BLOCK 2

FUNCTIONS OF ONE INDEPENDENT VARIABLE

BLOCK 2 INTRODUCTION

The first Block in the course, titled **Preliminaries** contains three units that deal with such topics as to make it easier for you to follow subsequent units and blocks in this course. The concepts that you will learn in the three units of this Block will enable you to understand the concepts that follow. These initial concepts lay the foundation for the whole course.

The first unit, **Sets and Set Operations**, begins by carefully defining a set and states the various ways of depicting a set. You learn about subsets, supersets and power sets, as well as various operations on sets like union, intersection, difference, as so on. The second unit, **Relations and Functions**, begins by defining the product of sets. This product is called Cartesian product. Then the concept of a relation as a subset of a Cartesian product is introduced. Subsequently the unit goes on to discuss functions and mappings. You also learn about real-valued functions, correspondences and set functions.

Unit 3, title simply **Logic**, takes you on a journey of acquiring skills in thinking deductively. You learn about statements, predicates, as well as implications and truth tables. You understand the meaning of a proof and see that there are several types of proof.

UNIT 4 ELEMENTARY TYPES OF FUNCTIONS*

Structure

4.0 OBJECTIVES

After going through the unit, you should be able to:

- define a variable, constant and a parameter;
- identify a dependent and an independent variable;
- describe a quadratic function;
- discuss the characteristics of a cubic and general polynomial functions;
- describe various other functions like exponential, logarithmic, power and hyperbolic functions; and
- discuss some simple applications of these functions in Economics.

4.1 INTRODUCTION

This unit should be read as a continuation of unit 2, particularly the section on functions. Unit 2 must have given you an idea of a *function*, and how functions can be seen as mappings of elements of a set called the *domain* into the elements of a set called the *range*. The elements of the range are called *images* of the elements of the domain. This unit discusses functions as a relationship among variables.

A relationship between the values of two or more variables can be defined as a *function* when a unique value of one of the variables is determined by the value of the other variable or variables. If the precise mathematical form of the relationship is not actually known, then a function may be written in what is called a *general form*. For example, a general form of a demand function is $Q_d = f(P)$. This particular general form just tells us that quantity demanded of a

* Contributed by Shri Saugato Sen, SOSS, IGNOU

good (Q_d) depends on its price (P). The 'f ' is not an algebraic symbol in the usual sense. $f(P)$ means 'is a function of P' and not 'f multiplied by P'. In this case P is what is known as the '*independent variable*' because its value is given and is not dependent on the value of Q_d, *i.e.*, it is exogenously determined. On the other hand Q_d is the '*dependent variable*' because its value depends on the value of P. Economic models link the value of exogenous variables to the value of endogenous variables. The variables studied in economics may be qualitative or quantitative. A *qualitative variable* represents some distinguishing characteristic, such as male or female, working or unemployed, *etc*. The relationship between values of a qualitative variable is not numerical. *Quantitative variables*, on the other hand, can be measured numerically. Familiar economic quantitative variables include— the rupee value of national income, the number of barrels of imported oil, the consumer price level, and the rupee-dollar exchange rate.

Functions may have more than one independent variable. For example, the general form production function $Q = f(K,L)$tells us that output (Q) depends on the values of the two independent variables capital (K)and labour (L). We will not study such functions in this course. You shall have occasion to study such functions in the mathematics-for-economics course in the next semester.

The *specific form of a function* tells us exactly how the value of the dependent variable is determined from the values of the independent variable or variables. In Economic applications of functions it may make sense to restrict the 'domain' of the function, *i.e.* the range of possible values of the variables. For example, variables that represent price or output may be restricted to positive values. Strictly speaking the *domain limits the values of the independent variables* and the *range governs the possible values of the dependent variable.*

When defining the specific form of a function it is important to make sure that only one unique value of the dependent variable is determined from each given value of the independent variable(s). Otherwise it will be a correspondence, as you have studied in unit 2. Consider the equation

$$y = 80 + x^{0.5}$$

This does not define a function because any given value of x corresponds to two possible values for y. For example, if $x = 25$, then 25 raised to power 0.5 = 5 or −5 and so $y = 75$ or 85. However, if we define $y = 80 + x^{0.5}$ for $x^{0.5} \geq 0$ then this does constitute a function. When domains are not specified then one should assume a sensible range for functions representing economic variables.

Inverse Functions

An inverse function reverses the relationship in a function. If we confine the analysis to functions with only one independent variable, x, this means that if y is a function of x, *i.e.*

$y = f(x)$ then in the inverse function, x will be a function of y, *i.e.*$x = g(y)$(The letter g is used to show that we are talking about a different function).

After this very general introduction about the nature of functions, let us go on to study various types of functions. The unit is organised in a way that each subsequent section deals with one type of function. We shall learn that some of these functions are what are called linear and the others are non-linear. Also, some are algebraic and others are non-algebraic. The next section deals with linear functions. The section, to set the ball rolling, sets out the definition of a variable, a constant and a parameter; although you may be familiar with these and we have in this introduction itself used these terms a little earlier. After that, the section takes up for discussion, the characteristics and properties of linear functions. You will come to know the difference between linear equations and linear functions. The next section deals with quadratic functions and equations. In the subsequent section, cubic and general polynomial functions are considered. The sections following these take up for discussion exponential and logarithmic functions, as well as power functions and hyperbolic functions. Let us begin the study of various types of functions.

4.2 LINEAR FUNCTIONS

In this section we begin our study of specific functions. At its simplest, a function relates one variable with another. So let us begin by defining a variable. An unknown value or an entity that can take different values is called a *variable*. The known value in an equation or a number that takes fixed values is called a *constant*. Thus a magnitude which does not change is a constant. In the context of an equation, when a constant is joined to a variable, then the constant is called a *coefficient*. Sometimes in an equation a coefficient is denoted by a letter of the alphabet, rather than by a numeral. This letter is supposed to represent a constant, but doesn't have a fixed numerical value. These types of coefficients are called *parameters*. An *equation* is any mathematical expression that contains an equality sign. Some equations represent statements that are conditionally true: for example, $x^2 = 9$. This statement would be true if $x = +3$ or -3. On the other hand, some equations may be true for all values of the variable(s). For example, $a + a + a + b + b + b = 3a + 3b$. These types of equations are called identities, and the relation is denoted by a sign of identity, which is ≡ rather than =, which is the equality sign.

Now that you know about an equation, let us see what a linear equation is. An equation is linear if the variables that appear in the equation are not raised to any power other than 1, and no two or more variables are multiplied. For example, the equation $x + 3 = 9$ is a linear equation because the x variable is raised only to the power 1. Similarly, $y = 3x – 5$ is also linear because the two variables y and x are raised to the power 1. On the contrary, $x^2 - 4 = 16$ is non-linear, and so is $y = x^{1/2} + 5$.

The general form of the linear equation in one variable is

$$ax + b = c$$

where$x$ is the unknown or the variable and a, b and c are unspecified parameters. In the context of a specific equation, a, b and c can acquire specific numerical values, but we are speaking of general equations. We can solve this equation by expressing x in terms of the parameters.

Solving it we get,

$$x = \frac{c-b}{a}$$

In the general form, the variable x is the endogenous variable, and a, b, c are the parameters. Solving the equation means expressing the endogenous variable in terms of the parameters.

Now that we know about a linear equation, let us discuss about a linear function. What is a linear function, and how is it related to a linear equation? To understand this, consider the following equation:

$$y = 2x + 1$$

The above equation is linear, but with two variables x and y, and two constants 2 and 1. Here, instead of a single value of x, there are many pairs of x and y values that satisfy this equation. For example, $x = 1$ and $y = 3$. In fact, we can assign any value we want to x, and find the value of y that satisfies the equation. The main point is that there is no unique solution of this equation. For this reason, these types of equations with two variables are called *functions*. A function does not limit the equation to a unique set of values, but describes a relationship between two (or more) variables. In this section, we discuss linear functions. Usually, in any function, the variable that appears on the left-hand side is called the dependent variable (or rather, the dependent variable is written on the left-hand side), and the variable that appears on the right-hand side is called the independent variable. In the above equation, we say that the values of y 'depend on' the values of x.

In the next unit, we shall discuss in great detail about the graphs of various types of equations (linear and non-linear). Section 5.5 will deal with graphs of linear functions. Here, we will briefly discuss some properties of linear functions. The general equation for a linear function is

$$y = ax + b \text{ , where } a \text{ and } b \text{ are two constants.}$$

Here, a is called the *gradient* or *slope* of the curve and b is the *y-intercept*. If we plot the x values on the horizontal axis, and y values on the vertical axis, then the two axes will cross at a point where the x and y values are both 0. The intersection results in four quadrants, and any point in any of the four quadrants gives a unique pair of points with x as the first value, and y the second value (recall our discussion of R^2— the real space of ordered pairs of numbers in unit 2).

Coming back to the equation for a straight line, $y = ax + b$, here b is the point where the equation of the linear function (a straight line) cuts the y-axis. You can see by putting $x = 0$, then $y = b$. If $y = 0$, then $x = -b/a$. Hence, the line (the graph of the linear function) cuts the x-axis at $-b/a$. The slope (which is 'a' here) is defined as the ratio of rise over run, that is the ratio of the difference between two y values to the difference between the corresponding two x values: $\frac{y_2 - y_1}{x_2 - x_1}$.

The crucial points to remember about linear functions are:

a) all variables are raised to the power 1 and no other power, and

b) the slope of the graph (the line) remains constant at all point.

These are the crucial characteristics of a straight line. You can jump straight to section 5.5 of unit 5 for supplementary material on linear functions and their graphs.

Example Given that the price of an item is Rs3.50 when 250 items are demanded, but when only 50 are demanded, the price rises to Rs5.50 per item, identify the linear demand function.

Solution: Let $p = a + bx$ be the linear demand function.

Substituting for $p = 3.5$ and $x = 250$, we get:

$$3.5 = a + 250b \qquad \ldots(1)$$

Substituting for $p = 5.5$ and $x = 50$, we get:

$$5.5 = a + 50b \qquad \ldots(2)$$

Subtracting equation (2) from (1) gives:

$$200b = -2 \text{ or } b = -0.01$$

Substituting for $b = -0.01$ in equation (1) gives:

$$3.5 = a + (250)(-0.01)$$

$$3.5 = a - 2.5$$

$\therefore a = 6$ and $b = -0.01$

The demand function is thus $p = 6 - 0.01x$

Notice that in the above example we considered the equation of the general linear function as

$y = a + bx$. Here a is the intercept and b is the slope. Just observe that earlier we had considered the equation of the function as $y = ax + b$. Just remember that the single constant is the intercept and the coefficient multiplied by x is the slope.

4.3 QUADRATIC FUNCTIONS

In the previous section, we looked at linear equations and functions. We saw that a linear equation has the general form $ax + b = c$, where x is the variable or unknown, and a,b, c are constants or parameters. This equation is called linear, because x is raised only to the power 1, and not to any lower or higher power. A linear function has the general form $y = ax + b$. The graph is a straight line and the slope remains constant. This means that a one-unit change in x *always* increases or decreases y by the same amount (the slope can be negative). This may hold for some relationships in Economics, like a linear demand curve or supply curve. But there are many situations in economics where we need to deal with non-linear relationships, where a given change in x does not always lead to a constant change in y. In this section, we deal with a simple non-linear function, called quadratic function. We shall also discuss quadratic equations. A quadratic function, when graphed, gives a U-shaped curve. We will describe how to solve quadratic equations. We shall give a couple of applications of quadratic functions in Economics.

A quadratic equation contains an unknown x that is raised to the power 2, that is, x^2. It may contain an x raised to the power of 1. The general form of a quadratic equation is:

$$ax^2 + bx + c = 0 \text{ , where } a, b \text{ and } c \text{ are constants.}$$

To solve this equation, we begin by taking a outside the bracket, and we get

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

Next, we search for factors of the term inside the bracket $x^2 + \frac{b}{a}x + \frac{c}{a}$

Let us suppose the factors are (x + C) and (x + D), where C and D are two numbers. If these are the factors, then by definition we have,

$$(x+C)(x+D)=x^2+\frac{b}{a}x+\frac{c}{a}$$

After multiplying both sides by a, this becomes

$$a(x+C)(x+D)\equiv ax^2+bx+c$$

Since this is an identity, when the left-hand side becomes zero, the right-hand side also becomes zero. The left-hand side becomes zero when $x = -C$ or $x = -D$. These values are the solution to the quadratic equation $ax^2+bx+c=0$. There is a general formula for solving quadratic equations. Here the solutions are called *roots*. Given any quadratic equation $ax^2+bx+c=0$, the solution (roots) are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Having discussed quadratic equations, let us now turn to quadratic functions. As we saw, a function has two variables in an equation. So let us see what a quadratic function looks like. The general form of a quadratic function is $y=ax^2+bx+c$ (where a, b, c are constants). If we plot the graph of a quadratic function, we shall find it is a U-shaped curve, specifically a parabola. You will learn about the features and properties of parabolas in the next unit. The U-shape arises from the fact that x^2 (and therefore y) is positive when x is either positive or negative. When x is zero, y is zero. When x is x_i or $-x_i$, y has the same value. Certain other features of the graph of a quadratic function are:

1) The term x^2 gives the function an approximate U-shape, called a parabola.

2) If the parameter a is positive, the graph looks like a U, whereas if the parameter a is negative, the graph is an inverted U. The absolute magnitude of a determines how steeply the curve slopes up or down.

3) The constant term, c, determines the intercept of the curve on the y-axis.

4) The x term shifts the parabola up or down. If the parameter b is positive, the curve shifts up when $x> 0$, and down when $x < 0$. If b is negative, these shifts are reversed. The absolute magnitude of b determines the strength of the shift effect.

Now let us discuss a couple of applications of quadratic functions in Economics. The first application is to the supply and demand analysis. Earlier we talked of a linear inverse demand function. Let us now consider a quadratic inverse demand function (while letting the linear inverse supply function). Suppose the inverse demand function is $p_d = 1.5q^2 - 15q + 35$ and the supply function be $p_s=2q+7$. In equilibrium, demand equals supply, and so we have

$$1.5q^2-15q+35=2q+7$$

$$1.5q^2-17q+28=0$$

Solving this quadratic equation gives us $q = 2$ or $9\frac{1}{3}$. We ignore the latter value because it occurs where the upward sloping branch of the quadratic demand function cuts the linear supply function for a second time (from below).

