



BLOCK 5
INTEGRATION

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BLOCK 5 INTRODUCTION

The fifth Block of the course, titled **Integration**, discusses an important branch of calculus. Block 3 and 4 studied differential calculus, while the current Block studies the other important branch of calculus: integral calculus. The current Block has two units. Unit 13, titled **Indefinite Integrals**, studies integration as the reverse of differentiation. In fact the indefinite integral is sometimes called anti-derivative. The unit discusses properties of integrals. You are also familiarised with techniques of integration—by substitution, by parts etc. Applications to economics are discussed, as for other units and blocks of the course.

Unit 14, whose title is **Definite Integrals**, takes another approach to integration — that of viewing an integral as the limiting value of a sum. Definite integrals are very useful in computing areas under curves. The unit on definite integrals discusses a particular type of integral—the Riemann integral. The unit also discusses properties of definite integrals as well as applications to some concepts in Economics like consumer Surplus, the Domar growth model, and so on.



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UNIT 13 INDEFINITE INTEGRALS*

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13.0 OBJECTIVES

After going through this unit, you will be able to:

- explain the concept of the integral;
- define the indefinite integral as the reverse of differentiation;
- describe some properties of Indefinite Integrals;
- discuss certain techniques of Integration: by substitution, by parts, and by partial fractions; and
- discuss certain applications of integration

13.1 INTRODUCTION

You have studied differential calculus in Units 7 to 10, and also you have used that for optimisation in Block 4. You have seen that differentiation of a function $F(x)$ gives another function $f(x)$. In other words, $F'(x) = f(x)$. Is this operation reversible? Or, given $f(x)$, can we get back the original function $F(x)$? In general, we can. The reverse operation to the derivation is called *anti-derivation* or *integration*. Historically, the concept of integration was developed before that of differentiation. Much later, it was pointed out that differentiation and integration are reverse of one another. In this unit we are introducing the concept of *Anti-derivative*. The present unit aims at introducing some elementary ideas about simple integration.

The next section explains the concept of the integration as the reverse of differentiation. Here it is important to note that the process is called *integration* while the concept itself is called the *integral*. This section also states the result of the integration of certain important functions. The following section deals with certain important techniques of integration, when the direct method of obtaining integration does not succeed. Three such important

* Contributed by Shri Saugato Sen, SOSS, IGNOU

techniques that are discussed are integration by *substitution*, integration by *parts* and integration by *partial fractions*. Finally, the unit provides a discussion of some economic applications of indefinite integrals.

13.2 INTEGRAL AS ANTI-DERIVATIVE

You can recall from unit 9 that if a function $F(x)$ is differentiated, you get $f(x)$, symbolically, $F'(x) = f(x)$. In this unit we shall consider the reverse process — given a function $f(x)$, find some function $F(x)$ such that its derivative is equal to $f(x)$, that is:

$$F'(x) = f(x)$$

Definition 1 : A function $F(x)$ is called an anti-derivative of the function $f(x)$ on the interval $[a, b]$, if at all points of the interval $F'(x) = f(x)$.

Example : Find anti-derivative of the function $f(x) = x^2$

Solution: From definition 1 it follows that the function $F(x) = \frac{x^3}{3}$ is an anti-derivative of the function $f(x)$, since $F'(x) = f(x)$.

$$\frac{d\left(\frac{x^3}{3}\right)}{dx} = x^2$$

It is easy to see that if for the given function $f(x)$ there exists an anti-derivative, then this anti-derivative is not the only one. In the above example, we could also take the following functions as anti-derivatives of $f(x) = x^2$:

$F(x) = \frac{x^3}{3} + 1$, $F(x) = \frac{x^3}{3} + 9$ or, generally, $F(x) = \frac{x^3}{3} + C$ (where C is an arbitrary constant).

$$\frac{d\left(\frac{x^3}{3} + C\right)}{dx} = x^2$$

Since, a function of the form $\frac{x^3}{3} + C$ exhaust all anti-derivatives of the function x^2 .

In general, if for a given function $f(x)$, one anti-derivative $F(x)$ is found, then any other anti derivative of $f(x)$ will have the form $F(x) + C$, where C is a constant.

As seen above, integration is the inverse of differentiation. It is the process of summation. There are two fundamental concepts of this process.

- a) Indefinite integration
- b) Definite integration

The latter we will discuss in the next unit. In this unit we limit ourselves to indefinite integral.

Definition 2 : Indefinite Integral— If the function $F(x)$ is an anti-derivative of $f(x)$, that is, if $F'(x) = f(x)$, then the expression $F(x)+C$ is

the indefinite integral of the function $f(x)$ and is denoted by the integral symbol \int . Thus, by definition

$$\int f(x) dx = F(x) + C$$

Note:

- i) \int is the integral sign, just as $\frac{d}{dx}$ or $\frac{dy}{dx}$ is for differentiation. It is a distorted form of 'S' meaning "sum". This is because originally integration was defined as the sum of a certain infinite series.
- ii) The function $f(x)$ is called integrand, meaning the function which is to be integrated. x is called the integration variable, and C is the constant of integration.
- iii) dx does not mean product of d and x but is a symbol to remind us that integration is being done with respect to x .
- iv) $\int f(x) dx$ means integral of the function $f(x)$ i.e. integral of the integrand.
- v) The process of finding integral of a function is called integration.
- vi) From the definition of integration as anti derivative, it is clear that

$$\frac{d}{dx} \int f(x) dx = f(x)$$

That is, the derivative/ differential coefficient of integration of a function is the function itself. Incidentally, this is the first rule of integration. Since $\frac{d}{dx}(x^2) = 2x$ $\frac{dy}{dx}(x^2) = 2x$, therefore

$$2x dx = 2 \int x dx \quad \text{or} \quad \int 2x dx = 2 \times \frac{x^2}{2} = x^2 + C$$

- vii) An indefinite integral is a family of functions of type $y = F(x) + C$. From geometrical point of view, an indefinite integral is a collection of curves, each of which is obtained by translating one of the curves parallel to itself, upwards or downwards (along y -axis).

A question may arise: Do anti-derivatives and indefinite integrals exist for every function $f(x)$? Answer is No. However, if the function is continuous over an interval $[a, b]$, then this function has an anti-derivative (and therefore indefinite integrals).

If the derivative of an elementary function is always an elementary function, then anti derivative of the elementary function may not prove to be able to be represented by a finite number of elementary functions.

To sum up:

- i) The derivative of an indefinite integral is equal to the integrand, that is, if

$$\left(\int f(x) dx \right)' = (F(x) + C)' = f(x)$$

$F'(x) = f(x)$ then

In simple words, derivative of an anti-derivative is equal to the integrand.

- ii) The differential of an indefinite integral is equal to the expression under the integrand sign:

Note that it is a direct result of expression contained in (i) above.

$$d\left(\int f(x) dx\right) = f(x)$$

- iii) The indefinite integral of the differential of some function is equal to this function plus an arbitrary constant:

One can prove the above assertion by differentiating the above expression on both the sides.

$$\int d F(x) dx = F(x) + C$$

13.3 SOME RULES OF INTEGRATION

Here we mention some basic rules of integration without proof. You are expected to understand these and apply wherever necessary.

Rule 1: The differential coefficient of the integral of a function is equal to the function itself. Symbolically, $\frac{d}{dx} \int f(x) dx = f(x)$.

Rule 2: Integral of a product of a constant 'k' and a function is equal to the product of the constant and the integral of the function.

Symbolically, $\int k f(x) dx = k \times \int f(x) dx$

Rule 3: The integral of the sum or difference of two functions is equal to the sum or difference of their integrals. Symbolically,

$$\int f(x) \pm g(x) \pm \psi(x) \pm \dots dx = \int f(x) dx \pm \int g(x) dx \pm \int \psi(x) dx \pm \dots$$

We now present some standard integrals in this section. These integrals will prove to be very useful in process of integration. These integrals are used frequently in the integration process, and therefore must be learnt by heart.

- 1) Since $\frac{d}{dx} \left[\frac{x^{n+1}}{n+1} \right] = x^n$, therefore $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
- 2) Since $\frac{d}{dx} [\log x] = \frac{1}{x}$, therefore $\int \frac{1}{x} dx = \log x + c$
- 3) Since $\frac{d}{dx} e^n = e^n$, therefore $\int e^n dx = e^n + c$
- 4) Since $\frac{d}{dx} \frac{e^{ax}}{a} = e^x$, therefore $\int e^{ax} dx = \frac{e^{ax}}{a} + c$ or $\int e^{ax} dx = \frac{e^{ax}}{\frac{d}{dx}(ax)} + c$
- 5) Since $\frac{d}{dx} \frac{(ax+b)^{n+1}}{a(n+1)} = (ax+b)^n$, therefore

$$(ax+b)^n dx = \frac{(ax+b)^{n+1}}{a \cdot (n+1)} + c = \frac{(ax+b)^{n+1}}{(n+1) \times \frac{d}{dx}(ax+b)} + c, \quad n \neq -1$$

6) Since $\frac{d}{dx} a^x = a^x \log a$, therefore $\int a^x dx = \frac{a^x}{\log a} + c, \quad a > 0$

7) Since $\frac{d}{dx} \frac{\log(ax+b)}{a} = \frac{1}{ax+b}$, therefore

$$\frac{1}{(ax+b)} dx = \frac{\log(ax+b)}{a} \quad \text{or} = \frac{\log(ax+b)}{\frac{d}{dx}(ax+b)}$$

8) Since $\frac{d}{dx}(x) = 1$, therefore $\int 1 dx = x + c$

[**Note:** In the above rules, we are assuming 'log' to the base 'e', which is also known as the *Natural log*, denoted by 'ln']

Let us take some examples based on above results.

1) $x^5 dx = \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c$

2) $e^{3x} dx = \frac{e^{3x}}{\frac{d}{dx}(3x)} + c = \frac{e^{3x}}{3} + c$

3) $e^{(3x+5)} dx = \frac{e^{(3x+5)}}{\frac{d}{dx}(3x+5)} + c = \frac{e^{(3x+5)}}{3} + c$

4) $\frac{1}{x^5} dx = x^{-5} dx = \frac{x^{-5+1}}{-5+1} + c = \frac{x^{-4}}{-4} + c = \frac{-1}{4x^4} + c$

5) $7 dx = 7 \int 1 dx = 7x + c$

6) Evaluate $3x^4 + 5x^{-3} - 3x^2 + \frac{1}{\sqrt{x}} + \frac{9}{x} dx$

$$I = 3 \int x^4 dx + 5 \int x^{-3} dx - 3 \int x^2 dx + \int x^{-\frac{1}{2}} dx + 9 \int \frac{1}{x} dx$$

$$= 3 \times \frac{x^{4+1}}{4+1} + 5 \times \frac{x^{-3+1}}{-3+1} - 3 \times \frac{x^{2+1}}{2+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + 9 \log x + c$$

$$= \frac{3}{5}x^5 - \frac{5}{2}x^{-2} - \frac{3}{3}x^3 + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + 9 \log x + c$$

$$= \frac{3}{5}x^5 - \frac{5}{2x^2} - x^3 + 2x^{\frac{1}{2}} + 9 \log x + c$$

7) Evaluate $I = (2-7x)(5-3x)(7-2x) dx$

$$I = (2-7x)(6x^2 - 31x + 35) dx = (12x^2 - 62x + 70 - 42x^3 + 217x^2 - 245x) dx$$

$$= 12 \int x^2 dx - 62 \int x dx + 70 \int dx - 42 \int x^3 dx + 217 \int x^2 dx - 245 \int x dx$$

$$= \left(12 \times \frac{x^3}{3}\right) - \left(62 \times \frac{x^2}{2}\right) + 70x - \left(42 \times \frac{x^4}{4}\right) + \left(217 \times \frac{x^3}{3}\right) - \left(245 \times \frac{x^2}{2}\right) + c$$

$$= 4x^3 - 31x^2 + 70x - \frac{21x^4}{2} + \frac{217x^3}{3} - \frac{245x^2}{2} + c$$

$$= -\frac{21x^4}{2} + \frac{229x^3}{3} - \frac{307x^2}{2} + 70x + c$$

8) Find the integration of

$$x^2 + \frac{1}{x^2}$$

$$I = \int \left(x^2 + \frac{1}{x^2}\right)^3 dx = \int \left(x^6 + \frac{1}{x^6} + 3x^4 \cdot \frac{1}{x^2} + 3x^2 \cdot \frac{1}{x^4}\right) dx$$

$$= \int \left(x^6 + \frac{1}{x^6} + 3x^2 + \frac{3}{x^2}\right) dx = \int x^6 dx + \int x^{-6} dx + 3 \int x^2 dx + 3 \int x^{-2} dx$$

$$= \frac{x^7}{7} - \frac{x^{-5}}{5} + \frac{3x^3}{3} - 3x^{-1} + c$$

$$= \frac{x^7}{7} - \frac{1}{5x^5} + x^3 - \frac{3}{x} + c$$

9) Evaluate $\int (e^{\log x^a} + e^{\log a^x}) dx$

$$I = \int (x^a + a^x) dx = \int x^a dx + \int a^x dx$$

[Note: $e^{\log_e x} = x$, and here we are assuming that 'log' has the base 'e']

