BLOCK 4
SINGLE-VARIABLE OPTIMISATION

BLOCK 4 INTRODUCTION

The fourth Block makes use of differential calculus to discuss one of the central concepts in economics, that of optimisation. The first three blocks developed concepts that made it possible to fully grasp the material in the present block, titled optimisation. The Block has two units. The first unit of the Block, Unit 11, titled **Concave and Convex Functions**. This unit discusses the nature of convexity and concavity of functions. This makes use of the concepts used in Unit 10 on higher-order derivatives. Concavity and convexity of functions are very useful in the study of optimistion, which is ealt with in the second unit of this Block.

Unit 12, titled **Optimisation Methods**, discusses as the name suggest, the optimisation of functions. Optimisation is a generic term that includes maximisation as well as minimisation of functions. The unit discusses among other things the geometric characterisation of local and global optima, and characterisation using calculus. The unit also provides applications of the techniques of optimisation to economics.

UNIT 11 CONCAVE AND CONVEX FUNCTIONS[*]

Structure

11.0 OBJECTIVES

After going through this unit, you will be able to:

- describe certain important geometric properties of Convex sets;
- understand the relation of a convex set with Convex and Concave functions;
- explain concave and convex function along with their characteristics;
- define the concept of Quasi-Concavity and Quasi-Convexity; and
- discuss some Economic applications of Convexity and Concavity.

11.1 INTRODUCTION

In the previous Unit, while discussing higher order derivatives, we discussed concave and convex function. We saw that concave and convex functions are defined in terms of their second-order derivatives. This unit carries forward that discussion of concavity and convexity. But along with functions, this unit carries the discussion forward to the case of convex sets. We will learn to appreciate the importance of convexity, although the crucial importance of convexity in Economics will become apparent to you only in the units on optimisation. In this unit we will focus on the idea that concavity and convexity are to do with the shapes that certain functions have. We shall see that concavity and convexity of functions are features decided by the second order derivatives of functions. We shall be relating the concept of convexity also to the related but rather important concept of quasi-concavity. Throughout, we shall also be describing some important economic applications of convexity. We shall take functions, mainly from Microeconomics since you have a course on Principles of Microeconomics in this semester, and see which of the

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important functions that you come across in that course are convex and which are concave, and what are the implications of these.

The unit begins with discussion about convex sets as well as convex functions and brings out the relationship between convex sets and convex functions. The unit discusses the relevant geometric properties of convex sets. Following this the unit discusses in detail the nature of concave and convex functions and their properties. The unit also explains carefully what we understand by quasi-concavity and how it relates to convexity. After these theoretical discussions, the unit describes some Economic applications of convexity.

11.2 CONVEX SET AND CONVEX FUNCTION

To understand the geometric properties of functions, we begin with some geometric definitions. Before that, recall the definition and properties of sets. Also recall various set-theoretic functions you have been familiarised with. In the previous unit, we saw the meaning of convexity as applied to functions. We must also understand the meaning of convexity as used to describe a set. Although convex functions and convex sets are related concept, they are also distinct concepts. We should not confuse among the two.

Let us understand the concept of convex sets. A set S of pointsin R^2 or R^3, or in other words, a set S in a real vector space, is definedas convex if for any two points in the set, the line segment connecting these two points lies entirely in the set. A straight line satisfies the requirement of a convex set. A set consisting of a single point is also a convex set. The geometric definition of convexity also applies to solid geometric figures. For example, a solid cube is a convex set, but a hollow cylinder is not. For higher dimensions, the geometric properties become less intuitively obvious.

11.2.1 Convex Combination and Convex Sets

Now we define the concept of a *convex combination* of points. A linear combination of two points,u and v can be written as $k_1 u + k_2 v$, where k_1 and k_2 are numbers. When k_1 and k_2 both lie in the closed interval [0,1] and add up to unity, the linear combination is said to be a *convex combination* and can be written as

$$k_1 u + (1-k_1)v, \ 0 \le k_1 \le 1$$

In view of the above concept, a convex set may be redefined as follows: a set S is said to be a convex set if and only if for any two points $u \in S$ and $v \in S$, and for every number $\theta \in [0,1]$, it is true that $w = \theta u + (1-\theta)v \in S$. Put another way, a set Q is convex if, for all points v_0 and v_1 in Q and for every real number λ in the unit interval [0,1], the point$(1 - \lambda) v_0 + \lambda v_1$is a member of Q. By mathematical induction, a set Q is convex if and only if every convex combination of members of Q also belongs to Q. By definition, a *convex combination* of an indexed subset$\{v_0, v_1, \ldots, v_D\}$ of a vector space is any weighted average $\lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_D v_D$, for some indexed set of non-negative real numbers $\{\lambda_D\}$ satisfying the equation $\lambda_0 + \lambda_1 + \ldots + \lambda_D = 1$.

The definition of a convex set implies that the *intersection* of two convex sets is a convex set. More generally, the intersection of a family of convex sets is a convex set.

Anything that is hollow or dented, for example, a crescent shape, is non convex (Refer figure 11.1). In Figure 11.1 (a), the set Q is convex, since for this, all the points of any line segment joining any pair of points of the set (here, v_0 and v_1) will lie inside the given set Q. Whereas, in 11.1 (b), we have a non-convex set R. In set R, not all the points of the line segment joining any pair of points of the set (here, v_2 and v_3) lies inside the set R. Trivially, the empty set is convex.

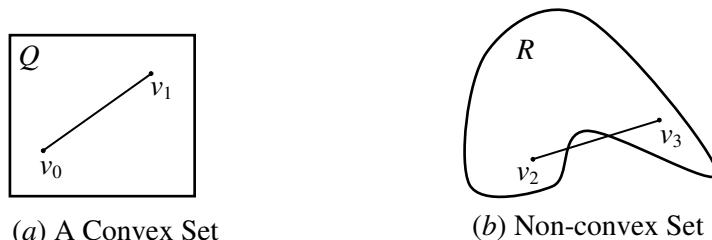(a) A Convex Set   (b) Non-convex Set

Figure 11.1

11.2.2 Convex Set, Convex and Concave Functions

Comparing the definition of a convex set with that of a convex function, we notice that even though the adjective 'convex' is used in both, the meaning of this word is different in the two contexts. In describing a function the word convex denotes how the curve or surface bends itself. *i.e.*, describes the bulge in the curve. In the context of a set, the word convex specifies how the points in the set are 'stacked' together, *i.e.*, how dense is the set. Mathematically, a convex set and convex function appear distinct. However, in some sense they are not unrelated. We know that for the definition of a convex function, we need a convex set for the domain. A convex function is a function with the property that the set of points which are on or above its graph is a convex function. In terms of the definition of a convex set that we just saw, a function is a convex function if it has the property that the chord joining any two points on its graph lie on or above the graph (refer Figure 11.2). For a function that is convex, the set above the function must be convex.

Convex functions are described algebraically as follows: a function *f* is convex if and only if

$$f\left(\alpha x_1 + (1-\alpha) x_2\right) \le \alpha f\left(x_1\right) + (1-\alpha) f\left(x_2\right),$$ for any real numbers $x_1, x_2, \alpha,$ such that $0 \le \alpha \le 1$. In the above expression, if the inequality is a strict one, the function is strictly convex. For convex functions, a general property is that the sum of two convex functions is also a convex function. Also, if a convex function is multiplied by a positive constant, we obtain another convex function.

For a concave function, the set below the function must be convex. Also, the inequality in the expression above is reversed in the case of a concave function. This we will see in more detail in subsequent sections.
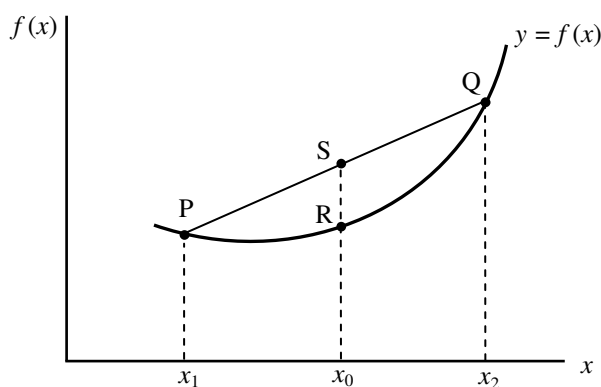
Figure 11.2: Convex Function

Consider the diagram above. For the function $y = f(x)$, let f be a differentiable convex function, and suppose $x_1 < x_2$ and $0 < \alpha < 1$, so that $x_o = \alpha x_1 + (1-\alpha) x_2$.

Now, let points P,Q, R and S in the xy-plane be defined as in the above figure. Since f is convex, R cannot lie above S; therefore the slope of the chord PR cannot be more than the slope of the chord PQ. This inequality holds even as α tends to 1, in which case R approaches P along the curve and the slope of PR approaches the slope of the curve at P. Hence the slope of the curve at P will be less than the slope of the chord PQ. Thus,

$$f'(x_1) \le \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Similarly, as α tends to 0, we obtain

$$f'(x_2) \ge \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We arrive at a very important property of differentiable convex functions:

Property 1: If f is a differentiable convex function, then $f(a+h) \ge f(a) + hf'(a)$ for all a and h.

