BLOCK 3

DIFFERENTIATION

BLOCK 3 INTRODUCTION

The third block continues with, and builds upon, the concepts introduced in the first two blocks. The basic theme in this block is the analysis of economic change. To be more accurate, the theme is to see how the dependent variable in a function changes as a result of change in the independent variable. The branch of mathematics from which techniques are used to study such changes is called differential calculus, and the process is called differentiation and this is the title of this Block. This Block has four units. The first unit in this Block, unit 7, titled **Limits,** extends the discussion of a sequence and explains what is meant by a sequence converging to a value, called its limiting value. The limit is a value that a sequence approaches but never equals. Even functions have limits. The unit discusses left and right-hand limits, as well as computation of limits. The study of limits is important because differential calculus studies, at its simplest, the change in a variable y as a result of the change in another variable x, when the change in x is negligible (tends to zero). This is where limits are useful.

The second unit in this Block, unit 8, **Continuity,** discusses Continuity of a function of one variable, and provides applications of continuous and discontinuous functions. The unit also discusses a very important theorem intermediate- value theorem. This is a very important theorem in differential calculus. Continuity is a crucial property in a function for it to be differentiable. Unless a function is continuous, it cannot be differentiated. Of course, there are examples of functions in economics which are discontinuous and the unit discusses these as well.

The final two units of this Block, units 9 and 10, titled **First-Order Derivatives** and **Higher-Order Derivatives** respectively, deal with the central theme of this Block. You are familiarised with what differentiation is and how to derive the derivate of a function. Of course, the derivative can itself be differentiated further (this is called higher-order derivative, discussed in unit 10). The two units discuss various concepts like definition of a tangent line as the derivative of the function of the curve; definitions of derivative and differential; conditions of differentiability; rules of differentiation; Convexity concavity and quasi-concavity*;* Taylor Series formula and Mean-Value Theorem.

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UNIT 7 LIMITS*

Structure

7.0 OBJECTIVES

After going through this unit, you will be able to:

- explain the idea of limit in a sequence;
- define limit of a function;
- evaluate left- and right-hand limits of a function; and
- apply numerical, algebraic and graphic approaches to finding limits.

7.1 INTRODUCTION

You will see in the following units that the concept of a limit is fundamental to Calculus. Many Calculus concepts like the derivative, the integral, etc. make use of limits. To understand the basic idea behind limit you may have to remember calculation of average and instantaneous speeds. Suppose you are travelling from point A to point B. In order to compute your average speed from A to B, you simply take the ratio of the distance between points A and B and the time it takes to travel this distance. For example, let us take a function $s(t)$ that determines the position of the moving body at time 't'. Assume that at time t_0, the moving body is at point A [= $s(t_0)$], whereas at time t_1 (where $t_1 > t_0$), it is at point B [= $s(t_1)$]. Time spent in travelling between these points will be given by Δt (= $t_1 - t_0$), where Δstands for "change in". Distance between points will be given by $s(t_1) - s(t_0)$. Then, the average speed over the time interval $[t_0, t_1]$ is given by

$$\frac{s(t_1) - s(t_0)}{\Delta t} \qquad \ldots(1)$$

This may also be interpreted as the average rate of change of the position function $s(t)$ over the interval $[t_0, t_1]$.

The instantaneous speed is the speed of the moving body at an instant. As we make the time interval, given by Δt (= $t_1 - t_0$), shorter and shorter, we approach nearer and nearer to the speed of the moving body at the instant in time t, that

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is the instantaneous speed. This is exactly the application of the concept of limit. In the above function (1), notion of instantaneous speed would mean the limiting value of the function (1), as Δt gets closer and closer to 0.

That is, the instantaneous speed at time *t* will be given by,

$\lim_{\Delta t \to 0} \frac{s(t_1) - s(t_0)}{\Delta t}$, provided such a limit exists.

Reformulate such a situation and suppose that between A and B there lies a point C through which you need to pass. To compute instantaneous speed at C, you may try to compute the average speed from C to points close to C. In such a situation, the distance between these points and C is very small as well as the time taken to travel from them to C. Then when you look at the ratio, the value you get will be the instantaneous speed at C. This is the way policeman's radar computes the driver's speed at traffic points in your city.

The concept of a limit involves approaching a point or a value arbitrarily close and still never reaching it. Intuitively, this idea may not seem very appealing. However, understanding this concept is essential to the study of differential calculus, and therefore very important. We will first discuss the concept of the limit of a sequence and then go on to discuss the concept of the limit of a function. Moreover, we will also discuss the basic concepts used for arriving at the limit to a function.

7.2 LIMIT OF A SEQUENCE

The idea of a limit follows from the behaviour of a moving point that comes closer and closer to a fixed point. Roughly speaking, when a moving point approaches a fixed point and the distance between them becomes progressively smaller (but never actually vanishes), we say that the point is tending to a *limit*. Let us try to understand the concept by discussing the *limit of a sequence*. Consider the following two sequences of numbers:

(i) $S_1 = \{1,2,3,4,5\ldots.\}$ (ii) $S_2 = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots\}$

In the first sequence, beginning with the first number, each succeeding number is obtained by adding 1 to the preceding number. In this way, the numbers in this sequence go on increasing without showing any tendency to stabilise at a fixed value. In the second sequence, beginning with the first number, each succeeding number is obtained by adding 1 to both the numerator and the denominator. In this sequence also the numbers seem to go on increasing but with a difference; here the sequence gradually tends to stabilise at a fixed value, which is 1. To fix the idea better, let us write the second sequence of numbers in the decimal form (upto three-decimal point). The sequence then takes the following form:

$$S_2 = 0.888,\ \ 0.900,\ \ 0.909,\ \ 0.916,\ \ 0.923,\ \ 0.928,\ \ 0.933,\ \ 0.937,...$$

It is clear that each succeeding number of the sequence is closer to 1 than the preceding number. The numerical difference between the first number and 1 is 0.112 and that between the eighth number and 1 is just 0.063. In fact, if we consider the number $\frac{999}{1000}$, which is also a member of the sequence and occurs much later, the numerical difference between it and 1 is only 0.001. It seems that the sequence can continue endlessly and each succeeding number can become progressively as close to 1 as we want it to be. The numbers in the sequence thus appear to steadily approach 1 without ever attaining it. In this case, the sequence is said to '*tend to*' 1 and this 1 is called the *limit* of the sequence. In the above example, the sequence approaches 1 from below, that is, from numbers lower than 1. However, a sequence of numbers may approach a limit in other ways also. For example, it may approach a limit from above, that is, from numbers higher than the limit; or, it may converge to a limit by oscillating around it also. The important point to note is that for the existence of a limit, a sequence of numbers should have a definite tendency to approach a finite value without quite attaining it.

In general, a sequence may

i) increase or decrease indefinitely at a constant or increasing rate. For example,

$S = \{0, -1, -3, -8, -15...\}$

or, $S = \{1, 5, 10, 17, 26, ...\}$

ii) increase or decrease at a decreasing rate. For example,

$S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}...\}$

or, $S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}...\}$

iii) fluctuate with wider margin. For example,

$S = \{-1, 5, -7, 17, -31...\}$

iv) fluctuate without any tendency for an increase or decrease. For example, $S = \{-1, 1, -1, 1, -1...\}$

v) fluctuate with a decreasing margin. For example,

$S = \{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}...\}$

Sequences given above show characteristics on the basis of which we can categorize them into:

i) heading towards infinity $(\infty, -\infty)$;

ii) heading towards some finite number (positive, negative or zero);

iii) leading nowhere (not settling at any value).

These observations help us explain the limit of a sequence.

Let us consider a sequence of numbers $\{x_n\}_{n\geq 1}$. If the numbers get closer and closer to a number *L*, (*i.e.*, $x_n \approx L$), then we say that the sequence $\{x_n\}$ is convergent and has a limit equal to *L*.

We will write

$$\lim_{n\to\infty} x_n = L$$

or, $$\underset{n\to\infty}{Lt}\ x_n = L$$

or, $$x_n \to L \text{ when } n \to \infty.$$

When we say $n \to \infty$, we imply n is getting larger and larger. Of course we do not have $n = \infty$. But perhaps in the neighbourhood of ∞. If a sequence is not convergent, it is called *divergent*. We will discuss a bit more on the convergent sequence to formally define the limit of a sequence.

We have said above that a sequence $\{x_n\}_{n \geq 1}$ is convergent if there exists a number L such that the numbers x_n get closer and closer to L as n becomes larger. That is,

i) we want to see that $x_n \approx L$;

ii) make sure that $|x_n - L| < \varepsilon$, where ε is a small number greater than zero;

iii) for (ii) to happen, we have $N \geq 1$ such that for every $n \geq N$, we have $|x_n - L| \leq \varepsilon$.

Note that ε, a Greek symbol called 'Epsilon' represents an arbitrarily small positive quantity. It measures the error between the number x_n and the L. The integer N on the other hand, measures how fast the sequence gets closer to the limit.

Thus, we can say that the sequence $\{x_n\}_{n \geq 1}$ converges to the number L, if and only if, for every $\varepsilon > 0$, there exists $N \geq 1$ such that for every $n \geq N$, $|x_n - L| \leq \varepsilon$.

7.3 LIMIT OF A FUNCTION

We can now extend the concept of the limit of a sequence of numbers to the *limit of a function*.Let us consider a single-variable function $y = f(x)$. We may be interested to know, if x approaches (or tends to) a value say 'a'without ever attaining it (symbolically, $x \to a$), whether y also approaches (or tends to) a finite value say 'b' (*i.e.*$y \to b$). If indeed that is the behaviour of y, we say, *as x tends to a* (note that a need not be a finite value), *the limit of y is b*. Symbolically, $\lim_{x \to a} f(x) = b$

Thus, a limit L is the value that the function $f(x)$ approaches to (*i.e.,* $f(x) \to L$) as x approaches a ($i.e., x \to a$). It is symbolically written as $\lim_{x \to a} f(x) = L$.

Left and Right Hand Side Limits

There exists two ways by which x may approach a number 'a'. It may approach a either from values smaller than a*i.e.*, from the left hand side or from values greater than a*i.e.*, from the right hand side. When $x \to a$ from right hand side, y approaches a finite value say, L_1, we call L_1 the *right hand limit* of $f(x)$, denoted by $\lim_{x \to a^+} f(x)$;whereas when $x \to a$ from left hand side,y approaches a finite value say, L_2, we call L_2 the *left hand limit* of $f(x)$, denoted by $\lim_{x \to a^-} f(x)$. For example, consider the graph of a function $f(x)$ in Figure 7.1,

where $f(x) = \begin{cases} 2, & x \leq 0 \\ 4, & x > 0 \end{cases}$

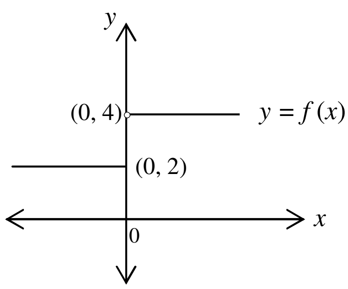

Figure 7.1

Here, it is clear that as $x \to 0$ from the left side, $y \to 2$. Thus, we get the left hand limit as 2, *i.e.*, $\lim_{x \to 0^-} f(x) = 2$. On the other hand, as $x \to 0$ from the right side, $y \to 4$, and thus we get the right hand limit as 4, *i.e.*, $\lim_{x \to 0^+} f(x) = 4$.

We just observed that, if y has a limit as x tends to a, that limit can be approached either when x tends to a from the left hand side or when it tends to a from the right hand side. Now a pertinent question arises? Does a given function have a limit or not? The answer to this question can be sought in terms of the following observations.

1) A function is said to have a limit if and only if the two limits (LHS and RHS) have a *common finite* value, *i.e.* they exist and are equal to each other.

2) If either of the limits is such that they equal to either $[+\infty]$ or $[-\infty]$, the function has *infinite limit and infinite limit is no limit.*

   Note: If the limit tends to either $+\infty$ or $-\infty$, take only one side limit as it will save time.

Thus, the condition for the existence of the limit of y as x tends to a is,

$$\lim_{x \to a^+} y = \lim_{x \to a^-} y = L \quad \text{(where } L \text{ is some finite number)}$$

With the above formulations we can say that, an increasing sequence with increasing x values leads to a left hand limit, on the other hand a decreasing sequence where x values are decreasing leads to a right hand limit.

Consider the various ways by which x may tend to a in following Figure 7.2 (a, b, c, d).

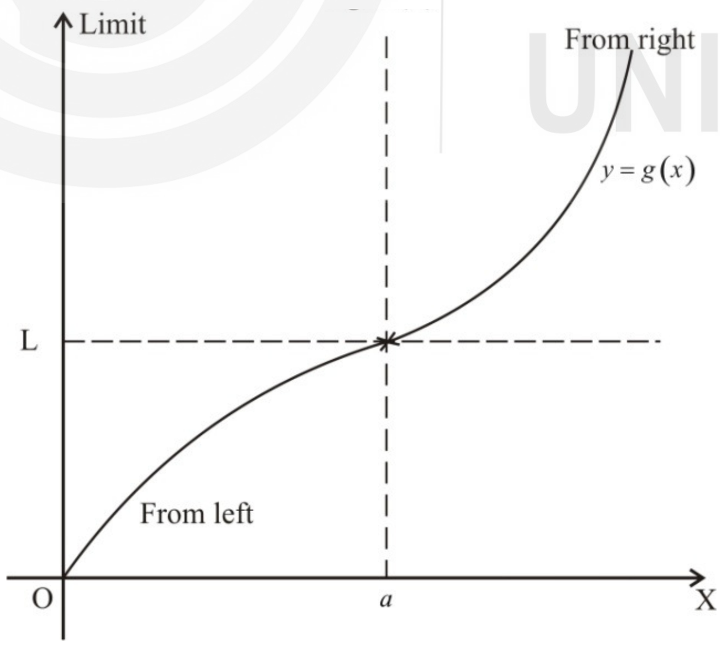

Figure 7.2 (*a*)

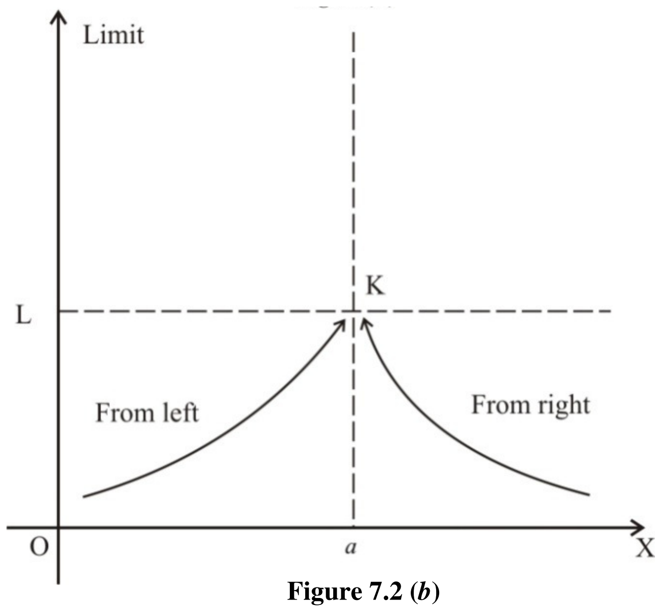

Figure 7.2 (*b*)

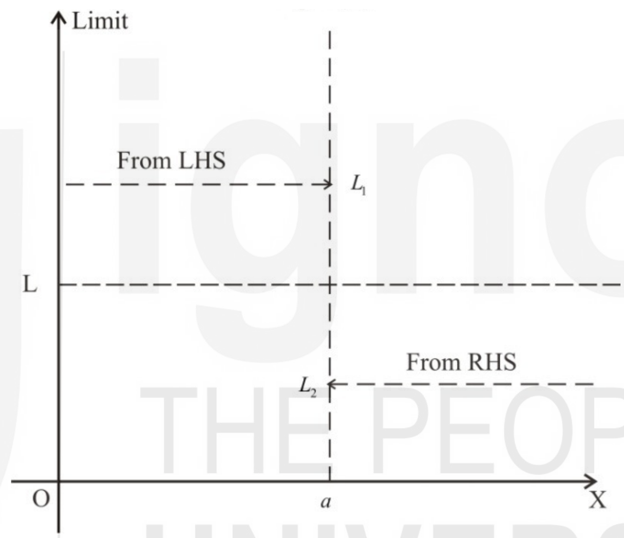

Figure 7.2 (*c*)

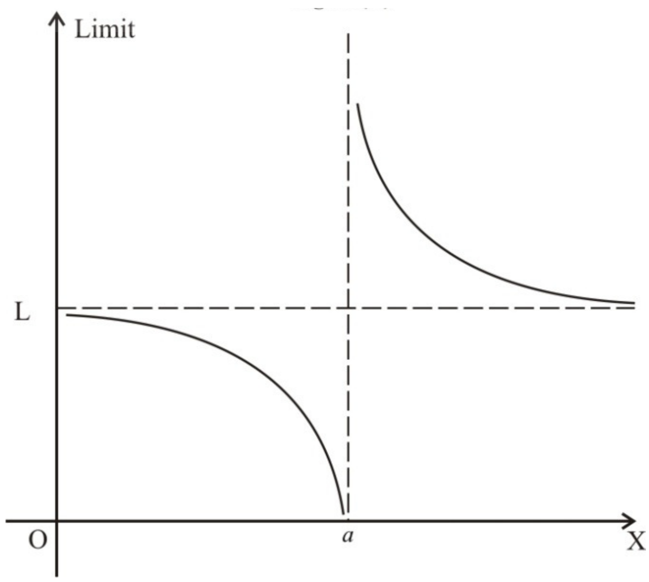

Figure 7.2 (*d*)

Part (a) of the diagram presents a smooth curve of the function $y = g(x)$. As variable x approaches the value a from either side on the horizontal axis, variable y approaches L on the vertical axis. Thus, both the left hand limit and the right hand limit exist and they are equal to each other. So in this case, as x tends to a, the limit of y exists and is equal to L.

The curve in Part (b) is not smooth. It has a sharp kink at point K directly above the point a on the horizontal axis. However, in this case also y approaches L as x approaches the value a from either side on the horizontal axis. Thus, as x tends to a, the limit of y again exists and is equal to L.

Part (c) is the diagram for a step function. Here we can see that as x approaches a from the left hand side, y approaches L_1, *i.e.,* the left hand limit of y is L_1. But, as x approaches a from the right hand side, y approaches L_2, *i.e.,* the right hand limit of y is L_2. In this case, although both the left hand and the right hand limits exist, they are not equal to each other. Hence, the condition for the existence of a limit is violated. Thus, as x tends to a, y does not tend to a limit here.

Finally, a rectangular hyperbola is shown in Part (d). Here, as x tends to a from the left hand side, y tends to $-\infty$ (minus infinity). On the other hand, when x tends to a from the right hand side, y tends to $+\infty$ (plus infinity). It is a hyperbolic curve with axis of symmetry being asymptotes. The curve has two branches, both falling and rising indefinitely. Here, both the right hand limit and the left hand limit do not exist. Therefore, in this case also y does not have a limit as x tends to a.

7.4 ALGEBRAIC APPROACH TO COMPUTATION OF LIMITS

We have studied the concept of a limit. Now we shall see the actual procedure of evaluating the limit of a function, with the help of certain rules that can be followed.

7.4.1 Rules for Evaluating a Limit

Limits allow themselves to be subjected to some algebraic manipulations such as addition, subtraction, multiplication and division. Consider the following rules based on two functions $y = f(x)$ and $y = g(x)$ such that $\underset{x \to a}{Lt}\ f(x) \to L_1$ and $\underset{x \to a}{Lt}\ g(x) \to L_2$

1) Limit of a constant (c) is the constant itself $\underset{x \to a}{Lt}\ c = c$

2) Limit of a sum and difference of two function is the sum and difference of their individual limits
$\underset{x \to a}{Lt}\left\{f(x) \pm g(x)\right\} = \underset{x \to a}{Lt}\ f(x) \pm \underset{x \to a}{Lt}\ g(x) = L_1 \pm L_2$

   Hence, limit of sum of a constant and a variable is the sum of their individual limits. $\underset{x \to a}{Lt}\left\{c + f(x)\right\} = \underset{x \to a}{Lt}\ c + \underset{x \to a}{Lt}\ f(x) = c + L_1$

3) Limit of product of two functions is the product of their individual limits.
$\underset{x \to a}{Lt}\left\{f(x) \cdot g(x)\right\} = \underset{x \to a}{Lt}\ f(x) \times \underset{x \to a}{Lt}\ g(x) = L_1 \times L_2$

Hence, limit of a product of a constant and a variable is constant times the limit of the variable $\underset{x \to a}{Lt}\ cf(x) = c\ \underset{x \to a}{Lt}\ f(x) = cL_1$

4) Limit of the quotient of two functions is the quotient of their individual limits, provided the limit of the function in the denominator is not zero.

$$\underset{x \to a}{Lt}\ \frac{f(x)}{g(x)} = \frac{\underset{x \to a}{Lt}\ f(x)}{\underset{x \to a}{Lt}\ g(x)} = \frac{L_1}{L_2}$$, provided $L_2 \neq 0$.

5) Limit of reciprocal of a function is the reciprocal of the limit of the function, provided the limit of the function is not *zero*.

$$\underset{x \to a}{Lt}\ \frac{1}{g(x)} = \frac{1}{\underset{x \to a}{Lt}\ g(x)} = \frac{1}{L_1}$$, provided $L_1 \neq 0$.

6) Limit of a root of a function times n is n times the root of the limit, provided the limit of the function is greater than or equal to *zero*.

$$\underset{x \to a}{Lt}\ n\sqrt{f(x)} = n \cdot \underset{x \to a}{Lt}\ \sqrt{f(x)} = n\sqrt{L_1}$$, provided $L_1 \geq 0$.