From standard results on integration, we have $\int a^x dx = \frac{a^x}{\log a} + c, a > 0,$

hence we get $I = \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + c$

10) Evaluate $(7x-2)\sqrt{3x+2} \, dx$

Express $7x-2$ in terms of $3x+2$ by multiplying $(3x+2)$ by $\frac{7}{3}$ and then balancing by adding $\frac{-20}{3}$.

$$\begin{aligned} I &= (3x+2) \cdot \frac{7}{3} - \frac{20}{3} (3x+2)^{\frac{1}{2}} \, dx \\ &= \frac{7}{3}(3x+2)(3x+2)^{\frac{1}{2}} - \frac{20}{3}(3x+2)^{\frac{1}{2}} \, dx \\ &= \frac{7}{3} (3x+2)^{\frac{3}{2}} \, dx - \frac{20}{3} (3x+2)^{\frac{1}{2}} \, dx \end{aligned}$$

$$\begin{aligned} &= \frac{7}{3} \times \frac{(3x+2)^{\frac{5}{2}}}{\frac{5}{2}} - \frac{20}{3} \frac{(3x+2)^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \frac{7}{3}(3x+2)^{\frac{5}{2}} \times \frac{2}{5} - \frac{20}{3}(3x+2)^{\frac{3}{2}} \times \frac{2}{3} + c \\ &= \frac{14}{15}(3x+2)^{\frac{5}{2}} - \frac{40}{9}(3x+2)^{\frac{3}{2}} + c \\ &= \frac{2}{9}(3x+2)^{\frac{3}{2}} \left[\frac{7}{5}(3x+2) - \frac{20}{3} \right] + c \end{aligned}$$

This transformation is meant to use the standard integration form $(ax+b)^n = \frac{(ax+b)^{n+1}}{a(n+1)}$

11) Evaluate $I = \frac{(3x+4)}{\sqrt{2x+7}} \, dx$

Express $(3x+4)$ in terms of $2x+7$ so as to use standard form of integration $(ax+b)^n$

$$\begin{aligned} I &= \frac{\frac{3}{2}(2x+7) - \frac{13}{2}}{(2x+7)^{\frac{1}{2}}} \, dx = \frac{\frac{3}{2}(2x+7)}{(2x+7)^{\frac{1}{2}}} \, dx - \frac{\frac{13}{2}}{(2x+7)^{\frac{1}{2}}} \, dx \\ &= \frac{3}{2} (2x+7)^{\frac{1}{2}} \, dx - \frac{13}{2} (2x+7)^{-\frac{1}{2}} \, dx \quad \text{Both are of the form } (ax+b)^n \\ &= \frac{3}{2} \times \frac{(2x+7)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{13}{2} \cdot \frac{(2x+7)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \end{aligned}$$

$$= \frac{1}{2}(2x+7)^{3/2} - \frac{13}{2}(2x+7)^{1/2} + c$$

12) Integrate $\frac{x^3}{2x+1}$

Sol. $I = \frac{x^3}{2x+1} dx$

$\frac{1}{2}x^2$	$-\frac{1}{4}x$	$+\frac{1}{8}$
x^3	x^3	
x^3	$+\frac{1}{2}x^2$	
$-$	$-$	
	$-\frac{1}{2}x^2$	
	$-\frac{1}{2}x^2$	$-\frac{1}{4}x$
	$+$	$+$
	$\frac{1}{4}x$	
	$\frac{1}{4}x$	$-\frac{1}{8}$
	$-$	$-$
	$-\frac{1}{8}$	

(Note that this is an algebraic expression wherein x in the numerator has higher power compared to x in the denominator. Hence, we can divide x^3 by $2x+1$ through long division method — a simple

and useful method $I = \frac{x^3}{2x+1} dx$

$$I = \int \left[\left(\frac{1}{2}x^2 - \frac{1}{4}x + \frac{1}{8} \right) + \left(-\frac{1}{8} \times \frac{1}{(2x+1)} \right) \right] dx$$

$$\left[\frac{1}{2}x^2 - \frac{1}{4}x + \frac{1}{8} - \frac{1}{8(2x+1)} \right]$$

$\therefore \frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \left(\text{Remainder} \times \frac{1}{\text{Divisor}} \right)$

$$= \frac{1}{2} \int x^2 dx - \frac{1}{4} \int x dx + \frac{1}{8} \int 1 dx - \frac{1}{8} \int \frac{1}{2x+1} dx$$

$$= \left(\frac{1}{2} \times \frac{x^3}{3} \right) - \left(\frac{1}{4} \times \frac{x^2}{2} \right) + \left(\frac{1}{8} x \right) - \left(\frac{1 \log(2x+1)}{8 \times 2} \right) + c$$

$$= \frac{x^3}{6} - \frac{x^2}{8} + \frac{x}{8} - \frac{\log(2x+1)}{16} + c$$

Check Your Progress 1

1) Evaluate the following integrals:

a) $\int (e^{-5x} + e^{5x}) dx$

b) $\int (x - 4x^{\frac{1}{4}}) dx$

c) $\int \frac{dx}{s-2x}$ (where s is any constant)

d) $\int 10x(1-x) dx$

2) Evaluate the following integrals:

a) $\int (x^3 + 15) dx$

b) $\int 15 x^5 dx$

c) $\int \frac{1}{x^5} dx$

d) $\int (3e^{-3x} + 2^x) dx$

e) $\int 5^x dx$

13.4 TECHNIQUES OF INTEGRATION

Above discussion relates to the integration of some standard form mentioned in the section 13.3. In such cases integration can be found out by using some standard formulae. But there are cases where this cannot be done. Here separate techniques have been developed. These are:

- i) Integration by Substitution
- ii) Integration by Parts
- iii) Integration by Partial Fractions

Let us take these one by one with some examples.

13.4.1 Integration by Substitution

Here the integrand, consisting of two functions, becomes easy by substitution of old variable by a new one (say from x to z) through some suitable relation. That is why it is called the method of integration by substitution. For example, if $y = f(x)$, and we have $x = g(z)$, then we apply the following formula:

$$f(x) dx = g(z) \cdot g'(z) dz$$

That is, the integration of a given function $f(x)$ is equal to the integration of another function $g(z)$ multiplied by its derivative $g'(z)$. Follow the following steps in integrating the following functions:

1) $I = (5x+7)^5 dx$

Sol:

Steps: 1) Put the inner expression equal to another variable, say, t

1) Put $5x+7 = t$

2) Find $\frac{dt}{dx}$

2) $\frac{dt}{dx} = 5$

3) Find dx in terms of dt

3) $dx = \frac{dt}{5}$

4) Substitute the value of $(5x + 7)$ and dx in the I (integrand)

$$4) \quad I = t^5 \cdot \frac{dt}{5} = \frac{1}{5} t^5 dt$$

5) Integrate I in terms of t

$$5) \quad = \frac{1}{5} \cdot \frac{t^6}{6} + c = \frac{t^6}{30} + c$$

6) Revert back to x values

$$6) \quad = \frac{1}{30}(5x+7)^6 + c \text{ will be the required answer.}$$

2) Integrate e^{8x+5}

Sol: Put $8x+5=t$ so that $\frac{dt}{dx} = 8$ and $dx = \frac{dt}{8}$

$$I = e^t \frac{dt}{8} = \frac{1}{8} e^t dt = \frac{1}{8} e^t + c = \frac{1}{8} e^{8x+5} + c$$

3) Evaluate (a) $\frac{x^3}{(x^2+1)^3} dx$ (b) $(4x+2)\sqrt{x^2+x+1} dx$

Sol: a) $I = \frac{x^3}{(x^2+1)^3} dx$

Put $t = x^2 + 1$ so that $t-1 = x^2$ and $\frac{dt}{dx} - 0 = 2x$

or $x dx = \frac{dt}{2}$. Let us write $\frac{x^3}{(x^2+1)^3} dx$ as $\frac{x^2}{(x^2+1)^3} \times x dx$

$$I = \frac{t-1}{t^3} \cdot \frac{dt}{2} = \frac{1}{2} \frac{t-1}{t^3} dt = \frac{1}{2} \left(\frac{t}{t^3} dt - \frac{1}{t^3} dt \right)$$

$$= \frac{1}{2} t^{-2} dt - \frac{1}{2} t^{-3} dt = \frac{1}{2} \frac{t^{-1}}{(-1)} + \frac{1}{2} \frac{t^{-2}}{2} + c = -\frac{1}{2t} + \frac{1}{4t^2} + c$$

or
$$= -\frac{1}{2t} \left(1 - \frac{1}{2t} \right) + c.$$

b) $(4x+2)\sqrt{x^2+x+1} dx$

Put $t^2 = x^2 + x + 1$ so that $2t \cdot \frac{dt}{dx} = 2x + 1$

or $(2x+1) dx = 2t \cdot dt$

We can write $I = 2 (2x+1)(x^2+x+1)^{\frac{1}{2}} dx$

$$\text{or } I = 2 t \cdot 2t dt = 4 t^2 dt = 4 \times \frac{t^3}{3} + c = \frac{4t^3}{3} + c$$

Substituting the value of $t = (x^2 + x + 1)^{\frac{1}{2}}$, we get
 $\frac{4}{3}(x^2 + x + 1)^{\frac{3}{2}} + c$.

4) Find $\int \left[\frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} \right] dx$

Sol: Note that numerator is the derivative of denominator and that

$$\frac{d}{dx}(10^x) = 10^x \log_e 10. \text{ Therefore,}$$

Put $t = x^{10} + 10^x$ so that $\frac{dt}{dx} = 10x^9 + 10^x \log_e 10$

$$\text{or } dt = (10x^9 + 10^x \log_e 10) dx$$

The Integrand becomes $I = \frac{dt}{t} = \log t + c = \log(x^{10} + 10^x) + c$

13.4.2 Integration by Parts

This method is more powerful than the method of substitution. It is based on the product formula used under differentiation. It transforms one integration problem into another which is easier to handle. Symbolically,

$$f(x).g(x)dx = f(x) \times dx - f'(x) \times g(x)dx \quad dx$$

where f and g are two differentiable functions.

In words, we can state it as under

Integral of the product of two functions is equal to the first function multiplied by the integral of the second function less integral of the product of differential coefficient of the first function and the integral of the second function.

Before, we attempt some question, let us note some important tips.

- 1) This method is useful when the integrand is given as the product of two differentiable functions f and g in a given variable, say, x .
- 2) Choose a power/ algebraic function of x as the first function, and the second function which is easy to integrate.
- 3) We can rearrange the two functions as first and second function, according to our convenience, even at a later stage.
- 4) When the integrand is not a product of two functions we can treat 1 (one/ unity) as the second function. For example:

$$f(x)dx = f(x) \times 1dx$$

- 5) When the product is of Inverse (I), Logarithmic (L), Algebraic (A), Trigonometric (T) and Exponential (E). Then the first function will be one which comes first in the order ILATE (To remember I, LATE)
- 6) If an integrand consists of functions that do not integrate, then see if its parts can be integrated.