Another important property of differentiable convex functions is the following:

Property 2: A differentiable function f is convex if and only if $f'(a) \le f'(b)$ whenever $a \le b$.

The discussion of convex and concave using calculus will be continued in the next section, but let us end this section with some discussion on concave functions from a geometric point of view.

We say that a function is concave if the set of all points which are on or below the graph of the function is a convex set. We may say that a function f is concave if and only if $-f$ is convex. Hence from this, we see that the properties of a concave function can be derived from the corresponding properties of convex functions. Thus, a function f is concave if and only if

$$f(\alpha x_1 + (1-\alpha) x_2) \ge \alpha f(x_1) + (1-\alpha) f(x_2)$$

If *f* is a differentiable concave function, then

$$f(a+h) \le f(a) + hf'(a)$$

Also, we say that a function *f* is strictly concave if –*f* is strictly convex.

In the next section we move to a discussion of convex and concave functions using properties from calculus, particularly second derivatives. We will see that if a differentiable function is strictly convex over some interval, then its second derivative is positive over that interval. We will also see that if a differentiable function is strictly concave over some interval, then its second derivative is negative over that interval.

Check Your Progress 1

1) What do you understand by a convex combination? Use the concept of convex combination to explain the concept of a convex set.

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2) In what way is the word 'convex' as used in the context of a convex set different from the way the word convex is used in the context of a convex function?

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3) What is the relation between a convex function and convex set?

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11.3 CONCAVE AND CONVEX FUNCTIONS AND THEIR CHARACTERISTICS

You will now be asked to recall the basic properties and techniques of differentiation that you studied in earlier units in order to understand this section. We briefly touched upon the meaning of a convex function in the previous section in the context of a convex set. In this section we discuss in

detail the concept of convex and concave functions using the tools of differential calculus and see what substantial mathematical meaning is imbued in the concept of a convex function and how it is useful to understand functions in Economics.

Let us, by way of preamble, mention that convex and concave functions have to do with the second derivative of functions. So as background let us recall that if we have $y = f(x)$ as a function, then $f'(x) = \dfrac{dy}{dx}$ is also a function of x.

Now if we take the derivative of $f'(x)$ then it is a derivative of the first derivative of y and is called the second derivative of y. It is denoted in several different ways: $\dfrac{d(dy/dx)}{dx}$ or $\dfrac{d[f'(x)]}{dx}$ or $\dfrac{d^2 y}{dx^2}$ or $f''(x)$.

Just as the first derivative is a function whose derivative can be found (called the second derivative), the second derivative, too, is a function whose derivative can be found (and is called the third derivative, denoted by $f'''(x)$.

If the first two derivatives of a function exist, we say that the function is twice-differentiable. Twice-differentiable functions are very useful in Economics.

11.3.1 Concave and Convex Functions

Let us first understand how the sign of the first-derivative determines whether a function is increasing or decreasing:

$$f'(x) \geq 0 \text{ on } (a,b) \Leftrightarrow f(x) \text{ is increasing on } (a,b).$$

Similarly, $\qquad f'(x) \leq 0 \text{ on } (a,b) \Leftrightarrow f(x) \text{ is decreasing on } (a,b).$

Where, symbol "$\Leftrightarrow$" stands for "if and only if " expression.

Now the second derivative is the derivative of the first derivative. The second derivative being non-negative in an interval, in analogy to the statement above, would mean that the first derivative would be increasing in that interval. We can state the relationship between the sign of the second derivative and whether the first derivative is increasing or not as below:

$$f''(x) \geq 0 \text{ on } (a,b) \Leftrightarrow f'(x) \text{ is increasing on } (a,b)$$

Similarly, $\qquad f''(x) \leq 0 \text{ on } (a,b) \Leftrightarrow f'(x) \text{ is decreasing on } (a,b).$

Let us recall that the first derivative measures the slope of the tangent at that point on the curve f. Hence, first derivative increasing means that the slope of the tangent line is increasing. This means that as x increases, the slope at the tangent to the curve $f(x)$ gets progressively steeper. The change in the steepness of the tangent line leads us towards the idea of convexity using second-order derivatives.

Let us consider a function $f(x) = x^2$. Its first derivative is $2x$ and the second derivative is 2. In the domain $x < 0$, we have $f'(x) = 2x$ but since $x < 0$, $2x < 0$. So in the domain $x < 0$, $f(x) = x^2$ falls. As x increases, the slope falls in absolute value which means that since the slope is negative, the value of the slope is actually increasing in x. Thus the second derivative is positive. We state that a

twice differentiable function $f(x)$ is convex if, at all points on its domain, $f''(x) \geq 0$. A twice differentiable function $f(x)$ is strictly convex if $f''(x) > 0$, except possibly at a single point. For example, consider the function $f(x) = x^4$. It's second derivative is $12x^2$, which is positive for all x except $x = 0$ where the second derivative becomes zero.

Now consider the function $f(x) = 10 - x^2$. We see that the first derivative is $-2x$ and the second derivative is -2. Thus this function is decreasing in x for all $x > 0$ and is increasing in x for all $x < 0$. The slope, however, is falling for all values of x. This means that when $f'(x) > 0$, it is becoming less steep, while when $f'(x) < 0$ the function is becoming more steep in absolute value, but the slope is becoming more negative. Since the function $f(x) = 10 - x^2$ has a negative second derivative, it has properties that are opposite of those of a convex function. It is a concave function. A twice differentiable function $f(x)$ is concave if $f''(x) \leq 0$ on all points of its domain. Also, a twice differentiable function $f(x)$ is strictly concave if $f''(x) < 0$ on all points of its domain except possibly at a single point.

Two points can be made here. First, a linear function, since it has a second derivative equal to zero, satisfies the condition for both a convex function as well as a concave function. Secondly, since multiplying by -1 reverses an inequality, we could say that $f(x)$ is concave if $-f(x)$ is convex, and that $f(x)$ is strictly concave if $-f(x)$ is strictly convex.

Suppose we consider an interval I and assume that $f(x)$ is continuous in I and is twice-differentiable in the interior of I (denoted by I^0 — interior means boundary points of the interval are excluded). Then we can state the following definitions:

A function f is convex on $I \Leftrightarrow f''(x) \geq 0$ for all x in I^0

A function f is concave on $I \Leftrightarrow f''(x) \leq 0$ for all x in I^0

Till now we have considered convex and concave functions over an interval. It would be interesting to see what can be said about convexity and concavity at a particular point. The sign of the second derivative at a point $x = a$ provides some useful information. If $f''(a)$ is positive, then $f(x)$ is changing at an increasing rate as x increases through a, and the slope of the tangent to the curve $y = f(x)$ increases as we pass through the point $x = a$. The tangent to the curve turns in an anticlockwise direction and the curve is convex from below when viewed at this point. On the other hand, if $f''(a)$ is negative, then $f(x)$ changes at a decreasing rate, the slope turns in the clockwise direction, and the curve is concave from below at the point where $x = a$. These results concerning the second derivative are independent of the value of $f'(a)$ and whether the tangent to the curve slopes upwards, downwards or is horizontal at the point $x = a$. Therefore

i) $f''(a) \geq 0$ implies that the function changes at an increasing rate as the function passes through point a and the function is convex from below at the point $x = a$.

ii) $f''(a) \le 0$ implies that the function changes at a decreasing rate as the function passes through point a and the function is concave from below at the point $x = a$.

The numerical value of $f''(a)$ shows how quickly the change in the value of $f(x)$ changes and how great is the curvature of the curve $y = f(x)$ at the point $x = a$.

11.3.2 Points of Inflection

The functions which we study in Economics sometimes have the property that they are convex in some part of the domain but concave in other parts. Such points where a function changes from being concave to being convex, or from being convex to being concave, are called *inflection* points. The curve changes from one side of the tangent to the other side of the tangent. In short, $x = k$ is an inflection point, if $f''(x)$ changes sign at $x = k$. We call the point $[k, f(k)]$ an inflection point on the graph.

The point k is an inflection point for a twice-differentiable function f if there is an interval (a, b) containing k such that one of the following two conditions holds:

i) $f''(x) \ge 0$ if $a < x < k$ and $f''(x) \le 0$ if $k < x < b$, or

ii) $f''(x) \le 0$ if $a < x < k$ and $f''(x) \ge 0$ if $k < x < b$

The most important property of a point of inflection is that it marks a change in curvature, from convex to concave as we move from left to right through the point of inflection, or conversely.

Apart from the change in curvature property, another property of inflection points is that an inflection point always corresponds to an extreme value (maxima or minima) of the slope of the tangent to the curve. You will of course, study maxima and minima in greater detail in the next unit. For any function $f(x)$, at the point where the maxima or minima of $f(x)$ occurs, the first derivative of $f(x) = 0$. In the case of inflection, we are talking of the maxima or minima of the slope of the tangent to the curve at that point. Now, if we have a function $f(x)$, the slope of the tangent to $f(x)$ is $f'(x)$, then *its* first derivative is $f''(x)$, and this has to be 0. Further the value of $f''(x)$ must change in sign as x increases through the point of inflection. We can thus restate the criteria for a point of inflection. If k is a point of inflection, then:

i)   If k is an inflection point for $f(x)$, then $f''(k) = 0$

ii) If $f''(k) = 0$ and $f''(x)$ changes sign at k, then k is an inflection point.

