to a proportionate change in price,

i.e., $E_d = (-)\frac{proportionate\ change\ in\ demand}{proportionate\ change\ in\ price}$

$= (-)\ \frac{\Delta Q/Q}{\Delta P/P}$

$= (-)\ \frac{P}{Q}\frac{\Delta Q}{\Delta P}$ or $=(-)\ \frac{P}{Q}\frac{dQ}{dP}$

For price elasticity of demand, we will use the relevant formula.

14) First, we need to calculate the percentage change in quantity demanded and percentage change in price. So,

% Change in Price = (Rs. 75-100)/(Rs.100) = -25%

% Change in Demand = (20,000-10,000)/(10,000) = 100%

Therefore, the Price Elasticity of Demand = 100%/-25% = -4.

This means the demand is relatively elastic.

15) The given demand function is

$Q = 50 - 5p$

Differentiating w.r.t.'$P$', we get

$\frac{dQ}{dP} = -5$

Hence, elasticity of demand is:

$E_d = (-)\frac{P}{Q}\frac{dQ}{dP}$

$= (-)\frac{P}{50-5P}.\frac{-5}{1} = \frac{P}{10-P}$

When p=5; then $E_d = \frac{5}{10-5} = \frac{5}{5} = 1$

16) Given the demand curve,

$x = \frac{9}{\sqrt{p}} = 9\ p^{-1/2}$

$\frac{dx}{dp} = -\frac{9}{2}p^{-\frac{3}{2}}$

$E_d = -\frac{p}{x}\frac{dx}{dp} = -\frac{p}{x}.-\frac{9}{2}p^{-\frac{3}{2}} = \frac{9p^{-\frac{1}{2}}}{2x}$

$= \frac{9p^{-\frac{1}{2}}}{2\times 9p^{-\frac{1}{2}}} = \frac{1}{2}$ which is a constant.

# UNIT 10 INTEREST RATES

**Structure**



## 10.0 OBJECTIVES

After studying this unit, you will be able to:

- understand the concept of interest;
- define simple interest and calculate simple interest;
- define compound interest and calculate compound interest;
- perform continuous compounding calculations; and
- perform changing compound rate calculations.

## 10.1 INTRODUCTION

In this unit, we will study about different kinds of interest rates and the methods of calculating interests. For that purpose, we will discuss the concepts of simple interest and compound interest along with their computations.

## 10.2 MEANING AND CONCEPT OF INTEREST

Money can be lent or borrowed from one entity to another. The price to be paid for the use of a certain amount of money for a certain period of time is known as interest. It is considered as an expense to the borrower and income for the lender. The money borrowed or lent is called principal. The interest may be payable yearly, half-yearly, quarterly or monthly.

The idea of interest is based on the time value of money. It means that the money available at the present point of time is worth more than the same amount of money at future point of time. One rupee of today is not equal to one rupee of tomorrow. So, interest is actually the amount charged for the borrowing of money to compensate the lender for the lost opportunity cost of not having the money and also for various risks of lending money such as default risk.

## 10.3 SIMPLE INTEREST

Simple interest (I) is the interest calculated on the original amount borrowed. Thus, for the entire period of computation, interest is calculated on the principal amount. Rates of interest are usually expressed as percentage.

Formula for calculation of simple interest-

**Simple interest** = **Principal** ×**Rate** ×**Time**

$$I = P \times r \times t$$

where

***I***: Interest, the amount of money that you pay to borrow money or the amount of money that you earn on a deposit.

***P***: Principal which is the original sum of money

***t***: Time duration for which the money is borrowed/ deposited.

***r***: Rate of interest which is usually expressed as percent that is paid for money borrowed, or earned for money deposited.

**Calculating Amount:**

Amount $(A) = P + I$

$$= P + (P \times r \times t)\ or, \quad P(1 + rt)$$

(Amount is the total amount due at the end of the period)

**Example 1**: To buy a new car, Ramesh borrowed Rs. 5,00,000 at 8% annual simple interest for 3 years. How much interest will he pay?

**Solution:** Here, $P$ = 5,00,000, $r$ = 8/100=0.08, $t$ = 3 years, $I$ =?

Using the formula$I = P \times r \times t$,

simple interest ($I$) = 500000 × 0.08 × 3

= Rs. 1,20,000.

Ramesh will pay Rs. 1,20,000 as interest for the car loan.

**Example 2**: Aditya deposited Rs. 1,50,000 to a bank at 9.8% interest p.a. Find the total interest that he will receive at the end of 5 years. Also find the amount he will get.

**Solution:** Here, $P$ = 1,50,000, $r$ = 0.098, $t$ = 5, $I$=?$A$ =?

Since $I=P\times r\times t$,

$I$= 150000 × 0.098 × 5 = Rs. 73,500.

Aditya will get Rs. 73,500 as interest.

Amount = $P + I$

or, $A$ = 150000 + 73500 = Rs. 2,23,500.

Aditya will receive Rs. 73,500 as interest and amount of Rs. 2,23,500 after 5 years.

**Note:** We can also calculate amount using formula $A = P\,(1 + rt)$

$A = 150000\,[1 + (0.098 \times 5)]$

*or,* $A$ = Rs. 2,23,500

**Example 3**: Himanshi lends Rs. 45,000 to Rama. Find the time required for this amount to yield Rs. 9,900 in simple interest at 11% per annum.

**Solution:** Here, $I = 9{,}900$, $P = 45{,}000$, $r = 0.11$, $t = ?$

*From* $I = P \times r \times t$, we get

$t = I/Pr$

*or,* $t = 9900/\,(45000)\,(0.11)$

= 2 (years).

**Example 4**: In how many years will a sum be double of itself at 5% p.a. simple interest?

**Solution:** Amount = 2 × Principal

*Given* $r = 0.05$, $t = ?$

*From* $A = P + P \times r \times t$,

$A = P\,(1 + rt)$.

According to question, $2P = P\,[1 + t(0.05)]$

or, $2 = 1 + t\,(0.05)$

*or,* $t = 1/0.05$

*or,* 20.

A sum will be double of itself at 5% p.a. simple interest in 20 years.

**Example 5**: A sum of Rs. 1,20,000 was lent out for 2 years at simple interest. The lender got Rs.1,53,600 in all. Find the rate of interest per annum (p.a).

**Solution:** Here, $P = 1{,}20{,}000$, $A = 1{,}53{,}600$, $t = 2$, $r = ?$

$A = P\,(1 + rt)$

That is, $153600 = 120000\,(1 + 2r)$

or, $153600 = 120000 + 240000\,r$

or, $240000r = 153600 - 120000$

or, $r = 33600/240000$

or, $r = 0.14$ or 14%.

The required rate of interest is 14% p.a.

**Example 6**: Find the principal that will yieldRs. 500.50 as interest in 2 years at 5% p.a. simple interest.

**Solution:**Here,$I$= 500.50,$t$ = 2, $r$ = 0.05,$P$ =?

*From* $I = P \times r \times t$, *we get*

$500.50 = P \times 0.05 \times 2$

or, $500.50 = 0.10P$

or,$P = 5{,}005$.

The required Principal isRs. 5,005.

**Example 7**:

In how much time will Rs. 17,000 become Rs. 22,100 at 10% p.a. simple interest?

**Solution:**

$P = 17{,}000$, $A = 22{,}100$,$r = 0.10$,$t$ =?

Since $22100 = 17000\,[1 + t\,(0.10)]$,

$22100 = 17000 + 1700\,t$

or,$1700\,t = 5100$

$t = 3$ years.

**Check Your Progress 1**

1) Why is interest paid?

2) How long will it take Rs. 10,000 to yield Rs. 2,200 at $5\frac{1}{2}$% per annum simple interest?

3) A certain sum of money at simple interest amounts to Rs. 1,120 in 3 years and to Rs. 1,200 in 5 years. Find the principal and the annual rate of interest.

## 10.4 COMPOUND INTEREST

Another type of interest is compound interest. Compound interest is calculated not only on the initial principal amount but also on the accumulated interest of previous periods. In other words, compound interest is calculated on interest as well as on principal. Therefore, compound interest is sometimes described as 'interest on interest'.

For example, if Rs. 5,000 is deposited at 5% per annum interest for 1 year, at the end of the year the interest is Rs. 250.The amount at the end of year is Rs. 5,000 + Rs. 250 = Rs. 5,250. If this amount is lent at 5% per annum interest for another year, the interest is calculated on Rs. 5,250 instead of the original Rs. 5,000.So the amount in the account at the end of the second year is Rs. 5,250 + (5,250 × 0.05 × 1)= 5,250 + 262.50 = Rs. 5,512.50. Note that simple interest would produce a total amount of 5,000 + (5,000 × 0.05 × 2) = 5,000+ 500 = Rs. 5,500. The additional Rs. 12.50 is the interest on Rs. 250 at 5% for one year.

Some concepts related to compound interest:

**Compound Interest –** It is the difference between the compound amount and original principal amount.

**Compound Amount -** The total amount due at the end of the last period.

**Frequency of Compounding -** It indicates the number of times the interest is compounded in one year.

**Compounding Period -** The time period between two consecutive points in time at which interest is compounded.

**Formula for calculation of compound amount**

$\boldsymbol{A = P(1+i)^n}$,

where$i = r/k$ or,

$$\frac{\text{annual rate of interest}}{\text{number of compounding period per year}}$$

and $n = k \times t$; (number of compounding period per year × number of years).

***A***: Amount at the end of $t$ periods

***P***: Principal which is the original sum of money

***r***: Annual rate of interest

***k***: Number of compounding periods per year

***t***: Time duration in years

***n***: Total number of compounding periods

***i***: Interest rate per compounding period

For example,if annual rate of interest is 10% and the compounding is quarterly, then there are 4 compounding periods per year.

Thus, $i = 0.10/4 = 0.025$

If annual rate of interest is 24% and the compounding is monthly, then there are 12 compounding periods per year. So,

$i = 0.24/12 = 0.02$.

**Note**: The value of $(1 + i)^n$can be found out using compound interest table

***Some Formulae for Compound Interest***

Compound Interest may be calculated yearly, half-yearly, quarterly, monthly or continuously.

Formulas for different cases have been given below:

| **Time** | **Amount** |
|---|---|
| Yearly | $A = P(1+i)^n$ where $i = r$ and $n = t$ |
| Half- yearly | $A = P(1+i)^n$where $i= r/2$ and $n = 2t$ |
| Quarterly | $A = P(1+i)^n$where $i = r/4$ and $n = 4t$ |
| Monthly | $A = P(1+i)^n$where $i= r/12$ and $n = 12t$ |

Compound Interest (in all cases) $I = A–P$.

**Example 8**: Bank pays compound interest at the rate of 5% p.a. Yogi deposited a principal amount ofRs. 5,000 in bank for 4 years. Find the interest that Yogi will receive.

**Solution:**Here,$P = 5{,}000$, $n = 4$, $i = 0.05$, $A = ?$

We have $A = P(1 + i)^n$

*So,* $A = 5000 (1+ 0.05)^4$

or, $A = 5000 (1.215506)$ (Using CompoundInterest Table)

or,$A$ = Rs. 6,077.53.

We get,$I$= 6077.53 - 5000

or, $I$= Rs. 1,077.53.

Yogi will get Rs. 1,077.53 as interest from bank.

**Example 9**: Mehak deposited Rs. 50,000 in a finance account that pays 8% interest, compounded annually. How much amount will be in her finance account after 10 years?

**Solution:** Here, $P = 50{,}000$, $i = 0.08$, $n = 10$, $A = ?$

As,$A = P(1 + i)^n$,

$A = 50000 (1 + 0.08)^{10}$

or, $A = 50000 (2.158925)$ (Using Compound Interest Table)

or, $A$ = 1,07,946.25

Mehak will have Rs. 1,07,946.25 in her finance account after 10 years.

**Example10**: Find the compound interest on Rs. 8,000 for 5 years at 6% per annum interest compounded (i) semi-annually and (ii) monthly

**Solution:** Here, $P = 8{,}000$,$t = 5$, $r = 0.06$,$A = ?$

$A = P(1 + r/k)^{kt}$

(i) $A = 8000(1+0.06/2)^{2\times5}$

or, $A = 8000(1+ 0.03)^{10}$

or, $A = 8000 (1.343916)$ (Using Compound Interest Table)

or, $A$ = Rs. 10,751.33.

We get,$I$= 10751.33 – 8,000

or, $I$= Rs. 2,751.33

(ii) $A = 8000(1+0.06/12)^{12\times5}$

*i.e.,* $A = 8000(1+ 0.005)^{60}$

or,$A$ =8000 (1.348850) (Using Compound Interest Table)

or, $A$ =Rs. 10,790.80.

So,$I$ = 10790.8 – 8000

or, $I$ = Rs. 2,790.80.

**Example 11**: If an amount of Rs. 86,400 is invested at 8% p.a. compounded quarterly, how long will it take to accumulate Rs. 2,06,500.60?

**Solution:** Here, $P = 86{,}400, r = 0.08, A = 2{,}06{,}500.60$, $t$ =?

*As* $A = P\,(1 + r/k)^{kt}$,

$206500.60 = 86400\,(1 + 0.08/4)^n$

or, $206500.60/86400 = (1.02)^n$

or, $(1.02)^n = 2.39005$

or, $n \log 1.02 = \log 2.390056$

or, $n(0.0086) = 0.378408$

or, $n = 0.378408/0.0086$

= 44 quarters or 11 years.

The required time is 11 years.

**Example 12**: If interest is compounded annually at an interest rate of 6% p.a., then how long will it take a principal to double itself?

**Solution:** Here, $P = P$, $A = 2P$, $r = 0.06$, $n$ =?

As $A = P\,(1 + r)^n$,

$2P = P(1 + 0.06)^n$

or, $2 = (1.06)^n$

or, $\log 2 = n \log 1.06$

or, $n = \log 2 / \log 1.06 = 0.3010/0.0253$

or, $n = 11.9$ years (approx.).

**Example 13**: A sum of money is deposited by Krishna which compounds interest annually. The amount at the end of 2 years is Rs. 5,000 and at the end of 3 years is 5,200. Find the money deposited and the rate of interest.

**Solution:** Here $A(1) = 5{,}000, n = 2$ and $A(2) = 5{,}200$, $n = 3$, $P$ = ?, $r$ = ?

Now $A = P\,(1 + r)^n$

That is, $5000 = P\,(1 + r)^2$ … eq. (i)

and $5200 = P(1 + r)^3$ …eq. (ii)

Dividing (ii) by (i)

$5200/5000 = P(1 + r)^3 / P\,(1 + r)^2$

or, $(1 + r) = 5200/5000$

or, $r = 5200/5000 - 1$

or, $r = 1.04 - 1 = 0.04$.