For the second application of quadratic function to Economics, let us talk about costs and revenue. Let us see the relation between a firm's total cost (denoted by *TC*) and its output *q*. Let us denote it by

$$TC = f(q)$$

What is the likely shape of this function, that is, the total cost curve? We can take a linear total cost curve $TC = aq + b$, where *b* is the fixed cost, and *aq* the total variable cost. For example, let us take a linear function $TC = 1.5q + 100$. The issue here is that it assumes that the firm can increase output indefinitely. But in the short run, a firm has a given plant size, stock of machinery, and so on. So let us take a quadratic cost function of the type $TC = aq^2 + bq + c$. As an example let us take $TC = 0.02q^2 + 1.5q + 100$. This is same as the linear function above but with an additional term q^2 with a small coefficient of 0.02. This additional term causes the curve to shift upwards, later rising more steeply as output increases. This shows that as the firm's productive capacity is utilised more and more intensively, costs rise more rapidly.

Now let us turn to the revenue side for this firm. Assume that this firm is a monopolist, that is, it is the sole supplier of output. We find that if the firm progressively lowers its price and thus sells more, its revenue will go up. After some time, revenue will increase but at a lower rate. Then beyond a certain point, revenue declines, because revenue loss from a lower price begins to outweigh the revenue gain from the increase in quantity sold. A suitable functional form to describe this kind of revenue function is a quadratic revenue curve, $TR = aq^2 + bq$. We assume that the parameter *a* is negative. We also assume there is no constant term (*i.e.*, $c = 0$), because the firm will not receive any revenue if no output is sold. Thus, if $q = 0$, then $TR = 0$. An example of a total revenue function of this type is

$$TR = -0.12q^2 + 10q.$$

If we were to plot this function, we will find that the Total revenue will reach its maximum when output *q* is approximately 42 (you may confirm the result by setting Marginal Revenue equal to 0).

Check Your Progress 1

1) What is the difference between a linear equation and linear function?

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2) What are the two features of a linear function that distinguish it from non-linear functions?

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3) Describe the features of the graph of a quadratic function.

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4.4 CUBIC AND POLYNOMIAL FUNCTIONS

In this section we consider cases where the highest power to which the variables are raised is 3 or higher. In subsection 4.4.1, we consider cubic functions, and in subsection 4.4.2 we consider general polynomial functions.

4.4.1 Cubic Functions

A Cubic function is expressed by a cubic equation which has the general form:

$$ax^3 + bx^2 + cx + d = 0$$

where *a, b, c* and *d* are constants. There is no general formula for solving a cubic function, but approximate solutions may be found graphically by plotting the graph and finding the points of intersection with the axes. Basic properties of the cubic function are:

1) The graph of cubic functions will have either no turning point or two turning points;

2) A cubic function will have either one root or three roots.

It is hard to factorise a cubic function, unlike a quadratic function. The shape of a cubic function is usually S-shaped. There are several contexts in which cubic functions are useful in Economics. Like we considered a quadratic cost function in the last section, we can have a cubic cost function as well. Consider a total cost function of the type, $TC = 2q^3 - 15q^2 + 50q + 50$. The curve of this equation will be upward sloping, with a slightly flat surface in the beginning for lower values of *q* (till about *q* = 1.5). Then there will be a range of *q* values where the *TC* curve will be less flat, that is, the slope is more than that at the lowest values of *q* (from about *q* = 1.5 to *q* = 2.5). It is in this middle range that the production is most efficient. At high levels of output, (particularly after *q* = 5) the slope is very steep.

4.4.2 General Polynomial Functions

Quadratic and cubic functions belong to a group of functions called *polynomials***.** The general form of a polynomial function of a single variable is:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots a_1 x^1 + a_0 x^0$$

where, $a_0, a_1, \ldots, a_n$ are constants. The 'degree' of the polynomial is given by the highest power of x in the expression. Therefore, a quadratic function is a polynomial of degree 2 and a cubic function is a polynomial of degree 3.The meaning of the word 'polynomial' is multi-term.

In the case of $n = 0, y = a_0$; $n = 1, y = a_0 + a_1 x$; $n = 2, y = a_0 + a_1 x + a_2 x^2$. Here, the first function is a constant function, the second a linear function and the third a quadratic function.

We are not going to study polynomials in depth, but you do need to be aware that they arecontinuous graphs, with no breaks or jumps. Therefore, it is quite safe to plot these graphs by joining the points as calculated and hence use the graphs to estimate roots and turning points.

Check Your Progress 2

1) What do you understand by a cubic function? What can you say about solution methods for cubic equations?

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2) What is a polynomial?

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3) What is the shape of a cubic function? What kind of economic relations can be described by cubic functions?

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4.5 EXPONENTIAL, LOGARITHMIC AND POWER FUNCTIONS

In this section we move to a different class of functions. These two functions, *exponential* and *logarithmic*, are non-algebraic functions, in that they are not linear or general polynomial functions. Sometimes exponential and logarithmic functions are called *transcendental* functions, since they transcend algebraic relationship. The third function considered in this section, that is the *power* function, is related to the exponential functions but is an algebraic function. Let us begin with exponential functions.

4.5.1 Exponential Functions

The argument or the independent variable of an *exponential function* appears as an exponent. The general form of the univariate exponential function is:

$$y= f(x) = b^x$$

where b is called the *base* and is assumed to be greater than 1 (*i.e.*, $b> 1$); and $x\in$ R (the set of real numbers). When $x= 0$, $y = b^0= 1$,for any base. When b >1 then b^x monotonically increases with x.

Note: $b > 1$ restriction has been imposed, since with $b = 0$ or 1, we get constant functions $y = 0$ or $y = 1$, respectively. Also, with $b< 0$, we may get a function involving a complex value of the form$\sqrt{-b}$, as $x \in$ R.Moreover, with $0 <b < 1$, we can always attain the base which is greater than 1 (for instance, consider $b = 0.5$ and $x = 2$, put these values in the exponential function to get $y = 0.5^2 \Rightarrow y = 2^{-2}$. Here, we have $b> 1$ and $x< 0$. Since our $x\in$ R, such a result holds for an exponential function).

Exponential functions have a special role in economic analysis because of their use in calculating the growth of variables over time. Exponential functions also play an important role in a related problem—the calculation of the present value of a future payment. Exponential functions are strictly monotonic and, therefore, one-to-one. One-to-one functions have an inverse. The inverse of an exponential function is called a *logarithmic* function. The properties of logarithmic functions are discussed in the next subsection. Logarithmic functions have a range of uses in economic analysis. These include the transformation of a non-linear relationship into a linear expression, which can be more easily evaluated; and the specification of an economic function with a constant elasticity.

4.5.2 Logarithmic Functions

Any exponential function has an inverse since it is strictly monotonic and, therefore, one-to-one. The classes of functions that are inverses to exponential functions are *logarithmic functions*. Logarithmic functions are used in many different ways in Economics. This section defines these functions and shows some of their useful properties by illustrating their application in Economic problems.

Any point (i, j) in the exponential function $y = b^x$, has a corresponding point (j, i) in the logarithmic function $y = log_b(x)$. Like the exponential function, the logarithmic function is strictly monotonic and increasing. Logarithmic functions are everywhere concave, while exponential functions are everywhere convex. You will learn about convex and concave functions in units 10 and 11.The domain of a logarithmic function is restricted to the set of positive real numbers, while the range of the function is the set of all real numbers, which is the converse of the case for exponential functions. Finally, since all exponential functions of the form $y = b^x$ cross the y-axis at the point (0,1), all logarithmic functions of the form $y = \log_b(x)$ cross the x-axis at the point (1,0).

The definition of a logarithmic function is as follows. The logarithmic function $y = \log_b(x)$, which is read as "y is the base b logarithm of x," satisfies the relationship $b^y = x$. This definition of logarithms implies $\log_b b = 1$ for any base, since $b^1 = b$, also that $\log_b b^x = x$. By the definition of an inverse function, we also have $b^{\log_b(x)} = x$.

Economic models often employ a *logarithmic transformation* of the variables of the model. A logarithmic transformation is the conversion of a variable that can take on different real positive values into its logarithm. In this section we demonstrate properties of logarithmic transformations and show why this is such a useful tool for economists. Economic models frequently include nonlinear relationships. For example, real money balances are represented by the quotient of nominal balances (M) over the price level (P)— (M/P), and the real exchange rate equals the product of the nominal exchange rate (E) and the foreign price level (P^*) divided by the domestic price level (P)— (EP^*/P). Nonlinear relationships among variables may be expressed as linear relationships among their logarithms. Multi-equation models that include products or quotients are more difficult to solve, than models that are linear in the variables of interest. Thus expressing these models in terms of the logarithms of their variables is often a useful strategy for making analysis more straightforward.

We begin by showing the relationship between the logarithm of two variables and the logarithm of their product. The revenue of a firm, R, is the product of the price of its good, P, and the quantity sold, Q. Define lowercase letters as the logarithms of their uppercase counterparts, so $q = \log_b(Q)$ and $p = \log_b(P)$. By the definitions of logarithms, we then have $Q = b^q$ and $P = b^p$, so

$$R = PQ = b^p b^q = b^{p+q}$$

Taking base b logarithm of both the sides, we get,

$$\log_b PQ = \log_b(b)^{p+q}$$

$$\log_b(PQ) = p + q$$

$$[\because \log_b b^x = x]$$

$$\therefore \log_b PQ = \log_b(P) + \log_b(Q)$$

The general rule is as follows.

Logarithmic Transformation of a Product: For any two positive variables X and Y,

$$\log_b(XY) = \log_b(X) + \log_b(Y).$$

The rule for the logarithm of a quotient is derived in a similar fashion. To illustrate this rule, we determine the relationship between real money balances, M/P, the logarithm of nominal balances, $m = \log_b(M)$, and the logarithm of prices, $p = \log_b(P)$. Noting that $M = b^m$ and $P = b^p$, we write real money balances as

$$\frac{M}{P} = \frac{b^m}{b^p} = b^{m-p}$$

which results in $\log_b(M/P) = \log_b(M) - \log_b(P)$

The general rule follows.

Logarithmic Transformation of a Quotient: For any two positive variables X and Y,

$$\log_b\left(\frac{X}{Y}\right) = \log_b(X) - \log_b(Y)$$

Exponents are frequently used in Economic modeling. The Cobb-Douglas production function that links output per worker (Q) to capital per worker (K), called the *intensive Cobb-Douglas production function*, is

$$Q = K^{\alpha}$$

The logarithmic transformation of this relationship can be determined by using the definition of logarithms and other results derived above. Define a variable Z such that it equals the base b logarithm of K^{α}; that is,

$$Z = \log_b\left(K^{\alpha}\right)$$

Now, let $k = \log_b(K)$

Then, from logarithm rules, we get, $K = b^k$

Now, consider$\log_b(K^{\alpha})$, and substitute the value of $K = b^k$in this expression to get,

$$\log_b(b^k)^{\alpha}$$
$$= \log_b(b^{\alpha k})$$
$$= \alpha k \qquad [\because \log_b b^x = x]$$
$$= \alpha \log_b(K)$$

Hence, we get $\log_b(K^{\alpha}) = \alpha \log_b(K)$

The general form of this rule is as follows.

Logarithmic Transformation of an Exponent: For any positive variable X and any values of λ

$$\log_b\left(X^{\lambda}\right) = \lambda \log_b(X)$$

These three rules can be combined as the need arises. For example, the logarithm of the real exchange rate (RER), $RER = EP^*/P$, is

$$\log_b(RER) = \log_b(E) + \log_b\left(P^*\right) - \log_b(P).$$

When output (Y), is related to technology (A), labor (L), and capital (C), by the function

$Y = AL^{\alpha}C^{\beta}$, logarithm of output is related to the logarithm of technology, labor, and capital by the function— $\log_b Y = \log_b A + \alpha \log_b L + \beta \log_b C$

Relationship between Logarithms with Different Bases

All of the rules presented so far include logarithms with the same base. We can show the relationship between logarithms with different bases by considering the value of the base J logarithm of a variable $H$and simultaneously base W logarithm of H, that is, $\log_J H$ and $\log_W H$, respectively. First define some number s such that $H = J^S$. Therefore $s = \log_J H$.

The base W logarithm of H, $\log_W H$, can be written as:

$$\log_W H = \log_W J^s$$

$$\log_W H = s \log_W J$$

Substituting for s, we find $\quad \log_W H = \log_W J . \log_J H$

which can be rewritten as $\quad \dfrac{\log_W H}{\log_J H} = \log_W J$

A general property derived from this rule is that if $W < J$,

$$\log_W J > 1$$

and therefore, $\quad \dfrac{\log_W H}{\log_J H} \geq 1$

That is $\quad |\log_W H| \geq |\log_J H|$

And where $|x|$ represents the absolute value of x.

Natural Logarithms

A *natural logarithm* is a logarithm that takes, as its base, the exponential, e. The natural logarithm of a variable x is written either as $\log_e(x)$ or, more commonly, as $\ln(x)$. The natural logarithm is particularly useful for many applications in Economics. In this section we present some of the properties of natural logarithms and show their use in a number of applications. All of the rules for logarithms presented above apply to the natural logarithm since it is just a logarithm with a particular base. Therefore we have the following rules for natural logarithms.

Rules for Natural Logarithms: For any variable Z and any positive variables X and Y,

i) $\ln\left(e^Z\right) = Z$

ii) $e^{\ln X} = X$

iii) $\ln XY = \ln X + \ln Y$

iv) $\ln\left(\dfrac{X}{Y}\right) = \ln X - \ln Y$

iv) $\ln X^Z = Z \ln X$

4.5.3 Power Functions

A *power function* takes the general form $f(x) = kx^r$ where k and r are any constants. The parameter r is the *exponent* of the function. Do not confuse a power function with an exponential function. In an exponential function, there is a fixed base while the exponent is a variable (the independent variable appears as an exponent); whereas in a power function, the base is variable and the exponent is a parameter. In other words, in a power function, the independent variable appears as the base. The use of power functions requires knowledge of the rules of exponents.