7.4.2 Some Standard Limits

You will find these forms useful in your subsequent study of this course, so try to learn these well. There proofs are not required as per the scope of this unit.

1) $\underset{x \to a}{Lt}\ \frac{x^n - a^n}{x - a} = na^{n-1}, a > 0$

2) $\underset{x \to \infty}{Lt}\ \left(1 + \frac{1}{x}\right)^x = e = 2.71828$

3) $\underset{x \to 0}{Lt}\ [1 + x]^{1/x} = e = 2.71828$

4) $Lt_{x \to 0} \frac{a^x - 1}{x} = \log_e a,\ a > 0$ [Note: $\log_e a$ is also denoted as '*ln a*']

5) $Lt_{x \to 0} \frac{e^x - 1}{x} = \log_e e = 1$

7.4.3 Finite and Infinite Limits

Limit whose values tends to a finite number are called finite limits. For example, $\underset{x \to 2}{Lt}(5x + 7) = 17$ where 17 is a finite number. On the other hand, limits whose values are not finite but tend to an infinite number such as $+\infty$ or $-\infty$ are infinite limits. For example,

$\underset{x \to 0}{Lt}\ \frac{5}{x^2} = \frac{5}{(0)^2} = \frac{5}{0} = \infty$ is an infinite limit. Also, keep in mind that an infinite limit is not considered as a limit.

Example1: Given the function $y = x^2 + 1$, find $\underset{x \to 0}{Lim}\ y$

Solution: To obtain the left hand limit, let us substitute the negative numbers $-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4} \ldots$ one by one for x. We find that x^2 (a positive number) becomes smaller and smaller and approaches zero. As a result, x^2+1 steadily falls and tends to 1. Thus, the left hand limit of the function is 1. Next, we obtain the right hand limit by substituting the positive numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$…. We find that x^2 (again a positive number) becomes smaller and smaller and approaches zero. As a result, x^2+1 steadily falls and tends to 1. Therefore, the right hand limit of the function is also 1. Thus, the two limits are equal. Hence, the limit exists and we write, $\underset{x \to 0}{Lim}\ y = 1$.

Example 2: Find $\underset{x \to 3}{Lim}\ \dfrac{x^3 - 2x^2 + 2x - 6}{5x^2 - 13x + 3}$

Solution: We shall solve this question by applying various limit theorems.

$$\underset{x \to 3}{Lim}\ \frac{x^3 - 2x^2 + 2x - 6}{5x^2 - 13x + 3}$$

$$= \frac{\underset{x \to 3}{Lim}\left(x^3 - 2x^2 + 2x - 6\right)}{\underset{x \to 3}{Lim}\left(5x^2 - 13x + 3\right)}$$

$$= \frac{\underset{x \to 3}{Lim}\left(x^3\right) - \underset{x \to 3}{Lim}\left(2x^2\right) + \underset{x \to 3}{Lim}\left(2x\right) - \underset{x \to 3}{Lim}\left(6\right)}{\underset{x \to 3}{Lim}\left(5x^2\right) - \underset{x \to 3}{Lim}\left(13x\right) + \underset{x \to 3}{Lim}\left(3\right)}$$

$$= \frac{\underset{x \to 3}{Lim}\ x^3 - 2\ \underset{x \to 3}{Lim}\ x^2 + 2\ \underset{x \to 3}{Lim}\ x - \underset{x \to 3}{Lim}\ 6}{5\ \underset{x \to 3}{Lim}\ x^2 - 13\ \underset{x \to 3}{Lim}\ x + \underset{x \to 3}{Lim}\ 3}$$

$$= \frac{27 - 2(9) + 2(3) - 6}{5(9) - 13(3) + 3}$$

$$= 1$$

Example 3: A man invests Rs. 1000 for 2 years at an annual rate of interest of 5 per cent compounded continuously. Determine the amount that the man will get after 2 years. What is the effective rate of interest for the investment?

Solution: To solve this problem, let us discuss the compound interest formula a bit. We know that if an initial amount of *P* is invested for 1 year at a rate of interest of 100 *r*% compounded annually, the final amount *A* that can be obtained at end of 1 year is given by *A*= *P* (1 + *r*). If the same annual rate of interest is compounded *m* times a year, the amount that can be obtained at the end of 1 year is given by $A = P\left(1 + \frac{r}{m}\right)^m$. Here, the expression $\frac{r}{m}$ signifies that for an annual rate of interest of 100 *r*%, a rate of only $\frac{100r}{m}$ % will be applicable for each compounding period. The exponent *m* in the formula denotes that there will be *m* compoundings in one year. Now, with the same annual compounding frequency of *m*, the amount that can be obtained after t years is $A = P\left(1+\frac{r}{m}\right)^{mt}$, where the exponent mt denotes the number of compoundings in t years. We can also write

$$A = P\left(1+\frac{r}{m}\right)^{mt} = P\left[\left(1+\frac{r}{m}\right)^{m/r}\right]^{rt} = P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt}, \text{where } n = \frac{m}{r}$$

. It is clear that when $m \to \infty$, n also $\to \infty$.

In the given question, interest is compounded continuously. As a result, the compounding frequency m approaches infinity. Hence, the required amount is the limit of $P\left(1+\frac{r}{m}\right)^{mt}$ as $m \to \infty$. This limit is

$$A = \lim_{m\to\infty} P\left(1+\frac{r}{m}\right)^{mt} = \lim_{m\to\infty} P\left[\left(1+\frac{r}{m}\right)^{m/r}\right]^{rt} = \lim_{n\to\infty} P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt}.$$

The limit of

$P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt}$ as $n \to \infty$ depends upon the limit of $\left(1+\frac{1}{n}\right)^{n}$ as $n \to \infty$

, that is, $\lim_{n\to\infty} P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt} = P\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{rt}$. From the standard formulae, we have $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$. Therefore,

$$A = P\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{rt} = Pe^{rt}$$

We have, P = Rs 1000,

$$r = \frac{\text{percentage rate of interest}}{100} = \frac{5}{100} = 0.05 \text{ and } t = 2 \text{ years}$$

Therefore, A = 1000 ($e^{0.05(2)}$) = 1000 $(2.71828)^{0.1}$ [Because, we have e = 2.71828..]

Taking common log on both the sides

$$\begin{aligned} \log A &= \log 1000 + 0.1 \times \log 2.71828 \\ &= 3 + 0.1 \times 0.4343 \\ &= 3.04343 \end{aligned}$$

Taking antilog on both the sides

A = antilog 3.04343

= 1105 (approximately)

Thus, the required amount is Rs. 1105.

The quoted annual rate of interest that is offered on an investment is called the *nominal rate*. However, the *annual rate* at which a given sum of money *actually* grows depends upon the frequency of compounding in a year. This rate is called the *effective rate of interest*. Thus, the effective rate is the equivalent annual compounding interest rate of a quoted rate that is compounded a given number of times (say, m) in a year. Now, if an amount A is due on a sum P after t years at a rate of interest 100 $r\%$ compounded continuously, we can write

$$A = Pe^{rt} \tag{1}$$

If 100 $i\%$ is the effective rate of interest, we can also write

$$A = P(1+i)^t \tag{2}$$

Equating (1) and (2)

$$(1+i)^t = e^{rt}$$

$$\text{or } 1+i = e^r \tag{3}$$

Equation (3) can be solved for the effective rate of interest 100 $i\%$.

We have $r = 0.05$, putting this value in (3)

$$1+i = e^{0.05}$$

Taking common log on both the sides

$$\log(1+i) = 0.05 \times \log 2.71828$$
$$= 0.021715$$

Taking antilog on both the sides

$$1+i = \text{ antilog}(0.021715)$$
$$\text{or } 1+i = 1.052$$
$$\text{or } i = 0.052$$

Thus, the effective rate of interest is 5.2%.

Check Your Progress 1

1) Evaluate the following limits

a) $\underset{x \to -2}{Lt}\left(x^2 + 5x\right)$

b) $\underset{x \to 0}{Lt}\ \dfrac{3x+1}{5x-1}$

c) $\underset{x \to 4}{Lt}\ \dfrac{2x^{3/2} - \sqrt{x}}{x^2 - 15}$

d) $\underset{x \to a}{Lt}\ Ax^n$

e) $\underset{x \to 1}{Lim}\ \dfrac{4x^2 - 3}{2x^2 + 1}$

f) $lim_{x\to -1} \frac{x^2+x-6}{x+2}$

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2) The total cost of production for a firm is given by $C = 4q + 2$. Does average cost approach a finite value when the output tends to infinity? If yes, what is that value?

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3) A trader has borrowed a certain sum of money from a moneylender. He has promised to pay Rs. 5000 at the end of 4 years. How much has the trader borrowed if interest is compounded continuously at 10 per cent per annum? What is the effective rate of interest that will be paid by him?

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In the above discussion we encountered L (the limit of a function) as a finite number. However, it could be infinite as well. In such a situation, we need to reconsider the left and right limits formulations given above.

If we have a situation of $\lim_{x\to N} f(x) = \infty$ (or $-\infty$), then the function does not have a limit in real sense of the term. As you may observe $f(x) \to \infty$ implies the function is ever increasing in terms of its values, hence without a limit point. In case of ∞ values, we also have to forget about deriving left and right hand limits seen above and work with one side limit. For example, if you are evaluating the limit of $f(x)$ as $x \to \infty^-$, only left hand limit of $f(x)$ is relevant. If you are having $x \to \infty^+$, from which higher point of ∞ is seen, you start to evaluate the right hand limit. Remember that in case of infinity, we do not have the benefit of usual arithmetic operations. For example, we have $(\infty + 1) = \infty$ or $(\infty - 1) = \infty$. A similar logic applies to $-\infty$ and consequently we do not have a lower value than $-\infty$ to evaluate a left hand limit.

Before looking into a few examples on limit of a function, it will be useful to note that while estimating a limit, we are interested only with the values of the function $f(x)$ as x takes values closer and closer to say, N, but not when $x = N$. If you are evaluating a function, $f(x) = \frac{x^2 - 1}{x - 1}$ for example, you cannot define it for $x = 1$, as the denominator becomes zero. However, the function can be defined for all other values of $x \in R$. The question in this problem before us, therefore, could be, what is its limit as $x \to 1$. See that we write x tends to 1 but not $x = 1$. We will find the limit value of the function in such a case as well. Please note that direct substitution of the limiting number into a function does not help getting the appropriate solution. So we have to be careful while evaluating such functions.

Now let us specify cases where the limits are either zero or infinity or become indeterminate and no general conclusion is possible.

1) When $Lt\, f(x) = A, Lt\, g(x) = \infty$, then $Lt\, f(x) + Lt\, g(x) = A + \infty = \infty$ (an indeterminate form)

2) When $Lt\, f(x) = A, Lt\, g(x) = +\infty$, then $Lt\, f(x) - Lt\, g(x) = A - \infty = -\infty$ (an indeterminate form)

3) When $Lt\, f(x) = \infty, Lt\, g(x) = \infty$, then $Lt\, f(x) + Lt\, g(x) = \infty + \infty = \infty$

4) When $Lt\, f(x) = \infty, Lt\, g(x) = +\infty$, then $Lt\, f(x) - Lt\, g(x) = \infty - \infty$. It is an *indeterminate case* and no general conclusion is possible.

5) When $Lt\, f(x) = A, Lt\, g(x) = 0$, then $\frac{Lt\, f(x)}{Lt\, g(x)} = \frac{A}{0} = \infty$ (Infinite Lt)

6) When $Lt\, f(x) = 0, Lt\, g(x) = 0$, then $\frac{Lt\, f(x)}{Lt\, g(x)} = \frac{0}{0}$. It is an *indeterminate case* and no general conclusion is possible.

7) When $Lt\, f(x) = A,\ Lt\, g(x) = \infty$, then $\frac{Lt\, f(x)}{Lt\, g(x)} = \frac{\infty}{\infty}$

8) When $Lt\, f(x) = \infty,\ Lt\, g(x) = \infty$, then $\frac{Lt\, f(x)}{Lt\, g(x)} = \frac{\infty}{\infty}$. It is an *indeterminate case* and no general conclusion is possible.

9) If $Lt\ f(x) = A, Lt\ g(x) = \infty$, then $Lt\ f(x) \times Lt\ g(x) = A \times \infty = \infty$

10) If $Lt\ f(x) = \infty, Lt\ g(x) = \infty$, then $Lt\ f(x) \times Lt\ g(x) = \infty \times \infty = \infty$

Some Rules for Limits to treat Indeterminate forms

Algebraically, the technique for evaluating limits involves certain operations such as

i) Factorisation
ii) Rationalisation
iii) Simplification
iv) Other types of manipulations

The aim of these operations is to avoid indeterminate cases such as

i) $0 \times \infty$

ii) $\frac{0}{0}$

iii) $\frac{\infty}{\infty}$

iv) $\infty - \infty$

v) $|0|+|\infty|$

Let us take some examples requiring above mentioned operations.

Example 4: Find $\lim_{x\to 1} f(x)$, where $f\ (x) = \frac{x^2-1}{x-1}$the limit of the function .

Solution: We know that for $x=1$, we cannot define the function. So simplify the function first. We write

$f(x) = \frac{(x+1)(x-1)}{(x-1)} = (x+1)$. Now, we have to find

$\lim_{x\to 1}(x+1)$. Inserting $x=1$ in the function $(x + 1)$, we get the limit as $(1+1)=2$.

Example 5: Find $\lim_{x\to\infty} f(x)$, where $f\ (x) = \frac{x}{1+x}$.

Solution: If you equate x with ∞, then $f(x)$ becomes $\frac{\infty}{\infty}$, which you cannot evaluate. So try to simplify the function. If you divide both numerator and denominator by x, then you get $\frac{1}{({}^{1}/_{x})+1}$. Now substitute the value of x as ∞. You will find that

$$f(x) = \frac{1}{(1/\infty)+1} = \frac{1}{(0+1)} = \frac{1}{1} = 1.$$

Hence, $\lim_{x\to\infty} \frac{x}{1+x} = 1$

Example 6: Find $\lim_{x\to\infty} f(x)$, where $f\ (x) = \frac{1+2x}{1+3x}$.

Solution: Direct substitution of $x=\infty$, makes the function indeterminate. So, simplify it first. Apply the trick we have adopted in the proceeding example and divide the numerator and denominator by x. Then substitute the value of x as ∞. You will find the limit to be 2/3.

Example 7: Find $\lim_{x\to 0} f(x)$, where $f\ (x) = \frac{[(x+1)^2-1]}{x}$.

Solution: Here again direct substitution will not yield the limit. So you need to simplify the function. Just write,

$$\frac{\left[(x+1)^2-1\right]}{x} \text{ as } \frac{1+x^2+2x-1}{x} = \frac{x(x+2)}{x} = (x+2)\ .$$

Then we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} (x+2)$. Direct substitution of $x=0$ will result in $\lim_{x\to 0} f(x) = 2$.

Example 8: Examine the behaviour of the following functions when (a) $x\to\infty$; (b) When $x\to-\infty$

(i) $\dfrac{5x^2+x+1}{x^2+2}$ (ii) $\dfrac{1-x^5}{x^4+x+1}$

Solution:

(i) $\underset{x\to\infty}{Lt} \dfrac{5x^2+x+1}{x^2+2} = \dfrac{5(\infty)^2+\infty+1}{(\infty)^2+2} = \dfrac{\infty}{\infty}$ (Indeterminate)

Therefore this limit requires the algebraic operation wherein both numerator as well as denominator will be divided by the highest power of x, that is x^2 in this case.

$$\therefore \underset{x\to\infty}{Lt} \frac{5x^2+x+1}{x^2+2} = \underset{x\to\infty}{Lt} \frac{\frac{5x^2}{x^2}+\frac{x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{2}{x^2}} = \underset{x\to\infty}{Lt} \frac{5+\frac{1}{x}+\frac{1}{x^2}}{1+\frac{2}{x^2}}$$

Now put $x=\infty$, to get $\dfrac{5+\frac{1}{\infty}+\frac{1}{\infty^2}}{1+\frac{2}{\infty^2}} = \dfrac{5+0+0}{1+0} = \dfrac{5}{1} = 5$

Similarly, $\underset{x\to-\infty}{Lt} \dfrac{5x^2+x+1}{x^2+2} = \dfrac{5+\frac{1}{-\infty}+\frac{1}{(-\infty)^2}}{1+\frac{2}{(-\infty)^2}} = \dfrac{5-0+0}{1+0} = \dfrac{5}{1}$

(ii) $\underset{x\to\infty}{Lt} \dfrac{1-x^5}{x^4+x+1}$ $= \underset{x\to\infty}{Lt} \dfrac{\frac{1}{x^5}-1}{\frac{1}{x}+\frac{1}{x^4}+\frac{1}{x^5}} = \dfrac{\frac{1}{(\infty)^2}-1}{\frac{1}{\infty}+\frac{1}{(\infty)^4}+\frac{1}{(\infty)^5}}$

$$= \frac{0-1}{0+0+0} = \frac{-1}{0} = -\infty$$

For $x\to-\infty$, we will get the same answer. The students may try this themselves.

Example 9: Evaluate $\underset{x\to 0}{Lt} \dfrac{1/x^2}{1/x^4}$.

Solution: When we put $x=0$ in the given expression, we get the value $\dfrac{1/0}{1/0} = \dfrac{\infty}{\infty}$, which is an indeterminate case. It requires some algebraic operation to simplify the expression and then put $x=0$:

$$\underset{x\to0}{Lt}\ \frac{1/x^2}{1/x^4} = \underset{x\to0}{Lt}\ \frac{1}{x^2}\times\frac{x^4}{1}\quad \underset{x\to0}{Lt}\ \left(x^2\right)=(0)^2=0$$

Example 10: If $f(x)=3+\dfrac{1}{x}$ and $g(x)=5-\dfrac{1}{x}$, examine the limits as $x\to0$ of $f(x)+g(x)$

Solution: $\underset{x\to0}{Lt}\ f(x)\quad g(x)\qquad \underset{x\to0}{Lt}\ 3\ \ \dfrac{1}{x}\qquad 5\ \ \dfrac{1}{x}$

$=\underset{x\to0}{Lt}\ 3+\dfrac{1}{x}\ +\underset{x\to0}{Lt}\ 5-\dfrac{1}{x}\ =\infty+\infty$ case. But a little careful observation will reveal that we can avoid this situation by just opening the brackets.

That is, $Lt_{x\to0}\left[3+\frac{1}{x}+5-\frac{1}{x}\right]=\underset{x\to0}{Lt}\ 3+\cancel{\frac{1}{x}}+5-\cancel{\frac{1}{x}}\ =\underset{x\to0}{Lt}\ 8=8$.

Hence the limit is 8.