That is, required Rate of interest = 4%.

Now, from eq (i), $P = 5000/\,(1 + 0.04)^2$

or, $P = 5000/(1.04)^2$

or, log $P$ = Log 5000 – 2 Log 1.04

= 3.69897 – 2 (0.0170) = 3.69897 – .0340 = 3.66497

or, $P$ = antilog (3.66497)

= Rs.4,623 (approx).

**Example 14**:

Vidya's savings account has a balance of Rs. 2,654.39. The annual interest rate is 3% compounded monthly. Find the original principal amount deposited two years ago.

**Solution:** Here, $A$= 2,654.39,$r$ = 3%,$t$=2, $P$ =?

*Since,* $i$= 3%/12 = 0.0025

*and* $n = 2 \times 12 = 24$, from

$A = P(1+i)^n$,

we have $2654.39 = P(1 + 0.0025)^{24}$

or, 2654.39 = $P$ (1.061757) (Using Compound Interest Table)

or, $P$ = 2654.39/1.061757

or, $P$ = Rs. 2,500.

## 10.5 SPECIAL CASES OF COMPOUND RATE OF INTEREST

**Continuous Compounding**

In the previous section, we discussed various cases where interest was compounded monthly, quarterly, semi-annually and annually. But what if interest is compounded daily or hourly? When the compounding period becomes very tiny and the number of compounding periods per year grows infinitely large, it is called a case of continuous compounding. In such cases, the interest is calculated and added to the principal amount every extremely small time period such as hourly or minutely.

**Formula for Continuous compounding**

$$A = Pe^{rt}$$

***A*:** Amount at the end of $t$ periods

***P*:** Principal amount which is original sum of money

***r*:** Annual rate of interest

***t*:** Time duration in years

**Compounding at Changing Rates**

We have discussed situations when the rate of interest is constant for the entire time period. However, in real life, interest rate can change from time to time. For example – Rashmi lends money to a friend for 2 years at 5% per annum rate of interest. Her friend fails to pay the money after 2 years. Rashmi increases the loan tenure for 2 more years but now at 7% per annum. So, the final amount she will receive is the product of original principal and two factors of form $(1 + i)^n$ or $e^{rt}$ (with proper values of $i$ and $n$ or $r$ and $t$ for each factor).

**Example 15**: If a businessman invests Rs. 50,000 at an annual interest rate of 5% per annum compounded continuously, calculate the amount he will have in the account after five years.

**Solution:** Here, $P = 50{,}000$, $r = 0.05$, $t = 5$, $A =?$

*As* $A = Pe^{rt}$,

*we have* $A = 50000\ e^{(0.05\times5)}$

or, $A = 50000\ (1.2840)$ (using $e^x$ value table)

or, $A$ = Rs. 64,200.

**Example 16**: The difference between simple and compound interest on a principal deposited for 8 years at 3% per annum is Rs. 267.7. Find the principal amount.

**Solution:** Here, $r = 0.03$, $t = 8$, C.I.- S.I. =Rs. 267.7. Let the principal amount be '$x$'

Simple interest on Rs.$x$ for 8 years at 3% p.a. $= x \times 8 \times 0.03 = 0.24\ x$

Compound Interest $= [\ x\ (1+ 0.03)^8 - x] = (1.03)^8 \times - x = 1.266770\ x$

$= 0.266770x$

According to question, $0.26677\ x - 0.24\ x = 267.7$

Or, $0.02677\ x = 267.7$

or, $x = 267.7/0.02677$

or, $x$ =Rs. 10,000.

**Example 17**: Find the principal amount that will become Rs. 16,000 in 9 years if money can be deposited at 2% p.a. compounded semi-annually.

**Solution:** Here, $A = 16{,}000$, $t = 9$, $r = 0.02$, $P =?$

Therefore, $i = 0.02/2 = 0.01$ and

$n = 2 \times 9 = 18$.

*From* $A = P\ (1 + i)^n$ we get $16000 = P\ (1 + 0.01)^{18}$

or, $P = 16000/(1.01)^{18}$

or, $P = 16000/\ 1.19614748$

or, $P$ = Rs.13,376.28.

**Example 18**: If interest is compounded continuously at an annual rate of 5%, how long will it take a principal to double itself?

**Solution**: Here, $P = P$, $A = 2P$, $r = 0.05$, $t =?$

*Since* $A = Pe^{rt}$, so

$2P = P\ e^{0.05t}$.

or, $e^{0.05t} = 2$

$0.05\ t \log e = \text{Log } 2$

or, $0.05\ (0.4343)t = 0.301$ (log e =0.4343)

or, $t = 0.301/0.021715$

or, $t = 13.86$ years = 14 years (approx.).

**Example 19**: If interest is compounded continuously, at what annual rate will Rs. 3,000 amount to Rs. 3,00,000 in 25 years?

**Solution:** Here, $P = 3{,}000$, $A = 3{,}00{,}000$, $t = 25$, $r =$?

*As* $A = Pe^{rt}$

We have $300000 = 3000\ e^{25r}$

or, $e^{25r} = 300000/3000 = 100$

or, $25\ r \log e = \log 100$

or, $25\ (0.4343)r = 2$ (log e =0.4343)

or, $r = 2/10.8575$

or, $r = 0.1842044$

or, $r = 18.42\ \%$ (approx.).

**Example 20**: Mr. Arvind deposited Rs. 20,000 in a bank savings account for 3 years. Bank pays 6% p.a. compounded semi-annually for the first year, 12% p.a. compounded quarterly for the second year and 13% per annum compounded continuously for the third year. Find the compound amount in his bank account at the end of 3 years.

**Solution:** Here, P =20,000, A=?

Amount at the end of 3 years = $20000\ (1 + 0.06/2)^2\ (1 + 0.12/4)^4 (e^{0.13})$, so

$A = 20000\ (1.03)^2\ (1.03)^4\ (e^{0.13})$

or, $A = 20000\ (1.03)^6\ (e^{0.13})$

or, $A = 20000\ (1.19405)\ (1.13882)$ (Using $e^x$ value table)

or, $A =$ Rs. 27,196.16.

---

**Check Your Progress 2**

1) Fill in the Blanks

   i) The time period between two consecutive points in time at which interest is calculated is called....................(frequency of compounding/compounding period).

   ii) When the number of compounding per year grows infinitely large, it is a case of....................(continuous compounding/ compounding at changing rates)

   iii) Interest on accumulated interest is called........................(simple interest/compound interest).

2) Find the compound amount in following cases:

   i) Rs. 1,000 at 10% per annum deposited for 5 years;

ii) Rs. 2,000 at 6% per annum for 4 years compounded quarterly; and

iii) Rs. 1,800 at 5 % per annum compounded semi-annually for 8 years.

3) Deepak deposited Rs.1,000 in bank account for 5 years at 12% simple interest per year. If the same amount had been deposited for the same period at 10% compound interest per year, how much more interest would he get?

4) In what time, will Rs. 1,000 amount to Rs. 5,000, if it is invested at 7% compounded continuously?

## 10.6 LET US SUM UP

In this unit, two basic concepts of interest, viz., simple interest and compound interest have been discussed. Starting with the intuitive idea that it is a payment made to lender by a borrower, it is observed that the difference between initial investment and the accumulated value at a future time point of time would show the amount of interest. While simple interest is paid only on the original principal, not on the interest accrued, in case of compound interest, the calculation would be not only on the initial principal but also the accumulated interest of prior periods. Two special cases of compounding- continuous compounding and compounding at changing rates have been explained.

## 10.7 KEY WORDS AND LIST OF SYMBOLS

**Compound Interest**: It refers to the interest on the principal amount as well as the interest due.

**Continuously Compounded Interest**: It is the interest on the principal amount that is constantly compounded, essentially leading to an infinite amount of compounding periods.

**Interest**: It is a payment for use of money. It is paid on borrowing money and is received on lending the money.

**Simple Interest**: It refers to the interest which is chargeable on principal amount only.

**Time Value of Money**: *Money* available at the *present time* is *worth* more than the identical sum in the future.

**List of Symbols**

The symbols used in writing different formulas in your study material are the one which are widely used. In the list given below, they are given as the first symbols under various words. Many textbooks have given symbols different from these. Most of the commonly used symbols are also given in the list below. Whenever you read a book it is important to clearly understand the meaning of various symbols used there. For writing different formulas you may use any set of symbols you like, but along with them it is always desirable to explain their meanings

| | |
|---|---|
| Amount at the end of t periods | A, S, $A_n$ |
| Annual rate of interest | r, i |

| | |
|---|---|
| Interest | I |
| Interest Rate per compounding period | i |
| Number of compounding periods per year | k, m |
| Principal which is the original sum of money | P, $A_0$ |
| Time duration in years | t |
| Total number of compounding periods | n |

## 10.8 SOME USEFUL BOOKS

Ayres, Frank Jr. *Theory and Problems of Mathematics of Finance.* Schaum's Outlines Series. McGraw Hill Publishing Co., New York, 1963.

Mizrahi and Sullivan, John. *Mathematics for businesss and Social Sciences.* Wiley and Sons., New Jersey, 1987.

Singh, J.K.,*Business Mathematics.* Himalaya Publishing House., New Delhi,2010.

Prasad, Bindra. and Mittal, P.K.,*Fundamentals of Business Mathematics.* Har-Anand Publications., New Delhi, 2007.

## 10.9 ANSWER OR HINTS TO CHECK YOUR PROGRESS

**Check Your Progress 1**

1) Towards the charge of using lender's money.
2) 4 years.
3) Rs. 1,000; 4% per annum

**Check Your Progress 2**

1) (i) compounding period, (ii) continuous compounding,(iii)compound interest
2) (i) Rs.1,610.51, (ii) Rs. 2,537.97, (iii) Rs.2,672.10.
3) Rs. 162.
4) 23 years (approx.)

## 10.10 EXERCISES WITH ANSWERS/HINTS

1) Differentiate between simple and compound interest with examples.
2) What is continuous compounding? Explain with the help of an example.
3) Find the simple interest on Rs. 12,000 borrowed for 8 months. The rate of simple interest is 9% p.a.

   (Ans: Rs. 720)
4) Monica paid Rs. 675 as interest on a loan of Rs. 5,000. Interest was charged at 9% per annum simple interest. Find the length of the loan.

   (Ans: 1.5 Years)

5) Find the compound amount and compound interest if Rs. 700 is lend for 15 years at 7% per annum compounded semi-annually.

   (Ans: Rs. 1964.76, Rs.1264.76)

6) Mohan deposits Rs. 600 in his bank account. He wants to accumulate Rs. 900 in his account. Find the time it will take to accumulate the amount at the rate of 8% per annum compounded quarterly.

   (Ans: 5.12 Years)

7) If invested for three years, which investment yields the largest compound amount? (a) Rs. 5,000 at 6% per annum compounded annually (b) Rs. 5,125 at 5% per annum compounded continuously or (c) Rs.4,950 at 6.5% per annum compounded annually?

   (Ans: (a) Rs. 5,978.09, (b) Rs. 5,954.40, (c) Rs. 5979.35).

8) How long would it take for a principal P to double if rate of interest is 14% per annum compounded monthly?

   (Ans: 4.98 Years)

9) A bank pays 5% per annum compounded continuously. Rs. 4,000 has been deposited for 6 years. Find the amount at the end of 6 years.

   (Ans: Rs.5399.44)