4.6 HYPERBOLIC FUNCTIONS

The basic idea about hyperbolic function is as follows. A function that is represented by the equation $y = \frac{a}{bx + c}$ is called a *hyperbolic* function. The simplest hyperbolic function is $y = \frac{1}{x}$ that is, $a = 1$, $b = 1$ and $c = 0$. This is called a *rectangular hyperbolic* function, or *rectangular* hyperbola. To plot the graph of $y = \frac{1}{x}$, we face a problem at the point where $x = 0$, because there is no number that gives a value for division by zero; no y value can be defined for the point $x = 0$. So in the graph, a point would be missing. Moreover, the graph increases or decreases sharply on the either side to the undefined point. We can see this if we evaluate y for values of x close to $x = 0$, as in the following table:

Table: Calculation of y values for a given x

x	$y = \frac{1}{x}$
− 2	−0.5
− 1.5	−0.67
− 1	− 1
− 0.5	− 2
0	undefined
0.5	2
1	1
1.5	0.67
2	0.5

We have pointed out a key feature of the rectangular hyperbola that the value of y at $x = 0$ is undefined. There is another key feature that is immediately obvious from the equation of the rectangular hyperbola, $y = \frac{1}{x}$. It is that y and x are inversely related. As x becomes smaller and smaller, y becomes larger and larger. This is true of negative values of x as well. This is because as x gets larger in absolute value, y becomes smaller in absolute value. When x is positive and gets larger and larger, y, as we have seen, gets smaller and smaller, and therefore the curve approaches the x-axis. But it never touches the x-axis because howsoever large x is, y is always greater than 0 and never becomes 0. To touch the x-axis y has to be 0. We can make y very very close to 0 but never equal to 0. Thus, we say that y approaches 0, or tends to 0. Zero is the limit, or limiting value of y (you will study about limits in unit 7). We say that as x tends to or approaches ∞, y approaches or tends to 0. The same would be true for negative values of x. In the example, the y-axis is a vertical asymptote and the x-axis is a horizontal asymptote. (Refer Figure 4.1)

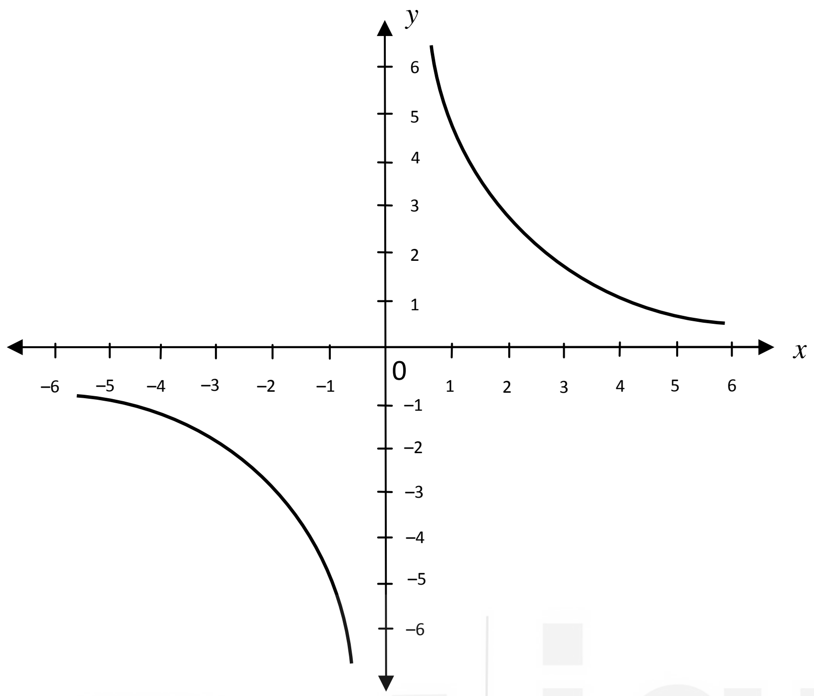

Figure 4.1

An asymptote is a line which the curve approaches at a distance from the origin, but never meets. We also say that the curve for rectangular hyperbola has a discontinuity at $x = 0$ (you will study about continuity and discontinuity in Unit 8).

The *general hyperbolic* function is of the form $y = \frac{a}{bx+c}$. The graph of this equation is similar to the graph of $y = \frac{1}{x}$, except that division by zero does not occur at $x = 0$, but at $bx + c = 0$, that is, at $x = \frac{-c}{b}$. Functions of the form $y = \frac{a}{bx+c}$ are used to depict average cost curves and supply and demand functions. The rectangular hyperbola often arises in Economic applications. Consider the inverse demand function $p = \frac{15}{q^D}$. This is a hyperbolic function. The demand function, with price on the vertical axis, and quantity demanded on the x-axis, is asymptotic. This implies three features of economic significance. The first is that howsoever high the price goes, consumers will never reduce their consumption to zero, and will buy some quantity of the commodity. The second is that on the other hand, the consumer will always buy a little more howsoever low the price comes down. The consumer is never satiated. The third is seen by looking at the equation of the demand curve:

$$p = \frac{k}{q^d}$$

Multiplying p with q^d we get k. So we see that the total expenditure, that is, price per unit multiplied by the quantity consumed, stays the same regardless of the price of the commodity and the amount consumed. Also, and you know this concept from the principles of microeconomics course, the price elasticity of demand at all points on the curve is a constant.

Check Your Progress 3

1) Explain the difference between an exponential function and power function.

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2) What do you understand by a logarithm? How is a logarithmic function related to an exponential function?

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3) That do you understand by a hyperbolic function? Discuss the features of a demand function whose graph is a hyperbola.

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4.7 LET US SUM UP

This unit had as its aim the exposition of various types of functions used in Mathematics and applied in Economics. As such, this unit carried the discussion forward from unit 2, which discussed about functions in a general way. This unit focused more on functions as relating variables with each other. To this end, the unit began by defining variables, constants and parameters. Equations and identities were also explained. Then the unit defined endogenous and exogenous variables, and made a clear distinction between an equation and a function.

Following this, the unit went on to discuss several types of functions. The unit made a distinction among the functions in two different ways: linear and non-linear functions, and algebraic and non-algebraic function. We saw what is meant by the slope and intercept of a function. Beginning with linear functions, we saw that the slope of a linear function stays constant throughout the function. This is one feature where non-linear functions are different from linear functions. In a linear function, the variables are raised to the power one and no other power. The unit subsequently went on to study functions where the variables are raised to the power of two (quadratic functions), power of three (cubic functions), and higher powers (general polynomial functions).

In the following sections, the unit went on to discuss exponential functions, where the independent variable appears as an exponent. In exponential functions, a fixed base is raised to a variable exponent. This is different from a power function which the unit discussed later. In power functions a variable base is raised to a fixed exponent. The unit also discussed logarithmic functions. Logarithmic functions are the inverses of exponential functions. The inverse of the exponential function $y = a^x$ is $x = a^y$. The logarithmic function $y = \log_a x$ is defined to be equivalent to the exponential equation $x = a^y$. Exponential and logarithmic functions are examples of what are called 'transcendental' functions, as they transcend algebra. Another example of transcendental function is trigonometric functions, which however, we did not consider in this unit. The unit finally discussed hyperbolic functions and gave certain Economic applications. You must read unit 5 along with this unit—read the two units together—to get the full flavor of the concepts and contents.

4.8 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) A linear equation, given by $ax + b = c$, when solved gives a unique value of x in terms of the parameters (a, b, c). A linear function on the other hand is given by, $y = ax + b$ .Here, instead of a single value, there are many pairs of x and y values that satisfy this equation. That is, there is no unique solution to this equation.

2)

Linear Functions	Non-linear Functions
1. *The variable* x is raised only to the power 1	1. The variable x is raised to any lower or higher power than 1.
2. Graph is a straight line with constant slope, that is a unit change in x increases or decreases y by the same amount.	2. Graph is not a straight line with constant slope, that is a unit change in x does not always lead to a constant change in y.

3) See section 4.3 and answer.

Check Your Progress 2

1) See subsection 4.4.1 and answer.
2) See subsection 4.4.2 and answer.
3) A cubic function is generally 'S'-shaped. Several Economic examples include Total product function, total cost function, etc.

Check Your Progress 3

1) A power function given by, $y = x^n$,has variable x as its base and constant n as the exponent, whereas an exponential function given by, $y = n^x$, has variable x as the exponent and constant n as its base.

2) See subsection 4.5.2 and answer the first part. The equation $y = \log_b x$ is equivalent to the equation $x = b^y$, where $b > 0$, $b \neq 1$.

3) See section 4.6 and answer.

UNIT 5 ANALYTICAL GEOMETRY*

Structure

5.0 OBJECTIVES

After studying this unit, you should be able to:

- explain the meaning of ordinate, abscissa and their graphical representation;
- state the formula of distance between two points;
- describe different expressions of a straight line; and
- define a parabola, a rectangular hyperbola and a circle, and discuss some economic applications of these.

5.1 INTRODUCTION

Co-ordinate geometry is that branch of mathematics which establishes a definite correspondence between the position of a point in a plane and a pair of algebraic quantities called co-ordinates. We can have co-ordinate geometry of higher dimensions, but for our purpose we will stick to two dimensions only. These co-ordinates are also known as Cartesian co-ordinates after the name of a well known mathematician Descartes who first introduced this idea. Coordinate geometry, which is also called analytical geometry, combines concepts of algebra and geometry. It gives a quantitative dimension to geometric points, lines and curves. As such, it provides a useful tool for depiction of economic variables and their magnitudes, but also helps to show relationships among these economic variables.

5.2 THE CARTESIAN COORDINATE SYSTEM

Just as the number line is useful for statements in one variable, the **Cartesian Co-ordinate system** is effective in working with a variety of statements in two variables. The Cartesian Co-ordinate system (see Figure 1) is constructed by placing a vertical number line and a horizontal number line on a plane so that their zero points coincide. The point where the two lines cross is called the

* Contributed by Shri Saugato Sen, SOSS, IGNOU

Origin. The horizontal line is called the ***x*-axis**, and the vertical line is usually called the ***y*-axis**. The arrows indicate the directions in which positive numbers increase. The two lines split the plane into four parts called **Quadrants** labeled counterclockwise I, II, III and IV, as shown in Figure 5.1 below. The lines themselves do not belong to any quadrant.

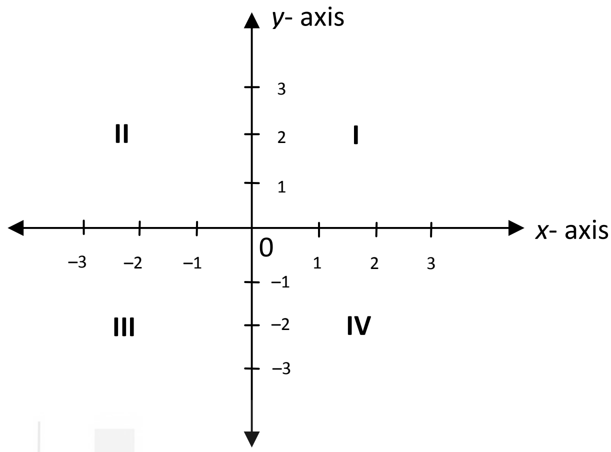

Figure 5.1

By means of a Cartesian coordinate system, any point in the plane is represented by exactly one ordered pair of real numbers, and conversely, any ordered pair of real numbers represents exactly one point in the plane.

The members of the ordered pair of real numbers representing a point are called the **coordinates** of the point. The first member is called the **first coordinate**, the ***x*-coordinate**, or the **abscissa.** The second member is called the **second coordinate**, the ***y*-coordinate**, or the **ordinate.**

The pair (*a, b*) is graphed or plotted on the Cartesian co-ordinate system by finding the position for *a* units on the *x*-axis and then moving vertically on the *y*-axis by *b* units, upward if *b* is positive and downward if *b* is negative.

In the first unit, you studied about functions. A real function, a function in which the dependent and the independent variable are all real numbers, represented on a Cartesian co-ordinate system by means of a graph. The **graph** of a function *f* is the set of all points (*x, y*) such that *x* is in the domain of *f* and *y = f(x)*. In other words, it is the set of all points associated with the ordered pairs of the function.

Check Your Progress 1

1) Plot the points (3, 4), (0,0), (–2,5), (4,3) and (2, –2) in the Cartesian co-ordinate system.

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2) Plot the points (2, 5), (0, –4), (7, 0) and (–6, –3) on a Cartesian co-ordinate system.

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3) Let f be the function such that

$f(x) = x^2$, and the domain is {–2, –1, 0, 1, 2, 3}

Sketch the graph of f.

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5.3 DISTANCE BETWEEN TWO POINTS

We shall now use the co-ordinate system to determine the distance between two departmental stores, between two warehouses, or between two cities. To answer questions of this nature, we must first identify the co-ordinates of cities A and B. If the two cities, say A(x_1, y_1) and B(x_2, y_2) have the same abscissa, i.e., $x_1 = x_2$, then the distance between them is $y_2 - y_1$ if $y_2 > y_1$ and $y_1 - y_2$, if $y_2 < y_1$. Thus, the distance between city A (20, 25) and city B (20, 40) is simply 40 – 25 = 15 km and distance between A (15, 20) and B (15, –5) is 20 – (–5) = 25 km. Similarly if the cities A and B have the same ordinate, i.e., $y_1 = y_2$, then the distance between them is $x_2 - x_1$ if $x_2 > x_1$ and $x_1 - x_2$ if $x_2 < x_1$. Thus the distance between A (10, 15) and B (25, 15) is 25–10 = 15 km. And the distance between A (10, 15) and B (–5, 15) is 10 – (–5) = 15km. Note that the distance measured is always positive.

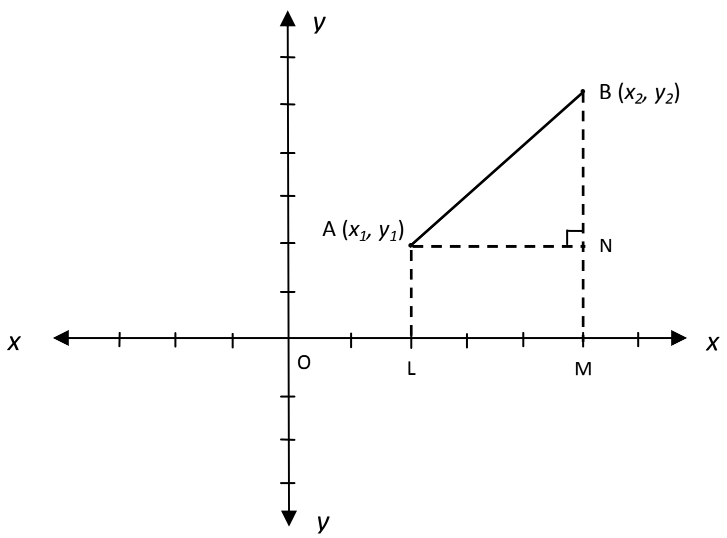

Figure 5.2

We now extend the concept of distance to two points A (x_1, y_1) and B (x_2, y_2) that have neither the same abscissa nor the same ordinate.

It is easy to find the distance between two points in a plane by constructing a right triangle and making a direct application of the theorem from geometry called Pythagoras's Theorem N the *Pythagorean Theorem*. This theorem states that in any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Consider figure 5.2, where A (x_1, y_1) and B (x_2, y_2) are the two points in the co-ordinate system. Now draw AL and BM perpendicular to the x-axis and AN perpendicular to BM. The triangle ANB so formed is a right triangle with a right angle at N. Note that the sides AN and NB are of lengths:

$$AN = LM = OM - OL = x_2 - x_1$$

$$\text{And } NB = MB - NM = y_2 - y_1$$

By the Pythagorean Theorem,

$(AB)^2 = (AN)^2 + (NB)^2$, where

$$AB = \sqrt{(AN)^2 + (NB)^2}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Hence, the distance from A(x_1, y_1) to B(x_2, y_2), where $x_1 \neq x_2$ and $y_1 \neq y_2$, written as d (A, B) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example : Find the distance between A (18, 8) and B (10, 2).