Example 11: Evaluate the following limits

a) $\underset{x\to1}{Lt}\ \dfrac{x-1}{2x^2-7x+5}$ b) $\underset{x\to1}{Lt}\ \dfrac{x^4-3x^3+2}{x^3-5x^2+3x+1}$

Solution: a) If we put $x=1$ in the given expression, then

$\underset{x\to1}{Lt}\ \dfrac{x-1}{2x^2-7x+5}$ becomes $\dfrac{1-1}{2(1)^2-7(1)+5}=\dfrac{0}{0}$ (Indeterminate)

Therefore, it requires some algebraic operation. Let us try factorisation method where $2x^2-7x+5=(x-1)(2x-5)$

$\therefore$ we get, $\underset{x\to1}{Lt}\ \dfrac{(x-1)}{(x-1)(2x-5)}=\underset{x\to1}{Lt}\ \dfrac{1}{(2x-5)}=\dfrac{1}{2(1)-5}=\dfrac{-1}{3}$

b) If we put $x=1$ in the expression $\dfrac{x^4-3x^3+2}{x^3-5x^2+3x+1}$

We get $\dfrac{(1)^4-3(1)^3+2}{(1)^3-5(1)^2+3(1)+1}=\dfrac{1-3+2}{1-5+3+1}=\dfrac{0}{0}$ (Indeterminate)

Therefore, it requires some algebraic operation. Here, we factorise both numerator as well as denominator, to get

$$\underset{x\to1}{Lt}\ \frac{x^4-3x^3+2}{x^3-5x^2+3x+1}=\underset{x\to1}{Lt}\ \frac{(x-1)\left(x^3-2x^2-2x-2\right)}{(x-1)\left(x^2-4x-1\right)}$$

(**Hint:** Given a function $f(x)$ and $x=a$ as one of its roots, that is, $f(a)=0$, then $(x-a)$ will be one of the factors of $f(x)$. On dividing $f(x)$ with $(x - a)$ by way of long division, other factors can be ascertained.)

$$= \underset{x \to 1}{Lt} \; \frac{x^3 - 2x^2 - 2x - 2}{x^2 - 4x - 1}$$

$$= \frac{(1)^3 - 2(1)^2 - 2(1) - 2}{(1)^2 - 4(1) - 1} = \frac{1 - 2 - 2 - 2}{1 - 4 - 1} = \frac{-5}{-4} = \frac{5}{4}$$

Example 12: Find the limit of the following

(a) $\underset{x \to 0}{Lt} \frac{\sqrt{2-x} - \sqrt{2+x}}{x}$ and (b) $\underset{x \to \infty}{Lim}\left(\sqrt{x^2 + 5x + 4} - \sqrt{x^2 - 3x + 4}\right)$

Solution: (a) This question involves irrational expression and, therefore, the algebraic operation here is rationalisation.

$$\therefore \quad \underset{x \to 0}{Lt} \; \frac{\sqrt{2-x} - \sqrt{2+x}}{x} = \frac{\sqrt{2} - \sqrt{2}}{0} = \frac{0}{0}$$ (Indeterminate case)

On rationalization expression becomes

$$= \frac{\sqrt{2-x} - \sqrt{2+x} \quad \sqrt{2-x} + \sqrt{2+x}}{x\left(\sqrt{2-x} + \sqrt{2+x}\right)}$$

$$= \frac{2 - x - 2 - x}{x\left(\sqrt{2-x} + \sqrt{2+x}\right)} = \frac{-2x}{x\left(\sqrt{2-x} + \sqrt{2+x}\right)} = \frac{-2}{\sqrt{2-x} + \sqrt{2+x}}$$

Now take the limit to get

$$\underset{x \to 0}{Lt} \frac{-2}{\sqrt{2-x} + \sqrt{2+x}} = \frac{-2}{\sqrt{2-0} + \sqrt{2+0}} = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

(b) Before rationalization we get

$$\underset{x \to \infty}{Lt} \sqrt{x^2 + 5x + 4} - \sqrt{x^2 - 3x + 4} = \infty - \infty$$ (Indeterminate case)

Therefore we rationalise and make use of the formulae $(a+b)(a-b) = a^2 - b^2$

$$= \underset{x \to \infty}{Lt} \frac{\left[\sqrt{(x^2 + 5x + 4)} - \sqrt{(x^2 - 3x + 4)}\right]\left[\sqrt{(x^2 + 5x + 4)} + \sqrt{(x^2 - 3x + 4)}\right]}{\left[\sqrt{(x^2 + 5x + 4)} + \sqrt{(x^2 - 3x + 4)}\right]}$$

$$= \underset{x \to \infty}{Lt} \frac{x^2 + 5x + 4 - x^2 + 3x - 4}{\left[\sqrt{(x^2 + 5x + 4)} + \sqrt{(x^2 - 3x + 4)}\right]}$$

$$= \underset{x \to \infty}{Lt} \frac{8x}{\left[\sqrt{(x^2+5x+4)} + \sqrt{(x^2-3x+4)}\right]}$$

(Dividing numerator and denominator by x, and under the root x becomes x^2)

$$= \underset{x \to \infty}{Lt} \frac{8}{\left[\sqrt{\left(\frac{x^2}{x^2}+\frac{5x}{x^2}+\frac{4}{x^2}\right)} + \sqrt{\left(\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{4}{x^2}\right)}\right]}$$

$$= \underset{x \to \infty}{Lt} \frac{8}{\left[\sqrt{\left(1+\frac{5}{x}+\frac{4}{x^2}\right)} + \sqrt{\left(1-\frac{3}{x}+\frac{4}{x^2}\right)}\right]}$$

$$= \frac{8}{\sqrt{1+0+0}+\sqrt{1-0-0}} = \frac{8}{1+1} = 4$$

Example 13: Evaluate a) $\underset{x \to \infty}{Lt} \frac{x+1}{3x-1} + \frac{2x+1}{x-1}$ and b) $Lt_{x\to\infty}\left(\frac{x+1}{3x-1}\right)\left(\frac{2x+1}{x-1}\right)$

(a) $\underset{x \to \infty}{Lt} \frac{x+1}{3x-1} + \frac{2x+1}{x-1} = \underset{x \to \infty}{Lt} \frac{x+1}{3x-1} + \underset{x \to \infty}{Lt} \frac{2x+1}{x-1}$

$$\underset{x \to \infty}{Lt} \frac{1+\frac{1}{x}}{3-\frac{1}{x}} + \underset{x \to \infty}{Lt} \frac{2+\frac{1}{x}}{1-\frac{1}{x}} = \frac{1+\frac{1}{\infty}}{3-\frac{1}{\infty}} + \frac{2+\frac{1}{\infty}}{1-\frac{1}{\infty}} = \frac{1+0}{3-0} + \frac{2+0}{1-0}$$

$$= \frac{1}{3} + 2 = \frac{7}{3}$$

(b) $\underset{x \to \infty}{Lt} \frac{x+1}{3x-1} \quad \frac{2x+1}{x-1} = \underset{x \to \infty}{Lt} \frac{x+1}{3x-1} \cdot \underset{x \to \infty}{Lt} \frac{2x+1}{x-1}$

$= \frac{1}{3} \times 2 = \frac{2}{3}$ [see part (a)]

Example 14: Find the limit of $\frac{x^m - a^m}{x^n - a^n}$ when $x \to a$.

Solution: When we put $x = a$ in the given expression, we get $\frac{0}{0}$ an indeterminate case. Also we have a standard form as $\underset{x \to a}{Lt} \frac{x^n - a^n}{x - a} = n\,a^{n-1}$. Now express an expression in this form. That is,

$$= \underset{x \to a}{Lt} \frac{x^m - a^m}{x^n - a^n} = \underset{x \to a}{Lt} \left[\frac{x^m - a^m}{x - a} \times \frac{x - a}{x^n - a^n} \right]$$

$$= \underset{x \to a}{Lt} \left[\frac{x^m - a^m}{x - a} \times \frac{1}{\frac{x^n - a^n}{x - a}} \right]$$

$$= \underset{x \to a}{Lt} \left[\frac{x^m - a^m}{x - a} \div \frac{x^n - a^n}{x - a} \right] = m a^{m-1} \div n a^{n-1} = \frac{m}{n} a^{m-n}$$

Check Your Progress 2

1) Given the function: $y = \frac{x^2 + x - 56}{x - 7}$, $(x \neq 7)$, find the left hand limit and the right hand limit of y as x approaches 7. Can we conclude from these answers that y has a limit as x approaches 7?

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2) Given the function, $y = \frac{x^3 - 6x^2 + 12x - 8}{x^2 - 4x + 4}$, $(x \neq 2)$, find the limit of y as x approaches 2.

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3) Find(a) $\underset{x \to 3}{Lim} \frac{x^2 - 7x + 12}{x - 3}$ (b) $\underset{x \to 0}{Lim} \frac{\sqrt{2 + 3x} - \sqrt{2 - 5x}}{4x}$

(c) $\underset{x \to \infty}{Lim} \frac{2x^3 + 3x^2 + 5}{5x^3 + 8x - 17}$

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4) Consider a demand function $p = \frac{5}{q + 2}$. What is the total revenue when the quantity demanded approaches infinity?

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L' Hospital's Rule

When two differentiable functions are such that their quotient $\frac{f(x)}{g(x)}$ approaches indeterminate cases ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) as x tends to some value, say, a, then the L' Hospital's Rule implies that:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} \text{ and so on.}$$

This means:

i) If $\underset{x\to a}{Lt}$ of $\frac{f(x)}{g(x)}$ does not exist, then we can take the limit of quotient of their derivatives.

ii) If $\underset{x\to a}{Lt}\ \frac{f'(x)}{g'(x)}$ is also indeterminate, then use can find limit of quotient of their second order derivative and so on; the process can be continued.

Note: i) There are several types of indeterminate forms, a few of which are:

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 0^0, \infty^\infty$, and 1^∞

But, L' Hospital's Rule only applies directly to indeterminate forms of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The other types of indeterminate forms must first be reduced to one of these two types if we wish to apply L' Hospital's Rule.

ii) L' Hospital's Rule also applies to one-sided limits and limits where $x \to \pm\infty$.

Example 15

Evaluate the following limits using L' Hospital's Rule.

a) $\underset{x\to a}{Lt}\ \frac{x^4 - a^4}{x - a}$ b) $\underset{x\to 0^+}{Lt}\ \frac{a^x - b^x}{x}$ c) $Lt_{x\to 0^+}\, x^x$

Solution

a) $\underset{x\to a}{Lt}\ \frac{x^4 - a^4}{x - a}$ is of standard form $\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$

Here, $n = 4$, therefore, the limit is $4a^{4-1} = 4a^3$

b) $\underset{x\to 0^+}{Lt}\ \frac{a^x - b^x}{x} = \frac{0}{0}$.

Therefore, in order to apply L' Hospital's rule we first find

$\frac{d}{dx}\ \frac{a^x - b^x}{x}$ which is a standard form and equals

$\left(a^x \log a - b^x \log b\right)$.

Now using L' Hospital's Rule, we get

$$\underset{x\to 0^+}{Lt}\ \frac{a^x-b^x}{x} = \underset{x\to 0^+}{Lt}\ \frac{d}{dx}\ \frac{a^x-b^x}{x} = \underset{x\to 0^+}{Lt}\left(a^x\log a-b^x\log b\right)$$

$$= a^0\log a - b^0\log b = \log a - \log b = \log\ \frac{a}{b}\ .$$

c) Notice that $\mathrm{Lt}_{x\to 0^+}\, x^x$ is an indeterminate form of type 0^0. This can be simplified by writing

$$x^x = e^{lnx^x} = e^{xlnx} = xlnx$$

Now, $$Lt_{x\to 0^+}\, x^x = Lt_{x\to 0^+}\, xlnx$$

$$\underset{x\to 0^+}{Lt}\, xlnx = \underset{x\to 0^+}{Lt}\, \frac{lnx}{\frac{1}{x}} = \frac{\infty}{\infty}\text{type}$$

Now, we can apply the L'Hospital's rule:

$$\underset{x\to 0^+}{Lt}\, \frac{lnx}{\frac{1}{x}} = \underset{x\to 0^+}{Lt}\, \frac{\frac{1}{x}}{\frac{1}{x^2}} = \underset{x\to 0^+}{Lt}(-x) = 0$$

7.5 LET US SUM UP

This unit discussed the very important concept of limits of a function. We saw that the concept of a limit has to do with "approaching a value" eventually as some other quantity or variable 'tends to' a value. The unit discussed first the basic notion of a limit intuitively and subsequently provided a rigorous definition. Following this the unit took up for discussion the limit of a sequence, and that of a function.

The unit also made a detailed discussion of left-hand and right-hand limits. Following this, the unit discussed some important properties of limits, the rules that are followed to arrive at the limit of a function, including the very useful L'Hopital's rule. You were also familiarised with some standard limits. On the whole, with the help of various examples, the learner has been introduced to various techniques of approaching the limit of a function. Not only to the finite extent, but also the infinite extent of approaching to the limit of a function has been touched.

7.6 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1)

a) $$\underset{x\to -2}{Lt}\left(x^2+5x\right) = \underset{x\to -2}{Lt}\, x^2 + 5\underset{x\to -2}{Lt}\, x = (-2)^2+5(-2) = 4-10 = -6$$

b) $$\underset{x\to 0}{Lt}\ \frac{3x+1}{5x-1} = \frac{\underset{x\to 0}{Lt}(3x+1)}{\underset{x\to 0}{Lt}(5x-1)} = \frac{3(0)+1}{5(0)-1} = \frac{1}{-1} = -1$$

c) $\underset{x\to 4}{Lt}\ \dfrac{2x^{3/2}-\sqrt{x}}{x^2-15} = \dfrac{\underset{x\to 4}{Lt}\left(2x^{3/2}-\sqrt{x}\right)}{\underset{x\to 4}{Lt}\left(x^2-15\right)} = \dfrac{2(4)^{3/2}-\sqrt{4}}{(4)^2-15}$

$= \dfrac{2\times 4\times 2-2}{1} = 14$

d) $\underset{x\to a}{Lt}\ Ax^n = \underset{x\to a}{Lt}\ A\cdot \underset{x\to a}{Lt}\ x^n = A\cdot(a)^n$

e) $\frac{1}{3}$

f) -6

2) Average Cost (AC) = $4+\frac{2}{q}$. Now, $\lim_{q\to\infty} 4+\frac{2}{q} = 4$

3) Principal = Rs 3357 (Approx.); Effective rate of interest = 10.5%

Check Your Progress 2

1) 15

2) 0

3) (a) -1 ; (b) $\frac{1}{\sqrt{2}}$; (c) $\frac{2}{5}$

4) 5

UNIT 8 CONTINUITY*

Structure

8.0 OBJECTIVES

This unit is a continuation of the previous unit: that unit discussed limits and stressed their importance while the present one is concerned with continuity of functions. After going through this unit you will be able to:

- define continuity of a function;
- explain how continuity is related to limits;
- explain the concept of discontinuity;
- identify various types of discontinuity;
- discuss the intermediate-value theorem; and
- describe some applications of continuous and discontinuous functions.

8.1 INTRODUCTION

To understand the importance of continuous functions, let us pause a while to see where we are in the course, and where we will be travelling further in this course. In the first unit, you learnt about sets, which are collection of well defined objects. These objects can be real numbers or ordered pairs or ordered n-tuples. So the concept of sets in a way gave you the categories of object we deal with in Economics and their measurements. The next unit explained rules for multiplying sets, and also explained certain rules by which elements of one set are associated with elements of another set. These rules are functions. Unit three discussed logic, and the aim of that unit was to familiarise you with concepts like statements, theorems, proofs, necessary and sufficient conditions, implications, and so on.. In a way you learnt how to put forth arguments

* Contributed by Saugato Sen, SOSS, IGNOU

cogently. Unit four took up for discussion various specific kinds of functions such as linear, polynomial, exponential and logarithmic, and power and hyperbolic. The fifth unit extended the fourth one to combine equations and functions with their geometrical depiction on the Cartesian coordinates and planes. The sixth unit talked of special types of function, about objects that we are quite familiar with: sequences. We saw that when sequences are added in certain ways we get what are called series. The seventh unit, the unit previous to this one, dealt with limits, the way to see whether sequences and functions converge to a specific value.

The present unit carries the story from there. This unit discusses functions, and a specific property of functions: continuity. Intuitively a function is continuous if you can draw it without lifting your pen. And why is this seemingly mundane thing so important? Well, it so happens that a function that is not continuous is not differentiable. Differentiability is what you will study in the subsequent two units; and in unit 11, you will study an important property of differentiable functions, namely convexity. Supposing we have a variable y that is a function of a variable x, that is y 'depends on' x. Differentiation investigates how y changes due to very small 'incremental' or 'marginal' changes in x. To aid us in our investigation, function showing y depends on x has to be differentiable. And that function will not be differentiable unless it is continuous—that is the importance of continuity. Proceeding from differentiation, we get to optimization: for what value of x is the dependent variable maximised, or minimised. You will have learnt from your principles of microeconomics course that this is a crucial part of economics. So now you see the deep importance of continuity.

The unit is organised as follows: the next long section discusses extensive, intuitively as well as in a formal manner, the nature characteristics, features and properties of continuous functions. Unfortunately, in economics we do come across functions that are not continuous. It is just that we should be aware that they cannot be differentiated. The next section explains discontinuity and also talks about various types of discontinuities. The subsequent section mainly discusses some applications of discontinuous functions, since most of the functions used in economics, and those discussed in units four and five, are continuous. Finally, the unit discusses a very important theorem, the intermediate-value theorem, and discusses its application to demand and supply equilibrium and excess demand functions.

8.2 CONTINUITY OF A FUNCTION OF ONE VARIABLE

A continuous variable is one that is capable of a continuous change (or change without gaps) over the number line. Let us consider a function$y = f(x)$, where, $x$and $y$are two continuous variables. The idea of the continuity of this function requires that not only the two variables, x and y, should be separately capable of a continuous variation, but also a continuous variation in x should result in a continuous variation in y. It is clear that we should have a continuous curve (without a gap in it) for a continuous function.

8.2.1 Continuity of a Function at a Point

Continuity refers to changes which are extremely gradual rather than sudden. It is closely related with the idea of a continuous function. A function $y = f(x)$

is continuous when a small change in independent variable (here x) produces a small change in dependent variable (here y). Geometrically a function is continuous if its *graph is connected*, and has no breaks or jumps. In other words, its graph can be drawn without lifting a pen or pencil with which it is being drawn.

Curves with sharp turning are continuous whereas, those with gaps are not. If there is any jump in the graph at any point, the function is said to be discontinuous at that point.It implies that there is a sudden change in the value of the function as we pass through that point. This situation is shown in Figure 8.1.

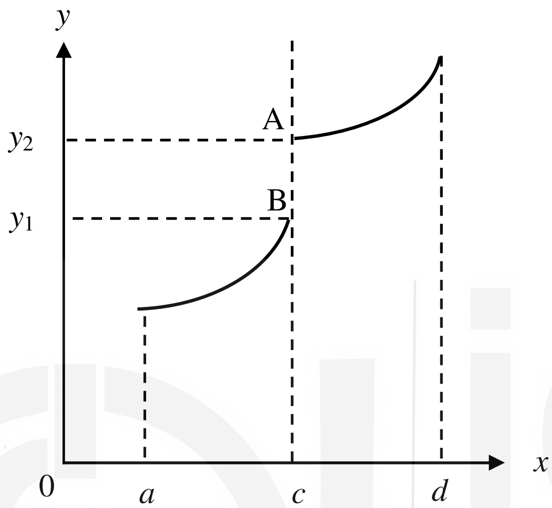

Figure 8.1

We note that when x takes a value that is slightly less than c, the value of the function is y_1, and the moment x becomes slightly greater than c, the value of the function suddenly jumps from y_1 to y_2, indicating thereby that it is discontinuous at $x = c$.

8.2.2 Criterion for Continuity at a Point

We will make use of the concept of limit to define the continuity of a function. If a function $y = f(x)$, has a limit as x approaches to the point a in its domain, and if this limit is also equal to $f(a)$ *i.e.*, equal to the value of the function at $x = a$, the function is said to be continuous at a.We shall discuss this criterion for continuity with the help of Figure 8.2.

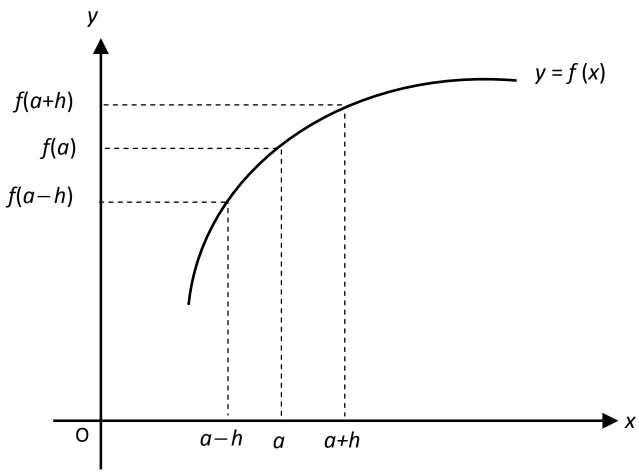

Figure 8.2

Consider a function $y = f(x)$. Let x change from a to $a + h$, where h is a small positive number. The corresponding change in the value of the function is given by $f(a + h)–f(a)$. In order that the function be continuous at $x = a$, we should have $f(a + h)–f(a)$ to be small for small values of h. That is,

$$\lim_{h\to 0}\{f\ (a+h) - f(a)\} = 0$$
$$\text{or} \qquad \lim_{h\to 0} f\ (a+h) = f(a) \qquad (1)$$

Equation 1 implies that the right hand limit of the function at $x = a$ should be equal to the value of the function at that point. Proceeding in a similar way, we can write the condition for continuity from the left hand side of $x = a$ as

$$\lim_{h\to 0} f\ (a-h) = f(a) \qquad (2)$$

From (1) and (2) we can conclude that the term continuity, essentially involves three requirements:

i) Existence of the limit at a point;

ii) Existence of the value at that point; and

iii) Equality between the limit and the value.

We may note that when right hand limit is equal to left hand limit at a point, the limit of the function is said to be existing at that point. Thus, more explicitly, a function $f\ (x)$ is said to be continuous at point a, when the following conditions hold true:

1) The point a must be in the domain of the function, *i.e.*, $f\left(a\right)$ is defined;

2) The function must have a limit as x approaches a, *i.e.*, $\lim_{x\to a} f(x)$ exists; and

3) That limit must be equal in value to $f\left(a\right)$, *i.e.*, $lim_{x\to a} f(x) = f(a)$.

8.2.3 Closed-form Functions

A *closed-form function* can be obtained by combining constants, powers of x, exponential functions, radicals, logarithms, etc., into a *single mathematical formula* by means of the usual arithmetic operations and composition of functions. Some examples of closed-form functions are:

$$3x^2 - x + 1; \qquad \frac{\left(x^2-1\right)^{1/2}}{6x-2}; \qquad e^{\left(x^2-1\right)^{1/2/x}}; \qquad \left(\log_3\left(4x^2 - e^x\right)\right)^{2/3}$$

Such functions can be made as complicated as we like. On the contrary, the following is not a closed form function.

$$f\left(x\right) = \begin{cases} -1 & \text{if } x < -1 \\ x^2 + x & \text{if } -1 \le x \le 1 \\ 2 - x & \text{if } 1 < x \le 2 \end{cases}$$

Why? Because $f\left(x\right)$ is not specified by a single mathematical expression.

Continuity of Closed Form Functions

A closed form function is always continuous in its domain.Therefore, *the limit of a closed-form function at a pointin its domain can be obtained by substitution.*

Example: $f(x) = \frac{x^2-3x}{4x+3}$is a closed-form function, and $x=2$ is in its domain. Therefore, we can obtain $\lim_{x\to 2} f(x)$ by substitution:

$$\lim_{x\to 2}\frac{x^2-3x}{4x+3} = f(2) = -\frac{2}{11}.$$

Let us now take a closed-form function $f(x)$ with the point $x=a$ not in its domain. Consequently, we will have to do the following:

a) Use simplification or some other technique to replace $f(x)$ by another closed-form function, which does have $x=a$ in its domain. This allows you to substitute $x=a$ in the new function to obtain the limit.

b) Try evaluating the limit numerically or graphically. That may give you an *estimate* of the limit.

Sometimes, you may need to use both (a) and (b).

8.2.4 Evaluating a Limit Using Simplification

Let us evaluate$\lim_{x\to -2} f(x)$, where$f(x) = \frac{3x^2+x-10}{x+2}$.