## Appendix: Amount at Compound Interest Table

| *n* | 0.25% | 0.50% | 0.75% | 1.00% | 1.25% | 1.50% | 2.00% |
|---|---|---|---|---|---|---|---|
| 1 | 1.002500 | 1.005000 | 1.007500 | 1.010000 | 1.012500 | 1.015000 | 1.020000 |
| 2 | 1.005006 | 1.010025 | 1.015056 | 1.020100 | 1.025156 | 1.030225 | 1.040400 |
| 3 | 1.007519 | 1.015075 | 1.022669 | 1.030301 | 1.037971 | 1.045678 | 1.061208 |
| 4 | 1.010038 | 1.020151 | 1.030339 | 1.040604 | 1.050945 | 1.061364 | 1.082432 |
| 5 | 1.012563 | 1.025251 | 1.038067 | 1.051010 | 1.064082 | 1.077284 | 1.104081 |
| 6 | 1.015094 | 1.030378 | 1.045852 | 1.061520 | 1.077383 | 1.093443 | 1.126162 |
| 7 | 1.017632 | 1.035529 | 1.053696 | 1.072135 | 1.090850 | 1.109845 | 1.148686 |
| 8 | 1.020176 | 1.040707 | 1.061599 | 1.082857 | 1.104486 | 1.126493 | 1.171659 |
| 9 | 1.022726 | 1.045911 | 1.069561 | 1.093685 | 1.118292 | 1.143390 | 1.195093 |
| 10 | 1.025283 | 1.051140 | 1.077583 | 1.104622 | 1.132271 | 1.160541 | 1.218994 |
| 11 | 1.027846 | 1.056396 | 1.085664 | 1.115668 | 1.146424 | 1.177949 | 1.243374 |
| 12 | 1.030416 | 1.061678 | 1.093807 | 1.126825 | 1.160755 | 1.195618 | 1.268242 |
| 13 | 1.032992 | 1.066986 | 1.102010 | 1.138093 | 1.175264 | 1.213552 | 1.293607 |
| 14 | 1.035574 | 1.072321 | 1.110276 | 1.149474 | 1.189955 | 1.231756 | 1.319479 |
| 15 | 1.038163 | 1.077683 | 1.118603 | 1.160969 | 1.204829 | 1.250232 | 1.345868 |
| 16 | 1.040759 | 1.083071 | 1.126992 | 1.172579 | 1.219890 | 1.268986 | 1.372786 |
| 17 | 1.043361 | 1.088487 | 1.135445 | 1.184304 | 1.235138 | 1.288020 | 1.400241 |
| 18 | 1.045969 | 1.093929 | 1.143960 | 1.196147 | 1.250577 | 1.307341 | 1.428246 |
| 19 | 1.048584 | 1.099399 | 1.152540 | 1.208109 | 1.266210 | 1.326951 | 1.456811 |
| 20 | 1.051206 | 1.104896 | 1.161184 | 1.220190 | 1.282037 | 1.346855 | 1.485947 |
| 21 | 1.053834 | 1.110420 | 1.169893 | 1.232392 | 1.298063 | 1.367058 | 1.515666 |
| 22 | 1.056468 | 1.115972 | 1.178667 | 1.244716 | 1.314288 | 1.387564 | 1.545980 |
| 23 | 1.059109 | 1.121552 | 1.187507 | 1.257163 | 1.330717 | 1.408377 | 1.576899 |
| 24 | 1.061757 | 1.127160 | 1.196414 | 1.269735 | 1.347351 | 1.429503 | 1.608437 |
| 25 | 1.064411 | 1.132796 | 1.205387 | 1.282432 | 1.364193 | 1.450945 | 1.640606 |
| 26 | 1.067072 | 1.138460 | 1.214427 | 1.295256 | 1.381245 | 1.472710 | 1.673418 |
| 27 | 1.069740 | 1.144152 | 1.223535 | 1.308209 | 1.398511 | 1.494800 | 1.706886 |
| 28 | 1.072414 | 1.149873 | 1.232712 | 1.321291 | 1.415992 | 1.517222 | 1.741024 |
| 29 | 1.075096 | 1.155622 | 1.241957 | 1.334504 | 1.433692 | 1.539981 | 1.775845 |
| 30 | 1.077783 | 1.161400 | 1.251272 | 1.347849 | 1.451613 | 1.563080 | 1.811362 |
| 31 | 1.080478 | 1.167207 | 1.260656 | 1.361327 | 1.469759 | 1.586526 | 1.847589 |
| 32 | 1.083179 | 1.173043 | 1.270111 | 1.374941 | 1.488131 | 1.610324 | 1.884541 |
| 33 | 1.085887 | 1.178908 | 1.279637 | 1.388690 | 1.506732 | 1.634479 | 1.922231 |
| 34 | 1.088602 | 1.184803 | 1.289234 | 1.402577 | 1.525566 | 1.658996 | 1.960676 |
| 35 | 1.091323 | 1.190727 | 1.298904 | 1.416603 | 1.544636 | 1.683881 | 1.999890 |
| 36 | 1.094051 | 1.196681 | 1.308645 | 1.430769 | 1.563944 | 1.709140 | 2.039887 |
| 37 | 1.096787 | 1.202664 | 1.318460 | 1.445076 | 1.583493 | 1.734777 | 2.080685 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 38 | 1.099528 | 1.208677 | 1.328349 | 1.459527 | 1.603287 | 1.760798 | 2.122299 |
| 39 | 1.102277 | 1.214721 | 1.338311 | 1.474123 | 1.623328 | 1.787210 | 2.164745 |
| 40 | 1.105033 | 1.220794 | 1.348349 | 1.488864 | 1.643619 | 1.814018 | 2.208040 |
| 41 | 1.107796 | 1.226898 | 1.358461 | 1.503752 | 1.664165 | 1.841229 | 2.252200 |
| 42 | 1.110565 | 1.233033 | 1.368650 | 1.518790 | 1.684967 | 1.868847 | 2.297244 |
| 43 | 1.113341 | 1.239198 | 1.378915 | 1.533978 | 1.706029 | 1.896880 | 2.343189 |
| 44 | 1.116125 | 1.245394 | 1.389256 | 1.549318 | 1.727354 | 1.925333 | 2.390053 |
| 45 | 1.118915 | 1.251621 | 1.399676 | 1.564811 | 1.748946 | 1.954213 | 2.437854 |
| 46 | 1.121712 | 1.257879 | 1.410173 | 1.580459 | 1.770808 | 1.983526 | 2.486611 |
| 47 | 1.124517 | 1.264168 | 1.420750 | 1.596263 | 1.792943 | 2.013279 | 2.536344 |
| 48 | 1.127328 | 1.270489 | 1.431405 | 1.612226 | 1.815355 | 2.043478 | 2.587070 |
| 49 | 1.130146 | 1.276842 | 1.442141 | 1.628348 | 1.838047 | 2.074130 | 2.638812 |
| 50 | 1.132972 | 1.283226 | 1.452957 | 1.644632 | 1.861022 | 2.105242 | 2.691588 |
| 51 | 1.135804 | 1.289642 | 1.463854 | 1.661078 | 1.884285 | 2.136821 | 2.745420 |
| 52 | 1.138644 | 1.296090 | 1.474833 | 1.677689 | 1.907839 | 2.168873 | 2.800328 |
| 53 | 1.141490 | 1.302571 | 1.485894 | 1.694466 | 1.931687 | 2.201406 | 2.856335 |
| 54 | 1.144344 | 1.309083 | 1.497038 | 1.711410 | 1.955833 | 2.234428 | 2.913461 |
| 55 | 1.147205 | 1.315629 | 1.508266 | 1.728525 | 1.980281 | 2.267944 | 2.971731 |
| 56 | 1.150073 | 1.322207 | 1.519578 | 1.745810 | 2.005034 | 2.301963 | 3.031165 |
| 57 | 1.152948 | 1.328818 | 1.530975 | 1.763268 | 2.030097 | 2.336493 | 3.091789 |
| 58 | 1.155830 | 1.335462 | 1.542457 | 1.780901 | 2.055473 | 2.371540 | 3.153624 |
| 59 | 1.158720 | 1.342139 | 1.554026 | 1.798710 | 2.081167 | 2.407113 | 3.216697 |
| 60 | 1.161617 | 1.348850 | 1.565681 | 1.816697 | 2.107181 | 2.443220 | 3.281031 |
| 61 | 1.164521 | 1.355594 | 1.577424 | 1.834864 | 2.133521 | 2.479868 | 3.346651 |
| 62 | 1.167432 | 1.362372 | 1.589254 | 1.853212 | 2.160190 | 2.517066 | 3.413584 |
| 63 | 1.170351 | 1.369184 | 1.601174 | 1.871744 | 2.187193 | 2.554822 | 3.481856 |
| 64 | 1.173277 | 1.376030 | 1.613183 | 1.890462 | 2.214532 | 2.593144 | 3.551493 |
| 65 | 1.176210 | 1.382910 | 1.625281 | 1.909366 | 2.242214 | 2.632042 | 3.622523 |
| 66 | 1.179150 | 1.389825 | 1.637471 | 1.928460 | 2.270242 | 2.671522 | 3.694974 |
| 67 | 1.182098 | 1.396774 | 1.649752 | 1.947745 | 2.298620 | 2.711595 | 3.768873 |
| 68 | 1.185053 | 1.403758 | 1.662125 | 1.967222 | 2.327353 | 2.752269 | 3.844251 |
| 69 | 1.188016 | 1.410777 | 1.674591 | 1.986894 | 2.356444 | 2.793553 | 3.921136 |
| 70 | 1.190986 | 1.417831 | 1.687151 | 2.006763 | 2.385900 | 2.835456 | 3.999558 |
| 71 | 1.193964 | 1.424920 | 1.699804 | 2.026831 | 2.415724 | 2.877988 | 4.079549 |
| 72 | 1.196948 | 1.432044 | 1.712553 | 2.047099 | 2.445920 | 2.921158 | 4.161140 |
| 73 | 1.199941 | 1.439204 | 1.725397 | 2.067570 | 2.476494 | 2.964975 | 4.244363 |
| 74 | 1.202941 | 1.446401 | 1.738337 | 2.088246 | 2.507450 | 3.009450 | 4.329250 |
| 75 | 1.205948 | 1.453633 | 1.751375 | 2.109128 | 2.538794 | 3.054592 | 4.415835 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 76 | 1.208963 | 1.460901 | 1.764510 | 2.130220 | 2.570529 | 3.100411 | 4.504152 |
| 77 | 1.211985 | 1.468205 | 1.777744 | 2.151522 | 2.602660 | 3.146917 | 4.594235 |
| 78 | 1.215015 | 1.475546 | 1.791077 | 2.173037 | 2.635193 | 3.194120 | 4.686120 |
| 79 | 1.218053 | 1.482924 | 1.804510 | 2.194768 | 2.668133 | 3.242032 | 4.779842 |
| 80 | 1.221098 | 1.490339 | 1.818044 | 2.216715 | 2.701485 | 3.290663 | 4.875439 |
| | | | | | | | |

| *n* | 3.00% | 4.00% | 5.00% | 6.00% | 7.00% | 8.00% |
|---|---|---|---|---|---|---|
| 1 | 1.030000 | 1.040000 | 1.050000 | 1.060000 | 1.070000 | 1.080000 |
| 2 | 1.060900 | 1.081600 | 1.102500 | 1.123600 | 1.144900 | 1.166400 |
| 3 | 1.092727 | 1.124864 | 1.157625 | 1.191016 | 1.225043 | 1.259712 |
| 4 | 1.125509 | 1.169859 | 1.215506 | 1.262477 | 1.310796 | 1.360489 |
| 5 | 1.159274 | 1.216653 | 1.276282 | 1.338226 | 1.402552 | 1.469328 |
| 6 | 1.194052 | 1.265319 | 1.340096 | 1.418519 | 1.500730 | 1.586874 |
| 7 | 1.229874 | 1.315932 | 1.407100 | 1.503630 | 1.605781 | 1.713824 |
| 8 | 1.266770 | 1.368569 | 1.477455 | 1.593848 | 1.718186 | 1.850930 |
| 9 | 1.304773 | 1.423312 | 1.551328 | 1.689479 | 1.838459 | 1.999005 |
| 10 | 1.343916 | 1.480244 | 1.628895 | 1.790848 | 1.967151 | 2.158925 |
| 11 | 1.384234 | 1.539454 | 1.710339 | 1.898299 | 2.104852 | 2.331639 |
| 12 | 1.425761 | 1.601032 | 1.795856 | 2.012196 | 2.252192 | 2.518170 |
| 13 | 1.468534 | 1.665074 | 1.885649 | 2.132928 | 2.409845 | 2.719624 |
| 14 | 1.512590 | 1.731676 | 1.979932 | 2.260904 | 2.578534 | 2.937194 |
| 15 | 1.557967 | 1.800944 | 2.078928 | 2.396558 | 2.759032 | 3.172169 |
| 16 | 1.604706 | 1.872981 | 2.182875 | 2.540352 | 2.952164 | 3.425943 |
| 17 | 1.652848 | 1.947900 | 2.292018 | 2.692773 | 3.158815 | 3.700018 |
| 18 | 1.702433 | 2.025817 | 2.406619 | 2.854339 | 3.379932 | 3.996019 |
| 19 | 1.753506 | 2.106849 | 2.526950 | 3.025600 | 3.616528 | 4.315701 |
| 20 | 1.806111 | 2.191123 | 2.653298 | 3.207135 | 3.869684 | 4.660957 |
| 21 | 1.860295 | 2.278768 | 2.785963 | 3.399564 | 4.140562 | 5.033834 |
| 22 | 1.916103 | 2.369919 | 2.925261 | 3.603537 | 4.430402 | 5.436540 |
| 23 | 1.973587 | 2.464716 | 3.071524 | 3.819750 | 4.740530 | 5.871464 |
| 24 | 2.032794 | 2.563304 | 3.225100 | 4.048935 | 5.072367 | 6.341181 |
| 25 | 2.093778 | 2.665836 | 3.386355 | 4.291871 | 5.427433 | 6.848475 |
| 26 | 2.156591 | 2.772470 | 3.555673 | 4.549383 | 5.807353 | 7.396353 |
| 27 | 2.221289 | 2.883369 | 3.733456 | 4.822346 | 6.213868 | 7.988061 |
| 28 | 2.287928 | 2.998703 | 3.920129 | 5.111687 | 6.648838 | 8.627106 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 29 | 2.356566 | 3.118651 | 4.116136 | 5.418388 | 7.114257 | 9.317275 |
| 30 | 2.427262 | 3.243398 | 4.321942 | 5.743491 | 7.612255 | 10.062657 |
| 31 | 2.500080 | 3.373133 | 4.538039 | 6.088101 | 8.145113 | 10.867669 |
| 32 | 2.575083 | 3.508059 | 4.764941 | 6.453387 | 8.715271 | 11.737083 |
| 33 | 2.652335 | 3.648381 | 5.003189 | 6.840590 | 9.325340 | 12.676050 |
| 34 | 2.731905 | 3.794316 | 5.253348 | 7.251025 | 9.978114 | 13.690134 |
| 35 | 2.813862 | 3.946089 | 5.516015 | 7.686087 | 10.676581 | 14.785344 |
| 36 | 2.898278 | 4.103933 | 5.791816 | 8.147252 | 11.423942 | 15.968172 |
| 37 | 2.985227 | 4.268090 | 6.081407 | 8.636087 | 12.223618 | 17.245626 |
| 38 | 3.074783 | 4.438813 | 6.385477 | 9.154252 | 13.079271 | 18.625276 |
| 39 | 3.167027 | 4.616366 | 6.704751 | 9.703507 | 13.994820 | 20.115298 |
| 40 | 3.262038 | 4.801021 | 7.039989 | 10.285718 | 14.974458 | 21.724521 |
| 41 | 3.359899 | 4.993061 | 7.391988 | 10.902861 | 16.022670 | 23.462483 |
| 42 | 3.460696 | 5.192784 | 7.761588 | 11.557033 | 17.144257 | 25.339482 |
| 43 | 3.564517 | 5.400495 | 8.149667 | 12.250455 | 18.344355 | 27.366640 |
| 44 | 3.671452 | 5.616515 | 8.557150 | 12.985482 | 19.628460 | 29.555972 |
| 45 | 3.781596 | 5.841176 | 8.985008 | 13.764611 | 21.002452 | 31.920449 |
| 46 | 3.895044 | 6.074823 | 9.434258 | 14.590487 | 22.472623 | 34.474085 |
| 47 | 4.011895 | 6.317816 | 9.905971 | 15.465917 | 24.045707 | 37.232012 |
| 48 | 4.132252 | 6.570528 | 10.401270 | 16.393872 | 25.728907 | 40.210573 |
| 49 | 4.256219 | 6.833349 | 10.921333 | 17.377504 | 27.529930 | 43.427419 |
| 50 | 4.383906 | 7.106683 | 11.467400 | 18.420154 | 29.457025 | 46.901613 |
| 51 | 4.515423 | 7.390951 | 12.040770 | 19.525364 | 31.519017 | 50.653742 |
| 52 | 4.650886 | 7.686589 | 12.642808 | 20.696885 | 33.725348 | 54.706041 |
| 53 | 4.790412 | 7.994052 | 13.274949 | 21.938698 | 36.086122 | 59.082524 |
| 54 | 4.934125 | 8.313814 | 13.938696 | 23.255020 | 38.612151 | 63.809126 |
| 55 | 5.082149 | 8.646367 | 14.635631 | 24.650322 | 41.315001 | 68.913856 |
| 56 | 5.234613 | 8.992222 | 15.367412 | 26.129341 | 44.207052 | 74.426965 |
| 57 | 5.391651 | 9.351910 | 16.135783 | 27.697101 | 47.301545 | 80.381122 |
| 58 | 5.553401 | 9.725987 | 16.942572 | 29.358927 | 50.612653 | 86.811612 |
| 59 | 5.720003 | 10.115026 | 17.789701 | 31.120463 | 54.155539 | 93.756540 |
| 60 | 5.891603 | 10.519627 | 18.679186 | 32.987691 | 57.946427 | 101.257064 |
| 61 | 6.068351 | 10.940413 | 19.613145 | 34.966952 | 62.002677 | 109.357629 |
| 62 | 6.250402 | 11.378029 | 20.593802 | 37.064969 | 66.342864 | 118.106239 |
| 63 | 6.437914 | 11.833150 | 21.623493 | 39.288868 | 70.986865 | 127.554738 |
| 64 | 6.631051 | 12.306476 | 22.704667 | 41.646200 | 75.955945 | 137.759117 |
| 65 | 6.829983 | 12.798735 | 23.839901 | 44.144972 | 81.272861 | 148.779847 |
| 66 | 7.034882 | 13.310685 | 25.031896 | 46.793670 | 86.961962 | 160.682234 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 67 | 7.245929 | 13.843112 | 26.283490 | 49.601290 | 93.049299 | 173.536813 |
| 68 | 7.463307 | 14.396836 | 27.597665 | 52.577368 | 99.562750 | 187.419758 |
| 69 | 7.687206 | 14.972710 | 28.977548 | 55.732010 | 106.532142 | 202.413339 |
| 70 | 7.917822 | 15.571618 | 30.426426 | 59.075930 | 113.989392 | 218.606406 |
| 71 | 8.155357 | 16.194483 | 31.947747 | 62.620486 | 121.968650 | 236.094918 |
| 72 | 8.400017 | 16.842262 | 33.545134 | 66.377715 | 130.506455 | 254.982512 |
| 73 | 8.652018 | 17.515953 | 35.222391 | 70.360378 | 139.641907 | 275.381113 |
| 74 | 8.911578 | 18.216591 | 36.983510 | 74.582001 | 149.416840 | 297.411602 |
| 75 | 9.178926 | 18.945255 | 38.832686 | 79.056921 | 159.876019 | 321.204530 |
| 76 | 9.454293 | 19.703065 | 40.774320 | 83.800336 | 171.067341 | 346.900892 |
| 77 | 9.737922 | 20.491187 | 42.813036 | 88.828356 | 183.042055 | 374.652964 |
| 78 | 10.030060 | 21.310835 | 44.953688 | 94.158058 | 195.854998 | 404.625201 |
| 79 | 10.330962 | 22.163268 | 47.201372 | 99.807541 | 209.564848 | 436.995217 |
| 80 | 10.640891 | 23.049799 | 49.561441 | 105.795993 | 224.234388 | 471.954834 |