Solution : Let $x_1 = 18$, $y_1 = 8$, $x_2 = 10$ and $y_2 = 2$ in the formula:

$$d(\text{A, B}) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(18 - 10)^2 + (2 - 8)^2}$$

$$= \sqrt{(8)^2 + (-6)^2}$$

$$= \sqrt{100}$$

$$= 10$$

Activity 1 : Find the distance between the following pair of points:

A (8, 10) and B (20, 15)

Activity 2 : Show that the distance between the points P ($\alpha, -\beta$) and Q ($-\alpha, \beta$) is:

$$d[P, \text{Q}] = 2\sqrt{\alpha^2 + \beta^2}$$

5.4 SECTION FORMULA

Now we proceed to find the co-ordinates of a point P(x, y) which divides the line AB in a given ratio.

Consider figure 5.3, where point P(x, y) divide the line joining the points A(x_1,y_1) and B(x_2, y_2) in the ratio $m : n$, that is, $\frac{AP}{PB} = \frac{m}{n}$, where m and n are some positive integers.

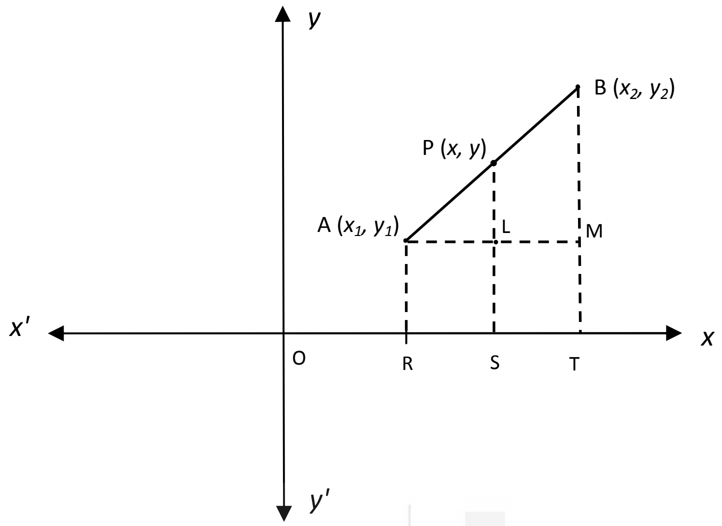

Figure 5.3

Points A and B are extended to intersect x-axis at points R and T, respectively. AM is drawn perpendicular to line BT intersecting line PS at L. In triangle AMB, we have LP parallel to MB.

Therefore,
$$\frac{AL}{LM} = \frac{AP}{PB} = \frac{m}{n} \text{ (given)}$$

(Note that AL = RS = $x - x_1$ and LM = ST = $x_2 - x$)

$$\text{or} \quad \frac{x - x_1}{x_2 - x} = \frac{m}{n}$$

Solving this for x, we get

$$x = \frac{nx_1 + mx_2}{m+n}$$

Similarly, it can be shown that

$$y = \frac{ny_1 + my_2}{m+n}$$

If P is the mid-point of AB, that is if we have $m = n$

In such a case, $x = \frac{x_1 + x_2}{2}$ and $y = \frac{y_1 + y_2}{2}$

Example : Find the co-ordinates of the point which divides the line joining points (3, 5) and (−1, 4) in the ratio 2 : 3.

Solution : Here x_1= 3, y_1= 5, x_2 = −1 and y_2= 4. Also, m = 2 and n = 3.

Let (x, y) be the coordinates of the required point.

Thus,
$$x = \frac{nx_1 + mx_2}{m+n}$$

$$= \frac{3 \times 3 + 2 \times (-1)}{2+3} = \frac{7}{5}$$

$$y = \frac{ny_1 + my_2}{m+n}$$

$$y = \frac{3\times5+2\times4}{2+3} = \frac{23}{5}$$

Similarly,

Hence, the co-ordinates are $\frac{7}{5}, \frac{23}{5}$

Activity : Find the coordinates of the point which divides the line joining the points (4, 7) and (3, −5) in the ratio 3:4.

5.5 THE STRAIGHT LINE

A straight line may be defined as the shortest distance between two points. A straight line can be specified if we know either the two points through which the line passes, or one point on the line and the slope (also called gradient) of that line.

The simplest position in which a straight line can be displayed on a co-ordinate system is that in which it is parallel to one of the axis. It is, therefore, reasonable that we begin with lines in these positions and find the corresponding algebraic equations.

Proposition 1: The equation of a straight line parallel to the x-axis and at a distance of $|b|$ units from it, is $y = b$.

Consider a straight line AB drawn parallel to the x-axis that meets the y-axis at point C so that OC = b units (see figure 5.4). Let P be any arbitrary point on AB such that PM is perpendicular to the x-axis. Here, MP = OC = b units. AB represents a locus of all the points which are at a distance of b units from the x-axis. Thus, it follows that $y = b$ is the required equation of the line parallel to the x-axis at a distance of b units from it.

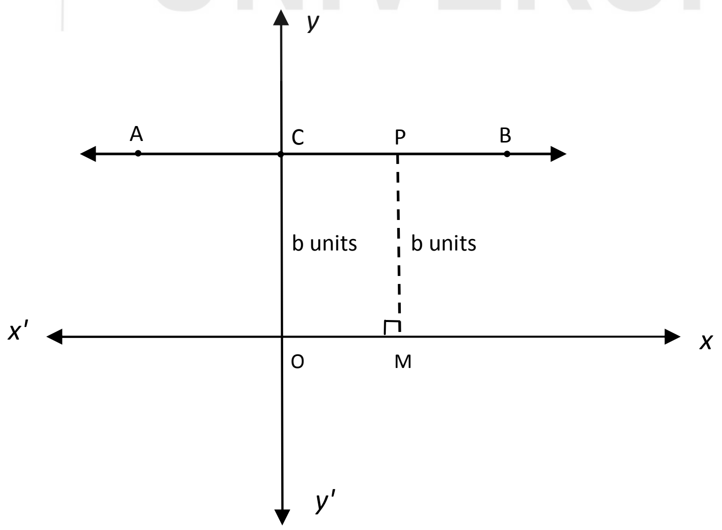

Figure 5.4

If b is positive, the line is above the x-axis; and if b is negative, the line is below the x-axis.

Proposition 2 : The equation of a straight line parallel to the y-axis and at a distance of $|a|$ units from it is $x = a$.

If the line is to the right of the y-axis, a is positive, and if it is to the left, a is negative.

Lines other than those of the form $x = Constant$ can be rewritten in the form $y = mx + b$, where m and b are constants. If $x = 0$, then $y = m.0 + b = b$, which indicates that the point (0, b) is on the graph of the line parallel to the x-axis at a distance of b units from it. Since (0, b) is the point where the line crosses the y-axis, it is called the *y-intercept*. The number m is called the *slope* of the line and is a measure of the inclination of the line. If (x_1, y_1) and (x_2, y_2) are the two points on the line $y = mx + b$, then equation for the line becomes

$$y_1 = mx_1+b$$

and $$y_2 = mx_2+b$$

Subtracting the top equation from the bottom equation:

$$y_2 - y_1 = (mx_2+b) - (mx_1+b)$$

or $$y_2 - y_1 = m(x_2 - x_1)$$

$$\therefore \quad \frac{y_2 - y_1}{x_2 - x_1} = m$$

Thus the slope of a line is given as:

$$slope = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{change\ in\ \ y}{change\ in\ \ x}$$

It is easy to see that if a line rises from left to right, its slope is positive, since

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{positive\ change\ in\ \ y}{positive\ change\ in\ x}$$

And, if a line falls from left to right, its slope is negative, since

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{negative\ change\ \text{in}\ y}{negative\ change\ \text{in}\ x}$$

For a horizontal line,

$$m = \frac{change\ in\ \ y}{change\ in\ \ x} = \frac{0}{change\ \text{in}\ x} = 0$$

So the slope is always 0 for a horizontal line, and for a vertical line the slope is undefined.

Slope-Intercept Form of the Equation of a Line:

$$y = mx + c$$

Where m is the *slope* and c is the *y-intercept.*

Example: a) Determine the slope and y-intercept of the line $2x - 3y = 6$.

b) What is the equation of a line whose slope is 3 and whose y-interceptis $\frac{6}{7}$?

Solution : a) Rewrite the equation in slope-intercept form:

$$2x - 3y = 6$$

or $$-3y = -2x + 6$$

$$\therefore \quad y = \frac{2}{3}x - 2$$

From this form it is easy to see that $\frac{2}{3}$ the slope is and the y-intercept is –2.

b) If $m = 3$ and $b = \frac{6}{7}$, the form $y = mx + b$ gives

$$y = 3x + \frac{6}{7}$$

Remarks: We need to be familiar with other forms of the equation of a line, as information about a line may be given in other ways too.

If we are given the slope m of a line and one point (x_1, y_1) on the line, we can return to the definition of slope of the line as the ratio of change in y-coordinates to change in x-coordinates. If (x, y) is a general point on a line and (x_1, y_1) is a particular (known) point on the line, then

$$\frac{y - y_1}{x - x_1} = m \text{ or } y - y_1 = m(x - x_1)$$

This gives a useful form of the equation of a line.

Point-Slope Form of the Equation of a Line:

$$y - y_1 = m(x - x_1)$$

Where (x_1, y_1) is a known point on the line and m is the slope of the line.

When dealing with the data from practical situations, we usually have two points or pairs (x_1, y_1) and (x_2, y_2). If the relationship between the variables x and y is linear, we calculate the slope of the line as the difference in y-coordinates divided by the corresponding difference in x-coordinates, and then use the point-slope equation form. This gives the two-point equation form of a line.

Two-Point Form of the Equation of a Line:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Where (x_1, y_1) and (x_2, y_2) are the two distinct points on the line with $x_1 \neq x_2$ and $m = \frac{y_2 - y_1}{x_2 - x_1}$ as the slope of the line.

Please note that all equations are of the most general form $Ax + By + C = 0$, where A, B and C are the three real numbers, and A and B cannot both be zero.

General form of the Equation of a Line:

$$Ax + By + C = 0; \quad A \neq 0, B \neq 0$$

Where *A, B,* and *C* have fixed values, representing a straight line.

Equation $Ax + By + C = 0$ may be written as

$$By = -Ax - C$$

$$or \qquad y = -\frac{A}{B}x - \frac{C}{B}$$

The slope of the line is given by

$$m = -\frac{A}{B} = -\frac{\text{Cofficient of } x}{\text{Cofficient of } y}$$

Example: What is the equation of a line with slope– 5 and passing through the point (3, 7)?

Solution : Using the point-slope form, we get

$$y - 7 = -5(x - 3)$$

$$\text{or} \qquad y - 7 = -5x + 15$$

$$\therefore \qquad y = -5x + 22$$

Example :XYZ Marketing Company arranges newspaper advertising for new consumer products. The relationship between rupees spent on door to door advertising and the initial sales of a new product is linear. If Rs. 500/- worth of advertising yields 100 sales and Rs. 1200/- worth of advertising yields 240 sales, how many sales would result from spending Rs. 750/- ?

Solution: Let x = the number of rupees spent on advertising and y = the initial sales of a product. Then (500, 100) and (1200, 240) are the two points on the line. Using the two-point form, we obtain the following equation of the line.

$$y - 100 = \frac{240 - 100}{1200 - 500}(x - 500)$$

$$or \qquad y - 100 = \frac{1}{5}(x - 500)$$

$$\text{or} \qquad y = \frac{1}{5}x - 100 + 100$$

$$\therefore \qquad y = \frac{1}{5}x$$

We now apply the advertising data in the equation to determine the corresponding number of sales, that is given $x = 750$, then y will be given by the following equation:

$$x = \frac{1}{5} \times 750 = 150 \text{ sales}$$

Activity 1 : What is the equation of a line with slope −9 and passing through the point (4, 9)?

Activity 2 : The sales of a departmental store are approximated by a straight line. The sales were Rs. 4,50,000 in January and Rs. 7,50,000 is May. Determine the equation of straight line representing the increase in sales. Assuming that this linear trend continues, estimate the sales in November.

5.6 THE CIRCLE

A *circle* is the set of all points (*x, y*)which are at a constant distance from a fixed point (*h, k*). The distance value is called the *radius* of the circle, and the point (*h, k*) is called the *centre* of the circle.

We can use the distance formula to find the equation of the circle with a given center and radius. That is, we can find an equation which is satisfied by every point on the circle and not by any other point. To demonstrate, let us find an equation of the circle with centre at (−3, 2) and radius 7 units. (Refer figure 5.5)

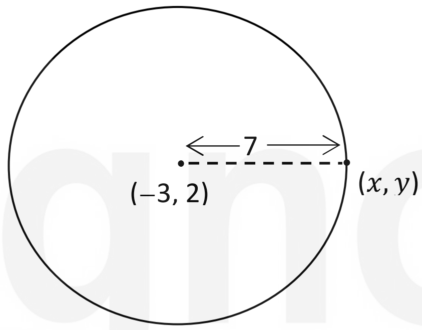

Figure 5.5

In figure 5.5, let the centre of the circle be at point (–3, 2) and (*x, y*) be any point on the circle. Then the distance from point (*x, y*) to the centre (–3, 2) must be 7. Using the distance formula;

$$\sqrt{(x+3)^2 + (y-2)^2} = 7$$

This is an equation of the circle with centre at (–3, 2) and radius 7. We can write the equation in an equivalent form by squaring both sides:

$$(x + 3)^2 + (y - 2)^2 = 49$$

Using precisely the same technique we just used for the specific circle in the previous paragraph, we can derive a general form for an equation of a circle with centre (*h, k*) and radius *r* where *h, k* and non-negative *r* represent specific real numbers. To do this, we again let (*x, y*) be any point on the circle, and we use the distance formula to express the fact that the distance from (*x, y*) to the centre (*h, k*) must be equal to the radius.

$$\sqrt{(x-h)^2 + (y-\mathrm{k})^2} = r$$

Squaring both sides, we obtain the *Standard* form:

$$(x - h)^2 + (y - k)^2 = r^2$$

Where, (*h, k*) represents the co-ordinate for the centre and *r* the radius.

If the centre is at the origin, then $h = 0$ and $k = 0$, so that the standard form becomes much simpler.

$$x^2 + y^2 = r^2, \text{ where } (0,0) \text{ is the centre and } r \text{ is the radius.}$$

If we simplify the standard form $(x - h)^2 + (y - k)^2 = r^2$ and collect terms, we obtain the equation

$$x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$$

or $$Ax^2 + By^2 + Cx + Dy + E = 0$$

Where A, B, C, D and E represent specific real numbers. In our special case

$A = B = 1$, $C = -2h$, $D = -2k$ and $E = h^2 + k^2 - r^2$. If we are given such a second degree equation with $A = B$ and A, B as non-zero, we can complete squares and write the equation in standard form as shown above.