The first is a necessary condition for a point of inflection, and the second is a sufficient one.

If $f(x)$ has a third derivative then an alternative criterion can be given in terms of the third derivative. If $f'''(k)$ is negative at a point where $f''(k) = 0$ then $f'(x)$ is maximum at $x = k$ (note that we are not saying that $f(x)$ is maximum but that $f'(x)$ is maximum). If the third derivative is positive, then $f'(x)$ is minimum at $x = k$.

11.4 QUASI-CONCAVITY AND QUASI-CONVEXITY

Let us now discuss the concept of quasi-concavity and quasi-convexity. In next section, we will get to know the application of concavity and convexity of a function in Economics. Before that we discuss here a weaker requirement for several Economic situations. This condition is called quasi-concavity and quasi-convexity. Quasi-concavity and quasi-convexity, just like concavity and convexity, can be either strict or non-strict. What do quasi-concavity and quasi-convexity mean? Let us look at it geometrically:

Let x and y be two distinct points in the domain of a function f, and let the segment xy in the domain of the function give rise to the arc CD on the graph of the function. Suppose point D is higher or equal in height to point C. Then the function is said to be quasi-concave if all points on the arc CD (other than C and D) are higher than or equal in height at point C. The function f would be a quasi-convex one if all points on the arc CD are equal to or lower in height at point D. The function f would be strictly quasi-concave (quasi-convex) if all points on the arc are strictly higher than point C (lower than point D). We may state here that any strictly quasi-concave (strictly quasi-convex) is quasi-concave (quasi-convex) but the converse is not true. Usually, a quasi-concave function that is not concave has a shape like a bell or like a portion of a bell, and a quasi-convex has a shape like an inverted bell. A concave function is a little like a dome and a convex function like an inverted dome.

Now let us convert these geometric characterisations into an algebraic definition of quasi-concavity and quasi-convexity. A function f is quasi-concave if and only if, for any distinct points x and y in the convex set domain of f, and for $0 < \lambda < 1$,

$$f(y) \geq f(x) \Rightarrow f[\lambda x + (1-\lambda)y] \geq f(x)$$

A function is quasi-convex if and only if, for any distinct points x and y in the convex set domain of f, and for $0 < \lambda < 1$,

$$f(y) \geq f(x) \Rightarrow f\left[\lambda x + (1-\lambda) y\right] \leq f(y)$$

To adapt this definition to strict quasi-concavity and strict quasi-convexity, change the weak inequalities into strict inequalities. From these definitions, we can state three results:

Result 1: If $f(x)$ quasi-concave (strictly quasi-concave), then $-f(x)$ is quasi-convex (strictly quasi-convex).

Result 2: Any concave (convex) function is quasi-concave (quasi-convex) but the converse is not true. Similarly any strictly concave (strictly convex) function is strictly quasi-concave (strictly quasi-convex) but the converse is not true.

Result 3: If a function $f(x)$ is linear, then it is quasi-concave as well as quasi-convex.

Check Your Progress 2

1) Comment upon the Convexity/concavity of the following functionsover the set of non-negative real numbers.

i) $f(x) = -\left(\frac{2}{5}\right)x^2 + 5x - 10$

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ii) $f(x) = 5x^2 - 7x$

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iii) $f(x) = x^{\frac{1}{2}}$

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2)   Describe the concavity and/or convexity of $f(x) = 7x^3 - 42x^2 + 12x + 97$ over its domain?

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3)   The notion of Quasiconcavity is weaker than the notion of Concavity, do you agree?

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4)   For what value of p and q will the graph of the function given by $f(x) = px^3 + qx^2$ pass through $(-1, 1)$ and has an inflection point at $x = \frac{1}{2}$?

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11.5 ECONOMIC APPLICATIONS OF CONVEXITY AND CONCAVITY

In Economics, convexity is one of the most important mathematical properties. For example, in utility maximisation problems, an optimal basket of goods occurs where the consumer's convex preference set is supported by his budget constraint. For a given utility function, and any reference bundle X (x, y), where x and y are the two goods— the set of all the bundles which are at least as good as bundle X, is usually assumed to be the upper level set which is the convex set. This upper level set consists of the consumer's indifference curve (IC) and all the set of x and y lying above it (Refer shaded region in Figure 11.3). This assumption of convexity simply means that consumer attains higher utility from consuming a convex combination of two goods than the extreme bundles, that is, if consumer is indifferent between bundles A and B then she prefers the average bundle C, given by $\lambda A + (1 - \lambda) B$ for $\lambda \in [0, 1]$ to either A or B. Convex preferences results in ICs which are convex to the origin. By convexity, $C = \lambda A + (1 - \lambda) B$ lies on a higher indifference curve, for $\lambda \in [0, 1]$.
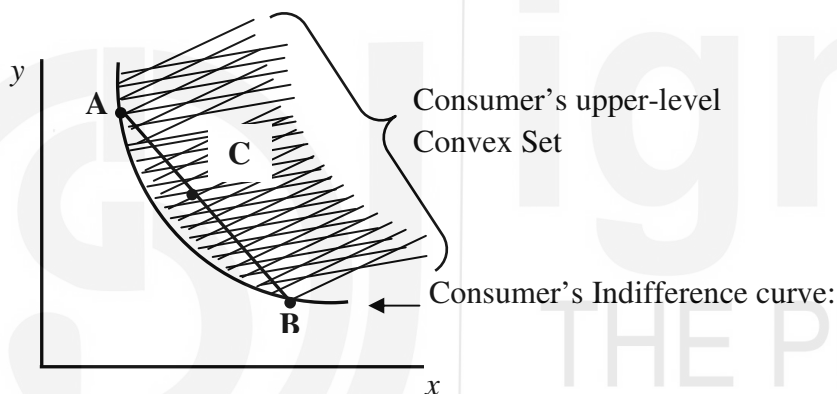

Figure 11.3: Indifference Curve and Convex Set

If the preference set is convex, then the consumer's set of optimal decisions is a convex set, for example, a unique optimal basket (or even a line segment of optimal baskets). If a preference set is non-convex, then some prices produce a budget supporting two different optimal consumption decisions (for instance, happens when consumer faces a choice between two substitutes). The indifference curve (IC) shows different combinations of two goods, here x and y, giving equal level of satisfaction (or utility) to the consumer. Notice it is downward (negative) sloping and convex-shaped. Slope of an IC is the Marginal Rate of Substitution (MRS)— the rate at which a consumer is willing to trade good x for good y, MRS $= \frac{\Delta y}{\Delta x}$ or $\frac{dy}{dx}$ or $\frac{MU_x}{MU_y}$, where MU_x and MU_y refer to the Marginal utility from consumption of additional units of good x and good y, respectively, $[MU_x = \frac{\partial U}{\partial x}$ and $MU_y = \frac{\partial U}{\partial y}$, for a utility function $U(x, y)]$ (refer Figure 11.4).Downward or a negative slope of IC implies that an increase in consumption of one good must be accompanied by decrease in consumption of another good, so as to keep the satisfaction level constant.

Convexity of the curve is a reflection of a diminishing MRS, which simply means that, as consumption of any one good increases more and more, the

individual will prefer to sacrifice lesser and lesser amounts of consumption of the other good, that is $\dfrac{d^2y}{dx^2} \geq 0$. This implies that slope of the tangent to the curve declines as we move down the indifference curve (refer Figure 11.5).
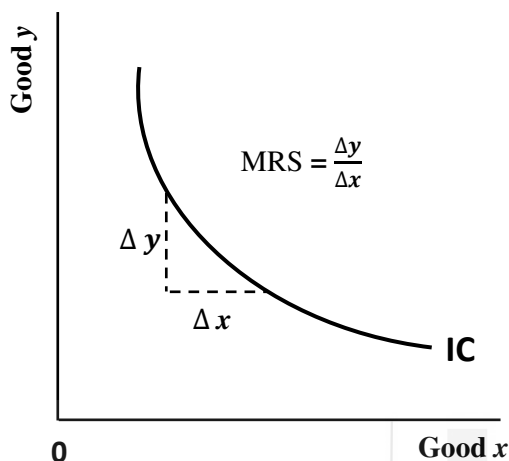

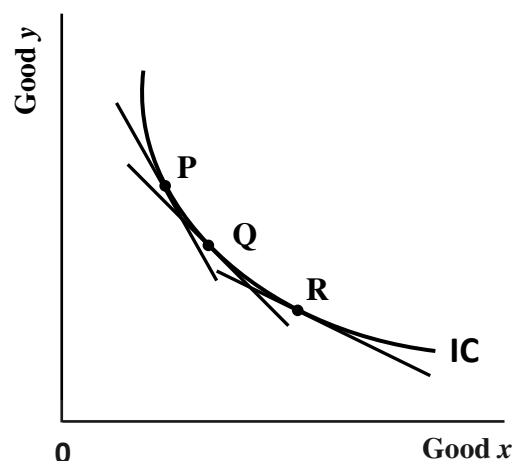Figure 11.4: Slope of an Indifference Curve   Figure 11.5: Diminishing MRS

The above property of a downward sloping and a convex-shaped curve also holds for an isoquant showing efficient combinations of two inputs (or factors) say— labor (L) and capital (K) that produce same level of output. In this case, slope is the Marginal Rate of Technical Substitution (MRTS)—the quantity of K that a firm is willing to sacrifice for employing an additional quantity of L to produce the same level of output. Downward slope results from the assumption of positive marginal productivities of the factors, which imply that an increase in quantity of a factor leads to positive increase in output. So in order to keep output level constant along an isoquant, increasing quantity of one factor must be accompanied by decreasing the quantity of other factor. Convexity of an isoquant results due to the principle of diminishing MRTS, *i.e.*, MRTS declines as we move down on the isoquant.

