BLOCK 6

DIFFERENCE EQUATIONS

BLOCK 6 INTRODUCTION

The final Block of the course, titled **Difference Equations,** slightly changes the approach from what had been followed in Blocks 3, 4 and 5, in that the present Block does not involve a discussion of functions and variables that are continuous. Rather the block discusses variables and functions that are discrete. Specifically, it discusses equations in which values of variables are related to lagged values of themselves. Actually you have to understand that basically we are looking for a way to depict, model and study the evolution of economic variables over time. We look at a variable say x, and see that if its value at time t (time measured as discrete periods)depends on past values. Difference equations study just this.

Te Block has two units, Unit 15, titled **Linear Difference Equations**, and Unit 16, titled **Non-Difference Equations.** It should be clear from the titles that unit 15 discusses difference equations that are linear. The unit discusses the concept of finite differences. The unit explains basic operators like lag and backward-shift operators. You learn about various types of equations like homogeneous and non-homogeneous difference equations; autonomous and non-autonomous equations. The unit studies linear difference equations of first- and second-degree, with relevant economic applications.

The final unit of the block, also the final unit of the course, Unit 16, **Non-Difference Equations,** discusses about difference equations that are non-linear. The unit discusses phase diagram and qualitative analysis. Solution methods for non-linear difference equations are discussed. Non-linear difference equations are applied to certain areas in economics like cycles, chaos, and so on

UNIT 15 LINEAR DIFFERENCE EQUATIONS*

Structure

15.0 OBJECTIVES

After studying the unit, you should be able to:

- understand finite differences;
- explain discrete economic processes;
- define a linear difference equation;
- identify the order of a difference equation;
- explain linear difference equations of first and second degree;
- differentiate between a homogeneous and non-homogeneous difference equation;
- describe methods to solve first order and second order difference equations; and
- apply difference equations to depict, set up and solve some dynamic economic models in discrete time.

15.1 INTRODUCTION

By now you are familiar with differential calculus and integral calculus and have learned how to solve problems in economics using techniques of differential calculus. You have also seen how integration is the reverse of differentiation. If we want to find whether a given function is increasing or decreasing, sign of the differential coefficient of the dependent variable with respect to the independent variable helps in determining it. If we want to decide about the maximum or minimum values of a function, again we take the help of the differential co-efficients of various orders and so on. So, the idea of

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differential coefficient is central to all the techniques developed in differential calculus. But if y is a function of x, we can find its derivative only if x is a continuous variable, that is, only if the change in x is infinitesimally small. In such a case, y is defined for every value of x in an interval. But this may not to the case always.

In Economics, we do come across situations in which the independent variable is not continuous but discrete is nature. That is, x may take two values, say, 2 and 4, without taking all the values between 2 and 4. In such a case, if y is a function of x, then y is not defined in the entire interval (2, 4), but only for the values $x = 2$ and $x = 4$. For example, suppose we are studying the savings of a family and collect the data at fixed time intervals, say every year. So if $S = S(t)$, where S is the savings at the end of time interval t, we get S as a function of time t which takes values 0, 1, 2, ...,etc. and hence is a discrete variable. We may be interested in studying changes in the dependent variable due to changes in the independent variable, in such cases as well. The concept of derivative no longer works here, as the independent variable involved takes discrete and not continuous values. The study of relations that exist between the values assumed by a function whenever the independent variable changes by finite jumps is the subject matter of the calculus of finite differences. In this unit and the next, we introduce difference equations and give methods to solve them. We also study some situations in economics where difference equations may prove to be useful. The present unit deals with difference equations.

15.2 PRELIMINARY CONCEPTS

We will be considering a function, with a discrete and not continuous independent variable, that is, in a function where y will be the dependent and x the independent variable, x will be taking integer values. The pattern of change in y, the dependent variable will then be determined by "differences" and not derivatives or differentials of the function $y(x)$.

Suppose that $y = f(x)$, that is, y is a function of x, where x takes equidistant integer values $x_0, x_0+h, x_0+2h, \ldots, x_0+nh$.

Then $f(x_0), f(x_0+h), f(x_0+2h), \ldots, f(x_0+nh)$ will be the corresponding y values denoted by $y_0, y_1, y_2, \ldots, y_n$.

The differences, $y_1 - y_0$, $y_2 - y_1, \ldots, y_n - y_{n-1}$ are known as the ***first forward differences*** of the function $y = f(x)$, denoted by $\Delta y_0, \Delta y_1, \ldots, \Delta y_n$ respectively. These differences are what we were looking for to represent the change in y with respect to the discrete variable x.