Ask yourself the following question:

1) Is the function $f(x)$ a closed-form function?

   Answer is yes, since $\frac{3x^2+x-10}{x+2}$ is a single mathematical formula.

2) Is the value $x = -2$ in the domain of $f(x)$?

   Answer is no, since $f(-2) = \left(3(-2)^2 + (-2) - 10\right) / \left((-2)+2\right)$ is not defined.

So we have to simplify the function to get

$$\frac{3x^2+x-10}{x+2} = \frac{(x+2)(3x-5)}{(x+2)}$$

$$= 3x-5.$$

Since we are now left with a closed form function that is defined when $x=-2$, we can now evaluate the limit by substitution:

$$\lim_{x\to -2}\frac{3x^2+x-10}{x+2} = \lim_{x\to -2} 3x-5 = 3(-2)-5 = -11.$$

Example: Let us now find if $f(x) = 3x-2$ is continuous at $x = 3$.

Sol. Given $f(x)=3x-2$

Condition 1:When we put $x=3$, we get $f(3)=3\times3-2=7$. Therefore, the function is defined at $x=3$.

Condition 2:Limit of $3x-2$ as $x\to3$ exists and is equal to 7.

Condition 3: Also, $\underset{x\to3}{Lt}\ (3x-2)=$ value of function $f(x)=7$.

Hence, the function is continuous at $x=3$. In fact, the function is continuous at all points. Students are expected to try other values.

Example: Examine the continuity of the function $f(x)=x^3+3x-4$ at $x=1$

Sol. Condition 1:For $x=1, f(1)=(1)^3+3(1)-4=1+3-4=0$. Therefore, the function is defined for $x=1$.

Condition 2: $\underset{x\to1}{Lt}\ (x^3+3x-4)=(1)^3+3(1)-4=0$. That is, the limit exists.

Condition 3:Thus, we find that $f(1)=\underset{x\to1}{Lt}\ f(x)=0$.

Since all the conditions are satisfied the function is continuous at $x=1$.

Example: Determine at which values of x, the functions given below are continuous.

a) $f(x)=\dfrac{x^4+3x^2-1}{(x-1)(x+2)}$

b) $f(x)=\left(x^2+2\right)\ x^3+\frac{1}{x}^{\ 4}+\dfrac{1}{\sqrt{x+1}}$

Sol. a) $f(x)=\dfrac{x^4+3x^2-1}{(x-1)(x+2)}$. This is a rational function and hence is continuous at all points except when $(x-1)(x+2)=0$ *i.e.* denominator vanishes (is zero). This gives us two cases:

(i) When $x-1=0$, $x=1$, $\lim\limits_{x\to1} f(x)\to\infty$, *i.e.* undefined.

(ii) When $x+2=0$, $x=-2$, $\lim\limits_{x\to-2} f(x)\to\infty$, *i.e.* undefined.

Other conditions will not be needed. Hence, the function is discontinuous being undefined at $x=1$ and $x=-2$.

b) $f(x)=\left(x^2+2\right)\ x^3+\frac{1}{x}^{\ 4}+\dfrac{1}{\sqrt{x+1}}$

Since there is $\dfrac{1}{x}$ term which tends to ∞ as $x\to0$, therefore, $x\neq0$. Similarly, $(x+1)>0$ or $x>-1$. Thus,the domain of continuity lies between $(-1,0)$ and $(0,\infty)$.

8.2.5 Continuity of a Function over an Interval

A function $y = f(x)$ is said to be continuous in an interval (a, b), if it is continuous at every value of x in that interval, *i.e.*, it is continuous

1) at $x = a$ 2) at $x = b$ 3) and at any point between a and b.

Example: Show that the function $\frac{1}{x-2}$ is continuous for values of x from $x = -2$ to $x = -1$ *i.e.*, in the interval [−2, −1].

Sol. In order to prove that the function is continuous over the range [−2, −1], we will prove that:

1) It is continuous at $x = -2$ .00

2) It is continuous at $x = -1$

3) It is continuous for any value between -2 and -1, say, -1.5 or $-\frac{3}{2}$.

Case 1: Let us first find out$\lim_{x \to -2} f(x)$

Assessing Right hand limit

$$\lim_{x \to -2^+} f(x) = \lim_{h \to 0} f(-2+h) = \lim_{h \to 0} \left[\frac{1}{-2+h-2}\right] = \frac{-1}{4}$$

Assessing Left hand limit

$$\lim_{x \to -2^-} f(x) = \lim_{h \to 0} f(-2-h) = \lim_{h \to 0} \left[\frac{1}{-2-h-2}\right] = \frac{-1}{4}$$

Since, $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^-} f(x) = \frac{-1}{4}$, we get $\lim_{x \to -2} f(x) = \frac{-1}{4}$, and so the function $f(x)$ is continuous at $x = -2$.

Case 2: Similarly, we can prove that the function is continuous at $x = -1$. We leave this exercise for the students to try themselves.

Case 3: Let us first find out$\lim_{x \to -\frac{3}{2}} f(x)$

Assessing Right hand limit

$$\lim_{x \to -\frac{3}{2}^+} f(x) = \lim_{h \to 0} f\left(-\frac{3}{2}+h\right) = \lim_{h \to 0} \left[\frac{1}{-\frac{3}{2}+h-2}\right]$$

$$= \frac{-2}{7}$$

Assessing Left hand limit

$$\lim_{x \to -\frac{3}{2}^-} f(x) = \lim_{h \to 0} f\left(-\frac{3}{2}-h\right) = \lim_{h \to 0} \left[\frac{1}{-\frac{3}{2}-h-2}\right]$$

$$= \frac{-2}{7}$$

Since, $\lim_{x\to -\frac{3}{2}^+} f(x) = lim_{x\to -\frac{3}{2}^-} f(x) = \frac{-2}{7}$, we get $\lim_{x\to -\frac{3}{2}} f(x) = \frac{-2}{7}$

Therefore, we find that the function is continuous at $x = -\frac{3}{2}$ which is a point lying between -2 and -1.

It can easily be checked that the function is continuous at every point between –2 and–1.

Check Your Progress 1

1) Given the function $f(x) = x^2 + 3$. Discuss the continuity of the function at $x = 1$.

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2) A bank charges interest at the rate of 10 per cent per year for a loan of up to Rs 10,000. It raises the interest rate by 5 per cent if the loan amount is more than Rs 10,000. Give an expression for the bank's (total) interest function. Examine this function for continuity.

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3) Find the limit of

i) $\lim\limits_{x\to 0} \dfrac{e^{x^2}}{2x-3}$

ii) $\lim_{x\to -3} \frac{2x+3}{x+3}$

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4) Find the limit of

$$\lim_{x\to -1} \frac{(x^3+1)(x-1)}{x^2+3x+2}.$$

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8.3 DISCONTINUOUS FUNCTIONS

Simply put, a function that is not continuous is discontinuous. To see whether a function is discontinuous, it is just enough to see if it fulfills the following conditions of continuity, as we know by now.

i) $f(a)$ is defined

ii) $\lim\limits_{x\to a} f(x)$ exists, that is, $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+}$ and

iii) $\lim\limits_{x\to a} f(x) = f(a)$

Therefore, if for a function any of the following conditions does not hold, the function will be discontinuous.

8.3.1 Types of Discontinuity

There are several types of discontinuity that we come across. First, the graph of a function may have an obvious "break" or "jump" at some point. For example, the function $f(x) = \begin{cases} +1, & x \le 0 \\ -1, & x > 0 \end{cases}$

is clearly discontinuous, as it has a break at the point $x = 0$. We get different values as we approach this point from the left or from the right. (Refer Figure 8.3)

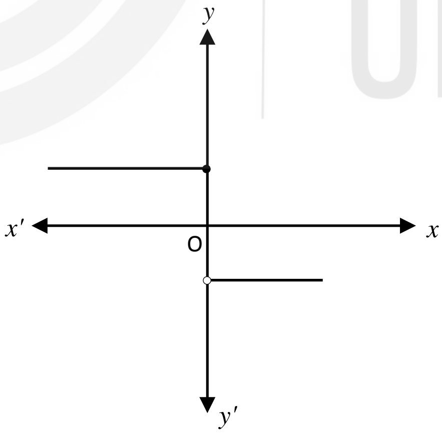

Figure 8.3

Another type of discontinuous function is the asymptotic function. Let us see what an asymptote of a function means. If the value of a function $f(x)$, $x \in$ R, becomes unbounded as x approaches some value $x = a$ either from the left or the right, then we say the line $x = a$ is an asymptote of the function $f(x)$. In such cases, the function is discontinuous. If any one or more of the following possibilities hold true, we say the line $x = a$ is a vertical asymptote: (Refer Figure 8.4)

$$\lim_{x \to a^+} f(x) = +\infty, \qquad \lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

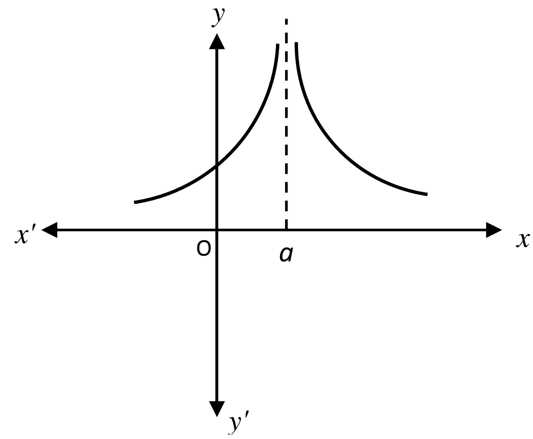

Figure 8.4

Another case of discontinuous function is one where there is a 'hole' in a function. Consider the function $f(x) = \begin{cases} x, & x \neq 2 \\ 5, & x = 2 \end{cases}$. Despite the fact that this function is defined at *x*= 2 and its left- and right-hand limits are equal at $x=2$, the function is not continuous because in this the criterion$\lim_{x\to 2} f(x) = f(2)$ is not satisfied. (Refer Figure 8.5)

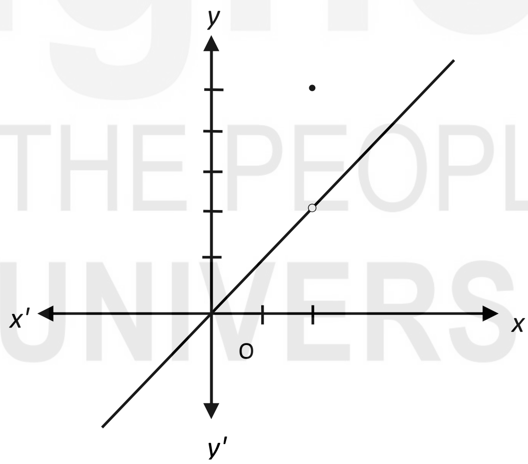

Figure 8.5

Figure 8.5 represents a function with discontinuity at *x* = 2. There is a hole in the graph of the function at *x* = 2.

Generally, we categorise the types of discontinuity into two categories:

a) ***Irremovable discontinuity***, whenthe second condition for continuity is not satisfied, *i.e.* the limit of the function as $x \to a$ does not exist.

b) ***Removable Discontinuity***, when$\lim_{x\to a} f(x)$ exists, but $\lim_{x\to a} f(x) \neq f(a)$.The discontinuity can be removed with the help of the Simplification techniques discussed before.

Example: Examine the continuity of the function $f(x) = \dfrac{x^2-1}{x-1}$ at $x=1$.

Sol. Condition 1:For $x=1$, $f(1) = \frac{1^2-1}{1-1} = \frac{0}{0}$, an indeterminate form.

The function seems to be discontinuous at $x=1$. However, a little thought will show that this discontinuity is removable. The numerator $\left(x^2-1\right)$ can be written as $(x-1)(x+1)$ so that the function becomes $f(x)=\dfrac{(x-1)(x+1)}{(x-1)}=(x+1)$

Now, $f(x) = x + 1$ and $f(1) = 1 + 1 = 2$. The function is defined, and hence, condition 1 is satisfied.

In condition 2, we find the limit *i.e.*, $\underset{x\to 1}{Lt}\ (x+1)=1+1=2$.

Therefore, condition 2 is satisfied. However, the condition 3 is not satisfied, because the value of the function is $\dfrac{0}{0}$ which is undefined. Hence the function is undefined at $x=1$.

Example Given the function

$$f(x) = x+2 \qquad \text{for } x \le 3$$
$$= x+3 \qquad \text{for } x > 3$$

Discuss the continuity of the function at $x = 3$.

Sol. When $x = 3$, the value of the function is given by

$$f(3) = x+2 = 3+2 = 5$$

For left hand limit, we will consider the function $f(x) = x + 2$, to get the limit as 5. Thus, when x approaches 3 from values less than 3, $f(x)$ approaches 5.

$$\therefore \underset{x\to 3^-}{Lim} = 5$$

For right hand limit, we will consider the function $f(x) = x + 3$, to get the limit as 6. Thus, when x approaches 3 from values greater than 3, $f(x)$ approaches 6.

$$\therefore \underset{x\to 3^+}{Lim} = 6$$

As $\lim\limits_{x\to 3^-} f(x) \neq \lim\limits_{x\to 3^+} f(x)$, the limit does not exist for the function when x approaches 3. Hence, the function is discontinuous at $x = 3$.

Example: Find the values of x for which the function $y=\dfrac{x+2}{(x+1)(x+3)}$ is discontinuous.

Sol. This is a rational function and hence is continuous at all points except when $(x+1)(x+3)=0$. This implies that,

$$\text{Either } x+1=0 \quad \Rightarrow \text{or} \quad x=-1$$
$$\text{or} \quad x+3=0 \quad \Rightarrow \text{or} \quad x=-3$$

Case 1 If $x=-1$, then $\underset{x\to -1}{Lt}\ f(x)=\infty$

Case2 If $x=-3$, then also $\underset{x\to -3}{Lt}\ f(x)=-\infty$

Both are undefined limits. Hence, for $x = -1$ and $x = -3$, the function $f(x)$ is discontinuous.

Check Your Progress 2

1) Given the function

$$f(x) = 4 - x \qquad \text{for } x \neq 3$$

$$= 0 \qquad \text{for } x = 3$$

Discuss the continuity of the function at $x = 3$.

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2) Examine the continuity of the function $y = |x|$ at $x = 0$.

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3) Define a discontinuous function. What are the different types of discontinuities?

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4) Construct some plausible Economic functions that are discontinuous.

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8.4 ECONOMIC APPLICATIONS OF CONTINUOUS AND DISCONTINUOUS FUNCTIONS

In this section we discuss some applications of continuous and discontinuous functions. Since you know what we understand by a continuous function, we might as well mention that most of the functions we came across in Units 2 and 4, particularly the latter, were continuous functions. So if you revisit Unit 4 and see the applications of functions given there, you will gain.

Continuity is usually discussed in the context of differentiation. Along with studying differentiation in the subsequent two units, you will be learning about continuous functions as well. So, here we are focusing on discontinuous functions. Economic theory has many natural examples of discontinuities. Usually, the assumption is made that the function is continuous, but that is for simplicity. In many cases, it would be more realistic to consider the function as discontinuous, asthere existsa possibility that the assumption of continuity might distort the true economic relationship.

Consider the case of an output *y* produced by a single input *x*. The production function is thus $y = f(x)$. To say that $f(x)$ is continuous at some point $x = a$, is to say that $f(x)$ is defined on some open interval of real numbers, including *a*. This means that *x* must be infinitely divisible. But there may be production conditions comprising discrete inputs and outputs. For instance, nuts and bolts used in car manufacture are discrete. Other examples can be considered. Suppose there is a firm employing salespersons. Also assume that the firm pays according to the rule— the salesperson's salary will consist of three parts: (a) a basic amount of Rs 800 (b) a commission of 10%, and (c) a lump-sum bonus of Rs 500 if the salesperson's sales reach or exceed Rs 20,000.

If we let *S*as the sales and *P* the pay of a salesperson, then the relationship between pay and sales can be represented as

$$P = 800 + 0.1S, \text{ if } S < 20{,}000$$

$$P = 800 + 0.1S + 500, \text{ if } S \geq 20{,}000$$

There is thus a discontinuity at the point 20,000.

To take another example, think of an income support programme. The government in some countries pays poor unemployed people a lump-sum monthly payment. Once the person start earning some income, the welfare payment is stopped. Suppose we have the following situation. A single-mother earns a monthly payment of Rs 2000, if she does not work, but if she gets a job, the payment is stopped. Suppose she could earn Rs 80 per day by working. The income *y* of this person as a function of the number of days worked (*d*) will be as follows:

$$y(d) = 2000 \text{ if } d = 0$$

$$y(d) = 80d \text{ if } d > 0$$

The above function has a discontinuity at point 0.

8.5 INTERMEDIATE-VALUE THEOREM

In this section we will be discussing a very important theorem pertaining to continuity of variables, called the *Intermediate-Value Theorem*. It has several applications in various disciplines, as also in Economics, where we can use it to study the important concept of *equilibrium*. Equilibrium, as you know from your study of Principles of Microeconomics, is one of the central concepts in

Economics. We now present a statement of the theorem itself and subsequently show how it is useful in discussing equilibrium.

The Statement of the Theorem

We begin with a simple exposition of the theorem. Suppose that the function $y = f(x)$is continuous on the interval [a,b],where b>a. It follows that the function must take on every value between $f(a)$ and $f(b)$, with them being the function values at the endpoints of the interval [a, b]. This result is called the intermediate value theorem because any intermediate value between $f(a)$ and $f(b)$ must occur for this function for at least one value of x between $x = a$and $x = b$. This result will not necessarily hold for a discontinuous function. We now present a formal statement of the theorem:

Suppose that $f(x)$is a continuous function on the closed interval $[a, b]$ and that $f(a) \neq f(b)$. Then for any value$\overline{y}$ between $f(a)$ and $f(b)$, there is some value of x, say $x = d$, between a and b, such that $\overline{y} = f(d)$.

Application of Intermediate Value Theorem to Equilibrium Analysis

Very often the Intermediate-Value Theorem is used to prove the existence of a special value of an Economic variable. Consider a simple demand-supply model. Let $q = D(p)$ represent the demand function, and $q = S(p)$ represent the supply function, where $p$is the price and q, the quantity. In such a model an equilibrium price is one at which quantity demanded equals quantity supplied, that is, the price at which the market clears. In other words, $p = p^* \geq 0$ is an equilibrium price if $D(p^*) = S(p^*)$. Substitution of p^* in any of the functions will give us the equilibrium quantity (*i.e.*, q^*). Graphically, we can say that when the demand curve intersects the supply curve at a point (q^*, p^*), with q^*, p^*both greater than zero, then this is an equilibrium.

Sometimes, instead of working with demand and supply functions separately, it is useful to combine the two into a single function, called the excess-demand function,$z(p) = D(p) - S(p)$. Notice that the value of $z(p)$ indicates the amount by which $D(p)$ exceeds $S(p)$ if $z(p) > 0$, while if $z(p) < 0$, the absolute value of $z(p)$ indicates the amount by which $S(p)$ exceeds $D(p)$. In terms of the excess-demand function, $z(p^*) = 0$ whenever,$D(p^*) = S(p^*)$, while $z(p) \leq 0$ if $D(p) \leq S(p)$. Thus, another way of describing the equilibrium price is that $p^* > 0$ is an equilibrium price if $z(p^*) = 0$, while $p^*=0$ is an equilibrium price provided $z(0) \leq 0$.

Let us talk of a set of sufficient conditions that will guarantee that there exists equilibrium price. This is where the Intermediate-Value Theorem comes in handy. Suppose we consider a commodity which at price zero will have zero supply. Assume demand is greater than zero. So at price zero, we have

$$z(0) = D(0) - S(0) > 0$$

Now suppose at a very high price, say $\hat{p}$, firms find it very profitable to produce the good, but at this price the consumers find the price too high, so that supply will exceed demand. Thus at this price, we have

$z(\hat{p}) = D(\hat{p}) - S(\hat{p}) < 0.$

Now notice how we use the Intermediate-Value Theorem here. If demand and supply functions are continuous on the interval of prices $[0, \hat{p}]$, then so will $z(p)$ be continuous on $[0, \hat{p}]$, as the sum or difference of two continuous functions is also a continuous function. Now, by the intermediate-value theorem, every function value between z(0) and $z(\hat{p})$ must be realised by some value of p between 0 and $\hat{p}$. Particularly, if $z(0) > 0$ and $z(\hat{p}) < 0$, there must be some value of p between 0 and $\hat{p}$, say $p = c$, such that $z(c) = 0$. But such a value of price is by definition an equilibrium price. That is, $p^* = c$. So we are assured of an equilibrium price.

We can state the above in the following manner.

If the demand and supply functions are continuous, and hence the excess demand function is continuous, and also the following two conditions are satisfied: (a) at zero price, excess demand is greater than zero and (b) there exists some price (greater than zero) at which supply exceeds demand, that is excess demand is less than zero.Then, there exists a price greater than zero at which excess demand is equal to zero, that is, there exists an equilibrium price. Of course this holds true only if excess demand function is continuous.

Check Your Progress 3

1) Explain the Intermediate-Value Theorem.

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2) Consider the following market demand and supply functions:

$D(p) = 50 - 2p$; $\quad S(p) = -10 + p$

Calculate the equilibrium price (p^*) and quantity (q^*) for this market and show that these demand and supply functions satisfy the requirements for the existence of a positive equilibrium price.

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8.6 LET US SUM UP

This unit extended the discussion of the previous unit, which was on limits, to continuous functions. We saw that a function is continuous if it does not have any break at any point of its graph. More formally, a function $f(x)$ is said to be continuous for $x = a$, if the limit exists, and is equal to $f(a)$. To estimate limits and continuity, we may use numerical or algebraic approaches.