Let us attempt some questions to illustrate the method.

1) Integrate the following functions by the method of integration by parts.

a) $x^3 e^x$ b) $x \log x$ c) $\frac{\log x}{x^2}$ d) $x^4 \log x$

Sol:

a) $I = \int x^3 e^x dx$

According to the ILATE principle, we choose x^3 as first function. Therefore,

$$I = x^3 \cdot e^x dx - \frac{d}{dx}(x^3) \cdot e^x dx$$

$$= x^3 e^x - 3x^2 \cdot e^x dx = x^3 \cdot e^x - 3x^2 \cdot e^x dx$$

Applying the rule once again on second part, we get

$$\begin{aligned} I &= x^3 e^x - 3 \int x^2 \cdot e^x - 2x \cdot e^x dx \\ &= x^3 e^x - 3[x^2 e^x - 2(xe^x - \int e^x)] \\ &= x^3 e^x - 3[x^2 e^x - 2(xe^x - e^x)] + c \\ &= x^3 e^x - 3x^2 e^x + 6x \cdot e^x - 6e^x + c \end{aligned}$$

b) $I = \int x \log x dx$

Choosing $\log x$ as first function

$$\begin{aligned} I &= \log x \cdot \frac{x^2}{2} - \frac{1}{x} \cdot x dx = \frac{1}{2} x^2 \log x - \frac{1}{x} \cdot \frac{x^2}{2} dx \\ &= \frac{1}{2} x^2 \log x - \frac{1}{2} x dx = \frac{1}{2} x^2 \log x - \frac{1}{2} \times \frac{x^2}{2} + c \\ &= \frac{1}{2} x^2 \log x - \frac{1}{4} x^2 + c. \end{aligned}$$

c) $I = \int \frac{\log x}{x^2} dx$ can be written as $\log x \cdot \frac{1}{x^2} dx$

Let us treat $\log x$ as the first function and $\frac{1}{x^2}$ as second.

$$I = \log x \cdot x^{-2} dx - \left[\frac{d}{dx}(\log x) \times x^{-2} dx \right] dx$$

$$= \log x \cdot \frac{x^{-1}}{-1} - \frac{1}{x} \cdot \frac{x^{-1}}{-1} dx$$

$$= -x^{-1} \log x + \int x^{-2} dx = -\frac{\log x}{x} + \frac{x^{-1}}{-1} + c$$

$$= \frac{-\log x}{x} - \frac{1}{x} + c = -\frac{1}{x}(\log x + 1) + c$$

d) $I = \int x^4 \log x \, dx$

Take $\log x$ as first function and x^4 as second.

$$\begin{aligned} I &= \log x \times \frac{x^5}{5} - \int \left[\frac{d}{dx}(\log x) \cdot x^4 dx \right] dx \\ &= \frac{1}{5} x^5 \log x - \int \left[\frac{1}{x} \times \frac{x^5}{5} \right] dx \\ &= \frac{1}{5} x^5 \log x - \frac{1}{5} \int x^4 dx \\ &= \frac{1}{5} x^5 \log x - \frac{1}{5} \times \frac{x^5}{5} + c = \frac{x^5}{5} \left[\log x - \frac{1}{5} \right] + c \end{aligned}$$

Let us now discuss some more Standard forms of Integrals.

1) $I = \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \frac{1}{(x-a)} - \frac{1}{(x+a)} dx$ [Identity: $x^2 - y^2 = (x-y)(x+y)$]

$$\begin{aligned} &= \frac{1}{2a} \frac{1}{x-a} dx - \frac{1}{x+a} dx \\ &= \frac{1}{2a} \log(x-a) - \log(x+a) + c \\ &= \frac{1}{2a} \log \frac{(x-a)}{(x+a)} + c. \end{aligned}$$

2) $I = \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \frac{1}{(a+x)} + \frac{1}{(a-x)} dx$

[Identity: $x^2 - y^2 = (x-y)(x+y)$]

$$\begin{aligned} &= \frac{1}{2a} \frac{1}{a+x} dx + \frac{1}{a-x} dx \\ &= \frac{1}{2a} \log(a+x) + \frac{\log(a-x)}{-1} + c \\ &= \frac{1}{2a} \log(a-x) - \log(a-x) + c \\ &= \frac{1}{2a} \log \frac{(a+x)}{(a-x)} + c. \end{aligned}$$

$$3) \quad I = \frac{1}{\sqrt{x^2 + a^2}} dx$$

Put $x + \sqrt{x^2 + a^2} = t$ so that

$$\frac{d}{dx} (x + (x^2 + a^2)^{\frac{1}{2}}) = \frac{dt}{dx}$$

$$1 + \frac{1}{2}(x^2 + a^2)^{-\frac{1}{2}} \cdot 2x \cdot \frac{dx}{dx} = \frac{dt}{dx}$$

$$\frac{\sqrt{(x^2 + a^2)} + x}{\sqrt{(x^2 + a^2)}} = \frac{dt}{dx} \quad \text{or} \quad dx = \frac{dt}{\sqrt{(x^2 + a^2)} + x} \times \sqrt{(x^2 + a^2)}$$

$$\therefore I = \frac{1}{\sqrt{(x^2 + a^2)}} \times \frac{dt \sqrt{(x^2 + a^2)}}{\sqrt{(x^2 + a^2)} + x} = \frac{dt}{t} = \log t + c$$

$$= \log (x + \sqrt{(x^2 + a^2)}) + c$$

$$4) \quad I = \frac{1}{\sqrt{(x^2 - a^2)}} dx$$

Put $t = x + \sqrt{(x^2 - a^2)}$ so that $\frac{dt}{dx} = 1 + \frac{1}{2}(x^2 - a^2)^{-\frac{1}{2}} \times 2x$

$$\frac{dt}{dx} = 1 + \frac{x}{\sqrt{(x^2 - a^2)}} \times \frac{\sqrt{(x^2 - a^2)} + x}{\sqrt{(x^2 - a^2)}}$$

$$\frac{dx}{dt} = \frac{\sqrt{(x^2 - a^2)}}{t} \quad \text{or} \quad dx = \frac{dt}{t} \cdot (x^2 - a^2)^{\frac{1}{2}}$$

$$I = \frac{1}{\sqrt{(x^2 - a^2)}} \times \frac{dt}{t} \times \sqrt{(x^2 - a^2)} = \frac{dt}{t} = \log t + c$$

$$= \log (x + \sqrt{(x^2 - a^2)}) + c$$

13.4.3 Integration by Partial Fractions

Let $f(x)$ and $g(x)$ be polynomials in x so $\frac{f(x)}{g(x)}$ that is a rational fraction. Integration of a rational

fraction can be easily found by splitting it into partial fractions (two or more than two). Let us trace the following steps and simultaneously take one

example $\frac{x+1}{x^2+4x-5} dx$.

Step 1: Factorise the denominator into two or more parts.	$x^2 + 4x - 5 = (x-1)(x+5)$
Step 2: Express the fraction (integrand) as the sum of two/more factors.	$\frac{x+1}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5}$
Step 3: Cross multiply.	$x+1 = \frac{A(x-1)(x+5)}{x-1} + \frac{B(x-1)(x+5)}{x+5}$ or $x+1 = A(x+5) + B(x-1)$
Step 4: a) Put 1 st factor equal to zero to get one value of x . b) Put 2 nd factor equal to zero to get another value of x .	a) $x-1=0, \quad x=1$ b) $x+5=0, \quad x=-5$
Step 5: a) Put 1 st value of x in the whole expression of step 3. b) Put 2 nd value of x in the whole expression of step 3.	a) $1+1 = A(1+5) + B(1-1)$ or $2 = 6A$ or $A = \frac{2}{6} = \frac{1}{3}$ b) $-5+1 = A(-5+5) + B(-5-1)$ or $-4 = -6B$ or $B = \frac{-4}{-6} = \frac{2}{3}$
Step 6: Now express the Integrand in the split form: $I = \frac{A}{\text{factor 1}} \pm \frac{B}{\text{factor 2}} dx$	$I = \frac{\frac{1}{3}}{x-1} + \frac{\frac{2}{3}}{x+5} dx$ $= \frac{1}{3(x-1)} + \frac{2}{3(x+5)} dx$
Step 7: Integrate each fraction separately.	$I = \frac{1}{3} \frac{1}{x-1} dx + \frac{2}{3} \frac{1}{x+5} dx$ $= \frac{1}{3} \log(x-1) + \frac{2}{3} \log(x+5) + c$

Example Evaluate $I = \frac{4x-2}{x^3-x^2-2x} dx$

Solution The denominator can be factorized into $x(x+1)(x-2)$

$$\text{Now, } \frac{4x-2}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$\text{or } 4x-2 = \frac{x(x+1)(x-2)A}{x} + \frac{x(x+1)(x-2)B}{x+1} + \frac{x(x+1)(x-2)C}{x-2}$$

$$4x - 2 = (x + 1)(x - 2)A + x(x - 2)B + x(x + 1)C \quad \dots (1)$$

a) Put $x = 0$ in equation (1) so that $-2 = (0 + 1)(0 - 2)A + 0 + 0$
 or $-2 = -2A$ or $A = 1$

b) Put $x + 1 = 0 \Rightarrow x = -1$ in equation (1):
 $4(-1) - 2 = 0 + -1(-1 - 2)B + 0$
 $-6 = 3B$ or $B = \frac{-6}{3} = -2$

c) Put $x - 2 = 0 \Rightarrow x = 2$ in equation (1): $4(2) - 2 = 0 + 0 + C \times 2 \times 3$
 or $6 = 6C$ or $C = \frac{6}{6} = 1$

Now,

$$\frac{4x - 2}{x(x + 1)(x - 2)} = \frac{1}{x} - \frac{2}{x + 1} + \frac{1}{x - 2}$$

$$\frac{(4x - 2)}{x(x + 1)(x - 2)} dx = \frac{1}{x} dx - 2 \frac{1}{x + 1} dx + \frac{1}{x - 2} dx$$

$$= \log x - 2 \log(x + 1) + \log(x - 2) + c$$

Example

Solution

Evaluate $\frac{x}{(x - 1)(2x + 1)} dx$

$$\text{Let } \frac{x}{(x - 1)(2x + 1)} = \frac{A}{x - 1} + \frac{B}{2x + 1} \quad \dots (1)$$

$$x = A(2x - 1) + B(x - 1) \text{ (By cross multiplication)} \quad \dots (2)$$

Now, put $x - 1 = 0$ or $x = 1$ in equation (2)

$$1 = A(2 \times 1 + 1) + B(0) \quad \text{or} \quad 3A = 1 \quad \text{or} \quad A = \frac{1}{3}$$

Now, put $(2x + 1) = 0$ or $x = -\frac{1}{2}$ in equation (2)

$$\text{or } -\frac{1}{2} = 0 + B \left(-\frac{1}{2} - 1\right) \quad \text{or} \quad -\frac{3}{2}B = -\frac{1}{2}$$

$$\text{or } B = \frac{1}{-2} \times \frac{-2}{3} \quad \text{or} \quad B = \frac{1}{3}$$

Therefore, $A = B = \frac{1}{3}$. Substituting these values in (1), we get

$$\frac{x}{(x - 1)(2x + 1)} dx = \frac{\frac{1}{3}}{x - 1} dx + \frac{\frac{1}{3}}{2x + 1} dx$$

$$= \frac{1}{3} \frac{1}{x - 1} dx + \frac{1}{3} \frac{1}{2x + 1} dx = \frac{1}{3} \log(x - 1) + \frac{\log(2x + 1)}{2} + c$$

$$= \frac{1}{3} \log(x-1) + \frac{1}{6} \log(2x+1) + c$$

Example Evaluate $\frac{7x^2 + 3x + 1}{x^2 + x} dx$.