Generally, first forward difference is defined as:

$$\Delta y_n = y_{n+1} - y_n, \text{ or}$$

$$\Delta f(x) = \Delta f(x+h) - f(x), \text{ where } h \text{ is the interval of differencing.}$$

Δy or $\Delta f(x)$ is basically the difference between two consecutive values of the discrete variable x.

For example, consider a sequence: $-5, -1, 3, 7, \ldots, (4x+3), \ldots$

Here, we have y or $f(x) = 4x + 3$ and $h = 4$ (calculated by subtracting any two consecutive values of the sequence). Then, the first forward difference will be given by:

$$\begin{aligned} \Delta y \text{ or } \Delta f(x) &= \Delta f(x+h) - f(x) \\ &= [4(x+h) + 3] - [4x + 3] \\ &= 4x + 4h + 3 - 4x - 3 \\ &= 4h = 16 \qquad [\because h = 4 \text{ (calculated)}] \end{aligned}$$

We define ***second forward difference***, $\Delta^2 y_n$ as $\Delta(\Delta y_n)$ so that

$$\begin{aligned} \Delta^2 y_n &= \Delta(\Delta y_n) = \Delta(y_{n+1} - y_n) \\ &= (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n) \\ &= y_{n+2} - 2y_{n+1} + y_n \end{aligned}$$

Similarly, we can define forward difference of higher order. Since our basic purpose is to study problems in Economics and since we are interested in finding a time path of a variable y from some given pattern of change over time, we take t as independent variable and write $\Delta y_t = y_{t+1} - y_t$. Thus, in the proceeding example, we may write $\Delta y_t = 16$. Such an equation is called a **difference equation**. The order of a difference equation is the order of the highest difference it contains. For example, $\Delta^2 y_t = 4$ is a difference equation of order 2.

The difference equation, $\Delta y_t = 16$ …(a)

can also be rewritten as $y_{t+1} - y_t = 16$ or $y_{t+1} = y_t + 16$ …(b)

Similarly, the difference equation $\Delta y_t = -0.3 y_t$ …(c)

can also be rewritten as $y_{t+1} - y_t = -0.3\, y_t$ or $y_{t+1} = 0.7\, y_t$ …(d)

A difference equation is **homogeneous** if its constant term, the term that does not contains y is zero, otherwise it is **non-homogenous**. Thus, we can define a difference equation as an equation involving an independent variable, a dependent variable and the successive difference of the dependent variable. Also, we have seen that the successive difference of a dependent variable can be expressed in terms of the successive values of the dependent variable. For example equation (a) can be rewritten as equation (b) and equation (c) as equation (d). So, we may also define a difference equation as an equation which expresses a relation between an independent variable and the successive values of the dependent variable. Difference equations expressed in this form [*i.e.* in the form (b), (d)] are more convenient to deal with.Note that $\Delta^2 y_t = 4$ can be rewritten as $y_{t+2} - 2y_{t+1} + y_t = 4$.

A solution of a difference equation is a relation between the independent variable and the dependent variable satisfying the given equation. The general solution of a difference equation of order n contains n arbitrary constants. For example $\Delta^2 y_t = 0$ is a homogeneous difference equation of order 2 and $\Delta y_t = 4$ is a non-homogeneous difference equation of order 1. A difference equation is said to be linear if every y present in the equation is at most of degree 1. We will first learn how to solve first order difference equations. Let us discuss different types of difference equations and their solution methods one by one.

15.3 FIRST ORDER DIFFERENCE EQUATIONS

In this sub-section we will learn to solve difference equations of order 1. We will first discuss the iterative method for solving a first order difference equation.

15.3.1 The Iterative Method

This method is explained with the help of an example.

Consider the difference equation $\Delta y_t = 4$

As seen earlier it can be rewritten as

$$y_{t+1} = y_t + 4 \quad \ldots(i)$$

Putting $t = 0, 1, 2, 3,$.....in equation (i) we get,

$y_1 = y_0 + 4$

$y_2 = y_1 + 4 = (y_0 + 4) + 4 = y_0 + 2(4)$

$y_3 = y_2 + 4 = y_0 + 2(4) + 4 = y_0 + 3(4)$

and so on.

In general, we have

$$y_t = y_0 + 4t \quad \ldots(ii)$$

Now suppose $y_0 = 20$. Inserting y_0 value in equation (ii), we get

$$y_t = 20 + 4t \quad \ldots(iii)$$

as the solution of the given difference equation $\Delta y_t = 4$, subject to the assumed value of y_0. Note that the solution in equation (iii) gives us the value of y corresponding to any value of t. If the initial value (y_0) is not specified, the solution would be obtained in terms of y_0.