Now we consider an Economic application of concavity property. A production function $Q = f(L, K)$ gives the amount of output Q that can be produced with given inputs (Labour, L and capital, K). The short run production function holds one of the factor as fixed (here, we consider K as fixed) and gives the total output produced by varying units of L, i.e. $Q = f(L)$. Slope of short-run production function, that is, $\dfrac{dQ}{dL}$ is the Marginal Product (MP) of labour, which equals the additional output produced by one more labour. The positive slope of the curve indicates the fact that MP is never negative (refer Figure 11.6). In the initial output range, i.e. till point A, labour additions results in increasing returns, that is, MP rises— this can be the result of increasing specialisation. However, after point A, additional units of labour input cause diminishing returns, that is, MP starts falling. This is depicted by the concave shape of the production function after point A. This is what is referred to as the law of diminishing marginal returns— as a firm increases more of any one input (here L) while holding other input fixed (here K), the MP of the input being added

will eventually decline. Reason for declining MP could be, as more labour is employed, increasingly more workers end up sharing fixed units of capital, so eventually each worker will be less productive.
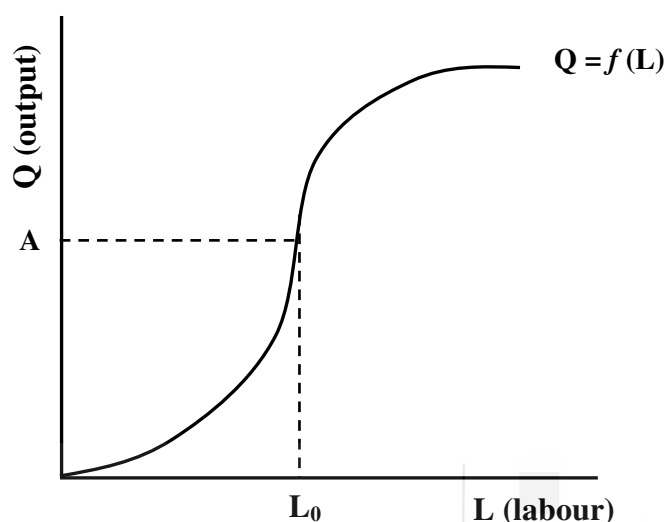

Figure 11.5: Short-runProduction function

Another important concave function is the Production possibility curve (PPC, also called a production possibility frontier, PPF), representing all the possible combinations of two goods (here, x and y) that an economy can produce given fixed amount of resources and technology, and efficient use of these resources. Such a curve is depicted in Figure 11.7. The slope of the PPF ($\frac{dy}{dx}$) at a given point is the amount of good y that would have to be sacrificed to produce an additional unit of good x. In other words, it is the opportunity cost of getting an additional unit of good x. This opportunity cost equals the absolute value of the slope of the PPF. The downward sloping PPF highlights the fact that there is a trade-off between the two goods. This demonstrates the principle of scarcity, as per which, producing more of good x requires shifting resources out of good y production and thus producing fewer units of good y. The concave shape of the curve illustrates the law of increasing opportunity cost, which holds that as an economy moves along its PPF in the direction of producing more of a particular good, the opportunity cost of additional units of that good will increase, that is, the sacrifice of the other good will be more and more. Notice in the figure, as we move from point A to B to C to D, the sacrifice for additional units of good x in terms of good y is increasing.
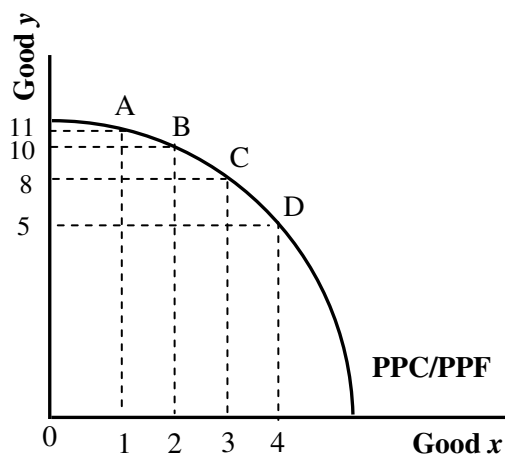
Figure 11.7: Production Possibility Curve

Check Your Progress 3

1) Given the following utility functions, determine whether they obey the assumption of diminishing MRS:

   i) $U(x, y) = 5x + y$

   ii) $U(x, y) = \sqrt{xy}$

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2) Consider an individual for whom extremes are better than averages. What would an indifference curve look like? Would it still imply diminishing marginal rates of substitution?

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3) Why is the production possibility downward sloping and concave?

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11.6 LET US SUM UP

This was the first of the two units in this fourth Block of your present course. The Block is on single-variable optimisation, and this unit has concepts and ideas to help you prepare for the next unit on optimisation methods.

The current unit discussed in detail the very important concept of Convex Sets and Convexity. Applications of Concave and Convex functions occur frequently in Economics and are a foundation for optimisation, which is a central and recurrent idea in Economics.

The unit began by explaining the idea of a convex combination and then used that to introduce you to the concept of a convex set. The unit then moved to a description of the relationship between a convex set and a convex function. In the next section a calculus based discussion of convex and concave functions using second derivatives was presented. It was seen that convex and concave functions can be either increasing or decreasing. The conditions and criteria of convex and concave functions were presented. This section also presented a discussion of inflection points which are points at which the second derivative is zero, and the curvature of the curve changes from convex to concave, or vice-versa.

The subsequent section presented a property that may be present in certain functions, and which provide a weaker requirement for certain Economic conditions. This was the concept of quasi-concavity and quasi-convexity. Both geometric and algebraic explanations were provided about the concept of quasi-concavity and quasi-convexity. Finally the unit presented the application of concavity and convexity in Economics citing examples of Indifference curves, isoquants, short-run production function, production possibility curve.

11.7 ANSWERS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) A *convex combination* of an indexed subset $\{v_0, v_1, \ldots, v_D\}$ of a vector space is any weighted average $\lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_D v_D$, for some indexed set of non-negative real numbers $\{\lambda_D\}$ satisfying the equation $\lambda_0 + \lambda_1 + \ldots + \lambda_D = 1$. A set for which every convex combination lies within the set is a convex set.

2) Although convex functions and convex sets are related concept, they are also distinct concepts. In describing a function the word convex denotes how the curve or surface bends itself, that is it talks about the bulge. In the context of a set, the word convex specifies how the points in the set are 'stacked' together, that is how dense is the set.

3) A convex function is a function with the property that the set of points which are on or above its graph is a convex set. In terms of the definition of a convex set that we just saw, a function is a convex function if it has the property that the chord joining any two points on its graph lies on or

above the graph. For a function that is convex, the set above the function must be convex.

Check Your Progress 2

1) i) Strictly Concave

   ii) Strictly Convex

   iii) Strictly Concave. Hint: $f\ ''(x) = -\dfrac{1}{4\sqrt{x^3}}$, has a negative second derivative for all non-negative real numbers.

2) **Hint:** $f'(x) = 21x^2 - 84x + 12$ and $f\ ''(x) = 42x - 84$

   Convexity requires $f\ ''(x) > 0 \implies 42x - 84 > 0 \implies x > 2$. Thus, function is strictly convex for all $x > 2$, and thus strictly concave for all $x < 2$.

3) Yes, the notion of quasiconcavity is weaker than the notion of concavity. This is because every concave function is quasiconcave, but a quasiconcave function may not be concave.

4) Given that graph pass through $(-1,1) \implies f(-1) = 1 \implies -p + q = 1$ …(1)

   Now, $f'(x) = 3px^2 + 2qx$ and $f''(x) = 6px + 2q$.

   $f(x)$ having inflection point at $\dfrac{1}{2} \implies \qquad f'(\dfrac{1}{2}) = 0$
   $$\implies 3p + 2q = 0 \qquad …(2)$$

   Solving equations (1) and (2), we get $p = -\dfrac{2}{5}, q = \dfrac{3}{5}$.