## Appendix: $e^x$ and $e^{-x}$ value table

| x | $e^x$ | $e^{-x}$ |
|---|---|---|
| 0.00 | 1.0000 | 1.0000 |
| 0.01 | 1.0101 | 0.99005 |
| 0.02 | 1.0202 | 0.98020 |
| 0.03 | 1.0305 | 0.97045 |
| 0.04 | 1.0408 | 0.96079 |
| 0.05 | 1.0513 | 0.95123 |
| 0.06 | 1.0618 | 0.94176 |
| 0.07 | 1.0725 | 0.93239 |
| 0.08 | 1.0833 | 0.92312 |
| 0.09 | 1.0942 | 0.91393 |
| 0.10 | 1.1052 | 0.90484 |
| 0.11 | 1.1163 | 0.89583 |
| 0.12 | 1.1275 | 0.88692 |
| 0.13 | 1.1388 | 0.87810 |
| 0.14 | 1.1503 | 0.86936 |
| 0.15 | 1.1618 | 0.86071 |
| 0.16 | 1.1735 | 0.85214 |
| 0.17 | 1.1853 | 0.84366 |
| 0.18 | 1.1972 | 0.83527 |
| 0.19 | 1.2092 | 0.82696 |
| 0.20 | 1.2214 | 0.81873 |
| 0.21 | 1.2337 | 0.81058 |
| 0.22 | 1.2461 | 0.80252 |
| 0.23 | 1.2586 | 0.79453 |
| 0.24 | 1.2712 | 0.78663 |
| 0.25 | 1.2840 | 0.77880 |

| | | |
|---|---|---|
| 0.26 | 1.2969 | 0.77105 |
| 0.27 | 1.3100 | 0.76338 |
| 0.28 | 1.3231 | 0.75578 |
| 0.29 | 1.3364 | 0.74826 |
| 0.30 | 1.3499 | 0.74082 |
| 0.31 | 1.3634 | 0.73345 |
| 0.32 | 1.3771 | 0.72615 |
| 0.33 | 1.3910 | 0.71892 |
| 0.34 | 1.4049 | 0.71177 |
| 0.35 | 1.4191 | 0.70469 |
| 0.36 | 1.4333 | 0.69768 |
| 0.37 | 1.4477 | 0.69073 |
| 0.38 | 1.4623 | 0.68386 |
| 0.39 | 1.4770 | 0.67706 |
| 0.40 | 1.4918 | 0.67032 |

# UNIT 11 COMPOUNDING AND DISCOUNTING

**Structure**



## 11.0 OBJECTIVES

After studying this unit, you should be able to:

- understand the concept of Nominal and Effective rates of interest;
- explain theinter-relationship of Nominal and Effective Rates of Interest;
- understand the concept and calculation of Present Value;
- use equation of value;
- understand the concept of discount; and
- perform calculation of various types of discounts.

## 11.1 INTRODUCTION

In the preceding, unit we have introduced the concept of interest and discussed the method of computing simple as well as compound interest. The present unit included some more themes useful for studying interest.

Often, we end up paying at an interest rate that is different from stated rate. Such scenario arises because nominal rate of interest may be different from effective rate of interest. In the previous unit, we learnt that although interest has been quoted as a percentage per annum but it can be compounded more than once a year. But what if we want to find whether interest rate of 5% per annum (p.a.) compounded biannually is higher or lower than an interest rate of 5% p.a. compounded quarterly? Calculation of effective rate of interest facilitates comparison between different alternative opportunities. This unit also elaborates the concept of present value and equation of value. Discounting which is opposite to the concept of compounding introduced and its various types have been explained. Compounding is used to the find the future value of a present amount whereas discounting is used to find value of an amount which is to be received in future at an earlier date.

## 11.2 NOMINAL AND EFFECTIVE RATES OF INTEREST

An interest rate is only meaningful in the context of time. In general it is understood as – *per year* **rate**- which may be called *the nominal interest rate*. With other periods of time than the year - like *month*, *week*, or *day* - the interest rate may be called *the effective interest rate.*

To begin, remember the way in which compound interest works. Suppose an amount $A$ is invested at interest rate $i$ per year and this interest is compounded annually. After 1 year, the amount will be $A + iA = A(1 + i)$, and this total amount will earn interest the second year. Thus, after $n$ years the amount will be $A(1 + i)^n$. The factor $(1 + i)^n$is called the **accumulation factor**. If interest is compounded daily after the same $n$ years the amount will be $A(1+\frac{i}{365})^{365n}$. This$i$ is the case of **nominal annual** rate of interest.

The **effective annual rate of interest** is the amount of money that one unit invested at the *beginning* of the year will earn during the year, when the amount earned is paid at the *end* of the year. In the daily compounding example, the effective annual rate of interest is $(1+\frac{i}{365})^{365} - 1$. This is the rate of interest which compounded annually would provide the same return. When the time period is not specified, both nominal and effective interest rates are assumed to be annual rates.

**Example 1**:What is the effective rate of interest corresponding to an interest rate of 5% per annum compounded quarterly?

The equation to be solved is $(1 + 0.05/4)^4 = 1 + i,$ where $i$is the effective rate of interest. Two different investment schemes with two different nominal annual rates of interest may in fact be **equivalent**, that is, may have equal value at any fixed date in the future.

**Formula for Calculation of Effective Rate of Interest:**

$$\boldsymbol{R = (1 + i)^k - 1 \; or, (1 + r/k)^k - 1}$$

***R***: Effective rate of interest

***r*:** Annual rate of interest

***k***: Number of compounding periods per year

***i*:** Interest rate per conversion period

**Note:** In case nominal interest is compounded continuously, the formula for effective rate will become

$$\boldsymbol{R = e^r - 1}$$

**Example2**: Find the effective rate which is equivalent to a nominal rate of 24% compounded monthly.

**Solution:** Here, $k = 12$, $r = 0.24$, $R$=?

$i = 0.24/12$

$R = (1 + r/k)^k - 1$

$= (1 + 0.24/12)^{12} - 1$

$= (1.02)^{12} - 1$

$= 1.268241-1$

$= 0.268241$

$= 26.82\ \%.$

The equivalent effective rate is 26.82%.

**Example 3**: Find the effective rate which is equivalent to a nominal rate of 12 % compounded quarterly.

**Solution:** Here, $k = 4$, $r = 0.12$, $R$=?

$i = 0.12/4$

$R = (1 + r/k)^k - 1$

$= (1 + 0.12/4)^4 - 1$

$=(1.03)^4 - 1$

$=1.1255088 - 1$

$=12.55\ \%.$

The equivalent effective rate is 12.55%

**Example 4**: Find the compound interest rate when compounded continuously is equivalent to an effective rate of 6%.

**Solution:** Here, $R$=0.06, $e^r$=?

$R = e^r - 1$

$0.06 = e^r - 1$

$e^r = 1.06$

$r \log e = \log (1.06)$

*or,* $r\ (0.4343) = 0.0253$

or, $r = 0.0253\ /0.4343$

$= 0.05825$ or, 5.82%.

The required rate of interest is 5.82%.

**Example5**: A local post office charges interest at the rate of 5 rupees per 100 rupees per quarter payable in advance. Find the effective rate of interest per annum.

**Solution:** Since the rate is charged in advance, it can be treated like 5 rupees on 95 rupees per quarter. Here, $i = 5/95$, $k = 4$, $R$=?

Now, $R = (1 + i)^{k} - 1$

$= (1 + 5/95)^4 - 1$

$= (1 + 0.053)^4 - 1$

$= (1.053)^4 - 1$

$=1.2295 - 1$

$=22.95\ \%.$

The effective rate of interest is 22.95% per annum.

**Example 6**: Radheshyam wants to borrow money for buying a motorcycle. His bank charges 8% per annum interest compounded semi-annually. A neighbourhood money lender charges 7.9% per annum interest compounded monthly. In which case will he pay the lesser amount of interest?

**Solution:** We will compare the effective rates in both cases

$R = (1 + i)^{k} - 1$ or, $(1 + r/k)^{k} - 1$

Case A:

$r = 0.08$, $k = 2$, $R =?$

$R= (1 + 0.08/2)^2 - 1$

$=(1.04)^2 - 1$

$= 1.081600 - 1$

$= 8.16\%.$

Case B:

$r = 0.079$, $k = 12$, $R =?$

$R= (1 + 0.079/12)^{12} - 1$

$=(1.00658)^{12} - 1$

$= 1.081924 - 1$

$= 8.19\%$ (Approx.).

The bank has lower effective rate of interest and thus charges lower effective interest than the money lender.

**Example 7**: Anita wants to deposit Rs. 75,000 for 3 years. She has two options: Option (A)- 10% per annum compounded semi-annually. Option (B)- 9.5% per annum compounded continuously. Which option should Anita choose?

**Solution:** We will calculate effective rate for both options:

$R = (1 + \mathrm{i})^{k} - 1$ or, $(1 + r/k)^{k} - 1$

Option (A), $r = 0.1$, $k = 2$, $R =?$

$R= (1 + 0.1/2)^2 - 1$

$=(1.05)^2 - 1$

$= 1.1025 - 1$

$= 0.1025$ or, 10.25%

Option (B), $\mathrm{e}^{r} = 0.095$, $R =?$

$R = e^r - 1$

$= e^{0.095} - 1$

$= 1.0997 - 1$

$= 0.0997$ or 9.97%.

Anita should choose option A.

Nominal interest rates are not comparable unless their compounding periods are the same.Effective rate of interest is the rate of interest per annum compounded only once in a year. Effective rate should not be confused with simple interest. When the conversion period is a year, the effective rate of interest and nominal rate of interest are same. So, when compounding is done annually, both rates are equal. But if compounding is done more than once in a year, effective rate of interest will be different from nominal rate of interest. For example – If the interest rate is 15% per year, compounded annually. In this case, the nominal interest rate is 15%, and the effective interest rate is also 15%. However, in case the compounding is more frequent than once per year, then the effective interest rate will be greater than 15%. The more often compounding occurs, the higher is the effective interest rate.

The nominal rate of interest compounded continuously and equivalent to a given effective rate of interest is called the**force of interest**.

For example, if interest is compounded $n$ times per year, then the amount after $t$ years is given by

$$\left(1 + \frac{i}{n}\right)^{nt}.$$

If we let $n \to \infty$,then this expression is $e^{it}$. That is the case of notion of instantaneous compounding of interest. In thiscontext denote by $\delta$the rate of instantaneous compounding which is equivalentto interest rate $i$. Here $\delta$is called the **force of interest**.

**Example 8**:Show that $\delta = \ln(1 + i)$.

Taking $e^{\delta} = e^{\ln(1+i)} = (1 + i)$.

**Example 9**: Find the force of interest which is equivalent to 5% compoundeddaily.

Here $e^{\delta} = (1 + 0.05/365)^{365}$, so that d= 0.4999.

So as a roughapproximation when compounding daily the force of interest is the same as thenominal interest rate.

***Note: Nominal rate is also known as Annual percentage rate and Effective rate is also known as Annual percentage yield.***

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**Check Your Progress 1**

1) State whether the following statements are True or False.

   i) Nominal interest rate is always less than or equal to effective rate of interest.

   ii) The formula for compound discount is same as formula of compound interest.

iii) Force of interest is effective rate of interest compounded annually.

iv) Equation of Value is used to equate value of obligations at a focal point of time.

2) Fill in the blanks with appropriate words given in the brackets.

i) When the conversion period is ..........................the effective rate is the same as the nominal rate. (year, continuous).

ii) ...................is good for comparison between alternative investment opportunities. (Nominal rate/ Effective Rate)

iii) Nominal rate is also known as.........................(Annual percentage yield/ Annual percentage rate)

3) Bank A pays a nominal interest rate of 10% compounded semi-annually. Bank B pays a nominal interest rate of 10.5% compounded quarterly. Which bank offers better effective interest rate?

4) What nominal rate has an effective rate of 8%, if compounded (a) Semi-annually (b) Quarterly and (c) Monthly

5) What is force of interest?