For example, the difference equation $\Delta y_t = -0.3\, y_t \quad \ldots(iv)$

can be written as $y_{t+1} - y_t = -0.3\, y_t$ or $y_{t+1} = 0.7 y_t \quad \ldots(v)$

and can be solved using the iterative method as follows:

Putting $t = 0, 1, 2,$ ....*etc.*, in equation (v), we get

$$y_1 = 0.7\, y_0$$
$$y_2 = 0.7\, y_1 = (0.7)^2\, y_0$$
$$y_3 = (0.7)^3\, y_0$$

and so on. In general, we get $y_t = (0.7)^t\, y_0$

as the solution to the difference equation (iv) in terms of y_0 .

We will now discuss the general method for solving a first order difference equation.

15.3.2 The General Method

Consider the first order difference equation

$$y_{t+1} + a_0 y_1 = a \qquad \ldots\text{(vi)}$$

wherea_0 and a are constants. The general solution of the difference equation (vi) consists of two parts, a particular integral y_p and a complementary function y_c. The complementary function y_c is the general solution of the homogeneous difference equation (vii) corresponding to equation (vi).

$$y_{t+1} + a_0 y_t = 0 \qquad \ldots\text{(vii)}$$

Whereas, a particular integral y_p is any solution of the given non-homogenous difference equation (vi). The general solution of (vi) is the sum of $y_c$andy_p.

This method is called the general method because it helps us in solving the most general linear difference equation of order 1. Not only that, the method applies to linear difference equations of higher order as well. Since we have also to study second order linear difference equations and their solutions, and the method is similar for the first order and second order difference equations (in fact for linear difference equations of any order for that matter), we postpone its discussion for a while to avoid unnecessary repetition.

Check Your Progress 1

1) What is the difference between homogeneous and non-homogeneous difference equations?

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2) Solve the following linear difference equation by Iterative method:

i) $y_{t+1} - 6y_t = 0$

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ii) $\Delta y_t = 0$

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iii) $y_{t+1} = 10 + y_t$

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iv) $\Delta y_t = 2y_t$

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15.4 SECOND ORDER DIFFERENCE EQUATIONS

A difference equation not involving successive values of y_t greater than $y_{t+2}$is said to be of order 2. Some examples of second-order difference equations are:

i) $u_{t+2} - 3u_{t+1} + 5u_t = 10t$

ii) $y_{t+2} - 4y_t = 0$

iii) $y_{t+2} - 3y_{t+1} + 4y_t = 6 + t$

iv) $y_{t+2} + y_{t+1} - 2y_t = 12^t$

In general, any second order linear, non-homogenous difference equation with constant coefficient takes the form,

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = \phi(t) \qquad \text{…(viii)}$$

wherea_1 and a_2 are constants and $\phi(t)$ is a function of t only. If $\phi(t) = 0$, we get

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0 \qquad \text{… (ix)}$$

which is a second order linear, homogeneous difference equation with constant coefficients.

A general solution of (viii) is, again, given by $y_c + y_p$, where y_c is the general solution of (ix) and y_p is any solution of (viii). We will first learn how to solve (ix).This method is quite general in the sense that it can be used to solve any homogeneous linear difference equation of any order including equation (vii) of order 1. The method is as follows:

Step 1: In the difference equation (ix) replace y_t by 1, y_{t+1} by a variable, say, b, and y_{t+2} byb^2 so as to obtain the following characteristic equation:

$$b^2 + a_1 b + a_2 = 0 \qquad \text{…(x)}$$

Step 2: Solve the characteristic equation (x). The values of b so obtained are called *characteristic roots*.

Step 3: The general solution of (ix) can be written using the characteristic roots obtained in step 2. The solution depends on the nature of the characteristic roots, giving rise to three possibilities.

Case I: When the characteristic roots obtained in step 2 are real and distinct, say b_1 and b_2, then the general solution to (ix) will be written as:

$$y = A_1 b_1^t + A_2 b_2^t \qquad \text{…(xi)}$$

Case II: When the characteristic roots obtained in step 2 are real but repeated say b_1 and $b_2 = b_1$, then the general solution to (ix) will be written as:

$$y = (A_1 + t\ A_2)b_1^t \qquad \text{...(xii)}$$

Case III: When the characteristic roots obtained in step 2 are complex, say $b_1 = \alpha + i\beta$, $b_2 = \alpha - i\beta$, then the general solution to (ix) will be written as:

$$y = A_1(\alpha + i\beta)^t + A_2(\alpha - i\beta)^t \qquad \text{... (xiii)}$$

This can also be written as

$$y = r^t(C_1\cos\theta t + C_2\sin\theta t) \qquad \text{...(xiv)}$$

Where, $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$, $C_1 = A_1 + A_2$, $C_2 = i(A_1 - A_2)$