Example : Find the equation of the circle

(a) with centre $(2, -3)$ and radius 5.

(b) with centre at the origin and radius 4.

Solution : (a) We will use the standard form with $(h, k) = (2, -3)$ and $r = 5$.

This gives $(x - 2)^2 + [y - (-3)]^2 = 5^2$

or $(x - 2)^2 + (y + 3)^2 = 25$

(b) Again, we use the standard equation with $(h, k) = (0, 0)$ and $r = 4$.

This gives $(x - 0)^2 + (y - 0)^2 = 4^2$

or $x^2 + y^2 = 16$

Remarks : Note that in part (a) we could expand

$(x - 2)^2 + (y^2 + 3)^2 = 25$ to get $(x^2 - 4x + 4) + (y^2 + 6y + 9) = 25$

or $x^2 + y^2 - 4x + 6y - 12 = 0$

Our next example shows how to verify that the graph of such an equation is a circle by reversing these steps.

Example : Show that the graph of $x^2 + y^2 - 2x + 6y - 6 = 0$ is a circle. Also find its centre and radius.

Solution : We group the like variables and complete the square for each variable:

$$(x^2 - 2x) + (y^2 + 6y) = 6$$

or $$(x^2 - 2x + 1) + (y^2 + 6y + 9) = 6 + 1 + 9$$

Now we rewrite, as in the general form:

$$(x - 1)^2 + (y + 3)^2 = 16$$

We can now see that the graph of $(x - 1)^2 + (y + 3)^2 = 16$ is a circle as it is a standard form of an equation of a circle with centre $(1, -3)$ and radius 4.

Check Your Progress 2

1) Find the equation of the circle

(a) with centre (3, – 4) and radius 7.

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(b) with centre at the origin and radius 5.

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2) Show that the graph of $2x^2 + 2y^2 - 16x - 20y + 64 = 0$ is a circle. Find its centre and radius.

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5.7 THE PARABOLA

A *parabola* is the set of all points (*x, y*) which are at equal distance from a fixed point F (*a, b*) and a fixed line *l*. The point F (*a, b*) is called the *focus*, and the fixed line *l* is called the *directrix* of the parabola. The point O (*h, k*) which is midway between the focus and directrix is on the parabola and is called its *vertex*. The line passing through the focus and the vertex is the *axis* of the parabola, whereas *latus ractum* (denoted as line *RR'*) is the chord drawn through the focus perpendicular to the axis of the parabola. (Refer figure 5.6)

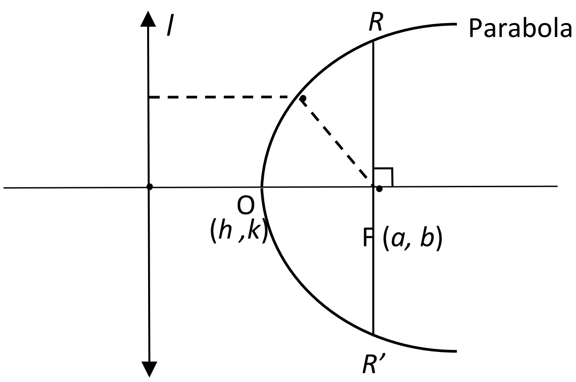

Figure 5.6

The general equation satisfied by a parabola with vertex (h, k) and directrix parallel to the x-axis is given by

$$(x-h)^2 = 4P\,(y-k) \qquad \ldots(1)$$

Where P is the distance from the focus to the vertex. Such a curve is symmetrical about y-axis, which is also called the axis of the parabola, with the equation of the directrix as $y = \pm a$. (Refer figure 5.7)

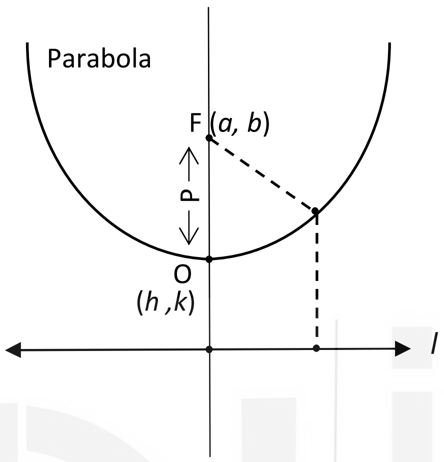

Figure 5.7

And when directrix is parallel to the y-axis, then the equation becomes

$$(y-k)^2 = 4P\,(x-h) \qquad \ldots(2)$$

Such a curve is symmetrical about x-axis, which is also called the axis of the parabola, with the equation of the directrix as $x = \pm a$ (as shown in figure 5.6).

If we expand the equation 2 and solve for y, we will recognize that it is a quadratic equation. Thus the graph will be a U-shaped curve which opens to the right if $P > 0$ and to the left if $P < 0$. (Refer figure 5.8)

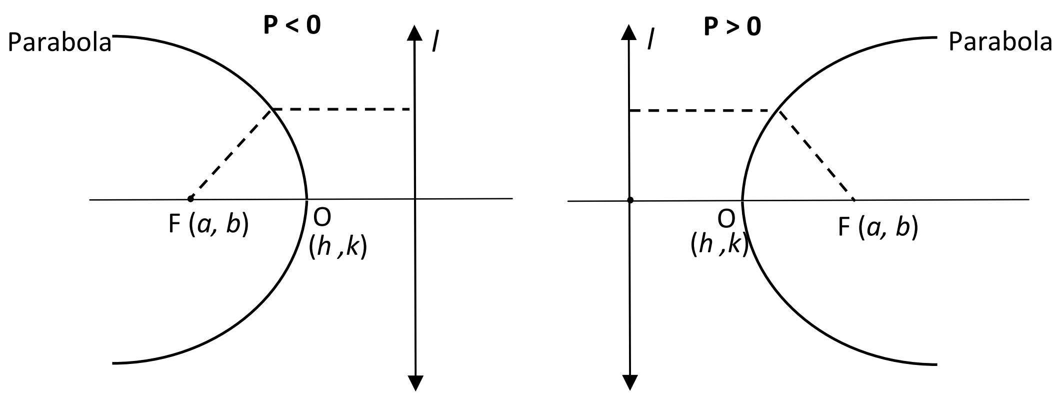

Figure 5.8

Similarly, equation $(x-h)^2 = 4P\,(y-k)$ will be quadratic in x variable and the graph of it will be a U-shaped curve which will open upwards if $P > 0$ and downwards if $P < 0$. (Refer figure 5.9)

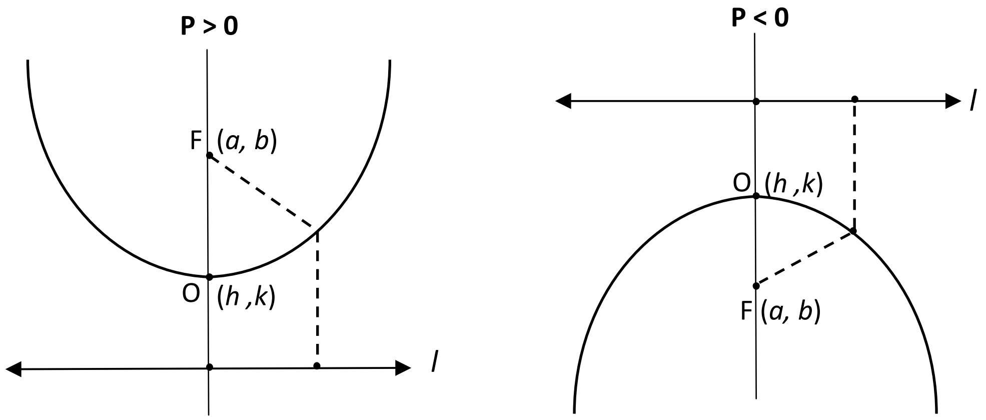

Figure 5.9

In the next example we use the definition of parabola to generate its equation.

Example: Write the equation of a parabola with focus (3, 2) and directrix $x = -1$ by using the definition.

Solution : We start by drawing a sketch of the given information. Plotting the focus at (3, 2) and the directrix at $x = -1$ (refer figure 5.10). We then locate the vertex of the parabola, which is midway between the focus and the directrix, i.e., (1, 2).

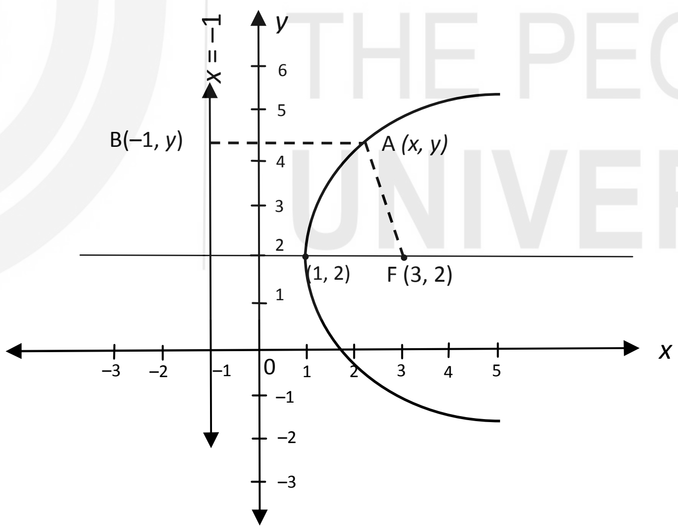

Figure 5.10

The definition of a parabola requires that a point (x, y) on it is at an equal distance from the focus F (3, 2) and the directrix $(x = -1)$. Hence, we have

$AF = AB$

Applying distance formula, we get,

$$(x-3)^2 + (y-2)^2 = [x-(-1)]^2 + (y-y)^2$$

Simplifying gives

$$x^2 - 6x + 9 + y^2 - 4y + 4 = x^2 + 2x + 1$$

or $$y^2 - 4y = 8x - 12$$

We complete the square on the quadratic expression and rewrite:

$$y^2 - 4y + 4 = 8x - 12 + 4$$

or $$(y - 2)^2 = 8x - 8$$

or $$(y - 2)^2 = 8(x - 1)$$

Note that the distance from the vertex (1, 2) to the focus (3, 2) is $P = 2$ units.

Thus our final result is

$$(y - 2)^2 = 4.(2)\,(x - 1)$$

Which is of the form $(y - k)^2 = 4P\,(x - h)$, where $P = 2$ and vertex $(h, k) = (1, 2)$.

Example : Show that $y^2 + 4x = -8$ is a parabola.

Solution : We rewrite the given equation as:

$$y^2 = -4x - 8$$

or $$y^2 = -4\,(x + 2)$$

or $$y^2 = 4(-1)\,(x + 2)$$

Thus the equation is a *parabola* symmetrical about x-axis with vertex at (– 2, 0) and $P = -1$. Since P is negative, the parabola will open to the left.

Example : Write the equation of the parabola whose axis is y-axis, vertex is at the origin, and which passes through point (–4, 2). Calculate the length of the latus rectum.

Solution : As per the given information, the equation must have the form:

$(x - h)^2 = 4P\,(y - k)$ with $(h, k) = (0, 0)$

The equation becomes, $x^2 = 4Py$

The point (–4, 2) lies on the parabola, thus will satisfy the equation, so that

$$(-4)^2 = 4P\,(2)$$

Solving the equation gives:

$$P = \frac{16}{8} = 2$$

Thus, the equation is: $x^2 = 8y$

The focus must lie on the y-axis and be at distance of 2 units from the vertex (0, 0). Thus the focus is (0, 2). Observe that *latus rectum,* which is a perpendicular through the focus, must intersect the parabola at points (– 4, 2) and (4, 2). This chord will have the length of 8 units, which we note is the value of $4P$. Thus, a conclusion can be made here that the *Length of Latus Rectum* equals $|4a|$.

Hence, the equation of the parabola is $x^2 = 8y$ and the length of the *latus rectum* is 8 units.

Check Your Progress 3

1) Write the equation of the parabola described. Sketch the graph. Label the focus, vertex and directrix:

   a) Focus = (0, 2), Directrix: $x = -2$

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   b) Focus = (2, 1), Directrix: $y = 4$

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2) Show that the graph of the following equations is a parabola. Find the vertex and focus. Sketch the graph.

   a) $x^2 - 8y = 0$

   b) $y^2 = 8x + 4$

   c) $y^2 - 4y - 2x = 0$

   d) $x^2 + 6x = 3 + 6y$

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5.8 THE RECTANGULAR HYPERBOLA

The rectangular hyperbola is a curve which can be defined as the locus of points such that, given two fixed perpendicular lines, the product of the distances of a point from these two lines is constant. These two fixed perpendicular lines are called the *asymptotes* of the rectangular hyperbola, and the intersection of these two fixed lines is called the *centre* of the rectangular hyperbola. Consider the following diagram:

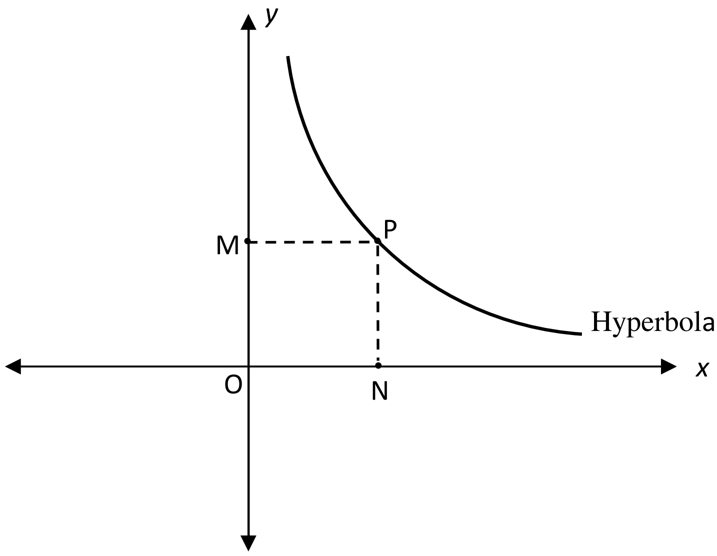

Figure 5.11

Consider figure 5.11 where the point *A* traces a locus in the *x-y* plane. The horizontal and vertical lines are the two asymptotes which are perpendicular to each other. As per the definition of the hyperbola, the distance *AN* from the horizontal asymptote to the point *A* declines in the same proportion as the distance *AM* from the vertical asymptote to the point A. The crucial point about the locus of points that traces out a rectangular hyperbola is that the area of the rectangle ONAM remains constant throughout. Keep in mind that a rectangular hyperbola could also be traced out on the (–*x*, –*y*) plane (that is, the south-western quadrant) Can you see why? (Hint: the product of two negative numbers is a positive numbers).