Solution Since both numerator and denominator contain quadratic functions, therefore, we first find the quotient by long division. Therefore,

$$\frac{7x^2 + 3x + 1}{x^2 + x} = 7 + \frac{1-4x}{x^2 + x} = 7 + \frac{1-4x}{x(x+1)} \quad \dots(1)$$

$$\text{Let us now break } \frac{1-4x}{x(x+1)} \text{ into } \frac{A}{x} + \frac{B}{(x+1)} \quad \dots(2)$$

On cross multiplication, we find that

$$1 - 4x = A(x+1) + Bx \quad \dots(3)$$

Now, put $x = 0$ in equation (3) so that

$$1 - 4 \times 0 = A(0+1) + B(0)$$

$$\text{or } 1 = A$$

Now, put $x+1=0$ or $x=-1$ in equation (3), we get

$$1 - 4(-1) = A(-1+1) + B(-1) \text{ or } 5 = -B \text{ or } B = -5$$

Consider values of A and B and equations (1) and (2). We get

$$\begin{aligned} \frac{7x^2 + 3x + 1}{x^2 + x} dx &= 7 + \frac{1}{x} - \frac{5}{x+1} dx \\ &= 7 \int 1 \cdot dx + \int \frac{1}{x} dx - 5 \int \frac{1}{x+1} dx \\ &= 7x + \log x - 5 \log(x+1) + C \end{aligned}$$

Example Integrate $\frac{1}{x^2 - x - 6} dx$ by using the method of partial fractions.

Solution First factorize the denominator:

$$x^2 - x - 6 = x^2 - 3x + 2x - 6 = x(x-3) + 2(x-3)$$

$$\text{or } (x+2)(x-3)$$

$$\text{Now, } \frac{1}{x^2 - x - 6} = \frac{1}{(x+2)(x-3)} = \frac{A}{(x+2)} + \frac{B}{(x-3)} \quad \dots(1)$$

Cross multiplying, we get

$$1 = A(x-3) + B(x+2) \quad \dots(2)$$

Now, put $x-3=0$ or $x=3$ in equation (2) so that

$$1 = A(3-3) + B(3+2)$$

$$\text{or } 1 = 5B \quad \text{or } B = \frac{1}{5}$$

Now, put $x+2=0$ or $x=-2$ in equation (2) so that
 $1 = A(-2-3) + B(-2+2)$

$$\text{or } 1 = -5A \quad \text{or } A = -\frac{1}{5}$$

Consider values of A, B and equation (1). We get,

$$\begin{aligned} \frac{1}{x^2-x-6} &= \frac{-\frac{1}{5}}{x+2} + \frac{\frac{1}{5}}{x-3} = \frac{1}{5(x-3)} - \frac{1}{5(x+2)} \\ \text{or } \frac{1}{x^2-x-6} dx &= \frac{1}{5} \frac{1}{x-3} dx - \frac{1}{5} \frac{1}{x+2} dx \\ &= \frac{1}{5} \log(x-3) - \log(x+2) + C \\ &= \frac{1}{5} \log \frac{(x-3)}{(x+2)} + C. \end{aligned}$$

Example

Evaluate $\frac{dx}{1-e^x}$ by using method of partial fractions.

Solution

We first convert e into some variable like t by putting $e^x = t$ or
 $\frac{dt}{dx} = e^x$ or $dx = \frac{dt}{t}$.

$$\therefore \frac{dx}{1-e^x} \text{ becomes } \frac{1}{1-t} \cdot \frac{dt}{t} \text{ or } \frac{1}{t(1-t)} dt$$

$$\text{Now, let } \frac{1}{t(1-t)} = \frac{A}{t} + \frac{B}{(1-t)} \quad \dots(1)$$

So that in cross multiplication we have

$$1 = A(1-t) + Bt \quad \dots(2)$$

Now, put $t=0$ in equation (2), so that $1 = A(1-0) + B(0)$

$$\text{or } A = 1$$

Now, put $1-t=0$, or $t = 1$ in equation (2), so that
 $1 = A(1-1) + B(1)$

$$\text{or } 1 = B \quad \text{or } B = 1$$

Consider values of A and B and equation (1). We get,

$$\begin{aligned} \frac{1}{t(1-t)} &= \frac{1}{t} + \frac{1}{(1-t)} \text{ so that } \int \frac{1}{t(1-t)} dx = \int \frac{1}{t} dx + \int \frac{1}{(1-t)} dx \\ &= \log t + \frac{\log(1-t)}{-1} + C \end{aligned}$$

$$= \log t - \log(1-t) + C = \log \frac{t}{1-t} + C$$

Substituting the value of $t = e^x$, we get

$$\frac{dx}{1-e^x} = \log \frac{e^x}{1-e^x} + C$$

Example Integrate: $\frac{(x-1)}{(x-2)(x-3)} dx$

Solution Let $\frac{(x-1)}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$ (1)

On cross multiplication, we get $(x-1) = A(x-3) + B(x-2)$... (2)

Now, put $x-3=0$ or $x=3$ in equation (2), we get

$$3-1 = A(3-3) + B(3-2) \quad 2 = B$$

Now, put $x-2=0$ or $x=2$ in equation (2), we get

$$2-1 = A(2-3) + B(2-2) \text{ or } 1 = -A \text{ or } A = -1$$

Consider values of A and B and equation (1). We get,

$$\begin{aligned} \int \frac{x-1}{(x-2)(x-3)} dx &= \int \frac{-1}{(x-2)} dx + \int \frac{2}{(x-3)} dx \\ &= -\log(x-2) + 2\log(x-3) + C \\ &= \log(x-3)^2 - \log(x-2) + C = \log \frac{(x-3)^2}{(x-2)} + C \end{aligned}$$

Example Evaluate: $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} dx$

Solution Let $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{A}{(x-4)} + \frac{B}{(x-5)} + \frac{C}{(x-6)}$... (1)

On cross multiplication, we get

$$\begin{aligned} (x-1)(x-2)(x-3) &= (x-4)(x-5)(x-6) + A(x-5)(x-6) + \\ &B(x-4)(x-6) + C(x-4)(x-5) \quad \dots (2) \end{aligned}$$

Now, put $(x-4) = 0$ or $x=4$ in equation (2), so that

$$\begin{aligned} (4-1)(4-2)(4-3) &= 0 + A(4-5)(4-6) + \\ &B(4-4)(4-6) + C(4-4)(4-5) \end{aligned}$$

$$6 = 2A \quad \text{or} \quad A = 3$$

Now, put $(x-5) = 0$ or $x=5$ in equation (2), so that

$$(5-1)(5-2)(5-3) = 0 + A(5-5)(5-6) + B(5-4)(5-6) + C(5-4)(5-5)$$

$$24 = -B \quad \text{or} \quad B = -24$$

Now, put $(x-6) = 0$ or $x=6$ in equation (2), so that

$$(6-1)(6-2)(6-3) = 0 + A(6-5)(6-6) + B(6-4)(6-6) + C(6-4)(6-5)$$

$$60 = 2C \quad \text{or} \quad C = 30$$

Consider values of A, B and C and equation (1). We get,

$$\int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} dx = \int \left[1 + \frac{3}{(x-4)} - \frac{24}{(x-5)} + \frac{30}{(x-6)} \right] dx$$

$$= \int 1 dx + 3 \int \frac{1}{(x-4)} - 24 \int \frac{1}{(x-5)} dx + 30 \int \frac{1}{(x-6)} dx$$

$$= x + 3 \log(x-4) - 24 \log(x-5) + 30 \log(x-6) + c$$

Example

Evaluate $\frac{1}{2x^2 + x - 1} dx$

Solution

Let $\frac{1}{2x^2 + x - 1} = \frac{1}{(2x-1)(x+1)} = \frac{A}{(2x-1)} + \frac{B}{(x+1)}$... (1)

On cross multiplication, we get

$$1 = A(x+1) + B(2x-1)$$

... (2)

Now, put $x+1=0$ or $x=-1$ in equation (2), so that

$$1 = A(-1+1) + B(2(-1)-1) \quad \text{or} \quad 1 = 0 + B(-3)$$

$$\text{or} \quad B = -\frac{1}{3}$$

Now, put $2x-1=0$ or $x = \frac{1}{2}$ in equation (2), so that

$$\therefore 1 = A \left(\frac{1}{2} + 1 \right) + B(0) \quad \text{or} \quad 1 = A \left(\frac{3}{2} \right)$$

$$\text{or} \quad A = \frac{2}{3}$$

Consider values of A and B , and equation (1). We get,

$$\begin{aligned}
\therefore \frac{1}{2x^2+x-1} dx &= \frac{\frac{2}{3}}{2x-1} dx - \frac{\frac{1}{3}}{x+1} dx \\
&= \frac{2}{3} \frac{1}{2x-1} dx - \frac{1}{3} \frac{1}{x+1} dx \\
&= \frac{\frac{2 \log(2x-1)}{2}}{3} - \frac{1}{3} \log(x+1) + C \\
&= \frac{1}{3} \log(2x-1) - \frac{1}{3} \log(x+1) + C \\
&= \frac{1}{3} \log(2x-1) - \log(x+1) + C \\
&= \frac{1}{3} \log \frac{(2x-1)}{(x+1)} + C
\end{aligned}$$

ExampleEvaluate $\int \frac{5}{(p+3)(p-2)} dp$ **Solution**

$$\text{Let } \frac{5}{(p+3)(p-2)} = \frac{A}{p+3} + \frac{B}{p-2} \quad \dots(1)$$

On cross multiplication, we get

$$5 = A(p-2) + B(p+3) \quad \dots(2)$$

Now, put $(p-2) = 0$ or $p = 2$ in equation (2), so that

$$5 = A(2-2) + B(2+3) \quad \text{or} \quad B = 1$$

Now, put $p+3 = 0$ or $p = -3$ in equation (2), so that

$$5 = A(-3-2) + B(-3+3) \quad \text{or} \quad A = -1$$

Consider values of A and B , and equation (1). We get,

$$\begin{aligned}
\int \frac{5}{(p+3)(p-2)} dp &= \int \frac{-1}{p+3} dp + \int \frac{1}{p-2} dp \\
&= \int \frac{1}{p-2} dp - \int \frac{1}{p+3} dp \\
&= \log(p-2) - \log(p+3) + C \\
&= \log \frac{(p-2)}{(p+3)} + C.
\end{aligned}$$

Check Your Progress 2

- 1) Using the method of substitution evaluate the following:
 - a) $e^{3x+4} dx$
 - b) $\frac{x}{\sqrt[3]{x^2+1}} dx$
 - c) $\frac{2x-7}{3(x^2-7x+6)^2} dx$
 - d) $\frac{1}{\sqrt{x+x}} dx$
- 2) Evaluate $x(x^2+4)^5 dx$
- 3) Using anyone of the various methods of Integration, evaluate the following:
 - a) $x^2 e^{3x} dx$
 - b) $(4x+7)^{1/2} dx$
 - c) $\frac{dx}{(7+3x)^5}$
- 4) Integrate the following by method of substitution:
 - a) $\frac{1}{x\sqrt{(x^2-a^2)}}$
 - b) $x\sqrt{(1+x)}$
- 5) Using the method of integration by parts, evaluate the following:
 - a) $\log(1+x)^{(1+x)} dx$
 - b) $x^2 e^{ax} dx$
 - c) $\int \frac{1+x^2}{e^x} dx$
 - d) $\log(x+2) dx$.
- 6) Split the following functions in partial fractions and then integrate.
 - a) $\frac{1}{x-x^3}$
 - b) $\frac{x^3-2x^2-13x-12}{x^2-3x-10}$

13.5 SOME ECONOMIC APPLICATIONS OF INTEGRATION

We are considering here a list of some of the applications of integration in the field of Economics, in our attempt to create its importance and thereby create an interest to learn it.

We know that marginal revenue is defined as the rate of change of total revenue. Thus MR is derivative of TR . We have a function $F(x) = TR$, such that its derivative $F'(x) = MR$, therefore we can say:

$$TR + C = \int MR dx$$

Where C is a constant and its value can be determined using the fact that $TR = 0$ when $x = 0$.