The unit discussed in detail the characteristics, features and properties of continuous functions. The necessity of a function to be continuous in order for it to be differentiable was highlighted. Of course, you will only study differentiation in the next two units. The unit discussed continuity in an intuitive as well as rigorous manner.The unit further considered the notion of discontinuity, and explored its various types. Some applications and use of continuous and discontinuous functions in Economics were discussed too. Although continuity of functions is usually assumed in Economics so that the tools of Economics can be used, we find that there exist many examples of discontinuity. Finally, the unit explained and discussed an important theorem pertaining to continuous functions, the intermediate value theorem, and gave an application of this theorem in equilibrium analysis.

8.7 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) $f(x)$ is continuous at $x = 1$.

2) See section 8.2 and answer

3) (i) $-\frac{1}{3}$; (ii) ∞

4) $lim_{x\to -1} \frac{(x^3+1)(x-1)}{x^2+3x+2} = \lim_{x\to -1} \frac{(x^3+1)(x-1)}{x^2+x+2x+2}$

$$= \lim_{x\to -1} \frac{(x^3+1)(x-1)}{(x+2)(x+1)} = \lim_{x\to -1} \frac{(x+1)(x^2+1-x)(x-1)}{(x+2)(x+1)}$$

$$= \frac{[(-1)^2+1+1](-2)}{1} = -6$$

Check Your Progress 2

1) The function $f(x)$ is discontinuous at $x = 3$ because $\lim_{x\to 3} f(x) \neq f(3)$.

2) The function is discontinuous at $x = 0$.

3) See section 8.3 and answer.

4) See section 8.3 and answer.

Check Your Progress 3

1) See section 8.5 and answer.

2) $q^* = 10$; $p^* = 20$.

UNIT 9 FIRST-ORDER DERIVATIVES*

Structure

9.0 OBJECTIVES

After going through the unit you will be able to:

- describe derivatives and differentials;
- define the limiting value of the tangent to a curve as its derivative;
- discuss conditions of differentiability and how these conditions relate to continuity; and
- state certain important rules of differentiation.

9.1 INTRODUCTION

We have studied about limits in Unit 7 and continuity in Unit 8. We saw under what conditions limits of a sequence and of a function exist. Building on the concept of limit, we went on to discuss continuity in the next unit. There we mentioned that continuity is important, indeed a necessary condition, for a derivative of a function to exist. This unit is concerned with finding the derivatives of functions. The process of finding the derivative of a function is called differentiation. We shall see, it is one of the most important or the central concepts used to understand Economic phenomena. Let us begin our study of Derivatives. We may mention here that this unit is concerned with first-order derivatives only. The next unit discusses derivatives of derivatives or even their derivatives, that is, higher-order derivatives.

The subsequent section of the unit discusses the geometric interpretation of derivatives. We shall see that a derivative can be interpreted as the limiting value of secant lines of a curve, as the secant gets to the limiting situation as a tangent to the curve. In the following section, the unit formally defines a derivative and also the important concept of a differential. In the unit after that, we talk about the conditions of differentiability, that is, what conditions are needed for a derivative to exist. Next, the unit discusses some rules of differentiating specific functions. Finally, the unit provides some examples of the use of first-order derivatives in Economics.

* Contributed by Shri Saugato Sen, SOSS, IGNOU

9.2 DERIVATIVES

We are by now familiar with the concepts of limits and continuity. These concepts form the background for the main theme of differential calculus, the derivative of a function. We shall devote this section to the study of derivatives.

Difference Quotient

Consider a function $y = f(x)$. Let us assume that when the independent variable x changes its value from $x_1$to x_2, the dependent variable y changes its value from $y_1 = f(x_1)$ to $y_2 = f(x_2)$. The change in x can be obtained by the difference $x_2 - x_1$. Now in Mathematics, any change is denoted by the Greek capital letter Δ (pronounced as *delta*). So, we can write the change in x as $\Delta x = x_2 - x_1$. Similarly, the change in y can be written as $\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$. Thus, when x changes by Δx, we find that y changes by Δy.

Therefore, the change in y per unit of change in x is given by the quotient (ratio) $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$. Since this quotient is the ratio of two differences, it is called the *difference quotient*. It is clear that this difference quotient measures the *average* rate of change of y over the interval (x_1, x_2). Let us consider a function $y = f(x)$ where y is a dependent variable and x an independent variable. Let there be in infinitesimally small change in x by Δx so that y also changes by infinitesimally small quantity Δy. Such a situation is widely prevalent in general as well as in dealing with Economic problems. For this, it is essential to assume continuity of function. A function is differentiable (*i.e.* its derivative exists and can be found out) only if it is continuous. And a function is continuous when it does not have a break. It can be drawn without raising pen/pencil from the paper on which it is being drawn.

Given continuity, we now attempt to find the effect of change in independent variable (x) on the dependent variable (y), when the function is $y = f(x)$. In other words, we want to find a function which is derived from a given function $y = f(x)$. Such a function gives relation between x and y so as to express the idea of change. This derived function is called *Derivative* of the given function and is expressed by various symbols such as— $y_1, y', \frac{dy}{dx}$ or $f'(x)$.

The process of obtaining a derivative $\frac{dy}{dx}$ (*i.e.* differential coefficient) is called *Differentiation* of the function. The branch of Mathematics which deals with/or studies this phenomenon is called *Differential Calculus*. Remember that a function has a derivative only when it is differentiable or simply, its derivative can be formed out. In order to be able to do so, the function should be continuous over a range of values called its *Domain*.

Differential Coefficient of a function $f(x)$ with respect to a variable x is the instantaneous rate of change in $f(x)$ due to a given change in x.

Let us take the function $\qquad y = f(x) \qquad \ldots(1)$

Now, let x change a small quantity Δx so that y changes by Δy such that,

$$y + \Delta y = f\left(x + \Delta x\right) \qquad \ldots(2)$$

Deduct (1) from (2) to get, $\Delta y = f\left(x + \Delta x\right) - f\left(x\right)$

Divide both sides by Δx to get, $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$

The ratio $\frac{\Delta y}{\Delta x}$ which is called the difference ratio, tends to $\frac{dy}{dx}$ as $\Delta x \to 0$. $\frac{dy}{dx}$ is called differential coefficient or derivative of y with respect to x.

Technically, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

Read as, $\frac{dy}{dx}$, the differential coefficient of $y = f(x)$, is the limiting value of the difference ratio $\frac{\Delta y}{\Delta x}$ or of $\frac{f(x+\Delta x)-f(x)}{\Delta x}$.

Derivation by First Principles

To demonstrate, let us take a very simple function

$$y = x^2 \qquad \ldots(1)$$

Let x change by Δx so that y also changes correspondingly by Δy and the function $y = f(x)$ becomes

$$y + \Delta y = \left(x + \Delta x\right)^2 \qquad \ldots(2)$$

Deduct (3) from (4) to get

$$\Delta y = \left(x + \Delta x\right)^2 - x^2$$

or $$\frac{\Delta y}{\Delta x} = \frac{\left(x + \Delta x\right)^2 - x^2}{\Delta x} \qquad \text{(Dividing both sides by } \Delta x\text{)}$$

$$= \frac{\cancel{x^2} + \Delta x^2 + 2x \cdot \Delta x - \cancel{x^2}}{\Delta x} = \frac{\Delta x^2}{\Delta x} + \frac{2x \cdot \Delta x}{\Delta x}$$

Or $$\frac{\Delta y}{\Delta x} = \Delta x + 2x$$

9.3 TANGENT LINE AS DERIVATIVE

Although derivative is essentially a concept from calculus, it has an interesting geometric interpretation as well. Let us consider Figure 9.1 and understand this geometric counterpart of derivative.

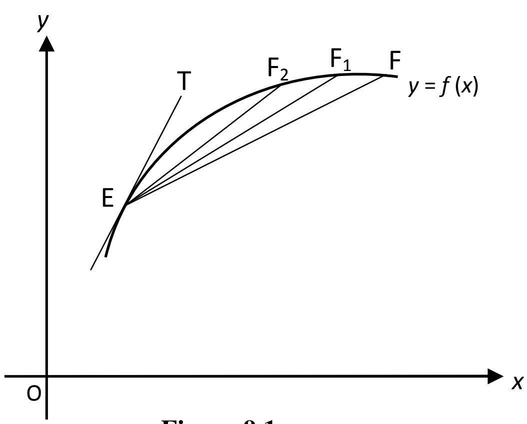


Figure 9.1

Suppose, *E* and *F* be two points on the curve described by a function $y = f(x)$.

Let the co-ordinates of *E* be (x_1, y_1) and that of *F*be $(x_1 + \Delta x, y_1 + \Delta y)$. Let us further assume that *E* is a fixed point and *F* is a moving point that is moving along the curve towards the fixed point *E*. If we join the points *E* and *F* by a straight line, then the line *EF* is called a *chord*. As the point *F* moves towards *E*, the chord *EF* changes its position. For example, at F_1, it takes the position EF_1 and at F_2, it takes the position EF_2. We can see from the diagram that as the point F moves towards the point *E*, the chord *EF* rotates about *E* in the anti-clockwise direction. During this process of rotation, the chord gets progressively shortened and, the area between the chord and the arc (segment of the curve) defined by it continuously falls. In fact, when the point *F* approaches *E*, the chord *EF* tends to become a tangent (say, *ET*) to the curve at the point *E*. Thus, the tangent *ET* can be described as the limiting position of the chord *EF* as *F* tends to *E*. However, this limiting position will only exist if the curve is continuous at the point *E*.

Let us now examine how the slope (or the gradient) of the chord *EF* behaves as the point *F* moves towards the point *E*. We know that the slope of a line between two points is given by the ratio of the change in the value of y to the change in the value of x. Thus when the moving point is at its original position *F*, the slope of the chord *EF* can be measured from the given co-ordinates of the points *E* and *F* respectively and it is $\frac{\Delta y}{\Delta x}$. From the diagram it is clear that as the point *F* moves towards the point *E*, Δx falls and whereas Δy rises, and thus the slope $\frac{\Delta y}{\Delta x}$ keeps changing for different positions of the chord *EF*. Finally, when the point *F* tends to the point *E*and $\Delta x \to 0$, the slope approaches a limit given by $\underset{\Delta x \to 0}{Lim} \frac{\Delta y}{\Delta x}$ and, as we have discussed earlier, this limit is $\frac{dy}{dx}$.

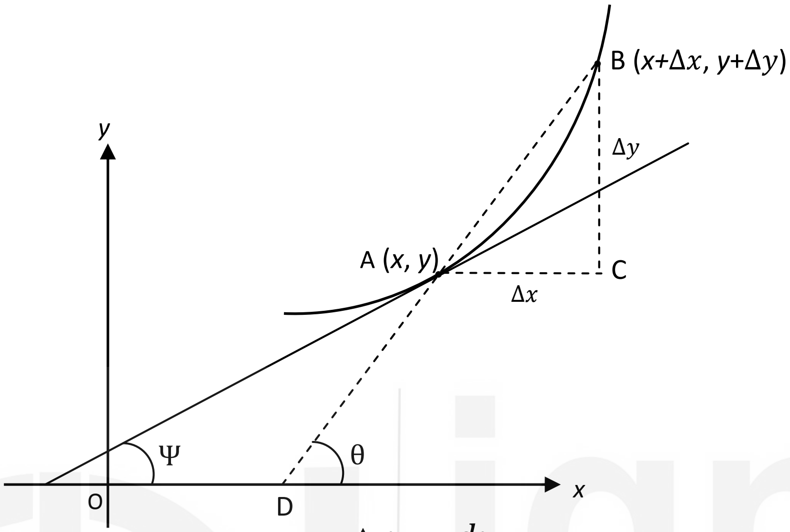
To sum up the above discussion, as the point *F* tends to the point *E*, two quantities approach their respective limits simultaneously. First, the chord *EF* approaches a limiting position given by a tangent *ET* to the curve at the point *E* and secondly, the slope, $\frac{\Delta y}{\Delta x}$ of the chord *EF* also tends to a limit given by $\frac{dy}{dx}$. Therefore, we can say that the tangent at the limit has a slope of $\frac{dy}{dx}$. We already know that $\frac{dy}{dx}$ is the derivative of some function*y* of*x*. Thus, the derivative of the function $y = f(x)$ at $x = x_1$ can be interpreted as the slope of the tangent at the corresponding point on the curve of the given function. However, the existence of the derivative and the tangent requires that both the function and its curve are continuous at the given value of x.

In Figure 9.2, let the curve be represented by a function $y = f(x)$. Take two points $A(x, y)$ and $B\ (x \quad x),(y \quad y)$ adjacent to each other. Join AB and extend it to meet x-axis at D, making an angle of θ with it.

The slope of AB joining points A and B is

$$\tan\theta = \frac{\Delta\, y}{\Delta\, x}\, i.e., \frac{(Perpendicular)}{(Base)}$$

Now let Δx tend to zero $(\Delta x \to 0)$ so that point B tends to points $A(B \to A)$ and slowly line AB becomes tangent to the curve at point $A(x, y)$ making a new angle ψ with the x-axis, when extended. Slope of this tangent at A is $\tan\psi$ which is the limiting value of $\tan\theta$ when $\Delta x (= AC) \to 0$. Technically,



$$\tan\Psi = \lim_{\Delta x \to 0} \frac{\Delta\, y}{\Delta\, x} = \frac{dy}{dx}$$

Figure 9.2

It is read as under

> Differential coefficient or derivative of y with respect to x ($\frac{dy}{dx}$) of a function $y = f(x)$ is the limiting value of the *difference ratio* $\frac{\Delta y}{\Delta x}$ as Δx tends to zero.

At any point, say A on the function $y = f(x)$, $\frac{dy}{dx}$ is the slope of the tangent (given by $\tan\Psi$) at that point. Further, if limit of the difference ratio $\frac{\Delta y}{\Delta x}$ exists for a particular value of x, say $x = a$, then we say a is the limit of the function $f(x)$, $f(a)$ is thevalue of the function at $x = a$, the function is differentiable at $x = a$, and the function is continuous at that point. We will study in more detail about continuity later on in section.

Continuity and Differentiability

We have seen above that the existence of the derivative of a function at some value of the independent variable requires that the function be continuous at that value of the independent variable. Geometrically, the condition states that for the existence of the tangent at the corresponding point on the curve for the given function, the curve should be continuous at that point. Continuity is a necessary condition for differentiability but it is not a sufficient condition. A function that is both continuous and differentiable is called a *smooth function*.

Check Your Progress 1

1) What is the difference between a secant to a curve and a tangent to a curve?

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2) Explain how the concept of derivatives can be understood as the tangent to a curve of the function $y = f(x)$.

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9.4 RULES OF DIFFERENTIATION

After understanding the concept of derivative, it is time for the evaluation of derivatives. The evaluation can be considerably simplified by following certain rules. Let us now state these rules, proofs of which are out of the scope of this unit.

Algebraic Function

To begin with, we are concentrating on algebraic functions. We are assuming that there are two given functions of x. They are $f(x)$ and $g(x)$, respectively. We are also assuming that these two functions are differentiable.

Rule 1: *Derivative of a constant.*

The derivative of a constant is zero. Let $f(x) = C$, where C is a constant. Then, $\frac{d}{dx}(C) = 0$

Rule 2: *Derivative of a power function.*

Let $f(x) = x^n$, where n is any real value. Then, $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

Rule 3:*Derivative of a Sum or Difference.*

The derivative of the sum (difference) of two functions is equal to the sum (difference) of the derivatives of the two functions.

That is, $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f`(x) \pm g`(x)$

Rule 4: *Derivative of a Product—Product Rule.*

The derivative of the product of two functions is equal to the first function times the derivative of the second function *plus* the second function times the derivative of the first function.

If $y = u.v$ where $u = f(x), v = g(x)$

Then, $\frac{dy}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx} = f(x)g`(x) \pm g(x)f`(x)$

Note: If there are more than two functions, we can treat any two as first functions and the other/s as the second function, and then apply the product rule. For example, let $y = f(x) \cdot g(x) \cdot L(x)$, then

$$\frac{dy}{dx} = \; f(x) \;\; g(x) \quad L'(x) \quad L(x) \; \frac{d}{dx} \; f(x) \;\; g(x)$$

where $\frac{d}{dx} \; f(x) \;\; g(x) \qquad f(x) \;\; g'(x) \quad g(x) \;\; f'(x)$

Rule 5: *Derivative of a Quotient—Quotient Rule.*

The derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator *minus* the numerator times the derivative of the denominator, all divided by the square of the denominator.

Let $y = \frac{u}{v}$, where $u = f(x), v = g(x)$

Then, $$\frac{dy}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x) \;\; ^2}$$

Rule 6: *Derivative of a function of a function— the Chain Rule.*

So far we have discussed the derivative of a function with respect to some independent variable that *directly* influences the function. Now we shall consider the derivative of a function with respect to some independent variable that *indirectly* influences the function. Suppose y is a function of u and u is a function of x, then x indirectly influences y through u. In this case, the derivative of y with respect to x is the product of the derivative of y with respect to u and the derivative of u with respect to x. That is,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Rule 7: *Derivative of an Inverse function.*

Suppose, $y = f(x)$ then from it if x is expressed as some function of y, it is called an *inverse* function of y. Symbolically, $x = f^{-1}(y)$.

The derivative of an inverse function is the reciprocal of the derivative of the original function provided both the functions exist. That is, $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$,

Rule 8: *Derivative of Logarithmic Functions*

It is convenient to present the expression for the derivative of the logarithmic function to the base e (natural logarithmic function) first. The symbol for this is $\log_e x$ (we should note here that sometimes an alternative symbol, ln$x$is also used, here 'ln' stands for natural logarithm).

Let $y = \log_e x$. Then, $\frac{dy}{dx} = \frac{d}{dx}(\log_e x) = \frac{1}{x}.\frac{d}{dx}(x) \ = \frac{1}{x}.$

Based on the above results, we have $\frac{d}{dx}(\log_e x^n) = \frac{1}{x^n} \cdot \frac{d}{dx}(x^n)$

$$= \frac{1}{x^n} \, .nx^{n-1} = \frac{n}{x}$$

Let us now consider the expression for the derivative of the logarithmic function with base other than e.

Let $y = \log_a x,$ with $a > 0$, where $\log_a x$ is the logarithm of x to the base a.

Then, $\frac{dy}{dx} = \frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e = \frac{1}{x\log_e a}$

Rule 9: *Derivative of Exponential Functions*

To begin with, let us consider the derivative of an exponential function with base e. Let $y = e^x$.

Then, $\frac{dy}{dx} = \frac{d}{dx}\left(e^x\right) = e^x$

Based on the above results, we have $\frac{d}{dx}\left(e^{ax}\right) = e^{ax} \cdot \frac{d}{dx}(ax) = e^{ax} \cdot a = ae^{ax}$

Now we shall consider the derivative of an exponential function with base other than e. Let $y = a^x$, with $a > 0$.

Then, $\frac{dy}{dx} = \frac{d}{dx}(a^x) = a^x \log_e a$

Let us take some examples covering these rules

Example 1 If $y = x^5$, then $\frac{dy}{dx} = 5x^{5-1} = 5x^4$ [Rule 1]

Example 2 If $y = 10x$, then $\frac{dy}{dx} = 10\frac{d}{dx}(x) = 10 \times 1 = 10$ [Rule 2]

Example 3 If $y = 100x^3$, then

$\frac{dy}{dx} = 100\frac{d}{dx}(x^3) = 100 \times 3x^{3-1} = 300x^2$ [Rule 3]

Example 4 If $y = 100$, then $\frac{dy}{dx} = 0$ [Rule 4]

Example 5 If $y = ax^5 + bx^4 - cx^2$, then

$\frac{dy}{dx} = a.\frac{d}{dx}(x^5) + b.\frac{d}{dx}(x^4) - c.\frac{d}{dx}(x^2)$ [Rule 3]

$= a.5x^{5-1} + b.4.x^{4-1} - c.2.x^{2-1}$

$= 5ax^4 + 4bx^3 - 2cx$

Example 6 If $y = 3x^6 + 2x^5 - 7x^3 + 3x^2 + 10$

$$\frac{dy}{dx} = 3\cdot\frac{d}{dx}\left(x^6\right) + 2\cdot\frac{d}{dx}\left(x^5\right) - 7\cdot\frac{d}{dx}\left(x^3\right) + 3\cdot\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(10)$$

[Rule 3]

$$= 3\times 6\times x^{6-1} + 2\times 5\times x^{5-1} - 7\times 3\times x^{3-1} + 3\times 2\times x^{2-1} + 0$$

$$= 18x^5 + 10x^4 - 21x^2 + 6x$$

Example 7 If $y = x^2 \log x$, using the product rule, we get

$$\frac{dy}{dx} = x^2.\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}(x^2) = x^2.\frac{1}{x} + \log x \times 2x$$

[Rule 4]

$$= x + 2x\log x$$

$$= x(1 + 2\log x)$$

Example 8 If $y = x^3 e^{2x}$, using the product rule, we get

$$\frac{dy}{dx} = x^3\cdot\frac{d}{dx}\left(e^{2x}\right) + e^{2x}\cdot\frac{d}{dx}\left(x^3\right) \qquad \text{[Rule 4]}$$

$$= x^3\cdot 2\cdot e^{2x} + e^{2x}\cdot 3\cdot x^{3-1} = 2x^3e^{2x} + 3x^2e^{2x}$$

$$= x^2e^{2x}(2x+3)$$

Example 9 If $y = e^x\cdot\log x$, using the product rule, we get

$$\frac{dy}{dx} = e^x\cdot\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}\left(e^x\right) \text{ [Rule 4]}$$

$$= e^x\cdot\frac{1}{x} + \log x\cdot e^x = e^x \quad \frac{1}{x} + \log x$$

Example 10 If $y = \dfrac{\log x}{x^2}$, using the quotient rule, we get

$$\frac{dy}{dx} = \frac{x^2\cdot\frac{d}{dx}(\log x) - \log x\cdot\frac{d}{dx}\left(x^2\right)}{\left(x^2\right)^2} \text{ [Rule 5]}$$

$$= \frac{x^2\cdot\frac{1}{x} - \log x\cdot 2\cdot x}{x^4} = \frac{x - 2\log x(x)}{x^4}$$

$$= \frac{x[1 - 2\log x]}{x^4} = \frac{1 - 2\log x}{x^3}$$

Example 11 If $y = \dfrac{20}{3x^4}$, using the quotient rule, we get

$$\frac{dy}{dx} = \frac{3x^4\times\frac{d}{dx}(20) - 20\times\frac{d}{dx}(3x^4)}{(3x^4)^2} = \frac{0 - 20\times 12x^3}{9x^8} \text{[Rule 5]}$$

$$= \frac{-240x^3}{9x^8} = \frac{-80}{3x^5}$$

Example 12 Differentiate $y = \sqrt{1 - x^2}$ $w.r.t\ x$ using chain rule.