5.9 LET US SUM UP

This unit is a crucial one in the course that we hope went a long way in equipping you with the 'language of mathematics' used in Economics. You learnt to study various geometrical curves and shapes and help to provide diagrammatic picture to various economic relationships. You learnt to 'plot' various economic relationships diagrammatically. But equally important, you learnt that there is an algebraic counterpart for the various geometric shapes and curves. Points in a plane can be expressed as coordinates, and there are *equations* for lines, circles, parabolas and hyperbolas. Distance between points in a plane can be quantified.

To this end, the unit began by explaining the concept of Cartesian co-ordinate system. The unit explains what abscissa and ordinate mean, and how points in a plane are plotted. After this the Unit went on to describe the concept of distance between points, as well as about section formula, that is, methods to find points that divide given lines in terms of given ratios.

After this the unit proceeded to take up, one by one, the formulae for expressing various geometric shapes. It discussed the equation for the circle, the parabola and hyperbola.

5.10 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) See section 5.2
2) See section 5.2
3) The graph of f will consist of the points (–2,4), (–1,1), (0,0), (1,1), (2,4) and (3,9). Note that we do not connect these points because the graph of f consists only of these six points. Since there are only six elements in the domain, there are only six points on the graph.

Check Your Progress 2

1) See section 5.6
2) See section 5.6

Check Your Progress 3

1) See section 5.7
2) See section 5.7

UNIT 6 SEQUENCES AND SERIES*

Structure

6.0 OBJECTIVES

After going through this unit, you will be able to:

- define a sequence;
- explain how sequences and series are related;
- discuss arithmetic and geometric progressions;
- discuss the concept of convergence of sequences;
- state the elementary idea of a limit of a sequence; and
- describe certain applications of sequences and series in Economics.

6.1 INTRODUCTION

Recall the notion of a function that you studied in Unit 2. You remember that a function has a domain and a codomain. Now think of a function whose domain consists of natural numbers. A function would associate each member of a domain (here each natural number) with an element of the codomain. Well, such a function is a *Sequence*. Sequence is when you say "first this, then that, after that, such and such"… and so on. The months in a year occur in a sequence, as do days in a week. So a sequence is about "first, second, third…"

You might wonder, "What is so deep or important about sequences?" You would be surprised to know that from this elementary notion, some astonishingly important ideas in Mathematics are built up, lots of which have

* Contributed by Shri Saugato Sen, SOSS, IGNOU

applications in Economics. This unit discusses sequences as well as a special type of sequence called *Series*. It discusses some important types of series. Properties of sequences and series are also discussed. We talk about the ideas of if and how and when a sequence converges; moreover we hint at the important concept of a limit. This last concept meets you in the form of a full unit, immediately after the present one. It is that important, and is the foundation of differential calculus, which helps you to carry out a lot of Economic analysis.

This unit is organised as follows. The next section explains and discusses in detail the concept of a Sequence. It hints at the possible important applications that sequences finds in Economics. It introduces sequences as a function, and provides both an intuitive and formal definition. The section after that discusses the concept of a Series. The section following that deals with progression—a special type of series where the terms are steadily increasing or decreasing. We discuss two important progressions in this section: Arithmetic progression and Geometric progression. Following this, the unit turns to a detailed discussion of the concept of the convergence or divergence of a sequence. Along with the idea of a convergent sequence, the unit begins a discussion of limits. Finally the unit discusses certain applications of sequences and series in Economics.

6.2 SEQUENCES

We often hear the phrase " these things will occur in a sequence" or " a sequence of cards". What does a sequence mean? A sequence is nothing but a succession of numbers. For example, the sequence 2, 4, 6, 8,… is the sequence of even numbers. The sequence 1, 4, 9, 16,… is the sequence of squares of natural numbers.

The formal definition of a sequence is that a sequence is a *function* whose domain is the set of positive integers. What does this mean? It simply means that when we order the successive terms of the sequence, we point out the first term, the second term, the third term, and so on. So we associate the integer +1 with the first term, +2 with the second term, and so on. For example in the sequence of even numbers 2, 4, 6, 8, … , we associate +1 with the first term, that is, 2, +2 with the second term which is 4, and so on; similarly for the sequence of cubes of the positive integers, 1, 8, 27, 64... , we associate +1 with the number 1 (the first term), +2 with 8 (the second term), +3 with 64, and so on.

To give a formal definition, if to each natural number $n \in N$, where N is the set of natural numbers, is assigned a real number a_n, then $\{a_n\} = a_1, a_2, ..., a_n, ...$ is called a *Sequence*. The elements $a_1, a_2, a_3, ...$ are called the *terms* of the sequence. For such a sequence, a_n is the n^{th} term.

Sequences are useful in analysis in Economics, where things happen over time successively. Sequences are useful for writing out the quantities or magnitudes of the variable(s) in question in various time periods. For example, the figures of annual wheat production in India over a ten-year period would be a sequence with 10 terms. Or, suppose we have the time period 2000 to 2008. Let us say we have the Gross National Product (GNP) of India for these nine years. So it will be a sequence of nine terms. If we denote the starting year as 1, then the last year in this case will be 9. Let us index year by t and denote GNP by Y. Let GNP in year t be denoted by Y_t. Thus GNP in year 2000 is Y_1 and GNP in year

2008 is Y_9. To put it more generally, suppose we have a variable x, and let its value in the first time period (or point of time) be x_1, let time be indexed by t and let there be T time periods. Then we have the sequence $x_1, x_2, \ldots x_t, \ldots, x_T$.

A sequence (with the first and last term specified) is compactly denoted as $\{x_t\}_1^T$. Here 1 and T outside the braces show that the first term is x_1 and the last term is x_T. The letter t is used as a general index for time (year). Sometimes the initial time period is taken as 0 (zero), simply for mathematical convenience. Then, the sequence can be written as $\{x_t\}_0^T$. If we want to show a dynamic processes extending indefinitely into the future, we show this by $\{x_t\}_0^\infty$.

We have seen in unit 2 that ordered pairs could be used to write functions. Let us consider a transformation of real numbers, R, from the set of natural numbers, N. Thus, we write $f : N \to R$. The set of such images is a subset of R and it is ordered. If we write the ordered subset of Ras $S = \{x_1, x_2, x_3, \ldots\}$, then S is called a sequence on R. What we are doing here is—we are generating a set of numbers which follow an ordering. A sequence is a set of real numbers with a possible natural ordering. Consider the sequence $\{2^n\}_{n\geq 1}$. For $n = 1, 2, \ldots$, we have 2, 4, 8, 16,… and ordering is such that $2 \leq 4 \leq 8 \leq 16 \leq 32\ldots$. The numbers are getting bigger with each successive step. Alternatively, consider a sequence such as $1 \geq \frac{1}{2} \geq \frac{1}{3} \geq \frac{1}{4} \geq \ldots,$ where the numbers are getting smaller.

In general, a Sequence may

i) increase or decrease indefinitely at a constant or at an increasing rate. For example,

$$S = \{0, -1, -3, -8, -15\ldots\}$$

or, $S = \{1, 5, 10, 17, 26, \ldots\}$

ii) increase or decrease at a decreasing rate. For example,

$$S = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}\ldots\}$$

or, $S = \{\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}\ldots\}$

iii) fluctuate with wider margin. For example,

$$S = \{-1, 5, -7, 17, -31\ldots\}$$

iv) fluctuate without any tendency for an increase or decrease. For example $S = \{-1, 1, -1, 1, -1\ldots\}$

v) fluctuate with a decreasing margin. For example,

$$S = \{-\tfrac{1}{2}, \tfrac{1}{4}, -\tfrac{1}{8}, \tfrac{1}{16}\ldots\}$$

So, you must not have the idea that a sequence is always monotonically increasing or decreasing.

We now discuss some basic definitions pertaining to sequences. First a sequence may be bounded or unbounded. We say that a sequence is *bounded* if there is some finite value K>0, such that for some N it follows that,

$a_n < K$for all n>N (bounded above) and$a_n > -K$for all n>N (bounded below).

A sequence is said to be bounded if and only if it is *both* bounded above (has an upper bound) and bounded below (has a lower bound). We shall later in the unit use the definition of a bound set to discuss the concept of convergence.

6.3 SERIES

To understand the concept of a series we need to combine the concept of a sequence with that of summation. We now turn to an explanation of summation. Summation is just a process of adding up values of numbers. The capital Greek letter sigma 'Σ' is universally taken to represent the summation of various values of a variable. Remember, we are concerned with various values of a variable, say, x, represented by x_1, x_2, x_3, and so on, rather than the addition of different variables like x, y, z. For example, if a firm has many plants, then the total production of the firm can be obtained as the sum of production at various plants.

To understand summation, let us take an example. Let x represent annual income, so that the subscript notation $x_1, x_2, \ldots x_T$ represent the annual incomes in years 1, 2,…,T, respectively. Then the total income for all the years combined will be denoted by:

$$\sum_{t=1}^{T} x_t = x_1 + x_2 + \ldots + x_T .$$

Some basic Rules of Summation are:

1) $\sum_{t=1}^{T} k = kT$,here k is a constant. The summation of a constant over T periods equals the product of the constant and T.

2) $\sum_{t=1}^{T} kx_t = k\sum_{t=1}^{T} x_t$. The summation of a constant times a variable is equal to the constant multiplied by the summation of that variable.

3) $\sum_{t=1}^{T} (x_t + y_t) = \sum_{t=1}^{T} x_t + \sum_{t=1}^{T} y_t$. The summation of the sum of values of two variables is equal to the sum of their summations.

Summations come in very handy when we study discrete dynamic processes and discrete dynamic optimisation. We will see later in Block 5 that the limiting case of summation in certain contexts can be depicted by using definite integrals. Definite integrals take the place of summation when we study continuous dynamic processes.

A series is the summation of the first n terms of a sequence. Thus, if $\{a_t\} = a_1, a_2, a_3, \ldots$ is a sequence, then $S_n = \sum_{t=1}^{n} a_t$ is a series. We now mention two important types of series, but first we mention the two corresponding types of sequences. These are the arithmetic and geometric sequences. An arithmetic sequence is one where the difference of any two successive terms is constant, *i.e.* $a_{n+1} - a_n = d$, for all $n \in \mathbb{N}$, where d is a constant. and a_n the n^{th} term.

6.4 PROGRESSIONS

A progression is another name for a series whose terms in the sequence are monotonically increasing or decreasing. Since a series is nothing but the sum of the terms of a sequence, it is correct to say that a progression is also a sequence.

We discuss two important progressions.

6.4.1 Arithmetic Progression

An Arithmetic Progression (A.P. for short) is a sequence of numbers, called *terms,* in which any term after the first can be obtained from its immediate predecessor by adding a fixed number, called the *common difference* (d). For example, the sequence4, 7, 10, 13,...is an arithmetic progression with the common difference equal to 3.

Observe that an arithmetic progression is completely determined if the first term and the common difference are known. In fact, if

$a_1, a_2, a_3, \ldots a_n, \ldots$

is an arithmetic progression with the first term given by a and common difference given by d, then by definition,

$$\begin{aligned}
a_1 &= a \\
a_2 &= a_1 + d = a+d \\
a_3 &= a_2 + d = (a+d)+d = a+2d \\
a_4 &= a_3 + d = (a+2d)+d = a+3d \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
a_n &= a_{n-1} + d = a+(n-2)d+d = a+(n-1)d
\end{aligned}$$

Thus, we see that the n^{th} term of an arithmetic progression with first term a and common difference d is given by

$$a_n = a+(n-1)d$$

Further, let $S_n$denote the sum of the first n terms of an arithmetic progression with first term $a_1 = a$ and common difference d. Then,

$$S_n = a+(a+d)+(a+2d)+ \ldots +[a+(n-1)d] \qquad \ldots(1)$$

Rewriting the expression for S_n with the terms in reverse order gives

$$S_n = [a+(n-1)d]+[a+(n-2)d]+\ldots+a \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$\begin{aligned}
2S_n &= [2a+(n-1)d] +[2a+(n-1)d]+\ldots+ [2a+(n-1)d] \\
&= n[2a+(n-1)d]
\end{aligned}$$

$$\therefore \qquad S_n = \frac{n}{2}[2a+(n-1)d]$$

Or

$$= \frac{n}{2}[a + a+(n-1)d]$$

$$S_n = \frac{n}{2}[a+a_n]$$

Thus, if we have an arithmetic sequence with first term as a andcommon difference as d, then the sum of the first n terms is given by$S_n = \frac{n}{2}[2a+(n-1)d]$ $= \frac{n}{2}[a+a_n]$.

Formulae for an Arithmetic Progression

If an arithmetic progression has starting term a and common difference d, then:

(a) the n^{th} term (a_n) is:

$$a_n = a + (n-1)d$$

(b) the sum of the first n terms (S_n) is:

$$S_n = \frac{n}{2}[2a + (n-1)\, d]$$

Illustration Suppose we invest Rs.100 at a simple interest of 15% per annum for 5 years. The amount at the end of each year is given by 115, 130, 145, 160, 175

This forms an arithmetic progression.

Example Write the first five terms of an arithmetic progression whose third and eleventh terms are 21 and 85, respectively.

Solution Using the formula: $a_n = a+(n-1)d$, we obtain

$a_3 = a+2d = 21$ and $a_{11} = a+10d= 85$

Subtracting the first equation from the second gives

$8d = 64$ or $d = 8$

Substituting this value of d into the first equation yields $a+16 = 21$ or $a = 5$. Thus, the required arithmetic progression is given by the sequence 5, 13, 21, 29, 37, ...

Example Mr. X borrows Rs.24,000 and agrees to repay it in twelve installments of Rs. 2000 per month plus interest of 1.5% per month on the unpaid balance. What is the amount of Mr. X's 10^{th} payment?

Solution 1^{st} payment = Rs. 2000 + (0.015)(Rs. 24000) = Rs. 2360

2^{nd} payment = Rs. 2000 + (0.015)(Rs. 22000) = Rs. 2330

3^{rd} payment = Rs. 2000 + (0.015)(Rs. 20000) =Rs. 2300

We see that the payment amounts form an arithmetic progression with first term a= Rs. 2360 and common difference (d) = −Rs. 30. Hence,

10^{th} payment = 10^{th} term = $a + (10-1)d$

= Rs. 2360 + 9(−Rs. 30) = Rs. 2090.

Example XYZ Electric Company had sales of Rs. 2,00,000 in its first year of operation. If the sales increased by Rs. 30,000 per year thereafter, find company's sales in the fifth year and its total sales over the first five years of operation.

Solution Since Mr.XYZ company's yearly sales follows an arithmetic progression with the first term given by a = 2,00,000 and the common difference given by d = 30,000. The sales in the fifth year are found by using the formula : $a_n = a+(n-1)d$ with $n = 5$. Thus,

$$a_5 = 2{,}00{,}000 + (5-1)30{,}000 = \text{Rs. } 3{,}20{,}000.$$

The Company's total sales over the first five years of operation are found by using the formula: $S_n = \frac{n}{2}[2a+(n-1)d]$ with $n = 5$. Thus,

$$S_5 = \frac{5}{2}[2(2{,}00{,}000)+(5-1)30{,}000]$$

$$= \text{Rs. } 13{,}00{,}000.$$

Example Suppose Mr. X repays a loan of Rs. 32,500 by paying Rs. 200 in the first month and then increases the payment by Rs. 150 every month. How long will it take to clear his loan?