## 11.3 PRESENT VALUE

Present value describes how much a future sum of money is worth today. To see the underlying idea, suppose the objective is to have an amount $A$after $n$ yearsfromtoday.If the interest rate is$i$, how much should be deposited now so as to achieve the set target? See that the amount required is $A(1+i)^{-n}$. Such a quantity iscalled the **present value** of $A$. The factor $(1+i)^{-1}$ is called the **discount factor.**

**Example10**:Suppose the annual interest rate is 5%. What is the present value ofa payment of Rs.2000 payable 10 years from now?

The present value is $Rs. 2000(1\ +0.05)^{-10}\ =\ Rs. 1227.83$.

This example depicts the following idea: If you were given thechoice of either receiving Rs. 1227.83 today or Rs. 2000 after10 years from now, then you should be indifferent between these two choices. Thus, under the assumption of an interest rate of 5% per annum, the payment of Rs. 2000 in 10 years can be replaced by a payment of Rs. 1227.83 today.

**Formula:**

$$\boldsymbol{P\ = A\,(1+i)^{-n} or, A\,(1+r/k)^{-tk}}$$

***P*** = Principal Amount

***A***= Amount at future point of time

***r*:** Annual rate of interest

***k***: Number of compounding periods per year

***t***: Time duration in years

***i*:** Interest rate per conversion period

***n*** = Number of Compounding periods

(**Note:** $(1+i)^{-n}$ or, $(1+r/k)^{-tk}$ is known as the discount factor. It represents the present value of Rs.1 due after $n$ periods at the interest rate of $i$ per period).

**Present Value: Continuous Compounding**

**Formula**

$$P = Ae^{-rt}$$

***P*** **=** Principal Amount

***A*****=** Amount at future point of time

***r*****:** Annual rate of interest

***t***: Time duration in years

**Example 11**: What is the present value of Rs. 1,000 due in 2 years at 5% per annum compound interest, if the interest is paid (a) yearly, (b) half-yearly?

**Solution:**(a) Here $A$ = 1,000, $i$ = 0.05, $n$ = 2, $P$ =?

$P = A\,(1+i)^{-n}$

$= 1000\,(1+0.05)^{-2}$

$= 1000\,(1.05)^{-2}$

$= 1000\,(0.907029)$

= Rs. 907.03

The present value is Rs. 907.03

b) Here $A$ = 1,000, $i$ = 0.05/2 = 0.025, $n$ = 2×2

$P = A\,(1+i)^{-n}$

$=1000\,(1+0.025)^{-2\times2}$

$= 1000\,(1.025)^{-4}$

$= 1000(0.905950)$

= Rs. 906.

The present value is Rs. 906.

**Note:** Value of $(1.05)^{-2}$ and $(1.025)^{-4}$ can be calculated using calculator or using Present value table.

**Example 12**: Monica deposits money in a bank paying 16% per annum interest rate compounded quarterly. If she wants to have Rs. 15,000 in her account after 5 years. How much should she deposit today?

**Solution:** Here, $A$ = 15,000, $i$ = 0.16/4 = 0.04, $n$ = 5 × 4, $P$ =?

$P = A\,(1+i)^{-n}$

$= 15000\,(1+0.04)^{-20}$

$= 15000\,(1.04)^{-20}$

$= 15000(0.456386)$ (Using Present Value Table)

= Rs. 6,845.80.

Monica should deposit Rs. 6,845.80 in her account today.

**Example13**: Find the present value of Rs.100 which is due at the end of 3 years. The rate of interest is 5% per annum compounded continuously.

**Solution:** $A$ =100, $t$ = 3, $r$ = 0.05, p = ?

$P = Ae^{-rt}$

$= 100\ e^{-(0.05)(3)}$

$= 100\ e^{-0.15}$ (Using $e^{-x}$ value table)

$= 100\ (0.86071)$

$= 86.07.$

The present value is Rs. 86.07.

**Example 14**: Rahul has won a prize. He was given two options either to receive Rs.8,000 today or Rs.10,000 after 2 years. The market interest rate is 12% and the interest is compounded on monthly basis. Which option should Rahul choose?

**Solution:** Option (A) Rs. 8,000

Option (B) Here, $A = 10{,}000, n = 2 \times 12 = 24, r = 0.12, i = 0.12/12 = 0.01$

$P = 10000\ (1 + 0.01)^{-24}$

$= 10000\ (1.01)^{-24}$

$= 10000\ (0.787566)$ (Using Present Value Table)

= Rs. 7,875.66- (Value of Option B).

Rahul should choose option A.

**Example 15**: Find the present value of Rs. 4,500 due after 3 years from now. The interest is compounded continuously at the interest rate of 6%.

**Solution:** Here, $A = 4{,}500, t = 3, r = 0.06, P = ?$

$P = Ae^{-r \times t}$

$= 4500\ e^{-0.06 \times 3}$

$= 4500\ e^{-0.18}$

$= 4500\ (0.83527)$ (Using $e^{-x}$ value table)

= Rs. 3,758.72.

The present value is Rs. 3,758.72.

---

**Check Your Progress 2**

1) What do you mean by the term present value?
2) What is the present value of Rs. 5,000 after 6 years from now. The rate of interest is 8% per annum compounded quarterly?
3) Tarunis going to receive return of Rs. 10,000on an investment three months from now. What would be the present value of return if the continuous compounding interest rate is 16% p.a.?

## 11.4 EQUATION OF VALUE

We have seen that in calculations of interest, the value of an amount of money at any given point in time depends upon the time length since the money was paid in the past or upon time which will elapse in the future before it is paid. This principle is often called the time value of money. We assume that this principle rejects only the effect of interest and does not include the effect of inflation. Inflation reduces the purchasing power of money over time.So, investors expect a higher rate of return to compensate for inflation.

As a consequence of the above principle, various amounts of money payable at different points in time cannot be compared until all the amount are accumulated or discounted to a common date, called the comparison date/focal date/valuation date, is established. The equation which accumulates or discounts each payment to the comparison date is called the **equation of value**.

The time diagram is helpful to solve the equation of value. Note that a time diagram is one-dimensional diagram where the only variable is time. We may show values of money intended to be associated with different funds on a time line. Note that a time diagram is not a formal part of a solution. However, it may be very helpful in visualizing the solution.

**Example 16**:In return for a payment of Rs.1,200 at the end of 10 years, a lender agrees to pay Rs.200 immediately, Rs.400 at the end of 6 years, and a final amount at the end of 15 years. What is the amount of the final payment at the end of 15 years if the nominal rate of interest is 9% per annum converted semiannually?

Ans.: The comparison date is chosen to be $t = 0$. The time diagram is given in Figure1.

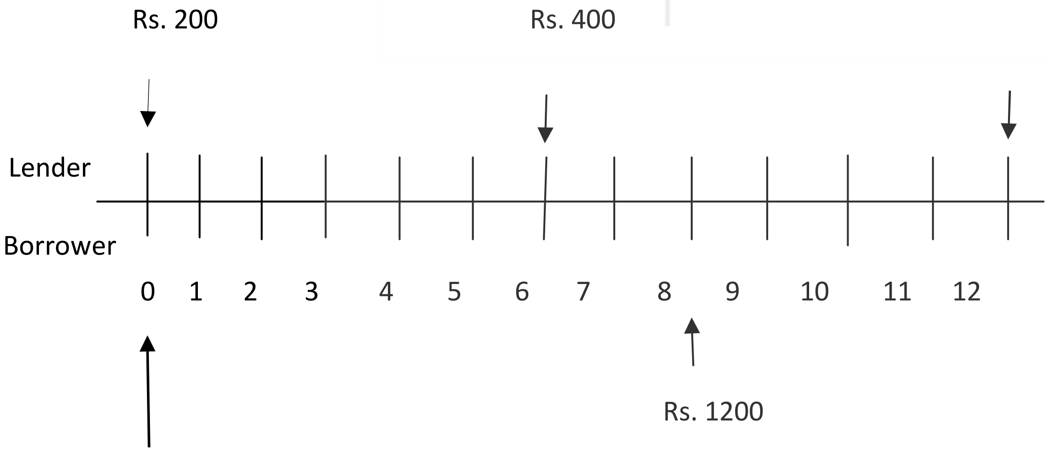


**Figure 1**: **Time Diagram for Payment**

The equation of value is $200 + 400(1 + 0.045)^{-12} + X(1 + 0.045)^{-30} = 1200(1 + 0.045)^{-20}$.

Solving this equation for $X$ we get $X = Rs. 231.11$ (approx.).

To solve equation of value problems, we need to

1) Select a focal date;

2) Calculate the value of all obligations on the focal date using the stated interest;

3) Set up the equation of value and then solve.

**Example 17**: You owe a debt that is to be paid by payments of Rs.1,800 after one month, Rs.1,600 after two months and Rs.1,000 after three months. Find the single payment that would repay the entire debt today. The simple rate of interest is 5.75% per annum.

**Solution:** Focal Point: Today

| Old obligation | Value at focal point | New obligation | Value at focal point |
|---|---|---|---|
| Rs. 1800 after 1 month | $1800(1+0.0575\times1/12)^{-1}$ | Rs. $x$ today | x |
| Rs.1600 after 2 months | $1600(1+0.0575\times2/12)^{-1}$ | | |
| Rs.1000 after 3 months | $1000(1+0.0575\times3/12)^{-1}$ | | |

Setting up Equation of value:

$x = 1800(1+0.0575\times1/12)^{-1} + 1600(1+0.0575\times2/12)^{-1} + 1000(1+0.0575\times3/12)^{-1}$

= 1791. 58 + 1584. 94 + 981.45

= 4357.97.

You have to pay Rs. 4,357.97 today to discharge the entire debt.

**Example 18**: Sita has two debts due: A debt of Rs. 800 is due in four months and another debt of Rs. 1,000 is due in nine months. She wants to settle both debts by a single payment at the end of six months. The rate of simple interest is 6% p.a. Find the amount of this payment using (a) the present date as the focal date, (b) the date of settlement as the focal date.

**Solution:** Let theamount of payment be $x$.

a) Focal point: Present date

| Old obligation | Value at focal point | New obligation | Value at focal point |
|---|---|---|---|
| Rs. 800 after 4 months | $800(1+0.06\times4/12)^{-1}$ | Rs. x after 6 months | $x(1+0.06\times6/12)^{-1}$ |
| Rs.1000 after 9 months | $1000(1+0.06\times9/12)^{-1}$ | | |

Setting up equation of Value:

$800(1+0.06\times4/12)^{-1} + 1{,}000(1+0.06\times9/12)^{-1} = x(1+0.06\times6/12)^{-1}$

$784.31 + 956.94 = 0.97087\,x$

*or,* $x = 1741.25/0.97087$

= 1793.49.

Hence, the amount required to pay today to settle both debts is Rs. 1,793.49.

b) Focal point: Six months

| Old obligation | Value at focal point | New obligation | Value at focal point |
|---|---|---|---|
| Rs. 800 after 4 months | $800(1+0.06 \times 2/12)^{1}$ | Rs. x after 6 months | $x$ |
| Rs.1000 after 9 months | $1000(1+ 0.06 \times 3/12)^{-1}$ | | |

Setting up equation of Value:

$$800\left(1+0.06\times\frac{2}{12}\right)^{1}+1000\left(1+0.06\times\frac{3}{12}\right)^{-1}=x$$

$808 + 985.22 = x$

*or,* $x= 1793.22$

Hence the amount required to pay after 6 months to settle both debts is Rs. 1,793.22.

**Example 19**: Vidya has a debt of Rs. 2,000 which is due after 12 years. Instead of a lump sum payment, she is willing to pay Rs. 500 now and Rs. 500 at the end of six years and a final payment at the end of 12 years. If the rate of interest charged is 2% per annum effectively, what should be her the final payment to pay off the debt?

**Solution:** Let $x$ be Vidya's final payment towards payment of debt. Focal point = 12 years

Calculation Table:

| Old obligation | Value at focal point | New obligation | Value at focal point |
|---|---|---|---|
| Rs. 2000 | $2000(1 + 0.02)^{12}$ | Rs.500 now | $500(1 + 0.02)^{12}$ |
| | | Rs. 500 at the end of 6 years | $500(1 + 0.02)^{6}$ |
| | | Rs. $x$ at the end of 12 years | X |

Equation of Value becomes

$500 (1 + 0.02)^{12} + 500(1 + 0.02)^{6} + x= 2000 (1 + 0.02)^{12}$

or, $500 (1.02)^{12} + 500(1.02)^{6} + x = 2000 (1.02)^{12}$

or, $500 (1.268242)+ 500(1.126162)+x = 2000 (1.268242)$

or, $634.121 + 563.081 + x = 2536.484$

*or,* $x =$Rs. 1339.282

Vidya should pay Rs. 1339.28 to settle her debt.

**Example 20**: Satnam borrowed Rs. 15,000 from a money lender. He pays Rs.4,000 at the end of 6 months and Rs. 6,000 at the end of one year. What should he pay at the end of two years to settle his debt? The rate of interest charged is 18% compounded continuously.

**Solution:** Let the final payment by Satnambe*x*. The focal point is present date i.e., 0.

Equation of Value:

$15000 = 4000\ e^{-0.09} + 6000\ e^{-0.18} + xe^{-0.36}$

or, $15000 = 4000\ (0.91393) + 6000\ (0.83527) + x\ (0.69768)$ (Using $e^{-x}$ value table)

or, $15000 = 3655.72 + 5011.62 + 0.69768\ x$

or, $0.69768\ x = 6332.66$

*or,* $x$ = Rs.9,076.74.

Satnam should pay Rs.9,076.74 to settle his debt.

**Check Your Progress 3**

1) What do you mean by equation of value?

2) State the meaning of a time diagram.

3) Write the meaning of focal date.

## 11.5 DISCOUNT

Discounting is the opposite of compounding. When we know the future value of some amount or obligation, we apply discount rate to find its value at some earlier date. So, discounting is about moving backwards in time.

### 11.5.1 Types of Discounts

**Simple Discount:**

Sometimes interest is paid or deducted upfront in the financial transactions. Such interest is called simple discount. For example, Anuj went to a money lender to ask for a loan ofRs. 2,000. The money lender agrees to provide him the loan and gives him Rs. 1,700 and says that Rs. 300 is the 15% interest (2000 × 0.15 =300). Now Anuj will have to pay back only Rs.2,000 to the money lender. Simple discount is different from simple interest as simple interest is a percentage of principal while simple discount is a percentage of the maturity amount.