Given $y = \left(1 - x^2\right)^{1/2}$.

Put $1-x^2=u$ so that $y=u^{\frac{1}{2}}$

and $\quad \dfrac{dy}{du}=\dfrac{1}{2}u^{-\frac{1}{2}}=\dfrac{1}{2u^{\frac{1}{2}}}$. Also $u=1-x^2$. Therefore $\dfrac{du}{dx}=-2x$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2u^{\frac{1}{2}}}\times-2x=\frac{-2x}{2\sqrt{1-x^2}}=\frac{-x}{\sqrt{1-x^2}}$$

Students can try the other method $\dfrac{dy}{dn}=\dfrac{d}{dx}\left(1-x^2\right)^{\frac{1}{2}}$

Example 13 Using chain rule/ function of a function rule, find the $\dfrac{dy}{dx}$ of $\left(3x^3-5x^2+x-1\right)^5$.

Let $u=3x^3-5x^2+x-1$ so that $y=u^5$ and $\dfrac{du}{dx}=9x^2-10x+1$

Also $\quad \dfrac{dy}{du}=5u^4=5\left(3x^3-5x^2+x-1\right)^4$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\left(9x^2-10x+1\right)\times5\left(3x^3-5x^2+x-1\right)$$

$$=5\left(3x^3-5x^2+x-1\right)\left(9x^{2-10}+1\right)$$

Example 14 If $y=a^{x\log x}$, find $\dfrac{dy}{dx}$ by using the formulae $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$

Put $u=x\log x$ so that $\dfrac{du}{dx}=x\cdot\dfrac{1}{x}+\log x=1+\log x$

Also $\quad y=a^u$ and $\dfrac{dy}{du}=a^u\log_e^a=a^{x\log x}\cdot\log e^a$

$$\frac{dy}{dn}-\frac{du}{dy}=a^{x\log x}\log a\left(1+\log x\right)$$

Differentiation of Implicit Functions

An implicit function is one which expresses mutual dependence between the two variables *x* and *y*. For every value of *x* there exists a predetermined or definite value of *y* and vice-versa. The value of one variable is implied or is derivable from the value of the other. An example of implicit function is:

$$f\left(x,y\right):2x^2+3xy+10y^2+5=0$$

Let us now find its derivative w.r.t *x*.

$$4x+3x\cdot\frac{dy}{dx}+y\times3+20y\cdot\frac{dy}{dx}+0=0$$

$$4x+3x\frac{dy}{dx}+20y\cdot\frac{dy}{dx}+3y=0 \text{ (collecting } \frac{dy}{dx} \text{ terms)}$$

$$\frac{dy}{dx}\left(3x+20y\right)+\left(4x+3y\right)=0$$

$$\frac{dy}{dx}(3x+20y)=-(4x+3y)$$

or $$\frac{dy}{dx}=-\frac{(4x+3y)}{(3x+20y)}$$

Let us take another example of differentiating, $xy+(x+y+6)^4=0$ *w.r.t* x treating y as function of x. Now, differentiating with respect to x, gives

$$x.\frac{dy}{dx}+y.1+4(x+y+6)^3\times\frac{d}{dx}(x+y+6)=0$$

$$x\frac{dy}{dx}+y+4(x+y+6)^3\times\left(1+\frac{dy}{dx}+0\right)=0$$

$$\frac{dy}{dx}[x+4(x+y+6)^3]=-[y+4(x+y+6)^3]$$

$$\frac{dy}{dx}=-\left[\frac{y+4(x+y+6)^3}{x+4(x+y+6)^3}\right]$$

Parametric Equations and their Derivatives

Two equations are called parametric if x and y are both functions of a third variable, say, t. That is, $x=f(t)$; $y=f(t)$. In such a case $\frac{dy}{dx}$ is expressed as

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{dy}{dt}\times\frac{dt}{dx}$$

Example Find $\frac{dy}{dx}$ when $x=8t^2+t+7$, $y=t^2+10t+2$

Sol. Here, $x=8t^2+t+7$ and $y=t^2+10t+2$

$$\frac{dx}{dt}=16t+1,\qquad \frac{dy}{dt}=2t+10$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2t+10}{16t+1}$$

Check Your Progress 2

1) Find the derivative of $y=3x^{m+1}+6x^m$

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2) Differentiate $y = \dfrac{1}{x^2 + 2x + 1}$

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3) If $y = \sqrt{(1 + x^2)}$, prove that $y\dfrac{dy}{dx} - x = 0$

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4) Find $\dfrac{dx}{dy}$ from $y = 4x^2 + 2$

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5) Find the derivative of $y = \log\sqrt{2 + x^2}$

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6) Differentiate $y = e^{2x}\log(2x + 1)$

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7) Find $\frac{dy}{dx}$ given $y = 2m^2 + 6m + 1$ and $x = m^2 + 5$

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9.5 USE OF FIRST ORDER DERIVATIVES IN ECONOMICS

In Economics the rate of change $\frac{dy}{dx}$ of a total quantity function represents its marginal function. That is, $\text{Marginal function} = \frac{d}{dx}(\text{Total function})$

For example,

1) Marginal Utility (MU) = $\frac{d}{dx}$ [Total Utility (TU)]

2) Marginal Cost (MC) = $\frac{d}{dx}$ [Total Cost (TC)]

3) Marginal Revenue (MR) = $\frac{d}{dx}$ [Total Revenue (TR)]

4) Marginal Product (MP) = $\frac{d}{dx}$ [Total Product (TP)]

Derivative and Marginal Analysis

In Economics and other related studies, we are often concerned with functional relationships between two quantities, say, y and x. For example, we may be interested in the relationship between cost and output. Generally two concepts— an average concept and a marginal concept are employed to study such kind of relationships. The *Average* concept focuses on the *total* quantities of y and x. Thus, given the total cost of production and total output, we can determine the average cost of production which in a sense gives us a rough measure of the cost of production of each unit of the total output. *Marginal* concept on the other hand, deals with *changes* in y and x *at the margin*. Thus, we often define marginal cost as the change in the total cost when the output increases by an additional unit. If change in the total cost is given by ΔC and the change in the output is given by Δq, then marginal cost is measured by $\frac{\Delta C}{\Delta q}$. However, a precise measure of marginal cost can be obtained by making the increase in the output as small as possible. In such a situation, marginal cost is given by the limit of $\frac{\Delta C}{\Delta q}$ as $\Delta q \to 0$. If we recapitulate our discussion in Section 9.2, it becomes clear that this concept of marginal cost is in fact a kind of instantaneous rate of change concept and can be interpreted as a derivative.

If our total cost function $C = f(q)$ is smooth, marginal cost is then defined as the derivative of total cost with respect to output and is denoted by $\frac{dC}{dq}$. Some other marginal concepts frequently used in Economics and Business Studies are given below:

Marginal Utility: If total utility (U) is a function of the quantity of a commodity possessed (q), then marginal utility is given by $\frac{dU}{dq}$.

Marginal Product: If a production function represents total product (Q) as a function of the quantity of some factor of production (x), then marginal product of that factor of production is given by $\frac{dQ}{dx}$.

Marginal Revenue: If total revenue (R) is a function of the output sold (q), then marginal revenue is given by $\frac{dR}{dq}$.

Example: If the demand law is given by $p = \frac{10}{q} - 5$, where p is the price and q is the quantity demanded, show that total revenue decreases as output increases, marginal revenue (MR) being a negative constant.

Sol. Let R be the total revenue. Then,

$$R = pq = \left(\frac{10}{q} - 5\right)q = \left(\frac{10-5q}{q}\right)q = 10 - 5q$$

Now as output q increases, $10 - 5q$ decreases. Hence, total revenue R decreases as output q increases.

$$\text{Marginal Revenue, } MR = \frac{dR}{dq} = \frac{d}{dq}(10 - 5q) = -5$$

Thus, marginal revenue (MR) is a negative constant.

Example: The total cost of producing a commodity is given by $C = 100 + 3q + \frac{1}{25}q^2$ where C is the total cost and q is the output. Find out the level of output at which the average cost is equal to the marginal cost.

Sol.

$$\text{Marginal Cost, } MC = \frac{dC}{dq} = \frac{d}{dq}\left(100 + 3q + \frac{1}{25}q^2\right) = 3 + \frac{2}{25}q$$

$$\text{Average Cost, } AC = \frac{C}{q} = \frac{100}{q} + 3 + \frac{q}{25}$$

To find q at which $AC = MC$, we write

$$\frac{100}{q}+3+\frac{q}{25}=3+\frac{2q}{25}$$

$$or \qquad \frac{100}{q}=\frac{q}{25}$$

$$or \qquad q^2=2500$$

$$or \qquad q=50 \qquad \text{(ignoring the negative value of } q\text{).}$$

Elasticity of a Function

Suppose, we have a function $y=f(x)$ and we are interested in measuring the responsiveness of y to a given change in x. The elasticity, η (Greek letter "*eta*") of y with respect to x can be conveniently used to measure this. The elasticity is defined as the ratio of a proportional change in the dependent variable to a proportional change in the independent variable (note that sometimes instead of the phrase *proportional change*, the phrase *percentage change* is used. It does not make any difference). Let us assume that the independent variable changes from its original value x by an amount Δx and as a result, the dependent value changes from its original value y by an amount Δy. The proportional change in the independent variable is given by $\frac{\Delta x}{x}$ and the resultant proportional change in the dependent variable is given by $\frac{\Delta y}{y}$.

The elasticity is then given by,

$$\eta_{yx}=\frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}}=\frac{x}{y}\times\frac{\Delta y}{\Delta x}.$$

This measure is called the *point elasticity* as it measures elasticity at a given point on the curve for the function. It is clear that we get a precise measure of elasticity by making the change in x as small as possible. In that case the point elasticity is,

$$\eta_{yx}=\underset{\Delta x\to 0}{Lim}\left(\frac{x}{y}\times\frac{\Delta y}{\Delta x}\right)=\frac{x}{y}\underset{\Delta x\to 0}{Lim}\frac{\Delta y}{\Delta x}=\frac{x}{y}\times\frac{dy}{dx}$$

Thus, the point elasticity of a function $y=f(x)$ can be defined as the product of the ratio of the initial value of x to the initial value of y and the derivative of y with respect to x. In terms of symbols,

$$\eta_{yx}=\frac{x}{y}\times\frac{dy}{dx}$$

We often use the concept of elasticity in Economics and Business Studies. Let us now discuss some of these elasticities.

1) **Price elasticity of Demand**

   Price elasticity of demand measures the responsiveness of the quantity demanded to a change in price. Normally by *elasticity of demand* we mean this *price elasticity of demand*. It is defined as the ratio of a proportional change in the quantity demanded of a commodity to a proportional change in its price. Suppose, we have a demand function $q = f(p)$, where q is the quantity demanded and p , the price. Then price elasticity of demand is measured by $\eta_{qp} = -\frac{p}{q}\times\frac{dq}{dp}$. Since there exists an inverse relationship between quantity demanded and price, the derivative $\frac{dq}{dp}$ is negative. In order to make the measure of elasticity positive, we put a minus sign before it.

Relationship between Average Revenue, Marginal Revenue and Price elasticity of Demand

Let us consider a demand function $q = f(p)$. Suppose, $R(=pq)$ is the total revenue, AR is the average revenue, MR is the marginal revenue and η_{qp} is the price elasticity of demand. Then we have,

$$MR = \frac{dR}{dq} = \frac{d}{dq}(pq) = p + q\frac{dp}{dq}.$$

Multiplying and dividing the right hand side by p,

$$MR = p\left(\frac{p}{p} + \frac{q}{p}\frac{dp}{dq}\right) = p\left(1 + \frac{q}{p}\frac{dp}{dq}\right) = p\left(1 + \frac{1}{\frac{p}{q}\frac{dq}{dp}}\right)$$

Now, we know $\eta_{qp} = -\frac{p}{q}\frac{dq}{dp}$ and $AR = \frac{R}{q} = \frac{pq}{q} = p$. Substituting these in the expression for MR above we have,

$$MR = AR\left(1 - \frac{1}{\eta_{qp}}\right)$$

$$or\ \frac{MR}{AR} = 1 - \frac{1}{\eta_{qp}}$$

$$or\ \frac{1}{\eta_{qp}} = 1 - \frac{MR}{AR} = \frac{AR - MR}{AR}$$

$$\therefore \eta_{qp} = \frac{AR}{AR - MR}$$

2) **Price elasticity of Supply**

Price elasticity of supply measures the responsiveness of the quantity supplied to a change in price. Sometimes we omit the term *price* and call it just *elasticity of supply*. Price elasticity of supply is defined as the ratio of a proportional change in the quantity supplied of a commodity to a proportional change in its price. Suppose, we have a supply function $q = f(p)$, where q is the quantity supplied and p, the price. Then, price elasticity of supply is measured by,

$$\eta_{qp} = \frac{p}{q}\times\frac{dq}{dp}.$$

3) **Income elasticity of Demand**

Income elasticity of demand measures the responsiveness of demand to a change in income. It is defined as the ratio of a proportional change in demand of a commodity to a proportional change in consumer's income. Suppose, q is the quantity demanded and y,the income. Then ignoring other determinants of demand, income elasticity of demand can be measured by,

$$\eta_{qy} = \frac{y}{q} \times \frac{dq}{dy}$$

4) **Elasticity of Cost**

Elasticity of cost measures the responsiveness of the total cost to a change in output. It is defined as the ratio of a proportional change in total cost of the output to a proportional change in output. Suppose, we have a total cost function $C = f(q)$, where C is total cost and q is output. Elasticity of cost is then measured by,

$$\eta_{cq} = \frac{q}{C} \times \frac{dC}{dq}$$

Example: Find the price elasticity of demand when the demand law is $q = \dfrac{20}{p+1}$ and $p = 3$.

Sol. Price elasticity of demand is given by $\eta_{qp} = -\dfrac{p}{q}\dfrac{dq}{dp}$. Now, we have

$$q = \frac{20}{p+1}$$

$$\frac{dq}{dp} = \frac{d}{dp}\left(\frac{20}{p+1}\right) = \frac{(p+1)\frac{d}{dp}(20) - 20\frac{d}{dp}(p+1)}{(p+1)^2} = \frac{-20}{(p+1)^2}$$

$$\therefore \eta_{qp} = -\frac{p}{q} \times \frac{-20}{(p+1)^2} = \frac{20p}{q(p+1)^2}$$

Putting $q = \dfrac{20}{p+1}$ in the above expression, we get

$$\eta_{qp} = \frac{20p}{q(p+1)^2} = \frac{(p+1)20p}{20(p+1)^2} = \frac{p}{p+1}$$

When $p = 3$, $\eta_{qp} = \dfrac{3}{3+1} = \dfrac{3}{4} = 0.75$

Thus at $p = 3$, price elasticity of demand is 0.75.

Example: A wholesaler supplies 500 kg of potatoes at a price of Rs 10 per kg. When the price increases by 10 per cent, he supplies 510 kg of potatoes. Assume that the producer's supply function is linear. Find his elasticity of supply at the initial price level.

Sol. Let the quantity supplied be q and the price be p. Since the supply function is linear, its equation is that of a straight line and is denoted by $q = a + bp$, where a and b are constants. Now, when the price goes up by 10 per cent, the new price becomes Rs.11 per kg.

When the price is Rs.10 per kg, quantity supplied is 500 kg. Putting these values in the supply function $q = a + bp$, we get

$$500 = a + 10b$$

When the price is Rs. 11 per kg, quantity supplied is 510 kg. Putting these values again in the supply function, we get

$$510 = a + 11b$$

Solving both the equations for a and b, we get a = 400 and b = 10. Therefore, the supply equation becomes

$$q = 400 + 10p$$

We know, elasticity of supply is given by $\eta_{qp} = \frac{p}{q}\frac{dq}{dp}$.From supply function, we get

$$\frac{dq}{dp} = \frac{d}{dp}(400 + 10p) = 10$$

At the initial price p = 10, q = 500. Putting these values in the expression for elasticity of supply, we get

$$\eta_{qp} = \frac{p}{q}\frac{dq}{dp} = \frac{10}{500} \times 10 = \frac{1}{5} = 0.2$$

Hence, at the initial price of Rs.10 per kg, elasticity of supply is 0.2.

Example: Demand for a good, denoted by x, in response to changes in its price p is expressed as $p = \beta - \alpha x$ (where $\alpha, \beta > 0$). Find price elasticity of demand.

Sol. Since demand (x) responds to price (p), therefore, $x = f(p)$

From $p = \beta - \alpha x$, we get $x = \frac{\beta}{\alpha} - \frac{p}{\alpha}$. $\Rightarrow x = \frac{1}{\alpha}(\beta - p)$

$$\therefore \frac{dx}{dp} = 0 - \frac{1}{\alpha} = -\frac{1}{\alpha}$$

By definition, elasticity of demand is $\eta_{xp} = - \frac{dx}{dp} \cdot \frac{p}{x}$

$$\eta_{xp} = - \; -\frac{1}{\alpha} \cdot \frac{p}{\frac{1}{\alpha}(\beta - p)} = \frac{1}{\alpha} \cdot \frac{\alpha p}{\beta - p} = \frac{p}{\beta - p}$$

Example: Supply function of a goodis given as $x = a\sqrt{p - b}$ where p is the price, x is the quantity demanded and a,b are positive constants.

Find the elasticity of supply. What relation do you see between price and elasticity of supply?

Sol. (i) From the given supply function $x = f(p)$, we find that:

$$x = a\sqrt{p-b} = a(p-b)^{1/2}$$

$$\frac{dx}{dp} = a \cdot \frac{1}{2}(p-b)^{-\frac{1}{2}} \times 1 = \frac{a}{2(p-b)^{1/2}}$$

$$\eta_{xp} = \frac{dx}{dp} \cdot \frac{p}{x} = \frac{a}{2(p-b)^{1/2}} \times \frac{p}{a(p-b)^{1/2}} = \frac{\cancel{a}p}{2\cancel{a}(p-b)} = \frac{p}{2(p-b)}$$

(ii) For establishing a clear relation between p and η_{xp}, we

manipulate $\frac{p}{2(p-b)}$ as $\frac{(p-b)+b}{2(p-b)} = \frac{1}{2}\ 1 + \frac{b}{p-b}$

Now as p (which is in the denominator) increases, $(p-b)$ increase, so $\frac{b}{p-b}$ falls and the whole expression $\frac{1}{2}\ 1 + \frac{b}{p-b}$ representing η_{xp} will fall. Thus, rise in price levels will lower elasticity of supply.

Example: The demand for sofa seta (x) is given as $x = 100 + 1.5M$, where M represents consumer income. Find income elasticity of demand, when $M = 10{,}000$.

Sol. We are given income demand curve $x = 100 + 1.5M$

$$\therefore \frac{dx}{dM} = 0 + 1.5 = 1.5$$

So, Income elasticity of demand $e_M = \frac{dx}{dM} \times \frac{M}{x} = 1.5 \times \frac{M}{100+1.5M}$

$$= \frac{1.5M}{100+1.5M}$$

For $M = 10{,}000$, $e_M = \frac{1.5(10{,}000)}{100+1.5(10{,}000)} = \frac{15000}{15100} = \frac{15}{151} = 0.0993$

Example: Verify for the linear demand law $p = 100 - .5x$ for the relation, $e = \frac{AR}{AR - MR}$.

Sol. The given demand function is $p = 100 - .5x$.

Average Revenue, $AR = p = 100 - .5x$

Total Revenue, $TR = px = 100x - .5x^2$

Marginal Revenue, $MR = \frac{d}{dx}(100x - .5x^2) = 100 - x$

So, $e = \frac{AR}{AR-MR} = \frac{100-.5x}{100-.5x-100+x} = \frac{100-.5x}{0.5x}$ …(5)

Now we find the elasticity of demand using $e = -\frac{dx}{dp} \cdot \frac{p}{x}$

From the given function, we have $\frac{dx}{dp} = \frac{1}{\frac{dx}{dp}} = \frac{-1}{.5}$

$$\Rightarrow e = \frac{1}{0.5} \times \frac{100-.5x}{x} = \frac{100-.5x}{0.5x} \qquad \ldots(6)$$

From (5) and (6), see that the relation $e = \frac{AR}{AR-MR}$ is verified for the given demand function.

Example: In respect of the demand function $p = ax^{\beta}\left(a > 0\right)$ answer the following questions:

i) What is the marginal revenue (MR)?

ii) Find the expression for elasticity of demand.

iii) Under what conditions, the demand function will become unit elastic?