Similarly, we can deduce for the total cost function that

$$TC + C = \int MC dx$$

Example 1: Let MR function be $\frac{60}{(x+3)^2} - 2$, find TR . Also derive the demand function.

Solution:

$$\begin{aligned} TR + C &= \int \left(\frac{60}{(x+3)^2} - 2 \right) dx = \int \frac{60}{(x+3)^2} dx - \int 2 dx \\ TR + C &= \frac{-60}{x+3} - 2(x) \end{aligned}$$

To find C , we use the condition that Total Revenue will be zero when x is nil:

$$\begin{aligned} \therefore 0 + C &= \frac{-60}{0+3} - 2(0) \text{ or } C = \frac{-60}{3} = -20 \\ \therefore TR &= \frac{-60}{(x+3)} - 2x + 20 = \frac{20x}{(x+3)} - 2x \end{aligned}$$

We know that demand function = average revenue (AR) function

$$\therefore \text{Demand function} = AR = \frac{TR}{x} = \frac{20}{(x+3)} - 2$$

Example 2 : Let Marginal Cost function be : $MC = 4 + 6x + 30x^2$. Find firm's total cost function if the fixed cost is Rs. 500.

Solution: Total Cost $+ C = \int (4 + 6x + 30x^2) dx$
 $= 4x + 3x^2 + 10x^3$

We know that total cost of production when output equals zero is equal to the fixed cost.

$$\therefore 500 + C = 4(0) + 3(0)^2 + 10(0)^3 \quad \text{or} \quad C = -500$$

Hence, Total Cost $= 10x^3 + 3x^2 + 4x + 500$

Note: When a total cost function is presented by a polynomial, the constant of integration represents Fixed Costs.

Example 3 : A company suffers a loss of Rs. 110 if its product does not sell at all. Its marginal revenue (MR) and marginal cost (MC) curve are given by

$$MR = 50 - 4x \quad \text{and} \quad MC = -10 + x$$

Find out its profits function and equilibrium output. Also find maximum profit and break-even output.

Solution: Profit Function: $\pi(x) = \text{Total Revenue} - \text{Total Cost}$

$$\begin{aligned} \therefore \pi(x) + C &= \int [(50 - 4x) - (-10 + x)] dx \\ &= \int (60 - 5x) dx = 60x - \frac{5x^2}{2} \end{aligned}$$

We know that at zero output, company suffers a loss of 110, that is,

$$\pi(0) = -110 \quad \therefore -110 + C = 0 \quad \text{or} \quad C = 110$$

Thus profit function will turn out to be $\pi(x) = 60x - \frac{5x^2}{2} - 110$

We know that equilibrium of the firm occurs when $MR = MC$

$$\therefore 50 - 4x = -10 + x$$

$$\text{or} \quad 5x = 60$$

$$\text{or} \quad x = 12 \text{ will be the equilibrium output.}$$

Maximum profit is earned by a firm at its equilibrium level of output

Break-even level of output is that level of output when profits are nil.

$$\begin{aligned} \therefore \pi(12) &= 60 \times 12 - \frac{5 \times 144}{2} - 110 = \text{Rs. } 250/- \\ \Rightarrow 60x - \frac{5x^2}{2} - 110 &= 0 \end{aligned}$$

Therefore, $\pi(x) = 0$ will give break-even level of output.

$$\text{or} \quad x^2 - 24x + 44 = 0$$

$$\text{or} \quad x = 2 \quad \text{or} \quad x = 22$$

Since firm starts getting positive profits when output exceeds 2, the Break-even level of output will be $x = 2$.

To Sum Up:

- a) Total Utility (TU) + C = \int Marginal Utility dx
- b) Total Cost (TC) = \int Marginal Cost dx
- c) Total Revenue (TR) = \int Marginal Revenue dx
- d) Total Product (TP) = \int Marginal Product dx

Check Your Progress 3

- 1) The marginal revenue (MR) function of firm is:

$$MR = \frac{ab}{(x-b)^2}$$

Prove that the demand curve in inverse form will be

$$p = \frac{a}{b-x}$$

- 2) If the marginal cost (MC) function is given by $MC = \frac{4}{\sqrt{3x+36}}$ and fixed cost = Rs. 20, find the average cost of producing 15 units of output.
- 3) Given that $MR = 20x - 2x^2$

$$MC = 81 - 16x + x^2$$

Assuming fixed cost as zero, find out profit maximising output, and amount of profits at that output.

13.6 LET US SUM UP

This unit was the first of the two units on integration. The next unit will deal with definite integrals. We shall explain what definite integrals are and how they differ from indefinite integrals when we come to it. This particular unit was on indefinite integrals. We saw that integrals can be considered as the reverse of derivatives; or to put it slightly differently, integration is the opposite of the process of differentiation. We learnt that if a function is differentiated, we get a new function. If, now the new function is integrated, we will get back the original function.

The unit then went on to discuss certain standard forms of integrals, as also certain properties or rules about integration. After that the unit took up for discussion some specific techniques of integration, namely, integration by substitution, integration by parts, and integration by partial fractions. Finally, we saw that integration finds several applications in economics, particularly in the way we can recover a 'total' function from a 'marginal' one. We saw examples like getting total utility from marginal utility and getting a total cost function from a marginal cost function.

13.7 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

- 1)
 - a) $-\frac{1}{5}e^{-5x} + \frac{1}{5}e^{5x}$
 - b) $\frac{x^2}{2} - \frac{16}{5}x^{\frac{5}{4}} + c$
 - c) $\frac{-1}{2}\log(s - 2x) + c$
 - d) $5x^2 - \frac{10}{3}x^3 + c$
- 2)
 - a) $\frac{x^4}{4} + 15x + c$
 - b) $\frac{5}{2}x^6 + c$
 - c) $\frac{-x^{-4}}{4} + c$
 - d) $e^{3x} + \frac{2^x}{\log 2} + c$
 - (e) $\frac{5^x}{\log 5} + c$

Check Your Progress 2

- 1)
 - a) $\frac{e^{3x+4}}{3} + c$
 - b) $\frac{3}{4}(x^2 + 1)^{\frac{2}{3}} + c$ (Hint: Put $t^3 = x^2 + 1$)
 - c) $\frac{-1}{3(x^2 - 7x + 6)} + c$ (Hint: Put $t = x^2 - 7x + 6$)
 - d) $2 \log(1 + \sqrt{x}) + c$ (Hint: $\frac{1}{\sqrt{x}+x} = 2 \cdot \frac{\frac{1}{2\sqrt{x}}}{1+\sqrt{x}}$, where, $\frac{1}{2\sqrt{x}}$ is the derivative of $(1 + \sqrt{x})$).

Hence, put $t = 1 + \sqrt{x}$. The integrand becomes of the form $2 \frac{f'(x)}{f(x)} dx$

2) $\frac{(x^2+4)^6}{12} + c$ (Hint: Put $t = x^2 + 4$)

3) a) $\frac{e^{3x}}{3} \left[x^2 - \frac{2}{3} \left(x - \frac{1}{3} \right) \right] + c$

b) $\frac{(4x+7)^{\frac{3}{2}}}{6} + c$ (Hint: Put $t^2 = 4x + 7$)

c) $\frac{-1}{12(7+3x)^4} + c$

4) a) $\frac{\log x^2}{2\sqrt{x^2-a^2}} + c$

(Hint: Put $t^2 = x^2 - a^2 \Rightarrow t = \sqrt{x^2 - a^2}$; also, $x^2 = t^2 + a^2$. Now, as we have $t^2 = x^2 - a^2$. Derivating both sides, we get $2t \frac{dt}{dx} = 2x \Rightarrow t dt = x dx \Rightarrow \frac{t dt}{x^2} = \frac{dx}{x} \Rightarrow \frac{t dt}{t^2+a^2} = \frac{dx}{x}$. Hence,

$\int \frac{1}{x\sqrt{x^2-a^2}} dx$ becomes $\int \frac{t}{(t^2+a^2)t} dt$

b) $2 \left[\frac{(1+x)^{\frac{5}{2}}}{5} - \frac{(1+x)^{\frac{3}{2}}}{3} \right] + c$ (Hint: Put $t^2 = 1 + x$)

5) a) $\frac{(1+x)^2}{2} \left[\log(1+x) - \frac{1}{2} \right] + c$

(Hint: Take $\log(1+x)^{(1+x)}$ as $(1+x) \log(1+x)$. Now, put $t = (1+x)$, and then integrate by parts taking the logarithm as the first function.)

b) $\frac{e^{ax}}{a} \left[x^2 - \frac{2}{a} \left(x - \frac{1}{a} \right) \right] + c$ (Hint: consider x^2 as first function)

c) $-x^2 e^{-x} - 2x e^{-x} - 3e^{-x} + c$

d) $(x+2) \log(x+2) - (x+2) + c$

(Hint: put $t = (x+2)$, and then integrate by parts taking the logarithm as the first function.)

6) a) $\log x - \frac{1}{2} \log(1-x) - \frac{1}{2} \log(1+x) + c$ [Hint: Convert $\frac{1}{x-x^3}$ into $\frac{1}{x(1-x)(1+x)}$]

b) $\frac{x^2}{2} + x - \frac{2}{7} \left[\log \frac{(x-5)}{(x+2)} \right] + c$

(Hint: Simplify the expression with the help of long division

$$\frac{x^3 - 2x^2 - 13x - 12}{x^2 - 3x - 10} = (x + 1) - \frac{2}{x^2 - 3x - 10} = (x + 1) - \frac{2}{(x+2)(x-5)}$$

Check Your Progress 3

- 1) See section 13.5 and answer.
- 2) Rs 1.4 (Approx.)
- 3) Profit maximizing output = 9 units; the firm breaks-even at this output level, that is profit earned will be zero.

UNIT 14 DEFINITE INTEGRALS*

Structure

- 14.0 Objectives
- 14.1 Introduction
- 14.2 Concept of Definite Integral
 - 14.2.1 Definite Integral as Area Under a Curve
 - 14.2.2 Riemann Integral
- 14.3 Properties of Definite Integral
- 14.4 Economic Applications of Definite Integral
 - 14.4.1 Consumer Surplus
 - 14.4.2 Producer Surplus
 - 14.4.3 Profit Maximisation
- 14.5 Let Us Sum Up
- 14.6 Answers/ Hints to Check Your Progress Exercises

14.0 OBJECTIVES

In the previous unit, you were acquainted with the basic idea of integration. We saw that integration can be looked at as the reverse of differentiation. This is the reason that integrals are sometimes called anti-derivatives. In this unit we look at what are called definite integrals. After going through this unit, you will be able to:

- explain the concept of definite integrals;
- note that the particular basic form of definite integrals we study are also called Riemann integrals;
- describe some properties of definite integrals; and
- discuss some applications in economics of definite integrals

14.1 INTRODUCTION

In the previous unit you were introduced to the idea of integration. There we saw that integration can be seen a process which is the reverse of differentiation. In this unit we shall continue with the same theme of integration but with a slight difference. The basic difference is that indefinite integrals are functions; you differentiate a function and get a new function; if you were to integrate this new function, you will recover the old function. Thus indefinite integrals are functions. Definite integrals, as we shall see, are, after they are computed, a single number. Definite integrals are very useful mathematical concepts and find several applications not only to better understand other area of mathematics, but also in economics.