Note:

1. A_1, A_2, C_1, C_2 used above are all constants.
2) The roots in case III have been chosen to be $\alpha + i\beta$ and $\alpha - i\beta$. Here, α and β are real numbers. In fact, whenever a quadratic equation with real coefficients has complex roots, they have to be of the form $\alpha \pm i\beta$.
3) The case of complex repeated roots has not been discussed since we are dealing with equations of order up to 2 only, and for an equation to have repeated complex roots, its order must be at least 4.
4) When we are dealing with a non-homogeneous difference equation of type (viii), then the general solution of the corresponding homogeneous equation (ix) is called the complementary function of (viii) and is denoted by y_c. So in that case, let the solutions given in (xi), (xii), (xiii) or (iv) be denoted by y_c.

Let us take some examples to illustrate the procedure.

Example 1 Solve $y_{t+2} - 4y_t = 0$.

Solution: The characteristic equation (with $y_t = 1$ and $y_{t+2} = b^2$) of the difference equation is

$$b^2 - 4 = 0 \Rightarrow b^2 = 4 \Rightarrow b = \pm 2$$

Thus, solutions are $b_1 = 2$ and $b_2 = -2$. These are real and distinct.

Therefore, the general solution to the given equation is:

$y = A_1 2^t + A_2 (-2)^t$, (where A_1 and A_2 are constants)
[using equation (xi)]

Example 2 Solve $y_{t+2} + 10\ y_{t+1} + 25y_t = 0$

Solution The characteristic equation (with $y_t = 1$, $y_{t+1} = b$ and $y_{t+2} = b^2$) for this difference equation is:

$$b^2 + 10b + 25 = 0$$

Solving this equation we get –5 as a repeated root. Therefore, the general solution to the given equation is:

$y = (A_1 + t\,A_2)\,(-5)^t$ [using equation (xii)]

Example 3 Solve $y_{t+2} - 4y_{t+1} + 13y_t = 0$

Solution The corresponding characteristic equation of the given equation is

$$b^2 - 4b + 13 = 0$$

On solving, you will realize that its roots are complex given by, $b_1 = 2 + 3i,\ b_2 = 2 - 3i.$

$\therefore \alpha = 2, \beta = 3$. This implies that $r = \sqrt{2^2 + 3^2} = \sqrt{13}$

$\therefore$ The general solution to the given equation is:

$y = \sqrt{13}\,(C_1 \cos\theta t + C_2 \sin\theta t)$, where $\theta = \tan^{-1}\left(\frac{3}{2}\right)$

[using equation (xiv)]

Now we come to the general solutions of a non-homogeneous linear difference equation of order 2, of the form (viii). As discussed earlier, the general solution to equation (viii) will consist of two parts, y_c and y_p. Here, y_c, the complementary function, is the general solution of the corresponding homogeneous difference equation (ix), and we have already learned how to find it. Now we will learn to find y_p, the particular integral for the difference equation (viii). In fact, y_p depends on the function $\phi(t)$ on the right hand side of the equation (viii). We will discuss methods for obtaining the particular integral for some special types of ϕ (t) only. To be specific, we will be discussing only the following cases:

Case 1: When $\phi\ (t)$ is a constant function.

Case 2: When $\phi\ (t)$ is any polynomial function.[Note that case 1 is just a special case of case 2]

Case 3: When $\phi\ (t)$ is an exponential function of the form a^t, for some constant a.

Case 4: When ϕ (t) is a polynomial function multiplied by a^t for some constant a.

We will discuss these cases one by one. Since we can choose any solution of equation (viii) as y^p , we suggest methods for choosing the function which can be tried for the particular integral in each of the above mentioned cases.

Case 1 When $\phi(t)$ is a constant, say a, we choose

$$y_t = k$$

as a possible solution of equation (viii), where k is a constant to be determined by substituting $y_t = k$ in (viii).

Let us illustrate the procedure with the help of an example.

Example 4 Solve $y_{t+2} - 4y_t = 5$

Solution: Note that we are being given a non-homogenous equation, and the homogenous equation corresponding to this equation is

$$y_{t+2} - 4y_t = 0$$

which is the equation that has been solved in example 1. We already know the characteristic roots of the homogenous equation, giving us the complementary function as the general solution.

$$\therefore y_c = A_1 2^t + A_2(-2)^t$$

Now we proceed to find the particular integral. Since $\phi(t) = 5$, in this case we choose

$$y_t = k$$

as a trial solution for finding the particular solution of the given equation.