Solution Since Mr. X increases the monthly payment by a constant amount of Rs. 150, therefore $d = 150$ and first monthly instalment is a = Rs. 200. This forms an A.P. Now if the entire amount be paid in n monthly instalments, we have

$$S_n = \frac{n}{2}[2a + (n-1)\, d]$$

or $$32500 = \frac{n}{2}[2(200) + (n-1)150]$$

or $$65000 = n(250+150n)$$

or $$15n^2 + 25n - 6500 = 0$$

$$\therefore \quad n = \frac{-25 \pm \sqrt{(25)^2 - 4\times15\times(-6500)}}{2\times15}$$

$$= \frac{-25 \pm 625}{30} = 20 \text{ or } -21.66$$

The value, $n = -21.66$ is meaningless as n is positive integer. Hence, Mr.X will pay the entire amount in 20 months.

6.4.2 Geometric Progression

Now we will be looking at geometric series (or progression), as this is useful in understanding the concepts of compounding and discounting that we will be discussing in the next section. A geometric progression (abbreviated as G.P.) is a sequence of numbers whose each term increases or decreases by a constant ratio called *common ratio* of G.P. and is denoted by r. In other words, each term of G.P. after the first, is obtained by multiplying the preceding term by a constant r. A geometric progression is completely determined if the first term and the common ratio are known. Thus, if

$$\begin{aligned}
a_1 &= a \\
a_2 &= a_1 r = ar \\
a_3 &= a_2 r = ar(r) = ar^2 \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \\
a_n &= a_{n-1} r = ar^{n-2}(r) = ar^{n-1}
\end{aligned}$$

Thus, a sequence $\{a_n\}$ where the ratio of any two successive terms is a constant (the constant $\neq$ 0) is called a geometric sequence, *i.e.*, $\frac{a_{n+1}}{a_n} = r, r \neq 0,$ for all $n \in \mathbb{N}$.. In such a sequence of n numbers $a, ar, ar^2, ar^3, ..., ar^{n-1}$, each term will be obtained by multiplying the previous one by a constant r. The sum (S_n) of the n terms is given as:

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-2} + ar^{n-1} \qquad ...(3)$$

We call the sum a geometric series or progression with quotient r.

To find the sum, multiply both sides of the above equation by r to get

$$rS_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n$$

Subtracting the second equation from the first, we get

$$S_n - rS_n = a - ar^n \qquad ...(4)$$

since all the other terms got cancelled. If $r = 1$, then from the first equation we will get $S_n = an$. For $r \neq 1$, we get from equation (4)above:

$$S_n = \frac{a - ar^n}{1 - r}$$

Thus, we can write the formula for the summation of a finite geometric series to be:

$$S_n = a + ar + ar^2 + ... + ar^{n-1} = a\frac{1 - r^n}{1 - r} \qquad ...(5)$$

If $r = 1$, then (3) gives

$$\begin{aligned}
S_n &= a + a + a + ... + a \\
&= na
\end{aligned}$$

What happens if the geometric series is infinite, that is, as n tends to infinity ($n \to \infty$)?In that case, the term r^n in equation (4) will $\to 0$ if $-1 < r < 1$ (*i.e.*, if $|r| < 1$). The sum (S_n) of the first n terms in this case will tend to $\frac{a}{1 - r}$. Whereas, r^n does not tend to any limit if $r > 1$ or $r \leq -1$.

Formulae for a Geometric Progression

If a geometric progression has starting term *a* and common ratio *r*, then

a) the n^{th} term is:

$$a_n = ar^{n-1}$$

b) the sum of the first *n* terms (S_n) is:

$$S_n = \frac{a(1-r^n)}{1-r} \quad ; \text{if } r \neq 1$$

$$na \quad ; \text{if } r = 1$$

Example Find the tenth term of a geometric progression whose third term is 16 and whose seventh term is 1.

Solution Using the formula: $a_n = ar^{n-1}$, we get

$$a_3 = ar^2 = 16 \qquad \ldots(6)$$

and $$a_7 = ar^6 = 1 \qquad \ldots(7)$$

Dividing (7) by (6) gives

$$\frac{ar^6}{ar^2} = \frac{1}{16}$$

From this we obtain $r^4 = 1/16$ or $r = 1/2$. Substituting this value of *r* into the expression for a_3, we obtain

$$a\left(\frac{1}{2}\right)^2 = 16 \qquad \text{or} \qquad a = 64$$

Finally with $a = 64$, $r = 1/2$, and $n = 10$ gives

$$a_{10} = 64\left(\frac{1}{2}\right)^9 = \frac{1}{8}$$

Example The XYZ Land Development Company had sales of Rs. 1 million in its first month of operation. If the sales increased by 10 per cent per month thereafter, find the company's sales in the fifth month and its total sales over the first five months of operation.

Solution The company's yearly sales follow a geometric progression with the first term given by $a = 1{,}000{,}000$ and the common ratio given by $r = 1.1$. The sales in the fifth month are found by using $a_n = ar^{n-1}$ with $n = 5$. Thus,

$$a_5 = 1{,}000{,}000(1.1)^4 = 1{,}464{,}100$$

The company's total sales over the first five months of operation are found by using the formula: $S_n = \frac{a(1-r^n)}{1-r}$ with $n = 5$. Thus,

$$S_5 = \frac{1{,}000{,}000[1-(1.1)^5]}{1-1.1} = \text{Rs. } 6{,}105{,}100$$

Example Mr. X deposits Rs. 1000 in a certain bank which pays interest at 10% per year compounded quarterly. How much money does he have after 5 years?

Solution The original principal is P = Rs. 1000. An interest rate of 10% annually amounts to 2.5% per quarter. Therefore, the amount at the end of the first quarter is

$$a_1 = 1000 + (1000)(0.025) = 1000(1.025)$$

The amount at the end of the second quarter is

a_2 = [amount at the end of first quarter] + [amount at the end of first quarter](0.025)

$$= a_1 + a_1(0.025) = a_1(1.025)$$

$$= 1000(1.025)(1.025) = 1000(1.025)^2$$

The amount at the end of the third quarter is

a_3 = [amount at the end of second quarter] + [amount at the end of second

quarter](0.025)

$$= a_2 + a_2(0.025) = a_2(1.025)$$

$$= 1000(1.025)^2(1.025)$$

$$= 1000(1.025)^3$$

We see that we have a geometric progression

$$1000(1.025),\ 1000(1.025)^2,\ 1000(1.025)^3,\ \ldots$$

With first term a = 1000(1.025) and common ratio r = 1.025. We want the amount at the end of the 20th quarter. This is simply the 20th term of the sequence. Hence,

$$a_{20} = ar^{20-1} = [1000(1.025)](1.025)^{19}$$

$$= 1000(1.025)^{20} \cong \text{Rs. } 1638.62$$

Check Your Progress 1

1) Find the 15th term of an A.P. whose first term is 15 and common difference is 3.

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2) A firm produces 1500 TV sets during its first year. The total production of the firm at the end of the 15th year is 24075 TV sets, then

i) estimate by how many units, production has increased each year.

ii) based on estimate of the annual increment in production, forecast the amount of production for the 10th year.

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3) The XYZ Furniture Company had sales of Rs. 1,50,000 during its first month of operation. If the sales increased by Rs. 16,000 per month thereafter, find the company's sales in the seventh month and its total sales over the first seven months of operation.

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4) Find the 11th term and the sum of the first 20 terms of the G.P.:

4, 8, 16, 32, 64, …

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5) Determine the common ratio of the G.P.

49, 7, 1, 1/7, 1/49, …

Also find the sum to first 10 terms of G.P.

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6.5 CONVERGENCE OF SEQUENCES

6.5.1 Convergence of a Sequence

Sequences show characteristics on the basis of which we can categorise them into:

i) heading towards infinity $(\infty, -\infty)$;

ii) heading towards some finite number (positive, negative or zero);

iii) heading nowhere (not settling at any value).

So the basic idea we have to pursue when dealing with a sequence is whether the values of subsequent terms (general term being a_n) getting closer and closer as the value of n increases. We know that a sequence is a function that maps elements of the set of natural numbers (N) to the elements of a set X. This set X can be the subset of R, the set of real numbers, or R itself. So, as n increases, that is, for subsequent terms of the sequence, we have to see (1) whether each successive term is greater than the previous term (the sequence is monotonically increasing), or each successive term is smaller than the previous term (the sequence is monotonically decreasing), or neither of the two situations are there; and (2) whether the sequence gets closer and closer to a definite value or not. If the sequence approaches a definite value, we say that the sequence *converges* to that value.

The idea of *convergence* of a sequence leads us to the concept of the *limit* of a sequence. Indeed the idea of convergence of a sequence is formally understood through the concept of a limit of a sequence.

6.5.2 An Elementary Introduction to Limits

These observations help us understand the concept of limit of a sequence.

Let us consider a sequence of numbers $\{x_n\}_{n\geq 1}$. If the numbers get closer and closer to a number L, ($i.e., x_n \approx L$), then we say that the sequence $\{x_n\}$ is *convergent* and has a limit equal to L. We will write,

$$\lim_{n\to\infty} x_n = L$$

or, $$\underset{n\to\infty}{Lt}\; x_n = L$$

or, $$x_n \to L \text{ when } n \to \infty.$$

When we say $n \to \infty$, we imply n is getting larger. Of course we do not have $n = \infty$. But perhaps in the neighbourhood of ∞. If a sequence is not convergent, it is called *divergent*. We will discuss a bit more on the convergent sequences to formally define the *limit of a sequence*.

We have said above that a sequence $\{x_n\}_{n\geq 1}$ is convergent if there exists a number L such that the numbers x_n get closer and closer to L as n becomes larger. That is,

i) we want to see that $x_n \approx L$;

ii) make sure that $|x_n - L| < \varepsilon$, where ε is a small number greater than zero;

iii) for (ii) to happen, we have $N \geq 1$ such that for every $n \geq N$, we have $|x_n - L| \leq \varepsilon$.

Note that ε measures the error between the number x_n and the L. The integer N on the other hand, measures how fast the sequence gets closer to the limit.

Thus we can say that the sequence $\{x_n\}_{n\geq 1}$ converges to the number L, if and only if, for every $\varepsilon > 0$, there exists $N \geq 1$ such that for every $n \geq N$, $|x_n - L| \leq \varepsilon$.

Check Your Progress 2

1) What are the conditions for a sequence to converge?

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2) What do you understand by the limit of a sequence?

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6.6 ECONOMIC APPLICATIONS OF SEQUENCES AND SERIES

6.6.1 Simple and Compound Interest

Simple interest: This is where any interest earned is *not added back* to the principal amount invested.

If amount *P* is invested at 100*i* % simple interest, then the amount accrued after *n* years is given by the following formula:

Accrued Amount Formula (Simple Interest)

$A_n = P\,(1 + i.n)$

where A_n = accrued amount at the end of the n^{th} year

P = principal amount

i = (proportional) interest rate per year

n = number of years

Generally, simple interest is of no great practical value in modern business and commercial situations, since in practice interest is always compounded.

Compound interest: This where interest earned is *added back* to the previous amount accrued.

For example, suppose that Rs.1000 is invested at 10% compound interest. The following table shows the state of the investment, year by year:

Year	**Amount on which interest is calculated**	**Interest earned**	**Cumulative amount accrued**
1	Rs. 1000	10% of Rs 1000 = Rs 100	Rs 1100
2	Rs. 1100	10% of Rs 1100 = Rs 110	Rs 1210
3	Rs. 1210	10% of Rs 1210 = Rs 121	Rs 1331
…	…	…	…
…	…	…	…
…	…	…	…

Accrued Amount Formula (Compound Interest)

$A_n = P\,(1 + i)^n$

where A_n = accrued amount at the end of n^{th} year

P = principal amount

i = (proportional) interest rate per year

n = number of years

Example A firm plans to invest an amount of money at the beginning of every year in order to accrue a sum of Rs.1,00,000 at the end of the five-year period. What is the value of the amount, if the investment rate is 14% compounded annually?

Solution Let the amount invested at the beginning of each of the five years be RsB.

The first payment of B accrues to $B(1.14)^5$, second payment to $B(1.14)^4$… and so on. But the sum of the separate accruals must add to 1,00,000, *i.e.*,

$$1{,}00{,}000 = B(1.14)^5 + B(1.14)^4 + \ldots + B(1.14)$$

$$= B[1.14 + (1.14)^2 + \ldots + (1.14)^5]$$

$$= B\left[\frac{1.14\{(1.14)^5 - 1\}}{1.14 - 1}\right] = B(7.5355)$$

$$1{,}00{,}000 = B(7.5355)$$

$$B = \frac{100{,}000}{7.5355} = \text{Rs}13270.52$$

Example A car is purchased for Rs. 80,000. Depreciation is calculated at 5% per annum for the first 3 years and 10% per annum for the next 3 years. Find the money value of the car after a period of 6 years.

Solution i) Depreciation for the first year = $80{,}000 \times \frac{5}{100}$

Thus, the depreciated value of the car at the end of first year

$$= 80{,}000 - \left(80{,}000 \times \frac{5}{100}\right)$$

$$= 80{,}000\left(1 - \frac{5}{100}\right)$$

ii) Depreciation for the second year = (Depreciated value at the end of first year) × (Rate of depreciation for second year)

$$= 80{,}000\left(1 - \frac{5}{100}\right)\left(\frac{5}{100}\right)$$

Thus, the depreciated value at the end of the second year is

= (Depreciated value after first year) – (Depreciation for second year)

$$= 80{,}000\left(1 - \frac{5}{100}\right) - 80{,}000\left(1 - \frac{5}{100}\right)\left(\frac{5}{100}\right)$$

$$= 80{,}000\left(1 - \frac{5}{100}\right)\left(1 - \frac{5}{100}\right)$$

$$= 80{,}000\left(1 - \frac{5}{100}\right)^2$$

Calculating in the same way, the depreciated value at the end of three years is

$$= 80{,}000\left(1 - \frac{5}{100}\right)^3$$

iii) Depreciation for fourth year= $80{,}000\left(1 - \frac{5}{100}\right)^3\left(\frac{10}{100}\right)$

Thus, the depreciated value at the end of the fourth year is

= (Depreciated value after three year) − (Depreciation for fourth year)

$$= 80{,}000\left(1 - \frac{5}{100}\right)^3 - 80{,}000\left(1 - \frac{5}{100}\right)^3\left(\frac{10}{100}\right)$$

$$= 80{,}000\left(1 - \frac{5}{100}\right)^3\left(1 - \frac{10}{100}\right)$$

Calculating in the same way, the depreciated value at the end of six years becomes

$$= 80{,}000\left(1 - \frac{5}{100}\right)^3\left(1 - \frac{10}{100}\right)^3$$

$\cong$ Rs. 50224

6.6.2 Compounding and Discounting

We usually prefer to consume in the present than in the future. A rupee today is psychologically worth more to us than a rupee tomorrow. This is even true when inflation is present. Investors will postpone current consumption and invest only if their future opportunities are larger due to the investment. Since one can invest money and start earning interest immediately, a rupee today must be worth more than a rupee tomorrow. Investors are investing money and getting returns in the future in terms of money. This is called cash flow. When cash comes to us it is called a cash *in*flow and is considered a positive cash flow, and when we pay out cash it is called a cash *out*flow and is considered a negative cash flow.