Check Your Progress 3

1) i) No, the given utility function portrays constant MRS. **Hint:** MRS = $\dfrac{MU_x}{MU_y} \implies \dfrac{5}{1}$

   ii) Yes, MRS is diminishing for this utility function. **Hint:** MRS = $\dfrac{MU_x}{MU_y} \implies \dfrac{y}{x}$. As we move down the IC, that is, as consumption of good x increases, the denominator gets bigger and MRS decreases.

2) The indifference curve for such an individual would bend away from the origin and not towards it, that is, it will be concave-shaped instead of convex. MRS is no longer diminishing along the indifference curve, that is, the indifference curve exhibits increasing MRS.

3) Refer section 11.5 and answer.

UNIT 12   OPTIMISATION METHODS[*]

Structure

12.0   OBJECTIVES

The primary objective of this unit is to characterise Optimal points, which are also called 'extremal points' and lay down the conditions that need to be satisfied to classify a point as an 'extremum', either a maximum or a minimum. Optimum (plural: optima) is the generic term for maximum and minimum. The process of finding an optimum is called optimisation. The basic idea is that a decision maker (for example a consumer or a firm) has an objective function that the decision-maker is attempting to optimise (that is, maximise or minimise). In this unit we shall focus only on objective functions with one single decision variable. Distinction will be made between the 'local optima' and the 'global optima'. After reading the unit you shall be able to:

- define the concept of an extremum point;

- explain the concepts of objective function and decision variable;

- discuss the conditions for a point to be classified as maxima or minima; and

- describe some applications of optimisation problems in economics.

12.1 INTRODUCTION

The question that needs to be answered at the outset is as follows: What is the importance of 'locating an extrema' in the context of Economics? The process is important because an economic unit (consumer, producer, *etc*.) is often faced with various different alternatives. For instance, a consumer has to choose from different commodity bundles, or a producer has to choose amongst various combinations of factor inputs, *viz*. Labour, capital, etc. The economic agent has to choose one particular alternative, which very often either maximise something (e.g. a producer will maximise profit, a consumer maximises her utility) or minimise something (e.g. cost of producing a given

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[*] Contributed by Shri Saugato Sen, SOSS, IGNOU

output). Economically, this process of maximisation or minimisation is characterised as a 'process of optimisation' or 'the quest for the best'. However, from the standpoint of a mathematician, the location of a 'maximum' or a 'minimum' does not carry forth any notion of optimality.

To solve an optimisation problem the first task of the economic agent is to construct an 'objective function'. The dependent variable of this function is the so-called 'object', which has to be either maximised or minimised. The independent variable(s) of the function are the choice/decision/policy variables that can be manipulated by the agent to achieve the desired goal. The optimisation process involves choosing a value of the independent variable that will yield an extreme value ('minimum' or 'maximum' as the case might be) for the requisite dependent variable of the objective function. The process of locating an extremum value (either a maximum or a minimum) is discussed in the following sections. Note, only classical technique for locating extreme positions, using differential calculus, will be discussed. For simplicity it will be assumed that the objective function consists of a single independent variable.

12.2 GLOBAL AND LOCAL OPTIMA

Consider the following objective function where 'f' is assumed to be a continuous differentiable function:

$$y = f(x) \qquad \qquad \text{...(i)}$$

Suppose the relationship between y and x can be graphically represented by figure 12.1. The point at which the graph of the function stops to increase and starts declining (see point A) looks like a little hilltop and the value that the function attains at this point is the largest it attains in its immediate vicinity. Conversely, the point on the graph (see point B) where the function stops decreasing and begins increasing look like a little valley and the value that the function attains at this point is the minimum in its immediate vicinity.

Definition 1: *If $f(x_0) \geq f(x)$ for all x sufficiently close to x_0 then $f(x_0)$ is said to be a relative maximum. If $f(x_0) \leq f(x)$ for all x sufficiently close to x_0 then $f(x_0)$ is said to be a relative minimum. Values that are either relative maxima or relative minima are referred to as relative extreme values or relative extrema.*
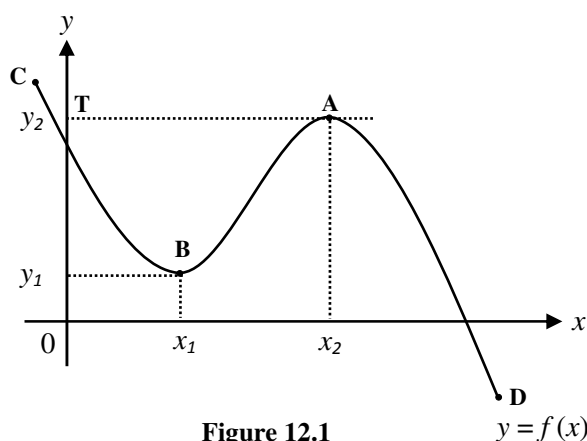

Figure 12.1

$y = f(x)$

Attention needs to be drawn to the term 'relative'. A 'relative' or 'local' maximum need not be the 'absolute' or 'global' maximum. Similarly, a 'relative' or 'local' minimum need not be the 'absolute' or 'global' minimum. This is also evident from the figure 12.1. Comparison between points A (x_2, y_2) and C in figure 12.1, reveals that although the former falls in the category of a relative maximum, it however is not the maximum value attained by the function $y = f(x)$ overall. Similarly, compare points B (x_1, y_1) and D, where point B is a relative minimum, but not the smallest value that the function attains overall.

12.2.1 Slope of a Function

The relative extreme values of a function can also be characterised in terms of its slope. Assume that the total cost (C) incurred by a producer depends on his output (Q) alone. The relationship between total cost and output is represented by the inverse S-shaped curve shown in figure 12.2.
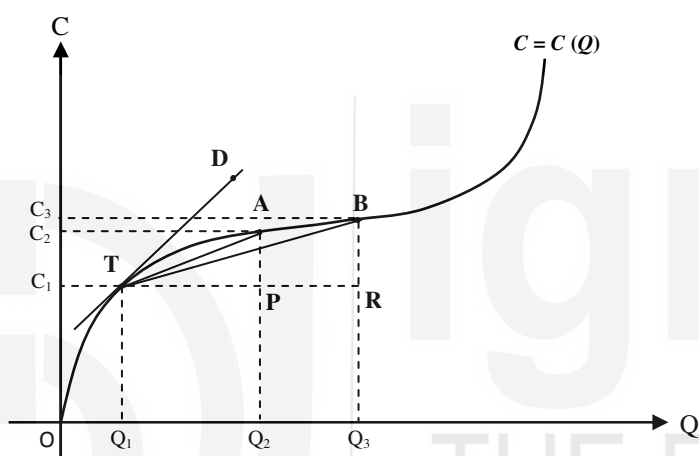

Figure 12.2

Suppose, to begin with, the producer is producing OQ_1 level of output at a cost of OC_1. This output-cost combination corresponds to point T at the total cost curve. To produce an additional output of Q_1Q_3, the producer has to increase his cost by the amount C_1C_3, thus $\Delta C/\Delta Q = C_1C_3 / Q_1Q_3$. Geometrically, this is the ratio of the two line segments, BR/TR and is equal to the slope of the chord TB. This ratio measures the average rate of change in cost for a particular change in output. If we vary the magnitude of change in output, reducing it to smaller margins, what happens to the total cost? For example if the producer instead of increasing his output to OQ_3, increases it to only OQ_2, then the cost increases by a lower margin of C_1C_2. In this case $\Delta C/\Delta Q = C_1C_2 / Q_1Q_2$ = slope of the new chord TA. Now, letting the interval, representing ΔQ grow progressively smaller (*i.e.*, $\Delta Q \to 0$), it more closely approximates a single point, in our case Q_1. The resultant increment in output is measured from the slope of the line segment TD, *i.e.*, the tangent to the total cost curve at point T.

In other words, at Q_1,

$$\lim_{\Delta Q \to 0} \frac{\Delta C}{\Delta Q} = \frac{dC}{dQ} = \text{Slope of the total cost curve at point T} = \text{Slope of the tangent TD.}$$

In general, the slope of a function $f(x)$ at any point is equivalent to the first derivative of the function [symbolically represented as $f'(x)$] at that point and is equal to the slope of the tangent to the function at the requisite point. The first derivative of a function plays a significant role in determining the extrema of the function without even plotting it graphically. This brings us to the second definition of relative maxima and minima outlined as follows:

12.2.2 First Derivative Test and Relative Optima

If the first derivative of a function $f(x)$ at $x = x_0$ i.e., if $f'(x_0) = 0$, then the value of the function at x_0 i.e., $f(x_0)$ will be

i) A relative *maximum* if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point x_0 to its immediate right.

ii) A relative *minimum* if the derivative $f'(x)$ changes its sign from negative to positive from the immediate left of the point x_0 to its immediate right.

iii) Neither a relative maximum nor a relative minimum if $f'(x)$ has the same sign on both the immediate left and right of the point x_0.

The value of the dependent variable, at which, the first derivative of the function is equal to zero i.e. at x_0 is referred to as the *critical* value of x. The value of the function at its critical point i.e. $f(x_0)$ is known as the *stationary* value. The point with the coordinates equal to x_0 and $f(x_0)$ is accordingly called the *stationary point*.