$y_t = k$ implies $y_{t+2} = k$ [y being a constant function]

Now substituting values of y_t and y_{t+2} in our non-homogeneous equation, we get $k - 4k = 5$

$$k = \frac{-5}{3}$$

$$\therefore y_p = \frac{-5}{3}$$

is a particular integral y_p of the given equation. Thus, the complete solution of equation $y_{t+2} - 4y_t = 5$ is given by

$$y_t = y_c + y_p$$

$$= A_1 2^t + A_2 (-2)^t - \frac{5}{3}$$

Case 2: When $\phi(t)$ is a polynomial of degree n in t, we choose the general polynomial $y_t = p_0 + p_1 t + \ldots\ldots p_n t^n$

of degree n as a possible candidate for the particular integral and find the constants $p_0, p_1, \ldots\ldots, p_n$ by substituting in the given difference equation.

Example 5 Solve $y_{t+2} + 10y_{t+1} + 25y_t = 1 + t^2$

Solution The complementary function $y_c = (A_1 + t A_2)(-5)^t$, has already been obtained in Example 2. To get the particular integral, we choose the general polynomial

$$y_t = p_0 + p_1 t + p_2 t^2$$

since we have $\phi(t) = 1 + t^2$, a polynomial of degree 2. The difference equation in the our example contains, besides y_t, the terms y_{t+1} and $y_{t+2.}$. We thus find these values from our general polynomial:

$$y_{t+1} = p_0 + p_1(t+1) + p_2(t+1)^2$$

$$= (p_0 + p_1 + p_2) + (p_1 + 2\,p_2)t + p_2\,t^2$$

$$y_{t+2} = p_0 + p_1(t + 2) + p_2(t + 2)^2$$

$$= (p_0 + 2\,p_1 + 4p_2) + (p_1 + 4\,p_2)t + p_2\,t^2$$

Now, on substituting the values of y_t, y_{t+1} and y_{t+2} in our difference equation, we get

$$[(p_0 + 2\,p_1 + 4p_2) + (p_1 + 4\,p_2)t + p_2\,t^2]$$

$$+ 10\,[(p_0 + p_1 + p_2) + (p_1 + 2\,p_2)t + p_2\,t^2] + 25\,[p_0 + p_1 t + p_2 t^2] = 1 + t^2$$

Collecting the coefficients of like terms on L.H.S, we get

$$(36p_0 + 12\,p_1 + 14\,p_2) + (36\,p_1 + 24\,p_2)t + (36\,p_2)t^2 = 1 + t^2$$

Now, equating the coefficients of both the sides,

$$36p_0 + 12p_1 + 14p_2 = 1$$
$$36p_1 + 24p_2 = 0$$
$$36p_2 = 1$$

Solving these equations, we get $p_2 = \frac{1}{36}$, $p_1 = \frac{-1}{39}$, $p_0 = \frac{-19}{8424}$

Thus we get $\quad y_p = \frac{-19}{8424} + \frac{-1}{39}t + \frac{1}{36}t^2$

as a particular integral for the given difference equation. Thus, a complete solution to the given difference equation $y_{t+2} + 10y_{t+1} + 25y_t = 1 + t^2$ is

$$y_t = y_c + y_p$$

$$y_t = (A_1 + t\,A_2)\,(-5)^t + [\frac{-19}{8424} + \frac{-1}{39}t + \frac{1}{36}t^2]$$

Case 3: When $\phi(t) = a^t$ for some constant a.

In this case the choice for the trial functions depends upon whether a is a root of the characteristic equation of the given difference equation or not. If a is not a root of the characteristic equation, then $y_t = ca^t$, where c is a constant to be determined, works as our trial function.

Whereas, if a is a root of the characteristic equation, then we check if it is repeating or not. If it is repeating m times, we choose $y_t = ct^m a^t$ as our trial equation. If a is a root of the characteristic equation and is not repeated, then m = 1, and we take $y_t = cta^t$ as our trial equation. If it is a root repeated twice, then m = 2, and we take $y_t = ct^2 a^t$ as our trial equation, and so on.

We take two examples to illustrate.

Example 6 Solve $y_{t+2} - 5y_{t+1} + 6y_t = 7^t$

Solution It is easy to see that in this case, the characteristics equation to the given difference equation will have $b_1 = 3$ and $b_2 = 2$ as the real and distinct characteristic roots. Thus the complimentary function ($C.F$) will be given by

$$y_c = A_1 3^t + A_2 2^t \quad \text{[using equation (xi)]}$$

To find y_p, note that $\phi(t) = 7^t$ and 7 is not a root of the characteristic equation, therefore we choose the following as our trial function:

$$y_t = c7^t$$

Now, we find y_{t+1} and y_{t+2} from the trial function.

$$y_{t+1} = c.7^{t+1}$$

$$y_{t+2} = c.7^{t+2}$$

Now, on substituting the values of y_t, y_{t+1} and y_{t+2} in our difference equation, we get

$$c.7^{t+2} - 5.c.7^{t+1} + 6.\ c.7^t = 7^t$$

Dividing both sides by 7^t, we get

$$49\ c - 35\ c + 6\ c = 1 \quad \Rightarrow \qquad c = \frac{1}{20}$$

$\therefore y_p = \frac{1}{20}.\ 7^t$ is the required particular integral (PI).