We will discuss below the concepts of compounding and discounting, although, in the previous paragraph we began by giving an idea about discounting. But before we discuss compounding, let us consider simple interest. We shall not use simple interest later, but we talk about it simply to contrast compound interest with it.

Let P be the principal, or the total amount of money borrowed (say from a bank as a loan) or invested (say in a bank as deposit) by an individual. Let the interest be r (r is expressed as a percentage). Let t be the period after which the loan is to be repaid, or the invested amount matures. Then if P is invested at a simple interest of r percent per annum for a period of t years, the interest charge I_n is given by

$$I_n = \mathrm{P} \times \mathrm{r} \times t \, .$$

The amount A owed at the end of t years is the sum of the principal borrowed and the interest charged:

$$A = P + I_n = P + Prt = P\,(1 + rt).$$

Now let us turn to the compound interest. Let us begin by compounding a single cash flow. Suppose you invest Rs 100 in 2008. If this investment earned simple interest at the rate of 10% per year, the future value of your investment would be Rs 100 plus Rs 10 per year for every year that the amount was invested at 10% per year. If you invested Rs 100 for 4 years you would have earned Rs 140 at the end of 4 years. In general if you invest Rs P (P stands for principal) at $100r\%$ per year for t years, at the end of t years you will get an amount

$$A = P + \Pr t = P(1 + rt) \text{ as we have seen.}$$

Compound interest is more interesting! When investments are made at compound interest rates, the investment earns "interest on interest". In other words, interest is paid on interest that has been earned in previous periods. In the above example, after the first year, amount would be Rs 100 + Rs 10 (interest =10% of 100, =10) = Rs 110 at the beginning of the second period, this Rs 110 is invested and the interest now is 10% of 110, that is Rs 11. So after the second year, the amount goes up to Rs 110+ Rs 11 = Rs 121.

In general, if a principal P is invested at $100r\%$ compound interest for t years, the amount obtained at the end of t years is

$$A = P\left(1 + r\right)^t .$$

Notice the difference with the simple interest formula. There, the term within parentheses was $1+rt$, that is t is multiplied by r and added to 1. In the case of compound interest, t appears not as a multiplicative term but as an exponent, so that $(1+r)$ is raised to the power t.

We can restate the above by calculating the future value of a single cash flow compounded annually as follows:

Let C_0 be the initial cash flow or investment

r be stated annual rate of interest or return

t be thelife of investment

C_t be the value of C_0 at end of t years.

Then, $C_t = C_0\ (1 + r)^t$

This is the equation for *compounding*, which converts a present cash value to the future cash value. $(1 + r)^t$is called the future value compound factor, denoted by $FVCF_{r,t}$, where the subscripts r and t have the meaning mentioned above. Thus,

$$C_t = C_0 FVCF_{r,t}$$

Now we turn to the inverse process, that is, we want to find out, if the future cash flow is of a given amount, what would be the value of the cash flow today. This is called *discounting*. We begin by considering a single period. To convert future cash value into present value, we use the procedure of discounting. To discount a future cash flow to the present, simply rearrange the equation for compounding, to get,

$$C_0 = \frac{C_t}{(1+r)^t}$$

Thus discounting is the opposite of compounding. In the above equation, $\frac{1}{(1+r)^t}$ is called the present value discount factor, denoted by$PVDF_{r,t}$. The discount factor is simply the reciprocal of compound factor.

So far, we have considered the case of compounding or discounting a single cash flow. Most financial problems, however, are concerned with multiple cash flows. Let us just take up discounting. Discounting multiple cash flows is simple: we can discount each individual cash flow and then add the present values (PV). The general case is present below [the cash flows (C_1, C_2, C_3… C_t) are unequal and uneven each year]:

$$PV_0 = \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_t}{(1+r)^t}$$

We can now consider the present value of an annuity. An annuity consists of a constant payment received each year. Letting P_0 equal the present value of an annuity which pays C rupees at the end of each year for t years. Then, we get,

$$P_0 = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^t}$$

$$= C\left[\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^t}\right]$$

The sequence of terms within the square brackets represents a geometric progression or geometric series. This geometric progression is called the present value annuity factor ($PVAF_{r,t}$) at the rate r and for t years. Using this notation we can represent the present value of any annuity as:

$$P_0 = C.PVAF_{r,t}$$

Where, C is the constant payment amount. It can be shown that the present-value annuity factor can be shown to be equal to

$$PVAF_{r,t} = \frac{1-\left[\frac{1}{(1+r)^t} \right]}{r}$$

We can derive the formula for the future value of an annuity (FVA_t). It can be shown to be

$$FVA_t = C(1+r)^{t-1} + C(1+r)^{t-2} + \ldots + C$$

This is equal to $= C\left[\frac{(1+r)^t - 1}{r}\right]$

By way of concluding this section let us consider compounding and discounting when cash flow is not on a yearly basis but occurs more than one time a year. Suppose r is the interest rate, t the number of years, as before, but now suppose it is compounded m times a year, that is, m is the number of compounding periods in a year.

To find out the relevant formula for compounding in such cases, we find it is equal to

$$C_t = C_0\left(1+\frac{r}{m}\right)^{mt}$$

For the case of discounting, if there is non-annual discounting the formula is:

$$C_0 = \frac{C_t}{\left(1+\frac{r}{m}\right)^{mt}}$$

6.6.3 Present Value

Suppose money can be invested at 10% compound interest, compounded annually. Then Rs. 100 could be invested and be worth Rs. 110 in one years time. Put another way, the value of Rs110 in *one year time* is exactly the same as Rs100 now (if the investment rate is 10% per annum). Similarly, Rs100 now has the same value as Rs. $100(1.1)^2$ = Rs. 121 in two years time, assuming the investment rate is 10%. This demonstrates the concept of the *present value* of a future sum. To state the above ideas more precisely, if the current investment rate is 10%, then

the present value of Rs. 110 in one years time is Rs. $\frac{110}{1.1}$ = Rs100.

Similarly, the present value of Rs121 in two years time is Rs. $\frac{121}{1.1^2}$ = Rs100 and so on.

The investment rate used in this context is sometimes referred to as the *discount rate.*

In the following section, the above technique for calculating present value is generalised to give a formula and then followed by an example.

Present Value Formula

The present value of an amount Rs A, payable in t years time, subject to a discount rate of $100i\%$ is given by:

$$P = \frac{A}{(1+i)^t}$$

where, P = present value

A = amount, payable in t year's time

i = discount rate (as a proportion)

t = number of time periods (normally years)

Example A firm has purchased an item on a fixed payment plan of Rs. 20,000 per year for 8 years. Payments are to be made at the beginning of each year. What is the present value of the total cash flow of payments for an interest rate of 20% per year?

Solution The present value of the payment this year = Rs 20,000

The present value of the 2nd yearpayment = Rs $\frac{20,000}{1.2}$

The present value of the 3rd yearpayment = Rs $\frac{20,000}{(1.2)^2}$

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The present value of the last year payment = Rs $\frac{20,000}{(1.2)^7}$

The present value of the total cash flow is

$$20,000\left\{1 + \frac{1}{1.2} + \frac{1}{(1.2)^2} + \ldots.. + \frac{1}{(1.2)^7}\right\}$$

This is a geometric series of common ratio $r = \frac{1}{1.2}$, and what we want is the sum of the first 8 terms, *i.e.*, S_8.

$$S_8 = \frac{a(1 - r^8)}{1 - r}$$

$$= 20,000\left\{\frac{1-\left(\frac{1}{1.2}\right)^8}{1-\frac{1}{1.2}}\right\}$$

$$= 1,20,000\left[1-\frac{1}{(1.2)^8}\right]$$

$$= 1,20,000(1–0.23)$$

= Rs. 92,400 nearly.

Net Present Value abbreviated as NPV is the sum of the present values of all cash flows associated with a project. The cash flows are normally tabulated in net terms per year and the standard presentation is shown in the following example.

Example A business project is being considered which has Rs12,000 initial costs, associated with revenues (*i.e.*, inflows) over the following four years of Rs8,000;Rs 12,000; Rs10,000 and Rs6,500, respectively. If the project costs (*i.e.*, outflows) over the four years are estimated as Rs8,500; Rs3,000; Rs1,500 and Rs1,500, respectively and the discount rate is 18.5%, evaluate the project's NPV.

Solution

Year	Cash Inflow (a)	Cash Outflow (b)	Net Cash Flow (a) – (b)	Discount Factor at 18.5%	Present Value
1	–	12000	(12000)	1.000 $(=\frac{1}{(1+0.185)^0})$	(12000)
2	8000	8500	(500)	0.8439$(=\frac{1}{(1+0.185)^1})$	(421.95)
3	12000	3000	9000	0.7121	6408.90
4	10000	1500	8500	0.6010	5108.50
5	6500	1500	5000	0.5071	2535.50
					Net Present Value= 1630.95

NPV can be interpreted *loosely* in the following way:

NPV > 0 $\longrightarrow$ project is in profit (*i.e.*, worthwhile)

NPV = 0 $\longrightarrow$ project breaks even, and

NPV < 0 $\longrightarrow$ project makes a loss (*i.e.*, not worthwhile)

6.6.4 Sinking Fund

A *sinking fund* can be defined as an annuity invested in order to meet a known commitment at some future date.

Sinking funds are commonly used for the following purposes:

a) repayment of debts.

b) to provide funds to purchase a new asset when the existing asset is fully depreciated.

For example, if Rs. 25,000 is borrowed over three years at 12% compound interest, the value of the outstanding debt at the end of the third year will be: Rs. $25{,}000(1.12)^3$ = Rs. 35123.20. If money can be invested at 9.5%, we need to find the value of the annuity$A$, which must be paid into the fund so that it matures to Rs. 35,123.20 in 3 years. Assuming that payments into the fund *are in arrears*, we need:

$$35{,}123.20 = A + A(1.095) + A(1.095)^2$$

Outstanding debt	3rd payment (*made at the end of year 3*)	2nd payment (*invested for 1 year*)	1st payment (*invested for 2 years*)

i.e., $35123.20 = A(1+1.095+1.095^2)$

$35123.20 = A(3.2940)$

Therefore, $A = \dfrac{35123.20}{3.2940}$

$= 10662.78$

Hence, the annual payment into the sinking fund is Rs10,662.78 (which will produce, at 9.5%, Rs35,123.20 at the end of 3 years).

Check Your Progress 3

1) What is the difference between simple and compound interest? Explain with the help of examples.

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2) What do you understand by compounding? In what way is the concept of present value related to discounting?

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3) The population of a country in 1985 was 50 crore. Calculate the population in the year 2000 if the compounded annual rate of increase is 2%.

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4) A departmental store advertises a good at Rs.700 deposit and three further equal annual payments of Rs.500. If the discount rate is 7.5%, calculate the present value of the good.

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5) A machine that costs Rs.1,00,000 is expected to have a life of 5 years and then a scrap value of Rs15,000. If expected net returns from the machine are: Rs.20,000— for year 1; Rs.50,000— for year 2; Rs.35,000—for year 3; Rs.35,000—for year 4; Rs.35,000—for year 5; and projects of this type are expected to return at least 18%, comment on whether the machine should be purchased.

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6) What do you understand by annuities? Explain what are sinking funds?

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6.7 LET US SUM UP

This unit, the sixth in the course, was concerned with a special type of function, called sequence, and a basic mathematical idea derived from sequences, called series. We saw that a sequence is a mapping fromthe set of natural numbers to a set of elements. The unit subsequently discussed the very important concept of series, which is derived by adding the various terms of the sequence.After this, the unit explained progressions, which play a very important role in

Economics. The unit discussed two very important progressions, namely arithmetic progressions and geometric progressions.

Following this, the unit went on to discuss the concept of convergence of a sequence. You came to know about conditions under which a sequence can be said to converge, and when we have a divergent sequence. Using this, the unit gave a rudimentary understanding of the concept of limits that you will study in the next unit. In the final part of the unit, the discussion moved to some applications of sequences and series in Economics. In particular, the unit discussed the applications of sequences, series and progressions to simple and compound interest, the idea of compounding and discounting, calculation of present values, and what sinking funds are.

6.8 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) 57

2) (i) 15; (ii) 1635

3) a_7= 246000; S_7 = 1386000

4) a_{11}= 4096; S_{20} = 4194300

5) Common ratio (r) = $\frac{7}{49} = \frac{1}{7} = \frac{\frac{1}{7}}{1} = \frac{1}{7}$; $S_{10} = \frac{343}{6}\left[1 - \left(\frac{1}{7}\right)^{10}\right]$

Check Your Progress 2

1) See section 6.5

2) See section 6.5.2

Check Your Progress 3

1) See section 6.6.1

2) See section 6.6.2

3) **Hint:** The population increasing at a geometric rate obeys the formula:

$P_n = P_0(1 + r)^n$

Where, P_n= Population at the end of the period, P_0 = Population at the beginning of the period, r = annual rate of increase, n = time period in years.

4) The present value of the cash flow in terms of three instalments

$$= 500\left\{1 + \frac{1}{(1+0.075)^1} + \frac{1}{(1+0.075)^2}\right\}$$

$$= 500\frac{\left[1-\left(\frac{1}{1.075}\right)^3\right]}{1 - \frac{1}{1.075}}$$

= Rs 1361.67 nearly

The present value of the total cash flow = 700 + 1361.67

= Rs 2061.67 nearly.

5)

Year	Cash Inflow (a)	Cash Outflow (b)	Net Cash Flow (a) – (b)	Discount Factor at 18.5%	Present Value
1	20000	–	20000	0.8474($=\frac{1}{(1+0.18)^1}$)	16948
2	50000	–	50000	0.7181($=\frac{1}{(1+0.18)^2}$)	35905
3	35000	–	35000	0.6086	21301
4	35000	–	35000	0.5158	18053
5	35000 + 15000 (Scrap value)	–	50000	0.4371	21855
				Present Value of cash inflows	114062

Net present value of the Machine Purchase =

Present Value of cash inflows – Initial cost of purchase

= 114062– 100000

= 14062

As, NPV > 0, machine should be purchased.

6) See section 6.6.4