**Note:** Another name for simple discount is bank discount.

**Formula for Simple Discount:**

$\boldsymbol{D = A \times d \times t}$,

*where*

***D*** =Discount Amount

***A*** = Amount at the end of *t* periods

***d*** = Discount percentage

***t*** = Time period

**Formula for Present Value at a Discount Rate:**

$$P = A(1 - d)^n,$$

where

$P$ = Present value of amount

$A$ = Amount at the end of $n$ periods

$d$ = Discount rate per period

$n$ = Number of time periods

Discount ($D$) = $A - P$

**Nominal rate of discount and Effective rate of discount**

The concept of nominal rate of discount and effective rate of discount is similar to compounding. The stated rate of discount, when the discount is converted more than once in a year is known as nominal rate of discount. On the other hand, effective rate of discount is the rate of discount per annum compounded only once in a year. Effective rate of discount is lower than the nominal rate of discount.The nominal discount rate converted continuously and equivalent to a given effective rate is known as force of discount. The value of nominal rate of discount is not dependent on principal amount or maturity amount.

The relationship between nominal discount rate and effective discount rate is given by

$$D_e = 1 - e^{-d},$$

where

$D_e$ = Effective discount rate

d = Nominal discount rate

**Compound Discount:**

Compound discount is the inverse of compound interest and is used to calculate the value of future value of an amount at an earlier date.

**Formula for Compound Discount:**

$$P = A(1 + i)^{-n},$$

where

$P$ = Present value of amount

$A$ = Amount at the end of $n$ periods

$i$ = Interest rate per period

$n$ = Number of time periods

**Continuous Discount:**

Continuous discount is used to find the present value of an asset or an obligation whose value is known at some future time when interest is compounded continuously. The concept of continuous discounting has wide application in valuation of futures and forwards contracts.

**Formula for Continuous Discount:**

$\boldsymbol{P = Ae^{-i\times n}}$,

where

$\boldsymbol{P}$ = Present value of amount

$\boldsymbol{A}$ = Amount at the end of $n$ periods

$\boldsymbol{i}$ = Interest rate per period

$\boldsymbol{n}$ = Number of time periods

**Example 21**: Find the present value ofRs. 5,000 due in 2 years at 4% rate of discount compounded quarterly.

**Solution:** Here, $A$ = 5000, $n = 2 \times 4$, $d = 0.04/4 = 0.01$, $P$=?

$P = A(1- d)^n$

$= 5000 (1- 0.01)^8$

$= 5000 (0.99)^8$

$= 5000 (0.9227446)$

= Rs. 4,613.72.

The required present value is Rs. 4,613.72.

**Example 22**: Find the effective discount rate when nominal rate of discount is 10% compounded continuously.

**Solution:** Here d = 0.10, $D_e$ =?

$D_e = 1 - e^{-d}$

$= 1- e^{-0.10}$

$= 1 - 0.90484$ (Using $e^{-x}$value table)

= 0.09516 or, 9.52%.

The required effective rate of discount is 9.52 %.

**Example 23**: Find the present value of Rs. 4,500 due after 3 years from now. The interest is compounded continuously at the interest rate of 6%.

**Solution:** Here, $A$ = 4,500, $t$ = 3, $r$ = 0.06, $P$ =?

$P = Ae^{-r\times t}$

$= 4500\ e^{-0.06\times 3}$

$= 4500\ e^{-0.18}$

$= 4500 (0.83527)$ (Using $e^{-x}$value table)

= Rs. 3,758.72.

The present value is Rs. 3,758.72.

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**Check Your Progress 4**

1) Spell out the difference between effective rate of discount and effective rate of interest.

2) What is the relationship between nominal rate of discount and effective rate of discount?

3) When would you use continuous discount?

## 11.6 LET US SUM UP

In this unit we have discussed the concepts of nominal and effective interest rates; present value, equation of value and discount that are used in course of interest rate computation. In the process we have learnt that nominal interest rate is annual percentage rate while effective annual rate of interest is the amount of money that one unit invested at the beginning of the year will earn during the year. The idea of present value is used to get the current value of a future sum of money at given rate of return. We have seen that equation of value is used to equate the value of obligations at a focal point of time and discounting is opposite to compounding.

## 11.7 KEY WORDS AND LIST OF SYMBOLS

**Discount Factor**: Present value of Re.1 due after *n* periods at the interest rate of *i* per period.

**Discount Rate**: The interest rate used to find the value of an amount at an earlier period than its given future value period.

**Effective Rate of Discount**: Rate of discount per annum compounded only once in a year.

**Equation of Value:** Equation of value equates value of different obligations at focal point. At the focal date, value of debts = value of payments.

**Focal Point of Time**: A common date to which various amounts of money payable at different points in time is fixed to compare all the amounts.

**Nominal Rate of Discount**: The stated rate of discount when the discount is converted more than once in a year.

**Present Value:** Current value of a future sum of money or obligation at given rate of return.

**List of Symbols**

The symbols used in writing different formulas in your study material are the one which are widely used. In the list given below, they are given as the first symbols under various words. Many textbooks have given symbols different from these. Most of the commonly used symbols are also given in the list below. Whenever you read a book it is important to clearly understand the meaning of various symbols used there. For writing different formulas you may use any set of symbols you like, but along with them it is always desirable to explain their meanings.

| | |
|---|---|
| Amount at the end of $t$ periods | A, S, $A_n$ |
| Annual rate of interest | r, R |
| Discount Amount | D |
| Discount rate per period | d |
| Effective rate of interest | R, $r^{eff}$, $r_E$, f |

| | |
|---|---|
| Effective Discount Rate | $D_e$ |
| Interest rate per conversion period | i |
| Number of compounding periods per year | k, m |
| Number of time periods | n |
| Principal Amount, Present Value of Amount | P, $A_0$ |
| Time duration in years | t |

## 11.8 SOME USEFUL BOOKS

Ayres, Frank Jr., *Theory and Problems of Mathematics of Finance.* Schaum's Outlines Series, McGraw Hill Publishing Co., New York, 1963.

Mizrahi and Sullivan, John, *Mathematics for Businesss and Social Sciences.* Wiley and Sons., New York, 1977.

Williams, W.E., and Reed, J.H. *Fundamentals of Business Mathematics*., Iowa 2014.

Singh, J.K. *Business Mathematics.* Himalaya Publishing House., New Delhi, 2018.

Prasad, Bindra, and Mittal, P.K. *Fundamentals of Business Mathematics.* Har-Anand Publications., New Delhi, 2007.

## 11.9 ANSWER OR HINTS TO CHECK YOUR PROGRESS

**Check Your Progress 1**

1) (i)True (ii) False (iii)False (iv) True
2) (i) year (ii) Effective Rate (iii) Annual Percentage rate
3) Bank A - 10.25%, Bank B - 10.92%
4) a) 7.85% b) 7.77% c) 7.72%
5) Nominal rate of interest compounded continuously and equivalent to a given effective rate of interest.

**Check Your Progress 2**

1) How much a future sum of money is worth today.
2) Rs. 3,108.60
3) Rs. 9,607

**Check Your Progress 3**

1) Equation that accumulates or discounts each payment to the comparison date.
2) One-dimensional diagram, where the only variable is time, shown on a single coordinate axis. In the coordinate of a point on the time-axis, values of money intended to be associated with different funds are shown.

3) A common date to which various amounts of money payable at different points in time is fixed to compare all the amounts of accumulation or discount.

**Check Your Progress 4**

1) The effective rate of discount, denoted by $d$, is a measure of interest where the interest is paid at the beginning of the period. It is a contrast to the effective rate of interest which is a measure of the interest paid at the end of the period.

2) Rate of discount, when the discount is converted more than once in a year is known as nominal rate of discount. The effective rate of discount is the rate of discount per annum converted only once in a year.

3) To find the present value of an asset whose value is known at some future time when interest is compounded continuously.

## 11.10 EXERCISES WITH ANSWERS/HINTS

1) Distinguish between the nominal rate of interest and effective rate of interest.

2) How compounding is different from discounting? Explain with the help of an example.

3) Interest on a savings account is quoted as 23% p.a. compounded monthly. Find the effective annual interest rate for the savings account.

   (Ans: 25.586%)

4) Find the effective annual interest rate if the nominal interest rate of 9% is:

   (a) compounded monthly (b) compounded quarterly (c) yearly (d) semi-annually

   (Ans: (a) 9.381%, (b) 9.308%, (c) 9%, (d) 9.202%)

5) A deposit of Rs. 1,300 earns Rs.339.45 as interest in 3years. If the interest is compounded monthly, what is the effective rate?

   (Ans: 8.04%)

6) Saloni deposits an amount today in her account that earns 5% interest, compounded annually. If her goal is to have Rs. 5,000 in the account at the end of six years, how much she must deposit in the account today?

   (Ans: Rs. 3731.08)

7) Sagar has to make payment of Rs. 500 due today and Rs.750 in four months from now. Instead he wants to settle the payments by paying Rs. 600 in three months and a final payment in seven months from today. The rate of interest is 7% p.a. Find the final payment using seven months from now as focal point.

   (Ans.:669.55)

8) An IT Company's bond will be worth Rs. 10,000 in 10 years. What should the investor pay for it today in order to earn 6.5% annually?

   (Ans: Rs. 5327.26)

9) Find the discount on Rs. 3000 due in 3 years at 6% rate of discount converted semi-annually.

   (Ans.: Rs. 501.09)

10) What is the effective discount rate when nominal rate of discount is 12% compounded continuously?

   (Ans.:11.31 %)