Now, the complete solution of the difference equation $y_{t+2} - 5y_{t+1} + 6y_t = 7^t$ turns out to be:

$$y_t = CF + PI$$

$$= \mathrm{A}_1 3^t + A_2 2^t + \frac{1}{20} 7^t$$

Example 7 Solve $y_{t+2} - 7\ y_{t+1} + 10 y_t = 3 \times 5^t$

Solutions In this case, the characteristic equation has solutions 2 and 5. Therefore, the complimentary function (CF) becomes:

$$y_c = A_1 2^t + A_2 5^t$$

Now $\emptyset(t) = 3 \times 5^t$, with 5 occurring in 5^t being one of the roots of the characteristic equation, but non-repeating. Therefore, we take $y_t = cta^t$ as our trial equation, with $m = 1$. Now,

$$y_t = c.t.5^t$$

$$\Rightarrow \qquad y_{t+1} = c.(t+1).5^{t+1}$$

$$\Rightarrow \qquad y_{t+2} = c.(t+2).5^{t+2}$$

Now, on substituting the values of y_t, y_{t+1} and y_{t+2} in our difference equation, we get

$$c.(t+2).5^{t+2} - 7.c.(t+1).5^{t+1} + 10.\ c.t.5^t = 3 \times 5^t$$

Dividing both sides by 5^t, we get

$$25.c.(t+2) - 35.c.(t+1) + 10.\ c.t = 3$$

$$\Rightarrow \qquad (25c - 35c + 10c).t + 50c - 35c = 3$$

$$\Rightarrow \qquad 15c = 3 \Rightarrow c = \frac{1}{5}$$

$\therefore y_p = \frac{1}{5} t\, 5^t$ is the required particular integral (*PI*).

Now, the complete solution of the difference equation $y_{t+2} - 7\, y_{t+1} + 10y_t = 3\times5^t$ turns out to be:

$$y_t = C.F. + P.I$$

$$= A_1 2^t + A_2 5^t + \frac{1}{5}\ t\, 5^t$$

Case 4: When $\emptyset(t) = (a_0 + a_1 t + \ldots + a_n t^n)\, a^t$, where $a, a_0, \ldots a_n$ are constants. In this case $\emptyset(t)$ is a product of functions considered in case 2 and case 3. The trial solution to be considered for y_p is also a product of the solutions considered in case 2 and case 3, respectively.

Thus if $\emptyset(t) = (a_0 + a_1 t + \ldots + a_n t^n)\, a^t$, we choose $y_t = (p_\circ + p_1 t + \ldots . p_n t^n) a^t$ as our trial equation, if a is not a root of the characteristic equation of the associated linear homogeneous difference equation and $y_t = t^m (p_\circ + p_1 t + \ldots . p_n t^n) a^t$ if a is a root of the characteristic equation repeated m times. For example, if the difference equation is

$$y_{t+2} - 5y_{t+1} + 6y_t = 3t^2 7^t$$

then we shall choose, $y_t = (p_0 + p_1 t + p_2\, t^2)\, 7^t$ as the trial function for the particular integral because $\emptyset(t)$ in this case is $3t^2 7^t$, which is the product of a second degree polynomial and 7^t, with 7 is not a root of the characteristic equation of the associated homogeneous linear difference equation.

But if we have the difference equation $\qquad y_{t+2} - 14y_{t+1} + 49y_t = 3t^2 7^t$

we shall choose $\qquad y_t = t^2 (p_0 + p_1 t + p_2 t^2) 7^t$

as the trial function, because in this case, the corresponding characteristic equation $b^2 - 14b + 49 = 0$ has 7 as a root repeated twice. This necessitated the induction of the factor t^2 in the trial equation. The values of the constants p_0, $p_1, \ldots, p_n$ in the trial functions above can be obtained by substituting the value of y_t of the trial function, in the respective difference equations. The calculation of the constants, of course, becomes more and more complicated with higher values of m and n in our trial equations. Even the difference equations of the type mentioned above need a lot of calculation. But the learner shall seldom encounter problem with higher values of n and m. We advise the learner to solve the above level of difference equations, along with some simple problems of this type given in the exercise that follows this unit. The techniques for solving difference equations of order 2, discussed here, are general in nature and can be applied to linear difference equations of any order. In particular, linear difference equations of order 1 can be easily solved using these techniques. We illustrate our point with the help of some examples.