Looking back at figure 12.1, it is evident that points A and B are stationary points. Point A is a relative maximum because for all values of x in the immediate left of x_2, the function is rising (the first derivative of $f(x)$ is positive) and for all values of x in the immediate right of x_2, the function is falling (the first derivative of $f(x)$ is negative). It is only at x_2, the critical point, the first derivative of the function is zero and $f(x_2)$ is the corresponding stationary value. Note, the slope of the tangent to the function at A i.e. AT is parallel to the x-axis and is equal to zero. Analogously, one can see that point B is a point of relative minimum.

To get a clearer understanding of the process of location of extremum values of a function let us turn to the following examples:

Example 1 $y = 50 + 90x - 5x^2$

Step 1: Find the first derivative of the function.

$$f'(x) = 90 - 10x \quad \text{or} \quad 10(9 - x)$$

Step 2: Equate the first derivative of the function to zero

$$10(9 - x) = 0$$

Step 3: Solve for x to obtain the critical value

The critical value of the function is $x = 9$ and the corresponding stationary value is $y = f(9) = 455$.

Let us plot the function in Example 1.

x	−6	−3	0	3	6	9	10	11
y	−670	−265	50	275	410	455	450	435

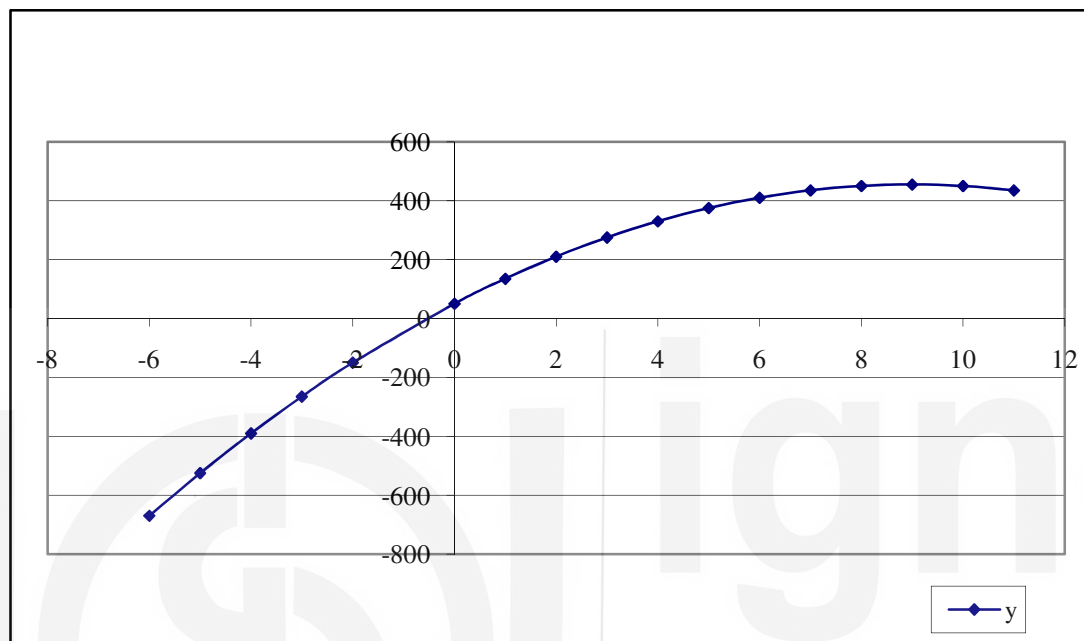The graph of this function is shown in Figure 3.

Figure 12.3: y = 50 + 90x − 5x^2

It is easily verified that for all points in the neighbourhood left of $x = 9$, the function is increasing, implying that its first derivative is positive. Similarly, for all points in the immediate neighbourhood right of $x = 9$, the function is decreasing, implying its first derivative is negative. This satisfies condition (i) of the first derivative test and establishes, the critical value of $x = 9$ (located in the peak of the hill!) as a relative maximum. The corresponding stationary value of the function is $y = 455$.

Example 2

$$y = x^3 - 3x + 5$$

Step1: Find the first derivative of the above function.

$$f'(x) = 3x^2 - 3$$

Step 2: Set the first derivative of the function equal to zero.

$$f'(x) = 3x^2 - 3 = 0 \quad \text{or} \quad 3(x^2 - 1) = 0$$

Step 3: Solve for x to obtain the critical value.

The critical values of x are $x = +1$ and $x = -1$, respectively. The corresponding stationary values of y are $f(+1) = 3$ and $f(-1) = 7$, respectively.

To distinguish between the relative maximum and relative minimum, let us plot the function in Example 2.

x	−3	−2	−1	0	1	2	3
y	−13	3	7	5	3	7	23

The relationship between x and y is graphically represented in Figure 12.4.
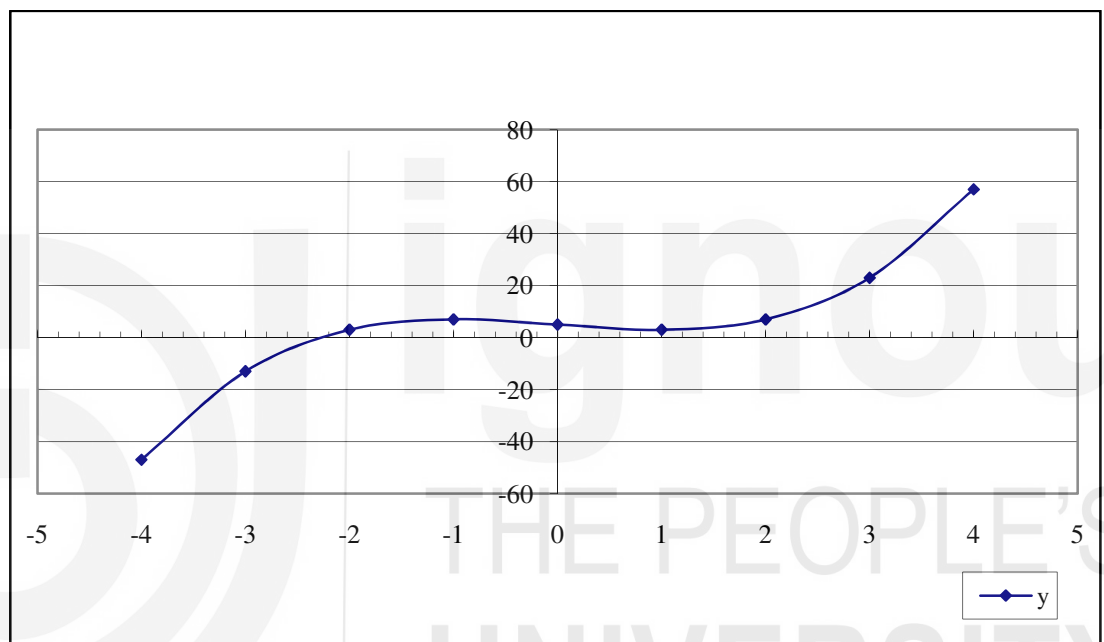

Figure 12.4: y = x³ − 3x + 5

It is easy to verify that $f'(x) > 0$ for $x < -1$ and $f'(x) < 0$ for $x > -1$ in the immediate neighbourhood of $x = -1$. Hence, the corresponding value of the function $f(-1)$ equal to 7 is established as a relative maximum. Similarly, note that $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$ in the immediate neighbourhood of $x = 1$. Consequently, the corresponding value of the function $f(1)$ equal to 3 is established as a relative minimum.

Caution: Zero slope while necessary is not sufficient to establish a relative extremum. If the first derivative of a function $f(x)$ is equal to zero at a value of x say x_0, then this does not automatically ensure that x_0 is a relative extremum of the function.

To get a clearer understanding of this issue let us turn to Example 3

Example 3: Consider the following function whose domain is assumed to be the interval $[0, \alpha)$.

$$y = (1/3)x^3 - x^2 + x + 10$$

Differentiating with respect to x we get the first derivative as

$$x^2 - 2x + 1 \quad \text{or} \quad (x-1)^2$$

Setting the first derivative of the function equal to zero yields $x = 1$ as a critical value of x. The corresponding stationary value of y is 10.33. To ascertain whether the stationary value is also a relative extremum we have to perform the first derivative test. The graphic representation of the function in Example 3 is as follows:

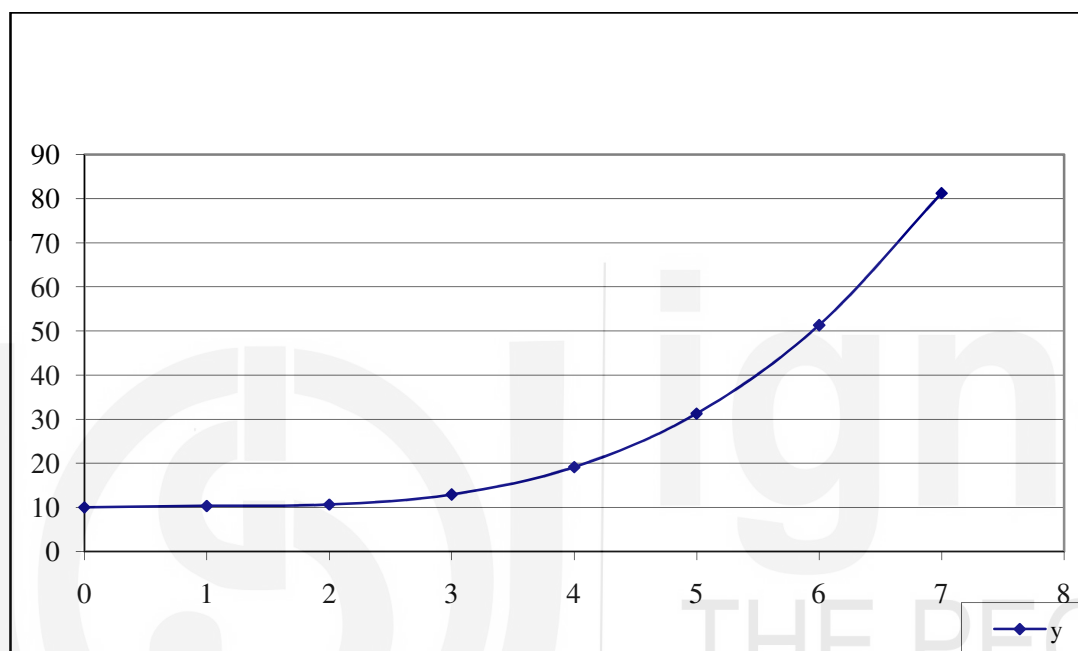| x	0	1	2	3	4	5	6
y	10	10.33	10.66	13	19.33	31.66	52

The graph of the function is presented in Figure 12.5.

Figure 12.5: y = (1/3)x³ - x² +x + 10

The function attains a zero slope at the point where $x = 1$. Even though $f'(1)$ is zero-which implies $f(1)$ is a stationary value— the derivative does not change its sign from the left -hand side neighbourhood of $x = 1$ to the other. In fact, as confirmed by the graph, the function is more or less flat in the immediate region of $x = 1$. On the basis of the first derivative test mentioned earlier it can be asserted that the stationary value $f(1) = 10.33$ is neither a relative maximum nor a relative minimum.

In sum, a relative extremum must be a stationary value (although the reverse is not necessarily true). To find the relative maximum or minimum of a given function, the first step would be to find the critical value of the dependent variable at which the first derivative of the function is equal to zero. This will enable us to find the stationary values of the function. To ascertain whether the stationary value is also a relative maximum or relative minimum one needs to apply the first derivative test.

Check Your Progress 1

1) Find the maximum and minimum values of

   a) $y = x^3 - 3x^2 + 2$

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b) $y = 3x^4 - 4x^3 - 12x^2 + 2$.

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12.3 THE PROBLEM OF NON-DIFFERENTIABILITY

So far we have assumed that the function is a continuous differentiable function. In this section, we will look into two cases where this restrictive assumption is relaxed.
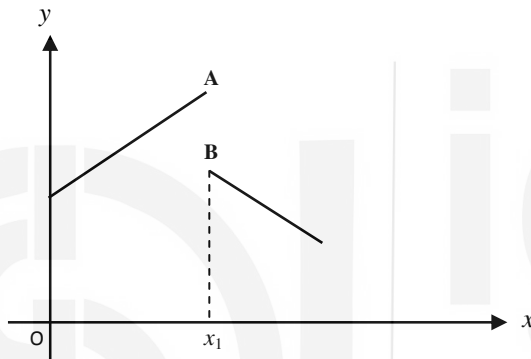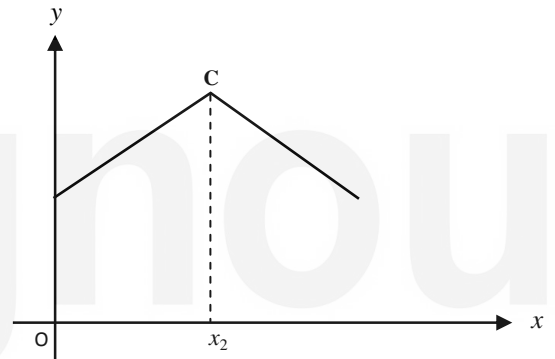

Figure 12.6

Figure 12.7

In figures 12.6 and 12.7, we represent relationships between x and y that exhibit two important types of irregularities. In figure 12.6, the function is discontinuous at x_1. At x_1, the graph has a complete break or discontinuity AB. The difficulty is that in this discontinuous stretch, the first derivative of the function is not even defined. It is not possible to draw a unique tangent to the curve at these points. However, note at A, the function attains a maxima *i.e.*, at x_1 the dependent variably y attains the largest possible value. A similar kind of problem is also encountered in case of the function exhibited in figure 12.7. In this case, the graph of the function has a kink at point C corresponding to x_2. The first derivative of the function is not defined at the kink and there is no unique tangent to the curve at x_2. Once again note that the function attains a maxima at point C corresponding to x_2. In other words, in the presence of discontinuities and kinks the derivative is not defined and hence it is not possible to employ the maximisation criteria outlined earlier.

Hitherto we have only considered first derivative of a function. In this section we will define the second derivative of the function and subsequently see the role played by it in determining the relative extrema of a function.

12.4 THE SECOND ORDER DERIVATIVE AND SECOND-ORDER CONDITION FOR OPTIMUM

Assuming that the first derivative $f'(x)$ is itself a function of x, the second derivative of the function is obtained by differentiating this function again with

respect to x. Symbolically, the second derivative is represented as $f''(x)$. The double prime indicates that the function $y = f(x)$ has been differentiated twice with respect to x. The expression (x) following the double prime indicates that the second derivative is also a function of x. If the second derivative $f''(x)$ exists for all values in the domain, the function $f(x)$ is said to be twice differentiable; if, in addition, $f''(x)$ is continuous, the function $f(x)$ is said to be twice continuously differentiable.

Example 4: Find the second derivative of the following function

$$y = f(x) = 4x^3 + 5x^2 - 3x + 10$$

Step 1: Differentiate the equation in Example 4 with respect to x to find the first derivative. We obtain the following equation:

$$f'(x) = 12x^2 + 10x - 3$$

Step 2: Now differentiate this equation with respect to x to obtain the second derivative of the original function:

$$f''(x) = 24x + 10$$

12.4.1 Interpretation of the Second Order Derivative

The first derivative of the function *i.e.*, $f'(x)$ measures the slope of the function or the rate of change of the function. If the first derivative is positive, *i.e.*, if $f'(x) > 0$, then the function is increasing; and if the derivative is negative *i.e.*, if $f'(x) < 0$, then the function is decreasing. Analogously the second derivative *i.e.*, $f''(x)$ measures the rate of change of the first derivative $f'(x)$. If the second derivative is positive *i.e.*, if $f''(x) > 0$, then there is an increasing rate of change; and when $f''(x) < 0$, then the rate of change is decreasing. In other words, the second derivative measures *the rate of change of the rate of change of the original function $f(x)$*. Note that if $f'(x) > 0$ and $f''(x) > 0$, then this means that the function has a positive slope which is changing at an increasing rate. In other words, the function is said to be increasing at an increasing rate. Conversely, if $f'(x) < 0$ and $f''(x) < 0$, then this means that the function has a negative slope which is changing at a decreasing rate. In other words, the function is said to be decreasing at a decreasing rate.

12.4.2 The Second Order Derivative Test

This test uses the second derivative of the function in question, hence the name.

Assume that $f'(x_0) = 0$, (so x_0 is a critical point!), then

1) If $f''(x_0) > 0$ then $f(x_0)$ is a relative minimum value.

2) If $f''(x_0) < 0$ then $f(x_0)$ is a relative maximum value.

As mentioned earlier, the zero slope condition, in other words $f'(x) = 0$ at $x = x_0$, was deemed to be a 'necessary condition' for $f(x_0)$ to be a relative extremum. Since this is based on the first derivative of the function, it is also known as the *first-order-condition*. Once we verify that the first order condition is satisfied, the negative (positive) sign of $f''(x)$ at $x = x_0$ is sufficient to ensure that x_0 corresponds to a relative maximum (minimum). Since the sufficiency condition is based on the second derivative of the function, it is also referred to as the *second-order-condition*.

To get a clearer understanding of how the second derivative enables us to determine whether the stationary value is a '*relative maximum*' or a '*relative minimum*', let us look back at figure 12.1. Recall, the extreme values of this function lies in the level stretch, either in the bottom of the hill (point B) or at the peak of the hill (point A). Around the point on the graph corresponding to the relative maximum value (point A), the graph is concave down. This makes sense, since A (x_2, y_2) is the highest point on the graph in an interval around x_2, so the graph must 'bend down' away from the peak, making it concave down. If we pick any two points in this region of the graph, then the straight line joining these two points will lie entirely below the graph except for the two end points on the curve.

This ensures that at x equal to x_2, $f''(x_2) < 0$ is satisfied. Similarly, around the point B (x_1, y_1) on the graph in figure 12.1, the graph is convex up. Once again, this makes sense. This point is the lowest point on the graph in an interval around x_1, so the graph has to 'bend up' away from the point (x_1, y_1), making it convex up. If we pick any two points in this region of the graph, then the straight line joining these two points will lie entirely above the graph except for the two end points on the curve. This ensures that at x equal to x_1, $f''(x_1) > 0$ is satisfied.