## Appendix: Present Value Table

| n | 0.25% | 0.50% | 0.75% | 1.00% | 1.25% | 1.50% | 2.00% |
|---|---|---|---|---|---|---|---|
| 1 | 0.997506 | 0.995025 | 0.992556 | 0.990099 | 0.987654 | 0.985222 | 0.980392 |
| 2 | 0.995019 | 0.990075 | 0.985167 | 0.980296 | 0.975461 | 0.970662 | 0.961169 |
| 3 | 0.992537 | 0.985149 | 0.977833 | 0.970590 | 0.963418 | 0.956317 | 0.942322 |
| 4 | 0.990062 | 0.980248 | 0.970554 | 0.960980 | 0.951524 | 0.942184 | 0.923845 |
| 5 | 0.987593 | 0.975371 | 0.963329 | 0.951466 | 0.939777 | 0.928260 | 0.905731 |
| 6 | 0.985130 | 0.970518 | 0.956158 | 0.942045 | 0.928175 | 0.914542 | 0.887971 |
| 7 | 0.982674 | 0.965690 | 0.949040 | 0.932718 | 0.916716 | 0.901027 | 0.870560 |
| 8 | 0.980223 | 0.960885 | 0.941975 | 0.923483 | 0.905398 | 0.887711 | 0.853490 |
| 9 | 0.977779 | 0.956105 | 0.934963 | 0.914340 | 0.894221 | 0.874592 | 0.836755 |
| 10 | 0.975340 | 0.951348 | 0.928003 | 0.905287 | 0.883181 | 0.861667 | 0.820348 |
| 11 | 0.972908 | 0.946615 | 0.921095 | 0.896324 | 0.872277 | 0.848933 | 0.804263 |
| 12 | 0.970482 | 0.941905 | 0.914238 | 0.887449 | 0.861509 | 0.836387 | 0.788493 |
| 13 | 0.968062 | 0.937219 | 0.907432 | 0.878663 | 0.850873 | 0.824027 | 0.773033 |
| 14 | 0.965648 | 0.932556 | 0.900677 | 0.869963 | 0.840368 | 0.811849 | 0.757875 |
| 15 | 0.963239 | 0.927917 | 0.893973 | 0.861349 | 0.829993 | 0.799852 | 0.743015 |
| 16 | 0.960837 | 0.923300 | 0.887318 | 0.852821 | 0.819746 | 0.788031 | 0.728446 |
| 17 | 0.958441 | 0.918707 | 0.880712 | 0.844377 | 0.809626 | 0.776385 | 0.714163 |
| 18 | 0.956051 | 0.914136 | 0.874156 | 0.836017 | 0.799631 | 0.764912 | 0.700159 |
| 19 | 0.953667 | 0.909588 | 0.867649 | 0.827740 | 0.789759 | 0.753607 | 0.686431 |
| 20 | 0.951289 | 0.905063 | 0.861190 | 0.819544 | 0.780009 | 0.742470 | 0.672971 |
| 21 | 0.948916 | 0.900560 | 0.854779 | 0.811430 | 0.770379 | 0.731498 | 0.659776 |
| 22 | 0.946550 | 0.896080 | 0.848416 | 0.803396 | 0.760868 | 0.720688 | 0.646839 |
| 23 | 0.944190 | 0.891622 | 0.842100 | 0.795442 | 0.751475 | 0.710037 | 0.634156 |
| 24 | 0.941835 | 0.887186 | 0.835831 | 0.787566 | 0.742197 | 0.699544 | 0.621721 |
| 25 | 0.939486 | 0.882772 | 0.829609 | 0.779768 | 0.733034 | 0.689206 | 0.609531 |
| 26 | 0.937143 | 0.878380 | 0.823434 | 0.772048 | 0.723984 | 0.679021 | 0.597579 |
| 27 | 0.934806 | 0.874010 | 0.817304 | 0.764404 | 0.715046 | 0.668986 | 0.585862 |
| 28 | 0.932475 | 0.869662 | 0.811220 | 0.756836 | 0.706219 | 0.659099 | 0.574375 |
| 29 | 0.930150 | 0.865335 | 0.805181 | 0.749342 | 0.697500 | 0.649359 | 0.563112 |
| 30 | 0.927830 | 0.861030 | 0.799187 | 0.741923 | 0.688889 | 0.639762 | 0.552071 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 31 | 0.925517 | 0.856746 | 0.793238 | 0.734577 | 0.680384 | 0.630308 | 0.541246 |
| 32 | 0.923209 | 0.852484 | 0.787333 | 0.727304 | 0.671984 | 0.620993 | 0.530633 |
| 33 | 0.920906 | 0.848242 | 0.781472 | 0.720103 | 0.663688 | 0.611816 | 0.520229 |
| 34 | 0.918610 | 0.844022 | 0.775654 | 0.712973 | 0.655494 | 0.602774 | 0.510028 |
| 35 | 0.916319 | 0.839823 | 0.769880 | 0.705914 | 0.647402 | 0.593866 | 0.500028 |
| 36 | 0.914034 | 0.835645 | 0.764149 | 0.698925 | 0.639409 | 0.585090 | 0.490223 |
| 37 | 0.911754 | 0.831487 | 0.758461 | 0.692005 | 0.631515 | 0.576443 | 0.480611 |
| 38 | 0.909481 | 0.827351 | 0.752814 | 0.685153 | 0.623719 | 0.567924 | 0.471187 |
| 39 | 0.907213 | 0.823235 | 0.747210 | 0.678370 | 0.616019 | 0.559531 | 0.461948 |
| 40 | 0.904950 | 0.819139 | 0.741648 | 0.671653 | 0.608413 | 0.551262 | 0.452890 |
| 41 | 0.902694 | 0.815064 | 0.736127 | 0.665003 | 0.600902 | 0.543116 | 0.444010 |
| 42 | 0.900443 | 0.811009 | 0.730647 | 0.658419 | 0.593484 | 0.535089 | 0.435304 |
| 43 | 0.898197 | 0.806974 | 0.725208 | 0.651900 | 0.586157 | 0.527182 | 0.426769 |
| 44 | 0.895957 | 0.802959 | 0.719810 | 0.645445 | 0.578920 | 0.519391 | 0.418401 |
| 45 | 0.893723 | 0.798964 | 0.714451 | 0.639055 | 0.571773 | 0.511715 | 0.410197 |
| 46 | 0.891494 | 0.794989 | 0.709133 | 0.632728 | 0.564714 | 0.504153 | 0.402154 |
| 47 | 0.889271 | 0.791034 | 0.703854 | 0.626463 | 0.557742 | 0.496702 | 0.394268 |
| 48 | 0.887053 | 0.787098 | 0.698614 | 0.620260 | 0.550856 | 0.489362 | 0.386538 |
| 49 | 0.884841 | 0.783182 | 0.693414 | 0.614119 | 0.544056 | 0.482130 | 0.378958 |
| 50 | 0.882635 | 0.779286 | 0.688252 | 0.608039 | 0.537339 | 0.475005 | 0.371528 |
| 51 | 0.880433 | 0.775409 | 0.683128 | 0.602019 | 0.530705 | 0.467985 | 0.364243 |
| 52 | 0.878238 | 0.771551 | 0.678043 | 0.596058 | 0.524153 | 0.461069 | 0.357101 |
| 53 | 0.876048 | 0.767713 | 0.672995 | 0.590156 | 0.517682 | 0.454255 | 0.350099 |
| 54 | 0.873863 | 0.763893 | 0.667986 | 0.584313 | 0.511291 | 0.447542 | 0.343234 |
| 55 | 0.871684 | 0.760093 | 0.663013 | 0.578528 | 0.504979 | 0.440928 | 0.336504 |
| 56 | 0.869510 | 0.756311 | 0.658077 | 0.572800 | 0.498745 | 0.434412 | 0.329906 |
| 57 | 0.867342 | 0.752548 | 0.653178 | 0.567129 | 0.492587 | 0.427992 | 0.323437 |
| 58 | 0.865179 | 0.748804 | 0.648316 | 0.561514 | 0.486506 | 0.421667 | 0.317095 |
| 59 | 0.863021 | 0.745079 | 0.643490 | 0.555954 | 0.480500 | 0.415435 | 0.310878 |
| 60 | 0.860869 | 0.741372 | 0.638700 | 0.550450 | 0.474568 | 0.409296 | 0.304782 |
| 61 | 0.858722 | 0.737684 | 0.633945 | 0.545000 | 0.468709 | 0.403247 | 0.298806 |
| 62 | 0.856581 | 0.734014 | 0.629226 | 0.539604 | 0.462922 | 0.397288 | 0.292947 |
| 63 | 0.854445 | 0.730362 | 0.624542 | 0.534261 | 0.457207 | 0.391417 | 0.287203 |
| 64 | 0.852314 | 0.726728 | 0.619893 | 0.528971 | 0.451563 | 0.385632 | 0.281572 |
| 65 | 0.850188 | 0.723113 | 0.615278 | 0.523734 | 0.445988 | 0.379933 | 0.276051 |
| 66 | 0.848068 | 0.719515 | 0.610698 | 0.518548 | 0.440482 | 0.374318 | 0.270638 |
| 67 | 0.845953 | 0.715935 | 0.606152 | 0.513414 | 0.435044 | 0.368787 | 0.265331 |
| 68 | 0.843844 | 0.712374 | 0.601639 | 0.508331 | 0.429673 | 0.363337 | 0.260129 |
| 69 | 0.841739 | 0.708829 | 0.597161 | 0.503298 | 0.424368 | 0.357967 | 0.255028 |
| 70 | 0.839640 | 0.705303 | 0.592715 | 0.498315 | 0.419129 | 0.352677 | 0.250028 |
| 71 | 0.837547 | 0.701794 | 0.588303 | 0.493381 | 0.413955 | 0.347465 | 0.245125 |
| 72 | 0.835458 | 0.698302 | 0.583924 | 0.488496 | 0.408844 | 0.342330 | 0.240319 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 73 | 0.833374 | 0.694828 | 0.579577 | 0.483659 | 0.403797 | 0.337271 | 0.235607 |
| 74 | 0.831296 | 0.691371 | 0.575262 | 0.478871 | 0.398811 | 0.332287 | 0.230987 |
| 75 | 0.829223 | 0.687932 | 0.570980 | 0.474129 | 0.393888 | 0.327376 | 0.226458 |
| 76 | 0.827155 | 0.684509 | 0.566730 | 0.469435 | 0.389025 | 0.322538 | 0.222017 |
| 77 | 0.825093 | 0.681104 | 0.562511 | 0.464787 | 0.384222 | 0.317771 | 0.217664 |
| 78 | 0.823035 | 0.677715 | 0.558323 | 0.460185 | 0.379479 | 0.313075 | 0.213396 |
| 79 | 0.820982 | 0.674343 | 0.554167 | 0.455629 | 0.374794 | 0.308448 | 0.209212 |
| 80 | 0.818935 | 0.670988 | 0.550042 | 0.451118 | 0.370167 | 0.303890 | 0.205110 |
| | | | | | | | |
| **n** | **3.00%** | **3.50%** | **4.00%** | **4.50%** | **5.00%** | **6.00%** | **7.00%** |
| 1 | 0.970874 | 0.966184 | 0.961538 | 0.956938 | 0.952381 | 0.943396 | 0.934579 |
| 2 | 0.942596 | 0.933511 | 0.924556 | 0.915730 | 0.907029 | 0.889996 | 0.873439 |
| 3 | 0.915142 | 0.901943 | 0.888996 | 0.876297 | 0.863838 | 0.839619 | 0.816298 |
| 4 | 0.888487 | 0.871442 | 0.854804 | 0.838561 | 0.822702 | 0.792094 | 0.762895 |
| 5 | 0.862609 | 0.841973 | 0.821927 | 0.802451 | 0.783526 | 0.747258 | 0.712986 |
| 6 | 0.837484 | 0.813501 | 0.790315 | 0.767896 | 0.746215 | 0.704961 | 0.666342 |
| 7 | 0.813092 | 0.785991 | 0.759918 | 0.734828 | 0.710681 | 0.665057 | 0.622750 |
| 8 | 0.789409 | 0.759412 | 0.730690 | 0.703185 | 0.676839 | 0.627412 | 0.582009 |
| 9 | 0.766417 | 0.733731 | 0.702587 | 0.672904 | 0.644609 | 0.591898 | 0.543934 |
| 10 | 0.744094 | 0.708919 | 0.675564 | 0.643928 | 0.613913 | 0.558395 | 0.508349 |
| 11 | 0.722421 | 0.684946 | 0.649581 | 0.616199 | 0.584679 | 0.526788 | 0.475093 |
| 12 | 0.701380 | 0.661783 | 0.624597 | 0.589664 | 0.556837 | 0.496969 | 0.444012 |
| 13 | 0.680951 | 0.639404 | 0.600574 | 0.564272 | 0.530321 | 0.468839 | 0.414964 |
| 14 | 0.661118 | 0.617782 | 0.577475 | 0.539973 | 0.505068 | 0.442301 | 0.387817 |
| 15 | 0.641862 | 0.596891 | 0.555265 | 0.516720 | 0.481017 | 0.417265 | 0.362446 |
| 16 | 0.623167 | 0.576706 | 0.533908 | 0.494469 | 0.458112 | 0.393646 | 0.338735 |
| 17 | 0.605016 | 0.557204 | 0.513373 | 0.473176 | 0.436297 | 0.371364 | 0.316574 |
| 18 | 0.587395 | 0.538361 | 0.493628 | 0.452800 | 0.415521 | 0.350344 | 0.295864 |
| 19 | 0.570286 | 0.520156 | 0.474642 | 0.433302 | 0.395734 | 0.330513 | 0.276508 |
| 20 | 0.553676 | 0.502566 | 0.456387 | 0.414643 | 0.376889 | 0.311805 | 0.258419 |
| 21 | 0.537549 | 0.485571 | 0.438834 | 0.396787 | 0.358942 | 0.294155 | 0.241513 |
| 22 | 0.521893 | 0.469151 | 0.421955 | 0.379701 | 0.341850 | 0.277505 | 0.225713 |
| 23 | 0.506692 | 0.453286 | 0.405726 | 0.363350 | 0.325571 | 0.261797 | 0.210947 |
| 24 | 0.491934 | 0.437957 | 0.390121 | 0.347703 | 0.310068 | 0.246979 | 0.197147 |
| 25 | 0.477606 | 0.423147 | 0.375117 | 0.332731 | 0.295303 | 0.232999 | 0.184249 |
| 26 | 0.463695 | 0.408838 | 0.360689 | 0.318402 | 0.281241 | 0.219810 | 0.172195 |
| 27 | 0.450189 | 0.395012 | 0.346817 | 0.304691 | 0.267848 | 0.207368 | 0.160930 |
| 28 | 0.437077 | 0.381654 | 0.333477 | 0.291571 | 0.255094 | 0.195630 | 0.150402 |
| 29 | 0.424346 | 0.368748 | 0.320651 | 0.279015 | 0.242946 | 0.184557 | 0.140563 |
| 30 | 0.411987 | 0.356278 | 0.308319 | 0.267000 | 0.231377 | 0.174110 | 0.131367 |
| 31 | 0.399987 | 0.344230 | 0.296460 | 0.255502 | 0.220359 | 0.164255 | 0.122773 |
| 32 | 0.388337 | 0.332590 | 0.285058 | 0.244500 | 0.209866 | 0.154957 | 0.114741 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 33 | 0.377026 | 0.321343 | 0.274094 | 0.233971 | 0.199873 | 0.146186 | 0.107235 |
| 34 | 0.366045 | 0.310476 | 0.263552 | 0.223896 | 0.190355 | 0.137912 | 0.100219 |
| 35 | 0.355383 | 0.299977 | 0.253415 | 0.214254 | 0.181290 | 0.130105 | 0.093663 |
| 36 | 0.345032 | 0.289833 | 0.243669 | 0.205028 | 0.172657 | 0.122741 | 0.087535 |
| 37 | 0.334983 | 0.280032 | 0.234297 | 0.196199 | 0.164436 | 0.115793 | 0.081809 |
| 38 | 0.325226 | 0.270562 | 0.225285 | 0.187750 | 0.156605 | 0.109239 | 0.076457 |
| 39 | 0.315754 | 0.261413 | 0.216621 | 0.179665 | 0.149148 | 0.103056 | 0.071455 |
| 40 | 0.306557 | 0.252572 | 0.208289 | 0.171929 | 0.142046 | 0.097222 | 0.066780 |
| 41 | 0.297628 | 0.244031 | 0.200278 | 0.164525 | 0.135282 | 0.091719 | 0.062412 |
| 42 | 0.288959 | 0.235779 | 0.192575 | 0.157440 | 0.128840 | 0.086527 | 0.058329 |
| 43 | 0.280543 | 0.227806 | 0.185168 | 0.150661 | 0.122704 | 0.081630 | 0.054513 |
| 44 | 0.272372 | 0.220102 | 0.178046 | 0.144173 | 0.116861 | 0.077009 | 0.050946 |
| 45 | 0.264439 | 0.212659 | 0.171198 | 0.137964 | 0.111297 | 0.072650 | 0.047613 |
| 46 | 0.256737 | 0.205468 | 0.164614 | 0.132023 | 0.105997 | 0.068538 | 0.044499 |
| 47 | 0.249259 | 0.198520 | 0.158283 | 0.126338 | 0.100949 | 0.064658 | 0.041587 |
| 48 | 0.241999 | 0.191806 | 0.152195 | 0.120898 | 0.096142 | 0.060998 | 0.038867 |
| 49 | 0.234950 | 0.185320 | 0.146341 | 0.115692 | 0.091564 | 0.057546 | 0.036324 |
| 50 | 0.228107 | 0.179053 | 0.140713 | 0.110710 | 0.087204 | 0.054288 | 0.033948 |
| 51 | 0.221463 | 0.172998 | 0.135301 | 0.105942 | 0.083051 | 0.051215 | 0.031727 |
| 52 | 0.215013 | 0.167148 | 0.130097 | 0.101380 | 0.079096 | 0.048316 | 0.029651 |
| 53 | 0.208750 | 0.161496 | 0.125093 | 0.097014 | 0.075330 | 0.045582 | 0.027711 |
| 54 | 0.202670 | 0.156035 | 0.120282 | 0.092837 | 0.071743 | 0.043001 | 0.025899 |
| 55 | 0.196767 | 0.150758 | 0.115656 | 0.088839 | 0.068326 | 0.040567 | 0.024204 |
| 56 | 0.191036 | 0.145660 | 0.111207 | 0.085013 | 0.065073 | 0.038271 | 0.022621 |
| 57 | 0.185472 | 0.140734 | 0.106930 | 0.081353 | 0.061974 | 0.036105 | 0.021141 |
| 58 | 0.180070 | 0.135975 | 0.102817 | 0.077849 | 0.059023 | 0.034061 | 0.019758 |
| 59 | 0.174825 | 0.131377 | 0.098863 | 0.074497 | 0.056212 | 0.032133 | 0.018465 |
| 60 | 0.169733 | 0.126934 | 0.095060 | 0.071289 | 0.053536 | 0.030314 | 0.017257 |
| 61 | 0.164789 | 0.122642 | 0.091404 | 0.068219 | 0.050986 | 0.028598 | 0.016128 |
| 62 | 0.159990 | 0.118495 | 0.087889 | 0.065281 | 0.048558 | 0.026980 | 0.015073 |
| 63 | 0.155330 | 0.114487 | 0.084508 | 0.062470 | 0.046246 | 0.025453 | 0.014087 |
| 64 | 0.150806 | 0.110616 | 0.081258 | 0.059780 | 0.044044 | 0.024012 | 0.013166 |
| 65 | 0.146413 | 0.106875 | 0.078133 | 0.057206 | 0.041946 | 0.022653 | 0.012304 |
| 66 | 0.142149 | 0.103261 | 0.075128 | 0.054743 | 0.039949 | 0.021370 | 0.011499 |
| 67 | 0.138009 | 0.099769 | 0.072238 | 0.052385 | 0.038047 | 0.020161 | 0.010747 |
| 68 | 0.133989 | 0.096395 | 0.069460 | 0.050129 | 0.036235 | 0.019020 | 0.010044 |
| 69 | 0.130086 | 0.093136 | 0.066788 | 0.047971 | 0.034509 | 0.017943 | 0.009387 |
| 70 | 0.126297 | 0.089986 | 0.064219 | 0.045905 | 0.032866 | 0.016927 | 0.008773 |
| 71 | 0.122619 | 0.086943 | 0.061749 | 0.043928 | 0.031301 | 0.015969 | 0.008199 |
| 72 | 0.119047 | 0.084003 | 0.059374 | 0.042037 | 0.029811 | 0.015065 | 0.007662 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 73 | 0.115580 | 0.081162 | 0.057091 | 0.040226 | 0.028391 | 0.014213 | 0.007161 |
| 74 | 0.112214 | 0.078418 | 0.054895 | 0.038494 | 0.027039 | 0.013408 | 0.006693 |
| 75 | 0.108945 | 0.075766 | 0.052784 | 0.036836 | 0.025752 | 0.012649 | 0.006255 |
| 76 | 0.105772 | 0.073204 | 0.050754 | 0.035250 | 0.024525 | 0.011933 | 0.005846 |
| 77 | 0.102691 | 0.070728 | 0.048801 | 0.033732 | 0.023357 | 0.011258 | 0.005463 |
| 78 | 0.099700 | 0.068336 | 0.046924 | 0.032280 | 0.022245 | 0.010620 | 0.005106 |
| 79 | 0.096796 | 0.066026 | 0.045120 | 0.030890 | 0.021186 | 0.010019 | 0.004772 |
| 80 | 0.093977 | 0.063793 | 0.043384 | 0.029559 | 0.020177 | 0.009452 | 0.004460 |
| n | 8.00% | 9.00% | 10.00% | 11.00% | 12.00% | 13.00% | 14.00% |
| 1 | 0.925926 | 0.917431 | 0.909091 | 0.900901 | 0.892857 | 0.884956 | 0.877193 |
| 2 | 0.857339 | 0.841680 | 0.826446 | 0.811622 | 0.797194 | 0.783147 | 0.769468 |
| 3 | 0.793832 | 0.772183 | 0.751315 | 0.731191 | 0.711780 | 0.693050 | 0.674972 |
| 4 | 0.735030 | 0.708425 | 0.683013 | 0.658731 | 0.635518 | 0.613319 | 0.592080 |
| 5 | 0.680583 | 0.649931 | 0.620921 | 0.593451 | 0.567427 | 0.542760 | 0.519369 |
| 6 | 0.630170 | 0.596267 | 0.564474 | 0.534641 | 0.506631 | 0.480319 | 0.455587 |
| 7 | 0.583490 | 0.547034 | 0.513158 | 0.481658 | 0.452349 | 0.425061 | 0.399637 |
| 8 | 0.540269 | 0.501866 | 0.466507 | 0.433926 | 0.403883 | 0.376160 | 0.350559 |
| 9 | 0.500249 | 0.460428 | 0.424098 | 0.390925 | 0.360610 | 0.332885 | 0.307508 |
| 10 | 0.463193 | 0.422411 | 0.385543 | 0.352184 | 0.321973 | 0.294588 | 0.269744 |
| 11 | 0.428883 | 0.387533 | 0.350494 | 0.317283 | 0.287476 | 0.260698 | 0.236617 |
| 12 | 0.397114 | 0.355535 | 0.318631 | 0.285841 | 0.256675 | 0.230706 | 0.207559 |
| 13 | 0.367698 | 0.326179 | 0.289664 | 0.257514 | 0.229174 | 0.204165 | 0.182069 |
| 14 | 0.340461 | 0.299246 | 0.263331 | 0.231995 | 0.204620 | 0.180677 | 0.159710 |
| 15 | 0.315242 | 0.274538 | 0.239392 | 0.209004 | 0.182696 | 0.159891 | 0.140096 |
| 16 | 0.291890 | 0.251870 | 0.217629 | 0.188292 | 0.163122 | 0.141496 | 0.122892 |
| 17 | 0.270269 | 0.231073 | 0.197845 | 0.169633 | 0.145644 | 0.125218 | 0.107800 |
| 18 | 0.250249 | 0.211994 | 0.179859 | 0.152822 | 0.130040 | 0.110812 | 0.094561 |
| 19 | 0.231712 | 0.194490 | 0.163508 | 0.137678 | 0.116107 | 0.098064 | 0.082948 |
| 20 | 0.214548 | 0.178431 | 0.148644 | 0.124034 | 0.103667 | 0.086782 | 0.072762 |
| 21 | 0.198656 | 0.163698 | 0.135131 | 0.111742 | 0.092560 | 0.076798 | 0.063826 |
| 22 | 0.183941 | 0.150182 | 0.122846 | 0.100669 | 0.082643 | 0.067963 | 0.055988 |
| 23 | 0.170315 | 0.137781 | 0.111678 | 0.090693 | 0.073788 | 0.060144 | 0.049112 |
| 24 | 0.157699 | 0.126405 | 0.101526 | 0.081705 | 0.065882 | 0.053225 | 0.043081 |
| 25 | 0.146018 | 0.115968 | 0.092296 | 0.073608 | 0.058823 | 0.047102 | 0.037790 |
| 26 | 0.135202 | 0.106393 | 0.083905 | 0.066314 | 0.052521 | 0.041683 | 0.033149 |
| 27 | 0.125187 | 0.097608 | 0.076278 | 0.059742 | 0.046894 | 0.036888 | 0.029078 |
| 28 | 0.115914 | 0.089548 | 0.069343 | 0.053822 | 0.041869 | 0.032644 | 0.025507 |
| 29 | 0.107328 | 0.082155 | 0.063039 | 0.048488 | 0.037383 | 0.028889 | 0.022375 |
| 30 | 0.099377 | 0.075371 | 0.057309 | 0.043683 | 0.033378 | 0.025565 | 0.019627 |
| 31 | 0.092016 | 0.069148 | 0.052099 | 0.039354 | 0.029802 | 0.022624 | 0.017217 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 32 | 0.085200 | 0.063438 | 0.047362 | 0.035454 | 0.026609 | 0.020021 | 0.015102 |
| 33 | 0.078889 | 0.058200 | 0.043057 | 0.031940 | 0.023758 | 0.017718 | 0.013248 |
| 34 | 0.073045 | 0.053395 | 0.039143 | 0.028775 | 0.021212 | 0.015680 | 0.011621 |
| 35 | 0.067635 | 0.048986 | 0.035584 | 0.025924 | 0.018940 | 0.013876 | 0.010194 |
| 36 | 0.062625 | 0.044941 | 0.032349 | 0.023355 | 0.016910 | 0.012279 | 0.008942 |
| 37 | 0.057986 | 0.041231 | 0.029408 | 0.021040 | 0.015098 | 0.010867 | 0.007844 |
| 38 | 0.053690 | 0.037826 | 0.026735 | 0.018955 | 0.013481 | 0.009617 | 0.006880 |
| 39 | 0.049713 | 0.034703 | 0.024304 | 0.017077 | 0.012036 | 0.008510 | 0.006035 |
| 40 | 0.046031 | 0.031838 | 0.022095 | 0.015384 | 0.010747 | 0.007531 | 0.005294 |
| 41 | 0.042621 | 0.029209 | 0.020086 | 0.013860 | 0.009595 | 0.006665 | 0.004644 |
| 42 | 0.039464 | 0.026797 | 0.018260 | 0.012486 | 0.008567 | 0.005898 | 0.004074 |
| 43 | 0.036541 | 0.024584 | 0.016600 | 0.011249 | 0.007649 | 0.005219 | 0.003573 |
| 44 | 0.033834 | 0.022555 | 0.015091 | 0.010134 | 0.006830 | 0.004619 | 0.003135 |
| 45 | 0.031328 | 0.020692 | 0.013719 | 0.009130 | 0.006098 | 0.004088 | 0.002750 |
| 46 | 0.029007 | 0.018984 | 0.012472 | 0.008225 | 0.005445 | 0.003617 | 0.002412 |
| 47 | 0.026859 | 0.017416 | 0.011338 | 0.007410 | 0.004861 | 0.003201 | 0.002116 |
| 48 | 0.024869 | 0.015978 | 0.010307 | 0.006676 | 0.004340 | 0.002833 | 0.001856 |
| 49 | 0.023027 | 0.014659 | 0.009370 | 0.006014 | 0.003875 | 0.002507 | 0.001628 |
| 50 | 0.021321 | 0.013449 | 0.008519 | 0.005418 | 0.003460 | 0.002219 | 0.001428 |
| 51 | 0.019742 | 0.012338 | 0.007744 | 0.004881 | 0.003089 | 0.001963 | 0.001253 |
| 52 | 0.018280 | 0.011319 | 0.007040 | 0.004397 | 0.002758 | 0.001737 | 0.001099 |
| 53 | 0.016925 | 0.010385 | 0.006400 | 0.003962 | 0.002463 | 0.001538 | 0.000964 |
| 54 | 0.015672 | 0.009527 | 0.005818 | 0.003569 | 0.002199 | 0.001361 | 0.000846 |
| 55 | 0.014511 | 0.008741 | 0.005289 | 0.003215 | 0.001963 | 0.001204 | 0.000742 |
| 56 | 0.013436 | 0.008019 | 0.004809 | 0.002897 | 0.001753 | 0.001066 | 0.000651 |
| 57 | 0.012441 | 0.007357 | 0.004371 | 0.002610 | 0.001565 | 0.000943 | 0.000571 |
| 58 | 0.011519 | 0.006749 | 0.003974 | 0.002351 | 0.001398 | 0.000835 | 0.000501 |
| 59 | 0.010666 | 0.006192 | 0.003613 | 0.002118 | 0.001248 | 0.000739 | 0.000439 |
| 60 | 0.009876 | 0.005681 | 0.003284 | 0.001908 | 0.001114 | 0.000654 | 0.000385 |
| 61 | 0.009144 | 0.005212 | 0.002986 | 0.001719 | 0.000995 | 0.000578 | 0.000338 |
| 62 | 0.008467 | 0.004781 | 0.002714 | 0.001549 | 0.000888 | 0.000512 | 0.000296 |
| 63 | 0.007840 | 0.004387 | 0.002468 | 0.001395 | 0.000793 | 0.000453 | 0.000260 |
| 64 | 0.007259 | 0.004024 | 0.002243 | 0.001257 | 0.000708 | 0.000401 | 0.000228 |
| 65 | 0.006721 | 0.003692 | 0.002039 | 0.001132 | 0.000632 | 0.000355 | 0.000200 |
| 66 | 0.006223 | 0.003387 | 0.001854 | 0.001020 | 0.000564 | 0.000314 | 0.000176 |
| 67 | 0.005762 | 0.003108 | 0.001685 | 0.000919 | 0.000504 | 0.000278 | 0.000154 |
| 68 | 0.005336 | 0.002851 | 0.001532 | 0.000828 | 0.000450 | 0.000246 | 0.000135 |
| 69 | 0.004940 | 0.002616 | 0.001393 | 0.000746 | 0.000402 | 0.000218 | 0.000118 |
| 70 | 0.004574 | 0.002400 | 0.001266 | 0.000672 | 0.000359 | 0.000193 | 0.000104 |
| 71 | 0.004236 | 0.002201 | 0.001151 | 0.000605 | 0.000320 | 0.000170 | 0.000091 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 72 | 0.003922 | 0.002020 | 0.001046 | 0.000545 | 0.000286 | 0.000151 | 0.000080 |
| 73 | 0.003631 | 0.001853 | 0.000951 | 0.000491 | 0.000255 | 0.000133 | 0.000070 |
| 74 | 0.003362 | 0.001700 | 0.000865 | 0.000443 | 0.000228 | 0.000118 | 0.000062 |
| 75 | 0.003113 | 0.001560 | 0.000786 | 0.000399 | 0.000204 | 0.000105 | 0.000054 |
| 76 | 0.002883 | 0.001431 | 0.000715 | 0.000359 | 0.000182 | 0.000092 | 0.000047 |
| 77 | 0.002669 | 0.001313 | 0.000650 | 0.000324 | 0.000162 | 0.000082 | 0.000042 |
| 78 | 0.002471 | 0.001204 | 0.000591 | 0.000292 | 0.000145 | 0.000072 | 0.000036 |
| 79 | 0.002288 | 0.001105 | 0.000537 | 0.000263 | 0.000129 | 0.000064 | 0.000032 |
| 80 | 0.002119 | 0.001014 | 0.000488 | 0.000237 | 0.000115 | 0.000057 | 0.000028 |