Example 8 Solve the difference equation $\qquad y_{t+1} - 4y_t = 3$

Solution In this case, the corresponding homogeneous difference equation is

$$y_{t+1} - 4y_t = 0$$

and therefore the characteristic equation $b - 4 = 0$ has $b = 4$ as the only root.

$\therefore y_c = A(4)^t$ as the required complimentary function (C.F).

To find particular integral (*P.I*), we take $y_t = k$ as the trial function, where k is to be determined, because $\emptyset(t) = 3$ is of the type discussed in case 1.

Now, $y_t = k \Rightarrow y_{t+1} = k$ [y_t being a constant function]

Substituting these values in the difference equation $y_{t+1} - 4y_t = 3$, we get

$$k - 4k = 3 \Rightarrow k = -1$$

$\therefore y_p = -1$ is the required particular integral (*P.I*).

Thus, the complete solution of the difference equation $y_{t+1} - 4y_t = 3$ will be given by

$$y_t = C.F + P.I \qquad \Rightarrow \qquad y_t = A(4)^t - 1$$

Example 9 Solve $y_{t+1} - 2y_t = 12t$

Solution Here $\emptyset(t) = 12t$, a polynomial of degree 1.

$\therefore$ we will take $y_t = p_0 + p_1 t$ as our trial function.

Now, $y_{t+1} = p_0 + p_1(t + 1)$

On substituting the values of y_t and y_{t+1} in the given difference equation, we get

$$[p_0 + p_1(t + 1) - 2(p_0 + p_1 t)] = 12\,t$$

$$\Rightarrow (-p_0 + p_1) - p_1 t = 12t$$

Now, compare coefficients of both the sides:

$$-p_1 = 12 \Rightarrow p_1 = -12$$

$$\Rightarrow p_0 = p_1 = -12$$

$\therefore$ we get $y_p = -12 - 12t$ as the particular integral (P.I)

Clearly, C.F. $= A.2^t$

$\therefore$ The complete solution of the given equation is

$$y_t = C.F. + P.I.$$

$$= A.2^t - 12(1 + t)$$

Example 10 Solve $y_{t+1} + 5y_t = 6^t$

Solution: In this case the characteristic equation of the associated homogeneous difference equation has -5 as the only root and $\emptyset(t) = 6^t$ is a function of the type discussed in case 3. Therefore, we choose $y_t = c.6^t$ as our trial equation because 6 is not a root of the characteristic equation. This gives $y_{t+1} = c.6^{t+1}$. On substituting the values of y_t and $y_{t+1}$is the given difference equation we get,

$$c.6^{t+1} + 5.\ c.6^t = 6^t$$

Dividing both sides by 6^t, we get

$$6c + 5c = 1$$

$$\Rightarrow\ c = \frac{1}{11}$$

$\therefore$ The particular integral (*P.I*), $y_p = \frac{1}{11}6^t$

Also, $C.F., y_c = A(-5)^t$

$\therefore$ The complete solution is $y_t = C.F. + P.I = A(-5)^t + \frac{1}{11}6^t$

Example 11 Solve $y_{t+1} - 3y_t = 4.(3)^t$

Solution In this case complimentary function (*C.F*) = $A.(3)^t$ as 3 is the only root of the characteristic equation and given that $\phi(t) = 4.3^t$, we choose $y_t = c.t.3^t$ as our trial equation, because 3 occurring in 3^t is a non-repeating root of the characteristic equation.

$$y_t = c.t.3^t \qquad \Rightarrow \qquad y_{t+1} = c.(t+1)3^{t+1}$$

Now, on substituting the values of y_t and y_{t+1} in the given difference equation $y_{t+1} - 3y_t = 4.(3)^t$, we get

$$c.(t+1)3^{t+1} - 3(c.t.3^t) = 4(3)^t$$

Dividing both the sides by 3^t, we get

$$3c\ (t+1) - 3ct = 4$$

3c = 4

3c =4

c =4/3

$\therefore$ P.I. = $\frac{4}{3}$ t.$(3)^t$

$\therefore$ The general solution is $y_t = A.(3)^t + \frac{4}{3}t.(3)^t$

Example 12 Solve $y_{t+1} + 7y_t = 2t5^t$

Solution In this case $C.F. = A(-7)^t$ and we choose $y_t = (p_0 + p_1t)5^t$ as our trial function for finding *P.I.*because $\phi(t)$ is the product ofpolynomial of degree 1 and 5^t and 5 is not a root of the associated characteristic equation we have.