The unit is organized as follows. The following section explains in detail and with great care, the concept of definite integral. We shall see that it can be helpful to understand definite integrals as areas under curves. Thus the fact that

* Contributed by Shri Saugato Sen, SOSS, IGNOU

a definite integral is a single number emerges, since an area measure is a number. This section also discusses a related conception of the definite integral, where it is known by a fancy name, the Riemann integral. The section after that states and briefly discusses certain properties of definite integrals. The subsequent section takes up for discussion certain applications of definite integrals in economics

14.2 CONCEPT OF DEFINITE INTEGRAL

Let us now try to explain in what substantive way a definite integral differs from an indefinite integral. Consider a continuous function $f(x)$, and let us integrate it (usual indefinite integral) to get a function $F(x)$ and a constant of integration c :

$$\int f(x)dx = F(x) + c \quad \dots(1)$$

If we choose two values of x in the domain of real numbers, say a and b ($a < b$), substitute them in succession into the right-hand-side of the equation, and form the difference

$$[F(b) + c] - [F(a) + c] = F(b) - F(a) \quad \dots(2)$$

We find that we get a specific numerical value that is free of the variable x as well as the constant of integration c . This specific value of the integral is called the definite integral of $f(x)$ from a to b . Here a is referred to as the lower limit of integration and b as the upper limit of integration. In order to depict the limits of integration, we modify the integral sign to the form \int_a^b . From equations (1) and (2), we get the value of the definite integral as:

$$\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a) \quad \dots(3)$$

The symbol with the vertical line with a and b as subscript and superscript respectively, is telling us to substitute b and then a successively for x in the solution to the integral to get $F(b)$ and then $F(a)$ and to compute their difference, as depicted in the right-hand side of the above equation. Of course, first we have to find the indefinite integral. We may leave out the c (constant of integration), since in any case c will drop away. Let us take a couple of examples:

Compute:

$$\int_1^5 3x^2 dx = x^3]_1^5 = (5)^3 - (1)^3 = 125 - 1 = 124$$

Take another example.

Compute:

$$\int_a^b ke^x dx$$

This does not have specific numbers but symbols as the limits of integration. Hence, the solution of the integration will also be in terms of these symbols:

$$\int_a^b ke^x dx = ke^x]_a^b = k(e^b - e^a)$$

14.2.1 Definite Integral as Area under a Curve

We saw above that integration may be defined as a process of summation. It literally means "adding small parts to make up the whole." We can define definite integral as the limit of a sum. As such it is useful in evaluation of areas, between two points, say, a and b (see shaded region in Figure 14.1). It is the area under the curve/ function $y = f(x)$ from $x = a$ to $x = b$ which is equal to area $abcd$.

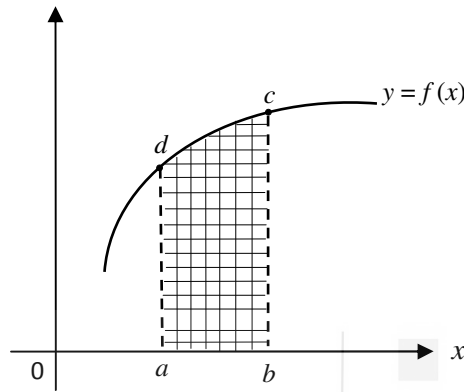


Figure 14.1

Let $\int f(x) dx = g(x)$, then definite integral of $f(x)$ over a given interval, say, $[a, b]$ is given by the area from zero to b minus area from zero to a . Technically, it is written as:

$$\int_a^b f(x) dx = g(x) \Big|_a^b \quad \text{or} \quad g(x) \Big|_a^b = g(b) - g(a), \text{ where } b > a$$

Let us see this point further. Let $y = f(x)$ be a continuous function and its curve be as shown in Figure 14.2. Suppose we want to find the area enclosed by the curve and the x -axis between two points a and b , in the domain of the function. This area can be divided into n strips of width Δx_i ($i = 1, 2, \dots, n$). Two such strips are shown in the figure. The approximate area under the curve can be written as the sum of the areas of n strips in the interval $[a, b]$. Note that smaller the width of the strips, better will be the approximation. Also note that smaller the width Δx_i , larger the number n .

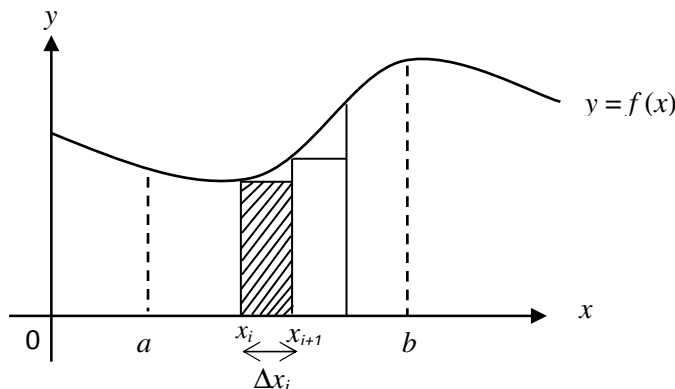


Figure 14.2

The area A_i of the i^{th} strip (the shaded region) can be approximated as the area of the rectangle of width $\Delta x_i (= x_{i+1} - x_i)$ and height $y_i = f(x_i^*)$, where x_i^* is the

$$A_i = \sum_{i=1}^n f(x_i^*) \Delta x_i$$

sample point in the interval $[x_i, x_{i+1}]$. So, the total area of all the strips will be:

Note that in the above expression, as n tends to infinity, or in other words, thinner the strips, we get closer to the exact area enclosed by the curve and the x -axis between two points a and b given by:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

This limit is known as the *Definite Integral* of function $f(x)$ from a to b , and is represented by $\int_a^b f(x) dx$. Symbolically,

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

- Note:** i) The limit is independent of the sample points x_i^* as long as the function is continuous.
- ii) The graph of the function $y = f(x)$ lies below the x -axis in the interval where $f(x)$ is negative. Then the definite integral takes a negative value.

14.2.2 Riemann Integral

The indefinite integral that we have studied in the previous unit, that is the one we consider as anti-derivative is called the *Leibniz-Newton integral*. In mathematics there are several kinds of integrals, if the function to be integrated is a continuous one, they all give the same kind of result. In this subsection, we will be discussing what is called the *Riemann integral*, and see that the definite integral that we are considering is the Riemann integral.

Let us consider a bounded function $g(x)$ in the interval $[a, b]$, and let n be a natural number. Divide $[a, b]$ into n parts by choosing points such that:

$$a = x_1 < x_2 < x_3 < \dots < x_{n-1} < x_n = b$$

Put $\Delta x_i = x_{i+1} - x_i$, $i = 1, 2, \dots, n$ and choose an arbitrary number μ_i in each interval $[x_i, x_{i+1}]$.

The sum

$$g(\mu_1)\Delta x_1 + g(\mu_2)\Delta x_2 + \dots + g(\mu_n)\Delta x_n$$

is called a *Riemann Sum* associated with the function. This sum will depend on the function $g(x)$ chosen, as well as on the subdivision. It will also depend on the choice of the μ_i 's.

Suppose when n approaches infinity and at the same time the largest of the numbers $\Delta x_1, \Delta x_2, \dots, \Delta x_n$ tends to zero, the limit of the sum exists. Then we say that the function $g(x)$ is *Riemann-integrable* in the interval $[a, b]$, and we put

$$\int_a^b g(x) dx = \lim_{i \rightarrow \infty} \sum_{i=1}^n g(\mu_i) \Delta x_i$$

The numerical value of the integral is independent of the choice of μ_i 's. It can be shown, although it is beyond the scope of the text that every continuous function is Riemann integrable.

Check Your Progress 1

1) Evaluate the following definite integrals:

a) $\int_4^9 (\frac{1}{\sqrt{x}}) dx$

.....

b) $\int_1^2 (\frac{1}{y^2}) dy$

.....

c) $\int_{-2}^{-1} \frac{1}{3-x} dx$

.....

d) $\int_0^1 e^{2x} dx$

.....

2) Evaluate the following.

a) $\int_0^2 (5x^2 - 3x + 1) dx$

.....

b) $\int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$

.....

c) $\int_{-2}^{-1} (z^2 + 2z + 2) dz$

.....

3) What do you mean by a Riemann Sum?

.....

.....

.....

.....

.....

14.3 PROPERTIES OF DEFINITE INTEGRAL

In the previous unit, we discussed the properties of the indefinite integral. All those properties would hold in the case of definite integrals as well. However, definite integrals have some specific properties involving largely the limits of integration. We state below some of these properties with examples but without providing proof.

Property 1 Interchange of limits changes the sign of the integral.

$$i.e. \int_a^b f(x)dx = - \int_b^a f(x)dx \quad b > a$$

Example 1

$$\int_1^2 x^2 dx = \left. \frac{x^3}{3} \right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\int_2^1 x^2 dx = \left. \frac{x^3}{3} \right|_2^1 = \frac{1}{3} - \frac{8}{3} = -\frac{7}{3}$$

$$\therefore \int_1^2 x^2 dx = - \int_2^1 x^2 dx = \frac{7}{3}$$

Property 2 Change of variable does not change the value of the integral.

$$i.e. \int_a^b f(x)dx = \int_a^b f(t)dt$$

Here, x or t are like dummy variables in the sense that integration is independent of its label.

Example 2

$$\int_1^2 x^2 dx = \left. \frac{x^3}{3} \right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\int_1^2 t^2 dt = \left. \frac{t^3}{3} \right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\therefore \int_1^2 x^2 dx = \int_1^2 t^2 dt$$

Property 3 A definite integral can be expressed as the sum of any number of definite integrals by splitting the range of integration (*i.e.* a to $b \Rightarrow a$ to c , c to b).

$$i.e. \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad (a < c < b)$$

Example 3 $\int_0^2 x^2 dx = \int_0^1 x^2 dx + \int_1^2 x^2 dx \quad (0 < 1 < 2)$

L.H.S: $\int_0^2 x^2 dx = \left. \frac{x^3}{3} \right|_0^2 = \frac{8}{3} + 0 = \frac{8}{3}$

R.H.S: $\int_0^1 x^2 dx + \int_1^2 x^2 dx = \left. \frac{x^3}{3} \right|_0^1 + \left. \frac{x^3}{3} \right|_1^2 = \frac{1}{3} + \frac{(2)^3}{3} - \frac{1}{3} = \frac{8}{3}$

Property 4 $\int_a^b [-f(x)] dx = - \int_a^b f(x) dx$

Example 4 Let $f(x) = x^2$

L.H.S: $\int_1^2 -f(x) dx = \int_1^2 -x^2 dx = - \left. \frac{x^3}{3} \right|_1^2 = -\frac{7}{3}$

R.H.S: $- \int_1^2 f(x) dx = - \int_1^2 x^2 dx = - \left. \frac{x^3}{3} \right|_1^2 = -\frac{7}{3}$

L.H.S. = R.H.S. Hence the result.

As a special case on above property, $\int_a^a f(x) dx = 0$

Property 5 $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

Property 6 $\int_0^a f(x) dx = \int_0^a f(a - x) dx$

Property 7 $\int_0^a f(x) dx = 2 \int_0^{a/2} f(x) dx,$ if $f(a - x) = f(x)$
 $= 0$ if $f(a - x) = -f(x)$

Property 8 $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx,$ if f is an even function, i.e., if $f(-x) = f(x)$
 $= 0,$ if f is an odd function, i.e., if $f(-x) = -f(x).$

Evaluation of Definite Integrals by Substitution

Following steps are involved while evaluating the definite integral using the method of substitution:

- i) As we learnt in unit 13, first make appropriate substitution in the given integrand in order to reduce it in a form which is comparatively easier to integrate.
- ii) Change the limits of the resultant integrand according to the substituted variable, and then integrate the changed form.
- iii) Find the difference of the resultant integral value at the upper and lower limits.

Let us consider an example.

Example 5 Evaluate $\int_2^3 \left(\frac{x}{x^2+1}\right) dx$

Solution: Put $t = x^2 + 1$. Now taking derivative of both sides, we get

$$\frac{dt}{dx} = 2x \implies \frac{dt}{2} = x dx$$

When $x = 2$, $t = 5$ and $x = 3$, $t = 10$. Therefore, 5 and 10 are the new limits when x variable is substituted by the t variable.

$$\begin{aligned} \text{Hence, } \int_2^3 \left(\frac{x}{x^2+1}\right) dx &= \frac{1}{2} \int_5^{10} \left(\frac{1}{t}\right) dt \\ &= \frac{1}{2} \log t \Big|_5^{10} \\ &= \frac{1}{2} (\log 10 - \log 5) \\ &= \frac{1}{2} \log 2 \end{aligned}$$

Check Your Progress 2

1) Evaluate $\int_{-2}^2 \frac{x e^{-x^2}}{1+x^2} dx$

.....