Let us apply this test to the function in example 2. We have the critical points $x = +1$ and $x = -1$, and the first derivative, $f'(x) = 3x^2 - 3$. The second derivative is then $f''(x) = 6x$. Applying the second derivative test to our two critical points we find that $f''(1) = +6 (> 0)$, making $f(1) = 3$ a relative minimum and $f''(-1) = -6 (< 0)$ making $f(-1) = 7$ a relative maximum.

Similarly, the second order condition proves to be a very useful test to determine whether the stationary value obtained in example 3 corresponds to a relative extremum without actually plotting the function. In this case, we obtained the first derivative, $f'(x) = x^2 - 2x + 1$ and the critical point at $x = +1$. The second derivative is then $f''(x) = 2x - 2$. Applying the second derivative test to our critical point, we find that $f''(+1) = 2 - 2 = 0$. In other words, the *second-order-condition* does not hold at the critical point $x = +1$, establishing that the stationary value $f(+1) = 10.33$ is neither a relative maximum nor a relative minimum.

Check Your Progress 2

1)   Find the relative maxima and minima of *y* by the second-derivative test:

a) $y = \dfrac{1}{3}x^3 - 3x^2 + 5x + 3$

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b) $y = \dfrac{(x-1)^2}{x}$ $(x \neq 0)$

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2) If a monopolist has a total cost of $C = ax^2 + bx + c$ and if the demand law is $p = \beta - \alpha x^2$, show that the output for maximum revenue is

$$x = \frac{\sqrt{a^2 + 3\alpha(\beta - b)}}{3\alpha} - \frac{a}{3\alpha}$$

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12.5 ECONOMIC APPLICATIONS OF OPTIMISATION

We can apply the first and second order conditions depicted earlier to show the determination of the optimal level of output a manufacturer needs to produce in order to maximise his profits.

Suppose, the demand faced by a monopolist is a function of the price of the product alone

i.e., $Q = f_1(P)$. Hence, the total revenue accruing to the monopolist is $R = P.Q = R(Q)$.

Also, the cost of production for the monopolist is a function of his output *i.e.* $C = C(Q)$.

The profit generated by the monopolist, by the definition, is his total revenue net of total cost of production *i.e.* $\pi(Q) = R(Q) - C(Q)$. The monopolist aims at producing that level of output that will maximise his profit. Question is how does he get to know the optimal level of output?

The *first-order-condition* necessitates that the profit function has a zero slope at the optimal point.

Step 1: Differentiate the profit function with respect to Q and equate it to zero.

$$\frac{d\Pi}{dQ} = \Pi'(Q) = \frac{dR}{dQ} - \frac{dC}{dQ} = 0$$

$$\frac{dR}{dQ} = \frac{dC}{dQ} \quad \text{or} \quad MR = MC$$

Where, *MR* shows the Marginal Revenue earned by the monopolist *i.e.*, the increment in revenue (*R*) due to an increment in his sales (*Q*). The Marginal Cost, denoted by *MC*, shows the increment in cost that has to be borne by the monopolist in order to produce an additional output. The first-order condition of profit maximisation shows that for an output level to be optimal it is necessary that $MR = MC$ at that point.

The *second-order-order* condition requires the following:

$$\frac{d^2\Pi}{dQ^2} = \Pi''(Q) = \frac{d^2R}{dQ^2} - \frac{d^2C}{dQ^2} < 0$$

$$\text{or } \frac{d^2R}{dQ^2} < \frac{d^2C}{dQ^2}$$

That is, Slope of *MR* < Slope of *MC* at the optimal level of output.

In sum, the necessary and sufficient condition of profit maximisation requires that the monopolist continue producing up to that point corresponding to which the Marginal Revenue is equal to the Marginal Cost and also the slope of Marginal Revenue function is less than that of his Marginal Cost function.

A Numerical Application

Problem: Determine the optimal level of production for a monopolist whose demand function is *Q = 50 – 0.5P* and his total cost function is *C = 50 + 40Q*. Also, what is the maximum profit earned by the monopolist at that point?

Solution: The linear demand function can be re-written as *P = 100 – 2Q*. Hence, the total revenue function of the monopolist is as follows:

$$R = P.Q = Q(100 - 2Q) = 100Q - 2Q^2$$

Hence, $MR = \dfrac{dR}{dQ} = 100 - 4Q$

The total cost function is *C = 50 + 40Q*

Hence, $MC = \dfrac{dC}{dQ} = 40$

To find the optimal *Q*, we implement the first order condition of profit maximisation and equate *MR* with *MC*

Hence, $100 - 4Q = 40$ or $Q = 15$

To ensure that the second-order-condition of profit maximization holds for this level of output we need to differentiate both the *MR* and *MC* function with respect to *Q*.

We obtain slope of $MR = \dfrac{d^2R}{dQ^2} = \dfrac{d}{dQ}(100 - 4Q) = -4$ and slope of *MC* =

$\dfrac{d^2C}{dQ^2} = 0$

This shows that the *second-order-condition* is satisfied at the optimal level of output because *–4 < 0*.

Substituting the optimal output level i.e. $Q = 15$ in the monopolist's Revenue and Cost function we obtain the profit at this point as

$$\Pi = R(Q) - C(Q) = 1050 - 650 = 400$$

Hence solution to the above problem is $Q = 15$ and $\pi = 400$.

Check Your Progress 3

1) Assume that an entrepreneur's short-run total cost function is given by $C = q^3 - 10q^2 + 17q + 66$. Determine the output level at which he maximises profit if price of the product is 5 per unit.

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2) A radio manufacturer produces x sets per week at a total cost of Rs $\frac{1}{5}x^2 + 13x + 500$. He is a monopolist and the demand of his market is $x = 75 - \frac{3}{5}p$, where the price is Rs. p per set. Show that the maximum net revenue is obtained when about 30 sets are produced per week. What is the monopoly price?

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3) A purely competitive firm produces output Q by incurring a variable cost Rs. V per unit. Its fixed inputs cost the firm a total of Rs. F per period. The price of the output is Rs. P.

   a) Write the revenue function, total cost function and the profit function.

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b) What is the first-order condition for profit maximisation? Interpret the condition economically.

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c) What economic circumstances would ensure that profit is maximised rather than minimised?

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12.6 LET US SUM UP

In this unit, we showed how to find stationary point of a function. We have tried to familiarise you with optimisation of a real valued function. We have taken the case of one independent variable. The unit discussed the concept of relative extreme values. First- order condition for optimisation was discussed. The problem of non-differentiability was taken up. Subsequently the unit went on to discuss second-order derivatives and second-order conditions. Finally, some economic applications of optimisation with one variable were discussed.

12.7 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERSISES

Check Your Progress 1

1) a) Critical points will be 0 and 2. After graphical analyses, you will get maximum at $x = 0$, with maximum value of the function as 2; and minimum at $x = 2$, with minimum value as -2.

b) Critical points will be 0, –1, and 2, with maximum at $x = 0$ and corresponding value of function as 2; minima at $x = -1$ and 2 with corresponding value of function as -3 and -30, respectively.

Check Your Progress 2

1) a) Critical points are $x = 1$ and $x = 5$; Function attains relative maxima at $x = 1$ with relative maximum value of the function being $\frac{-14}{3}$; and it attains relative minima at $x = 5$ with relative minimum value of the function being $\frac{-16}{3}$.

b) Critical points are $x = 1$ and $x = -1$; Function attains relative maxima at $x = -1$ with relative maximum value of the function being -4; and it attains relative minima at $x = 1$ with relative minimum value of the function being 0.

Note: Student should be clear with the fact that relative minimum is not the minimum possible value of the function, whereas relative maximum is not the maximum possible value of the function.

2) Read section 12.4 and answer. Also recall that a Monopoly firm attains equilibrium, or in other words an output level resulting in maximum revenue when Marginal Revenue (MR) = Marginal Cost (MC).

Check Your Progress 3

1) $q = 6$

2) $p = 75$

3)

a) Revenue function (R) : $R(Q) = QP$; Total cost function (C) : $C(Q) = F + QV$; Profit function (P): $P(Q) = R(Q) - C(Q) = QP - (F + QV) = Q(P-V) - F$

b) First order condition for profit maximization:

$$\frac{dP(Q)}{dQ} = 0 \implies \frac{dR(Q)}{dQ} - \frac{dC(Q)}{dQ} = 0 \implies MR = MC$$

Thus, profit maximizing competitive firm will choose Q so that marginal revenue equals marginal cost.

c) To make sure that profit is maximized and not minimized, we use the second-order condition, $\frac{d^2P(Q)}{dQ^2} < 0 \implies \frac{dMR}{dQ} < \frac{dMC}{dQ}$, that is slope of MC curve should be greater than that of MR curve. Recall that for a competitive firm, MR is constant and equal to the market price, so its curve will be a horizontal line at the market price, and thus the marginal cost curve should be upward sloping at the equilibrium point.