Sol. Given $AR: p = ax^{\beta}$ $(a > 0)$

i) Total Revenue, $TR = px = ax^{\beta+1}$ and $MR = \frac{d}{dx}(TR) = a(\beta+1)x^{\beta}$

$$= (\beta+1)ax^{\beta} = (\beta+1)p$$

ii) Elasticity of demand is obtained by, $e = -\frac{dx}{dp} \cdot \frac{p}{x}$.

Here, we have $p = ax^{\beta}$

Therefore, $\frac{dp}{dx} = a\beta x^{\beta-1}$

$$\Rightarrow e = -\frac{dx}{dp} \cdot \frac{p}{x}$$

$$= -\frac{1}{a\beta x^{\beta-1}} . \frac{p}{x} = -\frac{1}{a\beta x^{\beta-1}} . \frac{ax^{\beta}}{x} = -\frac{1}{\beta}$$

That is, elasticity of demand is $e = -\frac{1}{\beta}$.

iii) Demand function will be unitary elastic when $e = 1$. That is, $-\frac{1}{\beta} = 1$ or $\beta = -1$

Check Your Progress 3

1) A monopolist faces a demand function $p = 15 - \frac{1}{5}q$, where p is the price and q, the quantity. Obtain an expression for the marginal revenue. Does the relationship between the average revenue, marginal revenue and elasticity hold for such a market form? At what price will the marginal revenue be equal to zero?

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2) The cost function for a manufacturer is given by $C = 100q - \frac{10}{3}q^2 + \frac{q^3}{9}$, where C is the cost of production and q is the level of output. Find the level of output at which the marginal cost is equal to the average cost.

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3) The output X of a manufacturer is related to the size of the labour force L by the relation,

$$X = 91\,L + 16\,L^2 - L^3$$

i) Find marginal product of labour (MP_L) and average product of labour (AP_L) functions.

ii) Find at what size of the labour force does AP_Lreaches maximum.

iii) At what range of labour size do diminishing marginal returns occur?

iv) If wage rate is Rs 50 per labour, find marginal cost as a function of labour.

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9.6 LET US SUM UP

The derivative of a function is the central idea of differential calculus. It is essentially a limit and its evaluation crucially depends upon the continuity of the function. This unit began with a discussion on Derivatives. The concept of limit is applied to difference quotient to define the derivative of a function. The difference quotient for a function $y = f(x)$, is given by $\frac{\Delta y}{\Delta x}$. It measures the average rate of change of y, per unit change of x in some interval of x. When the limit of the difference quotient is taken as $\Delta x \to 0$, it defines the derivative of y with respect to x and is denoted by $\frac{dy}{dx}$. It measures the rate of change of y for an infinitely small change in x. In a sense, it refers to an instantaneous or marginal rate of change of y for some value of x. Geometrically, the derivative of a function $y = f(x)$ for some value of x is interpreted as the slope of the tangent that passes through the corresponding point on the curve for the given function. For the existence of the derivative of the function at the given value of x, the continuity of the function at that value of x is a necessary condition,but it is not a sufficient condition. An important counterpart of derivative is the marginal concept used in Economics and other related subjects.

Differential calculus deals in deriving another function from a given function. This derived function is called the derivative and is denoted variously. Thus symbolically, if $y = f(x)$ is the given function, then — $y_1, y', \frac{dy}{dx}$ or $f'(x)$ is the derived function called *derivative* or *differential coefficient*; and the process of obtaining the derived function or derivative is called *differentiation.* The unit proceeded further with discussing various standard rules for finding derivativesincluding— differentiation of sum and difference of different functions, the product rule, the quotient rule, the chain rule, etc. We finally saw in the unit how differentiation is applied in Economics, in the 'marginal' functions like marginal revenue, marginal cost, etc, as well as the concept of elasticity. The unit also discussed some other applications of first-order derivatives in Economics.

9.7 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) See section 9.3 and answer.
2) See section 9.3 and answer.

Check Your Progress 2

1) $3(m + 1)\, x^m + 6m\, x^{m-1}$
2) $\frac{-2}{(x+1)^3}$
3) See section 9.4 and answer.
4) $\frac{dx}{dy} = \frac{1}{8x}$
5) $\frac{x}{2+x^2}$
6) $2e^{2x}\left[\frac{1}{(2x+1)} + log\,(2x+1)\right]$
7) $\frac{2m+3}{m}$

Check Your Progress 3

1) See section 9.5 and answer.
2) See section 9.5 and answer.
3) i) $MP_L = 91 + 32\,L - 3\,L^2$; $AP_L = 91 + 16\,L - L^2$

   ii) For a Maximum, put $\frac{d(AP_L)}{dL} = 0 \Rightarrow 16 - 2L = 0 \Longrightarrow L = 8$

   iii) For a Maximum, put $\frac{d(MP_L)}{dL} = 0 \Rightarrow 32 - 6L = 0 \Longrightarrow L = 6$ (approx.). Now, take the second derivative $\frac{d^2(MP_L)}{dL^2} = -6 < 0$ (a negative value indicates a maximum). Diminishing marginal returns occur after the maximum MP_L, which will be when more than 6 people are employed.

   iv) Marginal Cost = $\frac{\text{Wage}}{MP_L} = \frac{50}{91 + 32\,L - 3\,L^2}$

UNIT 10 HIGHER-ORDER DERIVATIVES*

Structure

10.0 OBJECTIVES

After going through this unit, you will be able to:

- explain the concept of the derivative of a derivative;
- define Convexity and Concavity along with the Quasi-convexity and Quasi-concavity;
- get an insight into the Taylor Series formula; and
- develop an understanding for the Mean-Value Theorem.

10.1 INTRODUCTION

In the previous unit, we looked at a very important topic in Mathematical methods, namely *differentiation*. We saw it enabled us to speak about the change in the dependent variable as a result of the change in the independent variable. The change in the dependent variable as a result of an infinitesimally small change in the independent variable is called its *derivative*. If y is the dependent variable and x the independent variable, then $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is the derivative. It is a central concept in Mathematical methods, and is a bedrock of Economic analysis. Economics is full of situations where economic phenomena are explained through relationships among variables through functions, where, in the case of two variables, we say dependent variable y depends on independent variable x. But, it is not enough to just have knowledge about y's dependency on x. We better quantify the change. We better know that if x is increased by a unit (that is an infinitesimal small increase in x), by how much units will y change? Such further insights we get from the derivative. Moreover, the sign of the derivative tells us whether the function is an increasing or decreasing one.

This is all fine, but we should see what happens when we take the derivative of the derivative? When we take the derivative of a function for the first time, it is called the *first-order derivative*, and this is what you studied in the previous unit. You must remember that the derivative of a function is itself a function. In this unit we will study about the processes and outcomes of taking derivatives of derivative. These are called *higher order-derivatives*.

* Contributed by Saugato Sen, SOSS, IGNOU

Higher-order derivatives allow us to see the rate at which the change in y is taking place. The first-order derivative tells us by how much y will change when x changes by a unit, while the second-order derivative tells us if the rate at which y changes as x is changed by a unit itself changes, as x is successively changed. In other words, we want to see whether at higher values of x, as x is changed, y itself changes faster or not.

The next section, section 10.2 of the unit takes up the discussion of the derivative of a derivative (and also even higher-order derivatives) in detail. The section following that, that is, section 10.3 investigates certain geometric properties of curves *i.e.*, of functions, namely convex and concave functions. Finally, the unit discusses two very important ideas utilising higher-order derivatives— the Taylor Series and the Mean Value Theorem.

10.2 DERIVATIVE OF A DERIVATIVE

It is the derivative of the derivative of a given function. For example if $y = f(x)$, then First-order derivative is $\frac{dy}{dx} = f'(x)$ and the second-order derivative will be $\frac{d}{dx} f'(x) = f''(x)$, it can also be denoted by $f_2(x)$ or $\frac{d^2y}{dx^2}$.

For the second-order derivative, the same rules are used as in the case of first-order derivative. Let us now take some examples.

Example 1 If $y = x^5, \frac{dy}{dx} = 5x^4$ and $\frac{d^2y}{dx^2} = 5 \times 4 \times x^3 = 20x^3$

Example 2 If $y = 10x^3, \frac{dy}{dx} = 30x^2$ and $\frac{d^2y}{dx^2} = 30 \times 2 \times x = 60x$

Example 3 If $y = 100, \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = 0$

Example 4 If $y = ax^5 + bx^4 - cx^2, \frac{dy}{dx} = 5ax^4 + 4bx^3 - 2cx$

and $\frac{d^2y}{dx^2} = \frac{d}{dx} \quad \frac{dy}{dx} \quad = 20ax^3 + 12bx^2 - 2c$

Example 5 If $y = x^2 \log x$, then $\frac{dy}{dx} = x^2 \cdot \frac{1}{x} + \log x \cdot 2x$

$= x + 2x \log x$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}(x + 2x \log x) = 1 + 2x.\frac{1}{x} + 2.\log x$

$= 1 + 2 + 2\log x = 3 + 2\log x$

Example 6 If $y = e^x \cdot \log x$, then $\frac{dy}{dx} = e^x \cdot \frac{1}{x} + \log x \cdot e^x$

and $\frac{d^2y}{dx^2} = \frac{d}{dx} \quad e^x \cdot \frac{1}{x} + \log x \cdot e^x \quad = \frac{d}{dx} \quad e^x \cdot \frac{1}{x} \quad + \frac{d}{dx}(\log x \cdot e^x)$

$$= \left[e^x.\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}(e^x)\right] + \left[\log x.\frac{d}{dx}(e^x) + e^x.\frac{d}{dx}(\log x)\right]$$

$$= -\frac{e^x}{x^2} + \frac{2e^x}{x} + e^x \log x = -e^x \quad \frac{1}{x^2} - \frac{2}{x} - \log x$$

Derivatives of Higher Order

So long as the function is differentiable, we can go in finding third y_3 or $\frac{d^3y}{dx^3}$, fourth y_4 or $\frac{d^4y}{dx^4}$, fifth… , n$^{\text{th}}$ order derivative y_n or $\frac{d^ny}{dx^n}$.

All these are called higher order-derivatives. In Economics sometimes third-order derivatives are needed. For maximisation, and minimisation problems mostly, only first and second-order derivatives are used.

Check Your Progress 1

1) Find the third-order derivative of $y = 3x^5 + 23x^2 + 19x$.

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2) Find the second order derivative of $y = e^{3x+2}$

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10.3 CONCAVITY AND CONVEXITY

In this section we will discuss some geometric properties of functions that are second-order derivatives of some function. As you know, the derivative of a function is also a function. And since a second-order derivative is also a derivative, it is also a function. In other words, if *y* is a function of *x*, then so are $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

A rising or a falling function, with constant, increasing or decreasing rates of rising or falling can be either,

a) Linear

or b) Non-linear

We present below graphic representation of such functions (See Figure 10.1 – A, B, C, D, E and F)

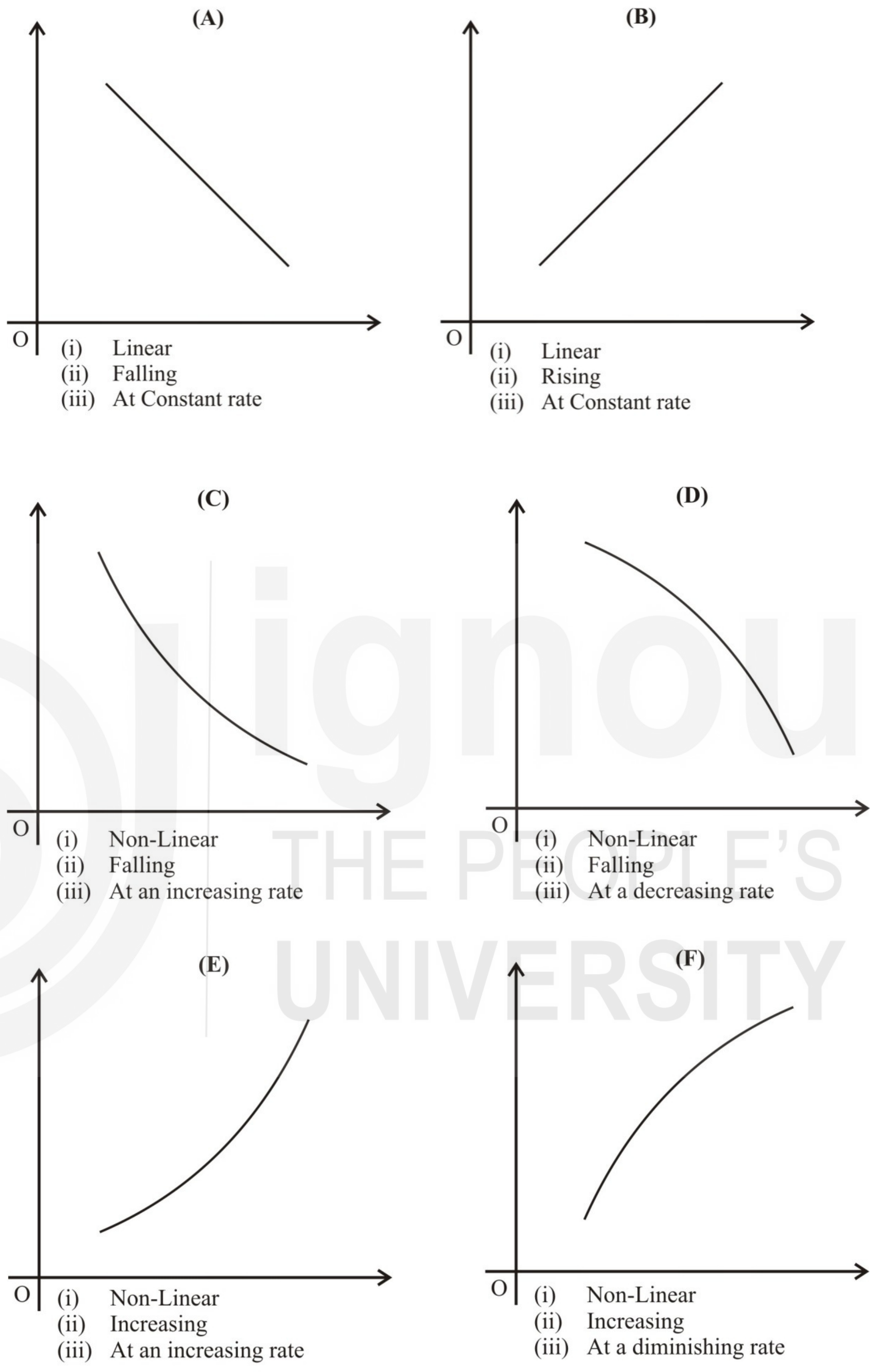

Figure 10.1

Sign of second-order derivative along with that of the first, helps us to identify different types of situations useful in Economic Analysis. They include:

a) A convex function such as indifference curve;

b) A concave function such as Production Possibility curve;

c) Maxima of a function such as maximisation of Total Product;

d) Point of inflexion, etc.

Note: You will learn in detail about the convex and concave functions in the next unit.

Let us now take up combinations of signs of first and second-order derivatives in detail.

Consider a function, $y = f(x)$, which is differentiable twice at least. The following cases depict the characteristics and the graph of function $f(x)$ on the basis of the signs of its first and second-order derivatives.

Case I: $\boxed{\begin{matrix} f'(x)>0 \\ f''(x)>0 \end{matrix}}$ Convex to x-axis

i) $f'(x)>0$ implies that the function is *increasing*.

ii) $f''(x)>0$ implies that the function is increasing at an *increasing rate* $(\theta_2 > \theta_1)$.

Together, it shows that the function is *increasing at an increasing rate*. (refer Figure 10.2 A)

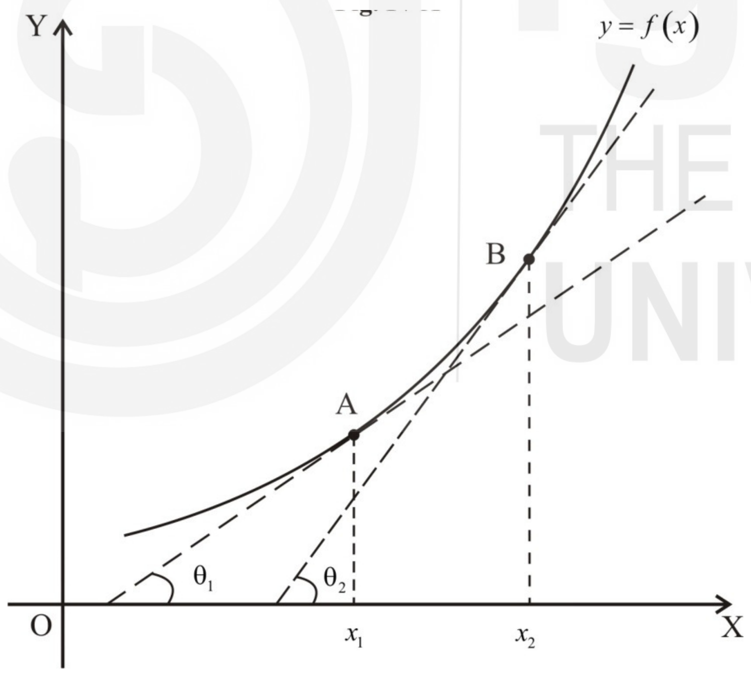

Figure 10.2 A

Case II: $\boxed{\begin{matrix} f'(x)>0 \\ f''(x)<0 \end{matrix}}$ Concave to x-axis

i) $f'(x)>0$ implies that the function is *increasing*.

ii) $f''(x)<0$ implies that the function is increasing at a *diminishing rate* $(\theta_2 < \theta_1)$.

Together, it shows that the function is *increasing at a diminishing rate*. (refer Figure 10.2 B)

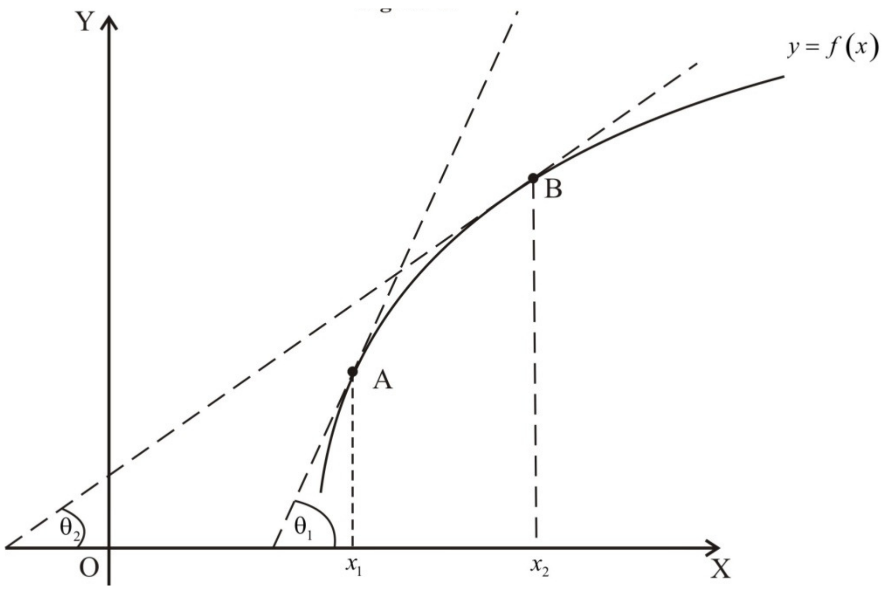

Figure 10.2 B

Case III: $\boxed{\begin{array}{l} f'(x) < 0 \\ f''(x) > 0 \end{array}}$ Convex to origin

i) $f'(x) < 0$ implies that the function is *decreasing*.

ii) $f''(x) > 0$ implies that the function is decreasing at an *increasing rate* $(\theta_2 > \theta_1)$.

Together it shows that the function is *decreasing at an increasing rate.*(refer Figure 10.2 C)

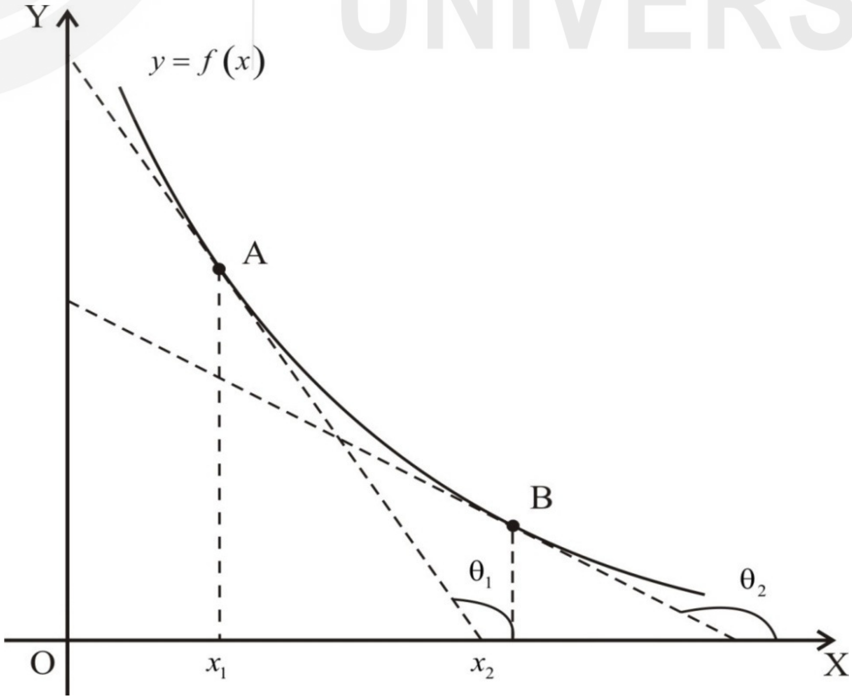

Figure 10.2 C

The curve is convex to origin (O) like an indifference curve which has a negative slope throughout and is convex to origin. An Isoquant, with variable capital coefficients, also slopes downward throughout (*i.e.* has a negative slope) and is convex to the origin.