2) Evaluate the following:

a) $\int_{-5}^5 |x + 2| dx$

.....

b) $\int_{-2}^3 f(x) dx$, Given $f(x) = \begin{cases} 6 & \text{if } x > 1 \\ 3x^2 & \text{if } x \leq 1 \end{cases}$

.....

3) Find the value of the definite integrals with the help of substitution method?

a) $\int_0^1 x e^{x^2} dx$

.....

b) $\int_0^1 \frac{x}{\sqrt{x^2+1}} dx$

.....

c) $\int_1^3 \frac{x+2}{\sqrt{x^2+4x+7}} dx$

.....

14.4 ECONOMIC APPLICATIONS OF DEFINITE INTEGRAL

In this section we will discuss some more applications of integral to Economics. You are familiar with the concepts of Consumer Surplus and Producer Surplus from the units on consumer behavior and production function analysis in Principles of Microeconomics course. We will use the concept of integral to throw some fresh light on these concepts.

14.4.1 Consumer Surplus

Concept of Consumer Surplus (CS) was introduced by Alfred Marshall as the monetary measure of the difference between the amount that the consumer is willing to pay for consuming a commodity or a service and the amount that he actually is required to pay, given his income and the prices he faces. Basically it is the net monetary benefit received by the consumer on consuming a good or a service.

$$\text{Consumer Surplus (CS)} = \text{Price willing to pay} - \text{Price actually paid}$$

Consumer Surplus and Marginal Utility: Willingness to pay is directly connected with the utility function of a consumer, it is a function of the marginal utility (MU).

$$\text{For a single unit, CS} = \text{MU} - \text{Price}$$

$$\text{Total CS} = \sum \text{MU} - (\text{Price} \times \text{Units consumed or purchased})$$

In integration form, given the Marginal Utility function, price (p) and units consumed/ demanded/ purchased (x).

$$CS = \int_0^x MU dx - px$$

Example 6

Given marginal utility function $MU(x) = 25 - 2x$, find the consumer surplus, if a consumer consumes 10 units and pays a uniform price of Rs 5 per unit.

Solution

$$CS = \int_0^x MU dx - px = \int_0^{10} (25 - 2x) dx - 5 \times 10$$

$$= \int_0^{10} 25 dx - 2 \int_0^{10} x \cdot dx - 50$$

$$= |25x|_0^{10} - 2 \left| \frac{x^2}{2} \right|_0^{10} - 50$$

$$= 250 - 25(0) - 2 \left(\frac{10^2}{2} \right) + 2 \left(\frac{0^2}{2} \right) - 50 = 100$$

Consumer Surplus and Demand Curves: Willingness to pay is simply the price reflected by the individual's demand curve for particular quantity of a good or a service consumed. Consumer surplus is the difference between the willingness to pay as shown by the demand curve, and the market price.

Consider Figure 14.3 where we have a straight line demand function AD. At price OP_0 , the consumer buys OQ_0 amount of good X. Consumer is willing to pay amount equal to area $OAEQ_0$ for Q_0 units of commodity which is the area between the demand curve and the x -axis. But since market price was OP_0 , he had to pay only OP_0EQ_0 . ΔAP_0E is the net gain the consumer attains from this purchase, which is known as the Consumer Surplus (given by the shaded region). It is equal to the area below the demand curve but above the price line. We can find it out through process of integration:

$$\text{Consumer Surplus (CS)} = \text{Area } \Delta AP_0E = \text{Area } AOQ_0E - \text{Area } P_0OQ_0E$$

$$= \int_0^{Q_0} f(x) dx - OP_0 \times OQ_0$$

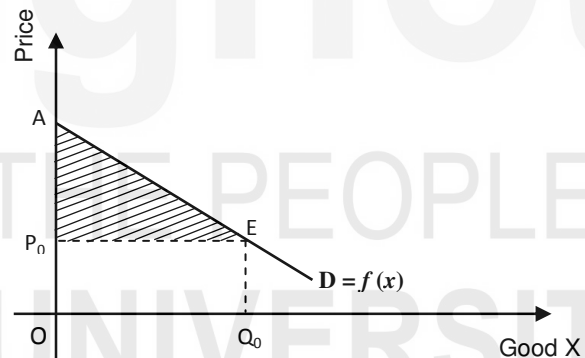


Figure 14.3

Example 7

A consumer's demand function (Dd) is given as $p = 90 - x^2$. Find consumer surplus if he purchases 5 units of x and pays a price of Rs 50 per unit.

Solution

$$CS = \int_0^5 (Dd) dx - px = \int_0^5 (90 - x^2) dx - px$$

$$= \int_0^5 90 dx - \int_0^5 x^2 dx - 50 \times 5$$

$$= 90 \int_0^5 1 \cdot dx - \int_0^5 x^2 dx - 250 = 90|x|_0^5 - \left| \frac{x^3}{3} \right|_0^5 - 250$$

$$= (90 \times 5 - 0) - \frac{1}{3}(5)^3 - 250 = 450 - \frac{125}{3} - 250 = 158.33$$

Producer Surplus

Producer Surplus (PS) like consumer surplus is the benefit received by the seller of the commodity or a service on participating in the market. It is determined as the difference between the total revenue that the producer or the seller receives on selling a good or a service, and the minimum amount he is willing to sell his commodity for, which is the producer’s cost of production given by his supply curve.

Consider the straight line supply curve [marginal cost function $g(x)$] given by line CS (Here we are assuming no fixed costs and rising marginal costs). P_0 is the price at which X_0 units of commodity are sold in the market, then producers’ surplus will be given by shaded area, which is the area above the marginal cost (supply) curve but below the price line. (Refer Fig 14.4)

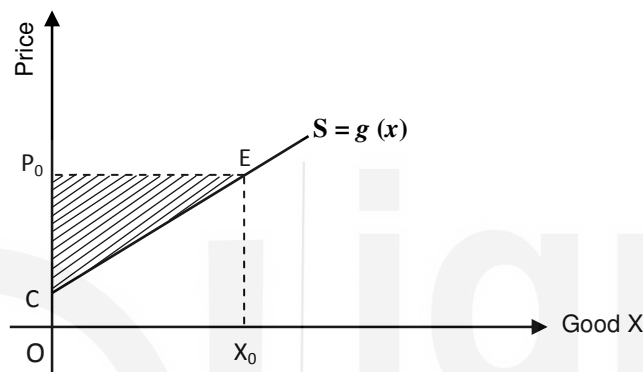


Figure 14.4

We know that area under the Marginal Cost function will be the Total Cost. Total Revenue is given by price (P_0) times the quantity (X_0). So the surplus to the producer will be given by:

$$\begin{aligned} \text{Producer Surplus (PS)} &= \text{Market Price} \times \text{Quantity sold} - \text{Total Costs} \\ &= \text{Area } OP_0EX_0 - \text{Area } OCEX_0 \\ &= P_0 \times X_0 - \int_0^{X_0} g(x)dx = P_0X_0 - \int_0^{X_0} MC dx \end{aligned}$$

Since we have assumed no fixed costs, $\int_0^{X_0} g(x)dx$ equals the total variable cost. Hence producer surplus can be regarded as difference between total revenue and total variable cost.

Example 8

Supply function of a commodity is given by $p = \sqrt{9 + x}$. Producer sells 7 units. Find his surplus.

Solution When $X_0 = 7$, price will be given by $P_0 = \sqrt{9 + 7} = 4$

\therefore Producers’ surplus,

$$\begin{aligned} PS &= 4 \times 7 - \int_0^7 (\sqrt{9 + x}) dx \\ &= 28 - \frac{2(9 + x)^{3/2}}{3} \Big|_0^7 = 28 - \frac{128}{3} + \frac{54}{3} \\ &= 10 \frac{2}{3} \text{ units} \end{aligned}$$

Example 9

We are given the following information. Find producer surplus.

- 1) Demand function : $p = 25 - 2x$
- 2) Supply function : $4p = 10 + x$ or $p = \frac{5}{2} + \frac{1}{4}x$

Solution: We will first find the equilibrium price and quantity from the given supply and demand functions. This means solving the demand and supply functions. Students can take up this exercise and find that $p = 5$ units and $x = 10$ units.

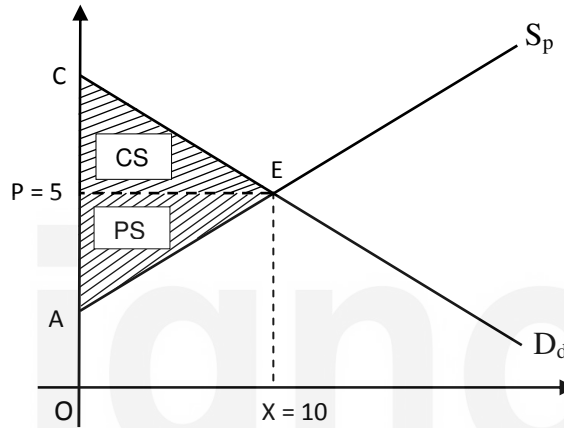


Figure 14.5

For finding producer surplus we will be using supply function only, along with the equilibrium price (5) and quantity (10) found above. Refer figure 14.5, where both producer and consumer surpluses have been marked. Thus, Producer surplus (PS) is given by:

$$\begin{aligned}
 PS &= px - \int_0^x Sp \cdot dx = 5 \times 10 - \int_0^{10} \left(\frac{5}{2} + \frac{1}{4}x \right) dx \\
 &= 50 - \left(\frac{5}{2} \int_0^{10} 1 dx + \frac{1}{4} \int_0^{10} x dx \right) \\
 &= 50 - \frac{5}{2} |x|_0^{10} - \frac{1}{4} \left| \frac{x^2}{2} \right|_0^{10} \\
 &= 50 - \frac{5}{2} \times 10 + 0 - \frac{1}{4} \times \frac{100}{2} + 0 \\
 &= 50 - 25 - 12.5 = 37.5 \text{ units}
 \end{aligned}$$

Example 10

Given the demand function (DD), $p = 20 - 5x$ and the supply function (Sp), $p = 4 + 3x$, find the consumer surplus as well as the producer surplus.

Solution

Solving the two functions, we find that $x = 2$, $p = 10$

$$\begin{aligned}
 \text{a) } CS &= \int_0^2 DD dx - px = \\
 &= \int_0^2 (20 - 5x) dx - 10 \times 2 = 20 \int_0^2 1 dx - 5 \int_0^2 x dx - 20
 \end{aligned}$$

$$= 20|x|_0^2 - 5 \left| \frac{x^2}{2} \right|_0^2 - 20 = 20 \times 2 - 0 - 5 \times \frac{2^2}{2} + 0 - 20$$

$$= 40 - 10 - 20 = 10$$

$$\text{b) } PS = px - \int_0^2 Sp \, dx = 10 \times 2 - \int_0^2 (4 + 3x) \, dx = 20 - 4 \int_0^2 dx - 3 \int_0^2 x \, dx$$

$$= 20 - 4|x|_0^2 - 3 \left| \frac{x^2}{2} \right|_0^2 = 20 - 4 \times 2 - 3 \times \frac{4}{2}$$

$$= 20 - 8 - 6 = 6$$

14.4.3 Profit Maximisation

We have done this type of analysis while studying the unit on differential calculus also. In differential calculus applications, we are given Total Revenue or Average Revenue along with Total Cost. But if these are not given and instead Marginal Revenue and Marginal Cost are given, then we have to first find Total Revenue (TR) and Total Cost (TC) with the help of integration. Differential calculus is then applied to maximise the profit. Thus, we first estimate the profit (π) function ($\pi = TR - TC$) and then apply the necessary (1st order) and sufficient (2nd order) conditions for profit maximisation.

Example 11

A firm has average revenue (AR) function as $p = 50 - x^2$ and marginal cost (MC) function as $1 + x^2$. Determine the consumer surplus if

- Perfect Competition conditions prevail in the market
- Monopolistic conditions prevail in the market.