## Appendix: $e^x$ and $e^{-x}$ value table

| x | $e^x$ | $e^{-x}$ |
|---|---|---|
| 0.00 | 1.0000 | 1.0000 |
| 0.01 | 1.0101 | 0.99005 |
| 0.02 | 1.0202 | 0.98020 |
| 0.03 | 1.0305 | 0.97045 |
| 0.04 | 1.0408 | 0.96079 |
| 0.05 | 1.0513 | 0.95123 |
| 0.06 | 1.0618 | 0.94176 |
| 0.07 | 1.0725 | 0.93239 |
| 0.08 | 1.0833 | 0.92312 |
| 0.09 | 1.0942 | 0.91393 |
| 0.10 | 1.1052 | 0.90484 |
| 0.11 | 1.1163 | 0.89583 |
| 0.12 | 1.1275 | 0.88692 |
| 0.13 | 1.1388 | 0.87810 |
| 0.14 | 1.1503 | 0.86936 |
| 0.15 | 1.1618 | 0.86071 |
| 0.16 | 1.1735 | 0.85214 |
| 0.17 | 1.1853 | 0.84366 |
| 0.18 | 1.1972 | 0.83527 |
| 0.19 | 1.2092 | 0.82696 |
| 0.20 | 1.2214 | 0.81873 |
| 0.21 | 1.2337 | 0.81058 |
| 0.22 | 1.2461 | 0.80252 |
| 0.23 | 1.2586 | 0.79453 |
| 0.24 | 1.2712 | 0.78663 |
| 0.25 | 1.2840 | 0.77880 |
| 0.26 | 1.2969 | 0.77105 |
| 0.27 | 1.3100 | 0.76338 |
| 0.28 | 1.3231 | 0.75578 |
| 0.29 | 1.3364 | 0.74826 |
| 0.30 | 1.3499 | 0.74082 |
| 0.31 | 1.3634 | 0.73345 |
| 0.32 | 1.3771 | 0.72615 |
| 0.33 | 1.3910 | 0.71892 |
| 0.34 | 1.4049 | 0.71177 |
| 0.35 | 1.4191 | 0.70469 |
| 0.36 | 1.4333 | 0.69768 |
| 0.37 | 1.4477 | 0.69073 |
| 0.38 | 1.4623 | 0.68386 |
| 0.39 | 1.4770 | 0.67706 |
| 0.40 | 1.4918 | 0.67032 |