Case IV: $\boxed{\begin{array}{l} f'(x)<0 \\ f''(x)<0 \end{array}}$ Concave to the origin

i) $f'(x)<0$ implies that the function is *decreasing*.

ii) $f''(x)<0$ implies that the function is decreasing at *diminishing rate* $(\theta_2<\theta_1)$.

Together it shows that the slope of the curve is negative and it decreases at diminishing rate. That is angle of inclination of the tangents falls. (refer Figure 10.2 D). The curve is concave downwards from origin or convex from above. Production possibility curve is a fit case for this type.

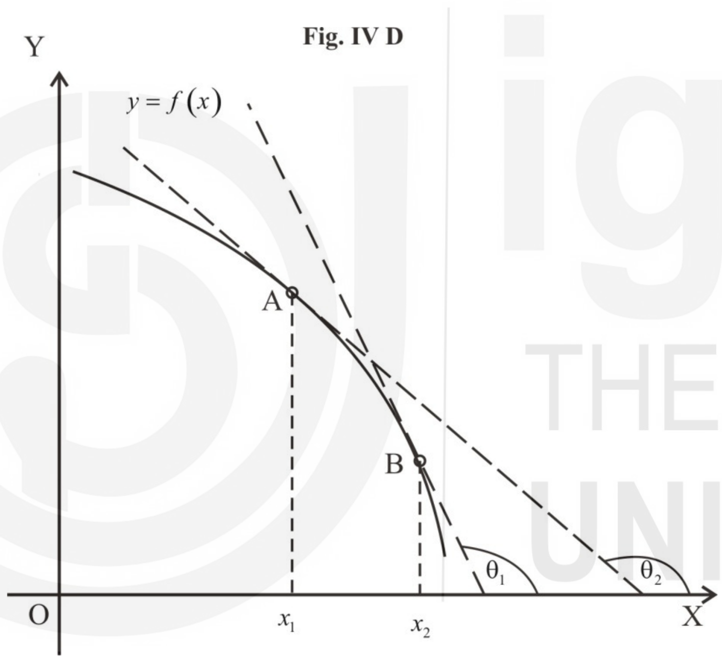

Figure 10.2 D

Case V: $\boxed{\begin{array}{l} f'(x)=0 \\ f''(x)>0 \end{array}}$ Minima

i) $f'(x)=0$ impliesthat the function is neither rising nor falling at a particular point. In Figure 10.2 E, the tangent is horizontal at point A which means that the slope is zero. It is a stationary point, also called a *critical* point. It could be either a minimum point, or a maximum point or a point of inflexion. To identify this, we need second-order derivative.

ii) $f''(x)>0$ implies that the curve has a minima at point A in a neighbourhood of A. The curve $y=f(x)$ is U-shaped having three phases, namely

a) A *falling phase* upto point A— the curve is falling (*i.e.* has a negative slope) implies that $f'(x) < 0$ on this portion of the curve.

b) A minimum and constant phase at point A, where $f'(x) = 0$.

c) A rising phase beyond point A— the curve is rising (*i.e.* has a positive slope) implies that $f'(x) > 0$ on this portion of the curve. (Refer Figure 10.2 E)

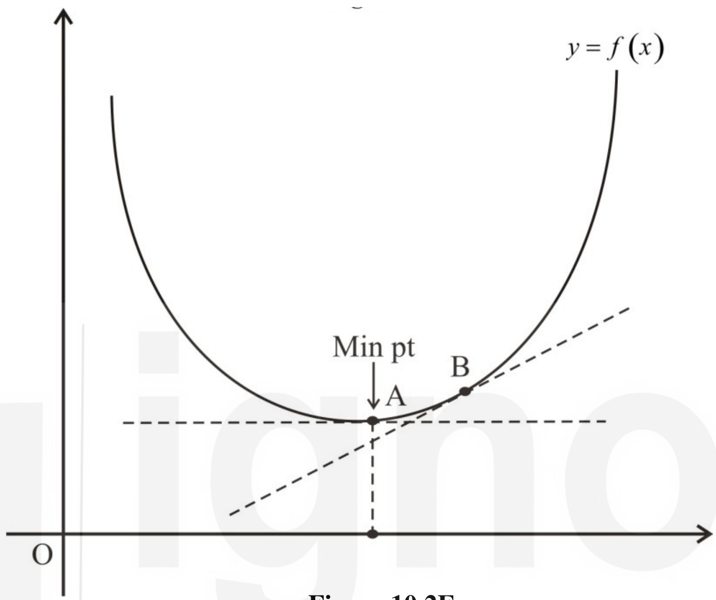

Figure 10.2E

The condition $f'(x) = 0$ is called the *first-order condition* or the *Necessary condition of Minima*. It is same for all extreme values (Minima, Maxima or Inflexion). In order to find which is thecase, we apply second-order condition. For instance, $f''(x) > 0$ in the case of minima. This is called a *Sufficient condition* and is confirmatory in nature.

Let us take one example from Economics. Let $y = f(x)$ be the average cost curve (AC). Let it be represented by a relation, AC $= 5x^2 - 20x + 170$. Let us find its minima.

For minimum AC, the first-order/necessary condition for finding the stationary point is,

$\frac{d}{dx}(\text{AC}) = 0.$

$$\frac{d}{dx}(5x^2 - 20x + 170) = 0$$

$$10x - 20 = 0$$

$$10x = 20$$

$$x = 2$$

Now, the second-order/sufficient condition to find whether the point is minima or maxima, is

$$\frac{d}{dx}\left(\frac{d}{dx}\text{AC}\right) > 0$$

$$\frac{d}{dx}(10x - 20) > 0$$

$$10 > 0$$

Thus, the function has a minima at $x = 2$.

Let us now find the minimum value of AC. For this we put $x = 2$ in the AC function.

$\therefore$ Minimum AC = $5(2)^2 - 20(2) + 170 = 151$

Case VI: $\boxed{\begin{array}{l} f'(x) = 0 \\ f''(x) < 0 \end{array}}$ Maxima

i) As in case V, $f'(x) = 0$ implies that the function is neither rising nor falling at a particular point. In Figure 10.2 F, the tangent is horizontal at point A, meaning that the slope is zero. It is a stationary/critical point which could be either a point of maxima or minima or an inflexion point. As before, to identify this, we need second-order/ sufficient condition.

ii) $f''(x) < 0$ implies that the curve has a maxima at point A in a neighbourhood of A. The curve representing such function is *inverted U-shape* having a maximum point at A.

To illustrate, Let us take an Economic function, say, a profit function (π) as $\pi = 500 + 160x - 2x^2$, where x stands for the advertising expense. We may be interested in finding that level of advertising expense (x) which can maximise our profit.

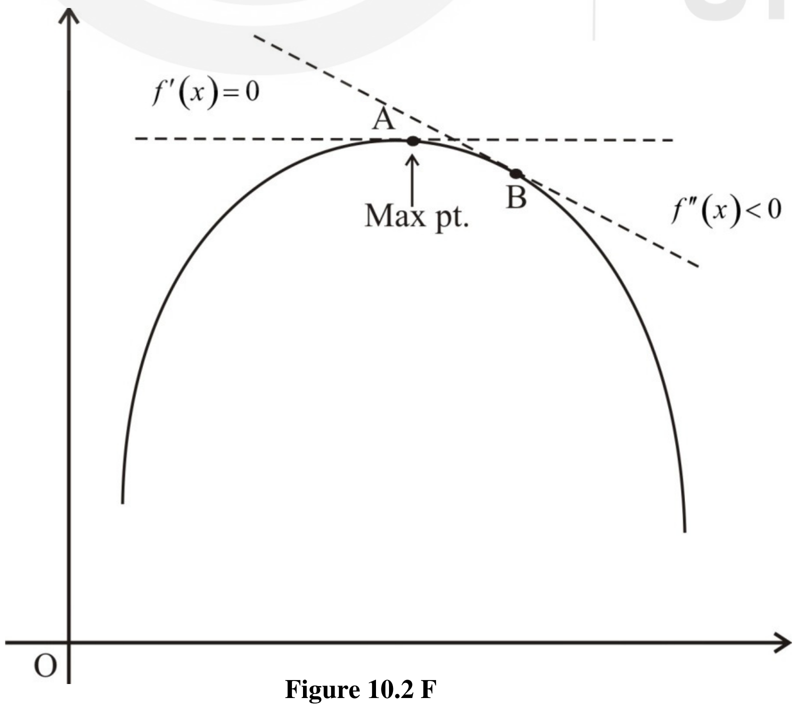

Figure 10.2 F

For maximum profit (π), the first-order/necessary condition for finding the stationary point is,

$\frac{d}{dx}(\pi) = 0.$

$$\frac{d}{dx}(500 + 160x - 2x^2) = 0$$

$$160 - 4x = 0$$

$$x = 40$$

Now, the second-order/sufficient condition to find whether the point is minima or maxima, is

$$\frac{d}{dx}\left(\frac{d}{dx}\pi\right) < 0$$

$$\frac{d}{dx}(160 - 4x) < 0$$

$$-4 < 0$$

∴ Profit is maximum when adverting expenses (x) equals 40 units.

Also, maximum amount of profit is given by,

$$\pi = 500 + 160(40) - 2(40)^2 = 500 + 6400 - 3200 = 3700$$

Thus maximum profit of Rs 3700 can be earned when advertising expenses are 40 units.

Case VII: $\boxed{\begin{array}{l} f'(x) = 0 \\ f''(x) = 0 \end{array}}$ Inflexion point

A single valued function $y = f(x)$ is defined to have an inflexional value at a point where the corresponding curve crosses from one side of its tangent to the other. The point so described is called *inflexion point*. A point of inflexion marks a change in *curvature*.The curvatureof the curve in such case changes from convex to concave or from concave to convex from below as we pass from left to right through the point. All panels of Figure 10.3 show this.

Two classes of Points of Inflexion

Class I: Change of curvature from *Convex to Concave,* irrespective of the slope of the tangent at the point of inflexion. [See Figure 10.3 (A) and (B)]

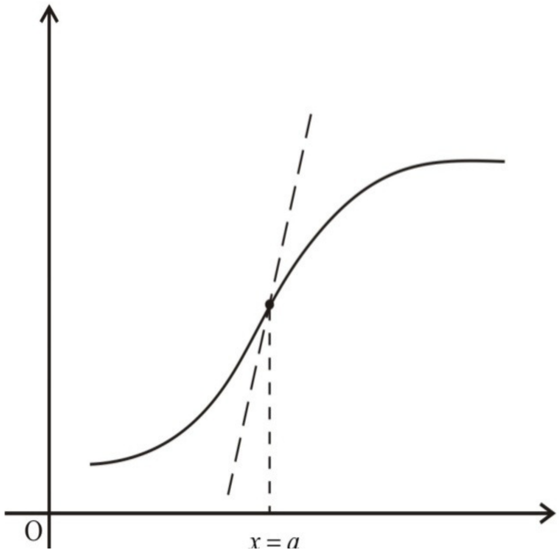

Figure 10.3 A

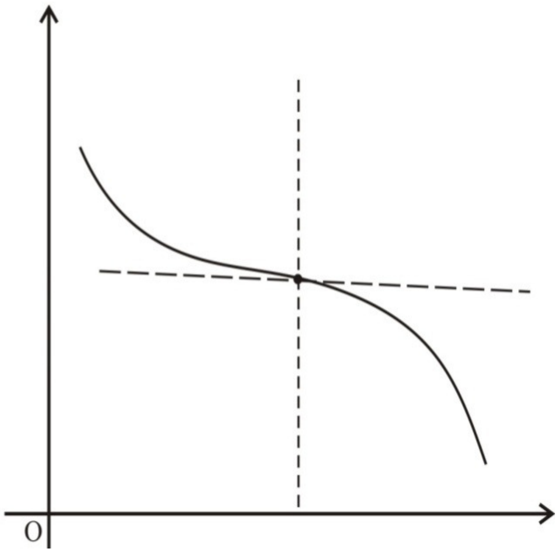

Figure 10.3 B

Class II: Change of curvature from *Concave to Convex*, irrespective of the slope of the tangent at the point of inflexion. [See Figure 10.3 (C) and (D)]

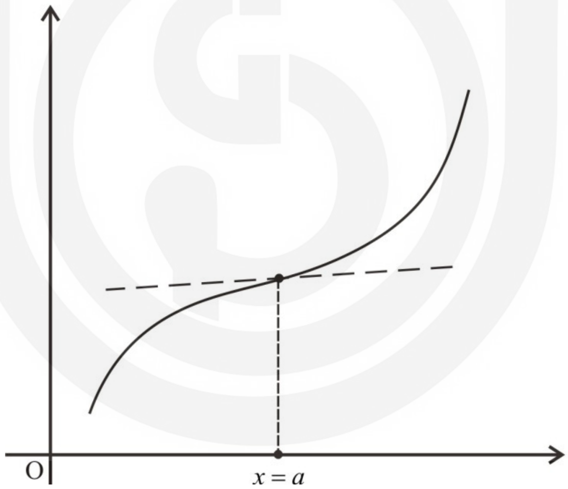

Figure 10.3 C

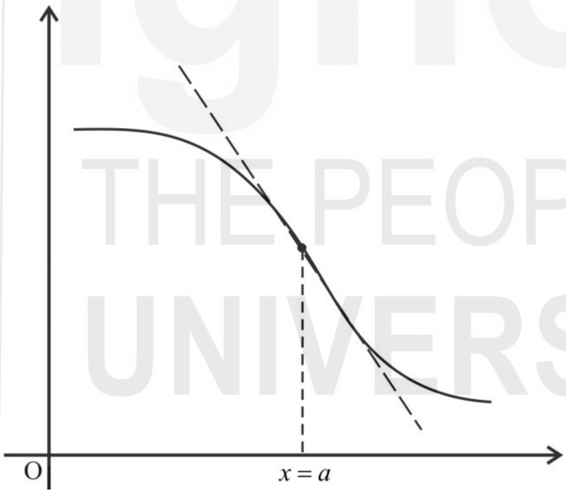

Figure 10.3 D

Equation of the Tangent

One very important use of differential coefficient $\frac{dy}{dx}$ of a function $y = f(x)$ is the equation of a tangent that can be drawn to it at any given point (x_1, y_1) (refer Figure 10.4).

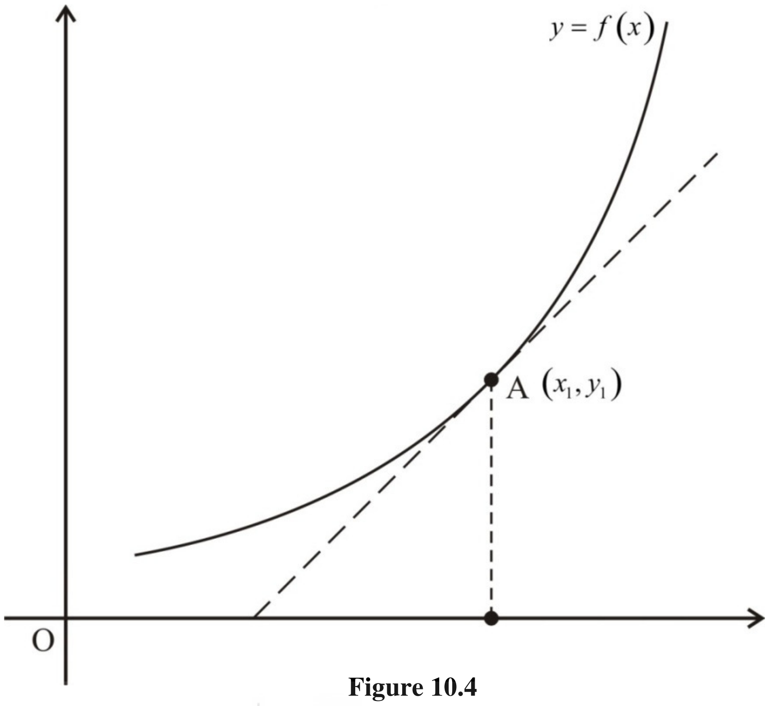

Figure 10.4

The equation takes the form of a point slope equation as discussed in coordinate geometry. It is given by:

$$y - y_1 = m(x - x_1)$$

where, m is the slope and $A(x_1, y_1)$ is the point of tangency with the curve of the given function $y = f(x)$. Slope of the tangent to the curve is the first derivative $\frac{dy}{dx}$ of the given function.

Now, the equation of the tangent can be written as:

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

If the point of tangency is $A(a, b)$, the equation of the tangent can be written as:

$$y - b = \frac{dy}{dx}(x - x_1)$$

Let us find the equation of the tangent to a parabola $y = x^2 + 3x - 2$ at point $A(2,3)$. We know slope of the tangent will be given by $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 2x + 3$$

Therefore, the required equation is:

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y - y_1 = (2x + 3)(x - x_1)$$

$$y = y_1 + (2x+3)(x - x_1)$$

Substituting the values of point A (2, 3) in the above equations, we get

Slope $\quad \dfrac{dy}{dx} = 2 \times 2 + 3 = 7$

Tangent $\quad y = 3 + 7(x-2)$

or $\quad y = 3 + 7x - 14 \quad$ or $\quad y = 7x - 11$

Check Your Progress 3

1) What are conditions for a function to have,

   a) A critical point

   b) A minimum value

   c) A maxima

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2) What can be said about the rate of change of a function on the basis of concavity or convexity of the function?

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10.4 TAYLOR SERIES FORMULA AND MEAN VALUE THEOREM

We know that the differential $dy = f'(x)dx$ can be used to give us an approximation to the change in the y variable, $dy \cong \Delta y$, for a given change in the x variable, $dx \equiv \Delta x$. The percentage error from using dy as an approximation to the actual change, Δy can be made arbitrarily small if we are willing to consider changes in x that are made arbitrarily smaller. However, we may not be satisfied with the restriction that Δx be infinitesimally small, and for non-infinitesimally small changes in the x variable, this approximation may not be very accurate.

The Taylor series expansion formula gives us to go deeper into this issue. The basic idea behind the Taylor series formula is to use information about the value of a function $y = f(x)$ at a specific point, $x = a$, together with information about the value of the derivative function of $f(x)$ at the point $x = a$, in order to obtain the value of the function at a different value of x,

$x = x_0$, within some neighbourhood of the point $x = a$. Let f be the function, differentiable $(n + 1)$ times, on an open interval containing points a and x. Then as per the Taylor theorem:

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f^{(3)}(a)(x-a)^3}{3!} + \frac{f^{(4)}(a)(x-a)^4}{4!} + \ldots + \frac{f^{n}(a)(x-a)^n}{(n)!} + R_n(x)$$

where, $R_n(x) = \dfrac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$ and c lies between a and x.The above formula is sometimes called the remainder form of the Taylor series expansion formula, where $R_n(x)$ is the remainder term.$f^{(n)}(a)$ is the n^{th} derivative of f at the point a, and is assumed to exist. We give the same expression by making use of the summation notation:

$f(x) = f(a) + \sum_{k=1}^{n-1}\left[\dfrac{f^{k}(a)(x-a)^k}{k!}\right] + R_n(x)$. This function is presumed to possess derivatives to the $(n+1)^{\text{th}}$ order. Since we have got the formula or equation, we can see that if we know the value of a function at some point $x = a$, and then use the formula to find the value of the function at some other point $x = x_0$, given by $f(x_0) = f(a) + \sum_{k=1}^{n-1}\left[\dfrac{f^{k}(a)(x_0-a)^k}{k!}\right] + R_n(x_0)$, it is the same as finding how the function f changes as a result of changing x by the amount $\Delta x = x_0 - a$. This we can see by shifting the term $f(a)$ of the expression above to the left side to get $f(x_0) - f(a) \equiv \Delta y$.

Restatement of the Mean Value Theorem from the Taylor's Series formula

We can illustrate the Mean Value theorem for the derivative by taking only one term in the Taylor series formula above with $n = 0$: $f(x) = f(a) + f'(c)(x-a)$ for some $c$between x and a. This brings us to the Mean Value theorem:

If the function $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there must be some $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

On rearranging terms, we get

$f(b) = f(a) + f'(c)(b - a)$, which is what the Taylor theorem claims for $n=0$.

Check Your Progress 3

1) Explain Taylor series formula. What is its significance?

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2) What is the connection between the mean-value theorem and the Taylor series formula?

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10.5 LET US SUM UP

This unit carried on from the previous unit the discussion on derivatives. However, the present unit focused on higher-order derivatives. We investigated what happens when we take derivatives of derivatives. The unit looked at how to compute derivatives of derivatives and also discussed some Economic applications of higher order derivatives. After this the unit moved on to the discussion of some very important geometric properties of functions, namely, concavity and convexity. We saw that the sign of the second-order being negative indicated concavity of the function, while it being positive indicated that the function is convex. The unit also discussed the implications of the function being concave or convex. Finally, the unit discussed in detail two important results that allow us to talk about approximations to a function around a point. The two important results were the Taylor Series and Mean-Value theorem.

10.6 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) $180x^2$

2) $9e^{3x+2}$

Check Your Progress 2

1) *a*) $f'(x) = 0$

   b) $f'(x) = 0$ and $f''(x) > 0$

   c) $f'(x) = 0$ and $f''(x) < 0$

2) Refer section 10.3 and answer.

Check Your Progress 3

1) Refer section 10.4 and answer.

2) Refer section 10.4 and answer.