Solution

- In pure competition, supply curve/ function is identified with MC. Therefore, we equate AR (the demand function) with MC (the supply function) for equilibrium quantity and price.

$$50 - x^2 = 1 + x^2 \quad 2x^2 = 49 \quad \text{or} \quad x^2 = \frac{49}{2} \quad \text{or} \quad x = \frac{7}{\sqrt{2}}$$

Neglecting negative value, we get $x = +\frac{7}{\sqrt{2}}$.

Corresponding value of p will be given by,

$$p = 50 - x^2 = 50 - \frac{7^2}{2} = 50 - \frac{49}{2} = \frac{51}{2}$$

Therefore, the equilibrium point is $x = \frac{7}{\sqrt{2}}, p = \frac{51}{2}$

$$CS = \int_0^{\frac{7}{\sqrt{2}}} (DD - p) \, dx = \int_0^{\frac{7}{\sqrt{2}}} (50 - x^2) \, dx - \frac{51}{2} \times \frac{7}{\sqrt{2}}$$

$$= 50 \int_0^{\frac{7}{\sqrt{2}}} dx - \int_0^{\frac{7}{\sqrt{2}}} x^2 dx - \frac{357}{2\sqrt{2}}$$

$$= 50 \left| x \right|_0^{\frac{7}{\sqrt{2}}} - \left| \frac{x^3}{3} \right|_0^{\frac{7}{\sqrt{2}}} - \frac{357}{2\sqrt{2}} = 50 \times \frac{7}{\sqrt{2}} - 0 - \frac{1}{3} \frac{7}{\sqrt{2}}^3 - \frac{357}{2\sqrt{2}} = \frac{343}{3\sqrt{2}}$$

- b) In case of a monopoly, profits are maximised or equilibrium occurs when $MR = MC$.

Now, Marginal Revenue (MR) = $\frac{d}{dx}$ (TR),

where Total Revenue (TR) = Price (AR) × Quantity (x)

$$MR = \frac{d}{dx}(x \cdot AR) = \frac{d}{dx}(50x - x^3) = 50 - 3x^2$$

Now, we have $MR = 50 - 3x^2$ and $MC = 1 + x^2$

For equilibrium $MR = MC$ $50 - 3x^2 = 1 + x^2$

$$4x^2 = 49 \quad \text{or} \quad x^2 = \frac{49}{4} \quad \text{or} \quad x = \sqrt{\frac{49}{4}} = \frac{7}{2}$$

Neglecting negative value, we get $x = \frac{7}{2}$.

Corresponding value of p will be given by,

$$p = 50 - x^2 = 50 - \frac{7^2}{4} = 50 - \frac{49}{4} = \frac{151}{4}$$

Therefore, $x = \frac{7}{2}$, $p = \frac{151}{4}$.

Now, $CS = \int_0^x (dd) dx - px$

$$\text{or } CS = \int_0^{\frac{7}{2}} (50 - x^2) dx - \frac{151}{4} \times \frac{7}{2} = 50 \int_0^{\frac{7}{2}} dx - \int_0^{\frac{7}{2}} x^2 dx - \frac{1057}{8}$$

$$= 50 \left| x \right|_0^{\frac{7}{2}} - \frac{1}{3} \left| x^3 \right|_0^{\frac{7}{2}} - \frac{1057}{8} = 50 \times \frac{7}{2} - 0 - \frac{1}{3} \left(\frac{7}{2} \right)^3 + 0 - \frac{1057}{8}$$

$$= 175 - \frac{343}{24} - \frac{1057}{8} = 175 - 14.291 - 132.125 = 28.584$$

Example 12

A firm's fixed cost is Rs 50 and marginal revenue is Rs 50. If its marginal cost function is $10 + \frac{x}{10}$, find (a) Profit maximising output and (b) Amount of Maximum profit.

Solution

$$TR = \int MR dx = \int 50 dx = 50x + C \quad [\text{Where } C = 0, \text{ as at } x = 0, TR = 0]$$

This implies, $TR = 50x$

$$TC = \int MC \, dx = \int \left(10 + \frac{x}{10}\right) dx = 10x + \frac{x^2}{20} + k$$

Now, when $x = 0$, the whole of TC is total fixed cost which is given as Rs 50.

$$\text{This implies, } TC = 10x + \frac{x^2}{20} + 50$$

a) Profit maximising output:

Profit function (π) will be given by:

$$\pi = TR - TC = 50x - 10x - \frac{x^2}{20} - 50 = 40x - \frac{x^2}{20} - 50$$

1st order condition for maximising Profits: $\frac{d\pi}{dx} = 0$

$$\frac{d\pi}{dx} = 40 - \frac{2}{20}x - 0 = 0 \quad \text{or} \quad 40 - \frac{x}{10} = 0 \quad \text{or} \quad x = 400$$

2nd order condition for maximising Profits: $\frac{d^2\pi}{dx^2} < 0$

$$\frac{d^2\pi}{dx^2} = -\frac{1}{10} < 0. \text{ Hence, profit is maximum when } x = 400$$

b) Maximum profit (π) = $40x - \frac{x^2}{20} - 50 = 40 \times 400 - \frac{(400)^2}{20} - 50$
 $= 16000 - 8000 - 50 = \text{Rs } 7950$

Example 13

A company's Marginal Cost (MC) function is $81 - 16x + x^2$. If its Marginal Revenue (MR) function is $20x - 2x^2$, find at what level of output it maximises profits, and what is the total amount of profits carved by it?

Solution

For equilibrium $MR = MC$ so that,

$$20x - 2x^2 = 81 - 16x + x^2 \Rightarrow 3x^2 - 36x + 81 = 0$$

or $x^2 - 12x + 27 = 0 \quad (x - 3)(x - 9) = 0 \quad \text{or} \quad x = 3, x = 9.$

We have to test whether profits are maximised at $x = 3$ or $x = 9$. For this we will be trying second order condition and that is:

$$\frac{d^2R}{dx^2} < \frac{d^2C}{dx^2}. \text{ Which is the same thing as:}$$

$$\frac{d}{dx}(MR) < \frac{d}{dx}(MC)$$

Case I: When $x = 3$, $\frac{d}{dx}(MR) = 20 - 4x = 20 - 4 \times 3 = 8$

$$\text{and } \frac{d}{dx}(MC) = -16 + 2x = -16 + 2 \times 3 = -10$$

Second order condition is not satisfied. Therefore profit is not maximum when $x = 3$.

Case 2: When $x = 9$, $\frac{d}{dx}(MR) = 20 - 4x = 20 - 4 \times 9 = -16$

$$\text{and } \frac{d}{dx}(MC) = -16 + 2x = -16 + 2 \times 9 = -16 + 18 = 2$$

For $x = 9$, the second order condition is satisfied. Therefore, profit is maximised at $x = 9$.

Calculation of maximum profit at $x = 9$

Profit (π) = TR - TC [where TR = $\int_0^9 MR \, dx$ and TC = $\int_0^9 MC \, dx$]

$$= \int_0^9 (20x - 2x^2) - (81 - 16x + x^2) \, dx$$

$$= \int_0^9 (20x - 2x^2 - 81 + 16x - x^2) \, dx = \int_0^9 (36x - 3x^2 - 81) \, dx$$

$$= 36 \int_0^9 x \, dx - 3 \int_0^9 x^2 \, dx - 81 \int_0^9 dx = 36 \left[\frac{x^2}{2} \right]_0^9 - 3 \left[\frac{x^3}{3} \right]_0^9 - 81 \left[x \right]_0^9$$

$$= 18(9)^2 - (9)^3 - 81 \times 9 = 1458 - 729 - 729 = 0$$

\therefore Total profit = Zero at $x = 9$

Check Your Progress 3

1) Demand function for a commodity is given by $p = 20 - 2x - x^2$. Find the consumer's surplus when he demands 3 units of the commodity.

.....

2) Find the producer surplus when equilibrium price is 36 units and the supply function is $p = x^2$.

.....

3) A monopolists demand function is $x = 210 - 3p$, where $p =$ price and $x =$ quantity demanded. The total cost function is: $x^2 + 6x + 10$, find consumer surplus at the equilibrium.

.....

- 4) Find both consumer as well as producer surplus at equilibrium, if the demand function is $x = \frac{25}{4} - \frac{1}{8}p$ and supply function is $p = 5 + x$.

.....

14.5 LET US SUM UP

This unit was the second unit on integration. This unit dealt with definite integrals. We saw that unlike indefinite integrals, computing definite integrals leads to numbers as solutions. That is because definite integrals have limits of integration. The unit explained how the definite integral is best viewed as the computation of the area under a curve. If we take the x -axis, and draw a curve in the x - y plane, then the area between the curve and the x -axis within certain specified closed interval on the x -axis can be computed as a definite integral. It was shown that the variable of integration is a dummy. Of course, the definite integral is initially solved in the same way as an indefinite integral, but then the variable with respect to which the function is integrated is replaced successively by the upper and lower limits of integration and then the difference is computed. The unit provided certain examples. The unit also explained the concept of a Riemann integral, and what does it mean for a function to be Riemann integrable.

Subsequently, you were familiarised with certain important properties of definite integrals. We saw that these properties have mainly to do with the limits of integration. The usual properties of indefinite integrals hold in the case of definite integrals as well. Finally the unit gave applications of definite integrals in economics, highlighting consumer surplus, producer surplus and profit maximisation.

14.6 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

- 1) a) 2
 b) $\frac{1}{2}$
 c) $\log \frac{5}{4}$
 d) $\frac{e^2}{2} - \frac{1}{2}$
- 2) a) $\frac{28}{3}$
 b) $\frac{64}{3}$
 c) $\frac{4}{3}$
- 3) See section 14.2.2 and answer.

Check Your Progress 2

1) 0

Here, $f(-x) = f(x)$, hence we have an odd function of x . From Property 8, we have $\int_{-a}^a f(x)dx = 0$, when $f(x)$ is odd.

2) a) 29

Solution: We have $|x + 2| = \begin{cases} x + 2, & \text{if } x \geq -2 \\ -x - 2, & \text{if } x < -2 \end{cases}$

$$\begin{aligned} \therefore \int_{-5}^5 |x + 2| dx &= \int_{-5}^{-2} |x + 2| dx + \int_{-2}^5 |x + 2| dx \quad (\text{Using property 3}) \\ &= \int_{-5}^{-2} (-x - 2) dx + \int_{-2}^5 (x + 2) dx \\ &= \left(-\frac{x^2}{2} - 2x\right) \Big|_{-5}^{-2} + \left(\frac{x^2}{2} + 2x\right) \Big|_{-2}^5 \\ &= 29 \quad (\text{On inserting respective values and taking the differences}) \end{aligned}$$

b) 21

Solution: We can write, $\int_{-2}^3 f(x)dx = \int_{-2}^1 f(x)dx + \int_1^3 f(x)dx$
(Using property 3)

$$\begin{aligned} &= \int_{-2}^1 3x^2 dx + \int_1^3 6 dx \\ &= x^3 \Big|_{-2}^1 + (6x) \Big|_1^3 \\ &= 21 \end{aligned}$$

(Hint: Put $x^2 = t$)

(Hint: Put $t = \sqrt{1 + x^2}$)

3) a) $\frac{1}{2}(e - 1)$

b) $\sqrt{2} - 1$

c) $2(\sqrt{7} - \sqrt{3})$

Check Your Progress 3

1) CS = 27 units.

(Hint: Market Price P_0 at $x = 3$ can be attained by substituting $x = 3$ in the given demand function, $\therefore P_0 = 20 - 6 - 9 = 5$ units)

2) PS = 144 units

(Hint: with $p = 36$, we can ascertain the quantity sold (x) from our supply function, hence, $x = \sqrt{36} = 6$.)

3) CS = 96

(Hint: On applying the equilibrium condition under monopoly, we get equilibrium output = 24 and equilibrium price = 62.)

4) CS = 100; PS = 12.5

(Hint: Here, demand function is given as $x = f(p)$. In order to apply our formula for CS and PS, we will first have to convert it into an inverse demand function, i.e. $p = f(x)$. Thus our demand function becomes $p = 50 - 8x$)