Now, from y_t, we can ascertain y_{t+1}: $y_{t+1} = [p_0 + p_1(t+1)]5^{t+1}$

Substituting values of y_t and y_{t+1} in the given difference equation $y_{t+1} + 7y_t = 2t5^t$, we get $[p_0 + p_1(t+1)]5^{t+1} + 7(p_0 + p_1 t)5^t = 2t5^t$

Dividing both sides by 5^t, we get

$$5p_0 + 5p_1 t + 5p_1 + 7p_0 + 7p_1 t = 2t$$

$$12p_0 + 5p_1 + 12p_1 t = 2t$$

On comparing coefficients of like terms, we get

$$12p_0 + 5p_1 = 0 \quad \text{and} \quad 12p_1 = 2$$

$$\Rightarrow p_1 = \frac{1}{6} \text{and} \quad p_0 = -\frac{5}{72}$$

$\therefore$ *P.I.* is given by: $y_p = (-\frac{5}{72} + \frac{1}{6}t)5^t$

Thus, the general solution is $y_t = A(-7)^t + (-\frac{5}{72} + \frac{1}{6}t)5^t$

Check Your Progress 2

1) How will you identify if a difference equation is a second-order one?

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2) Solve the following first-order linear difference equations:

i) $y_{t+1} + 2y_t = 5$

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ii) $y_{t+1} + 7y_t = 1 + t + t^2$

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iii) $y_{t+1} + y_t = 7t$

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iv) $y_{t+1} - 4y_t = 5(4)^t$

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3) Solve the following second-order linear difference equations:

i) $y_{t+2} - 5y_{t+1} + 6y_t = 0$

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ii) $y_{t+2} + 4y_{t+1} + 4y_t = 0$

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iii) $y_{t+2} - 11y_{t+1} + 28y_t = 6$

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iv) $y_{t+2} - 3y_{t+1} - 40y_t = 1 + t$

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v) $y_{t+2} - 6y_{t+1} + 9y_t = t + t^2$

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15.5 ECONOMIC APPLICATIONS

Difference equations have wide ranging applications in economics. In dynamic analysis, when the time variable t is allowed to take only discrete values and we are interested in finding a time path, given the pattern of change of a variable y over time variable t, the difference equations come out to be quite handy. We take up some situations in economics where the problem can be converted into a difference equation whose solution provides a solution to the problem in hand.

15.5.1 The Cobweb Model

Consider a situation in economic analysis where the demand function and the supply functions for a given commodity are given by

$Q_{st} = S(p_t) = -c + dp_{t-1}$ and $Q_{dt} = D(p_t) = a - bp_t$, respectively.

Q_{st} is the quantity supplied at time t and Q_{dt} the quantity demanded at time t.

p_t denotes the price of the commodity at time t. Constants a, b, c, and d are all taken to be positive, , with b representing the demand sensitivity to price at time t and d representing supply sensitivity to price at time $t-1$, respectively. In the supply function equation, the quantity to be supplied by the producer has been expressed in terms of the price in the time period $t-1$. Such situations do occur in Economics. For example, a producer may base his future output decision on the current price of the commodity (*i.e.*, Q_{t+1} may depend on p_t), or the current output decision on the price of the commodity in a previous time period (*i.e.*, Q_t may depend on p_{t-1}).

Assume that in each time period, the market price is always set at a level such that quantity demanded is equal to the quantity supplied. That is,

$$Q_{st} = Q_{dt}$$

The equations and the assumptions considered above, give us a market model for a single commodity. Substituting the values of Q_{st} and Q_{dt} in the above equation, we get

$$-c + dp_{t-1} = a - bp_t$$

$\Rightarrow bp_t + dp_{t-1} = a + c$

On replacing t by $t+1$ in the above equation, we obtain

$$\Rightarrow bp_{t+1} + dp_t = a + c$$

$$\Rightarrow p_{t+1} + \frac{d}{b}p_t = \frac{a+c}{b} \qquad \ldots\text{(xv)}$$

Note that the equation (xv) is nothing but a linear non-homogeneous difference equation of order 1. Thus, we have seen how a situation in Economics gets translated into a difference equation.

Let us now solve Equation (xv). The corresponding homogenous difference equation is

$$p_{t+1} + \frac{d}{b}p_t = 0$$

and the characteristic equation (with $p_{t+1} = x$ and $p_t = 1$) is

$$x + \frac{d}{b} = 0$$

The only root of the characteristic equation is $-\frac{d}{b}$

$$\therefore C.F. = A\left(-\frac{d}{b}\right)^t$$

where$A$ is a constant. To find *P.I.*, we observe that in equation (xv) $\phi(t) = \frac{a+c}{b}$, which is a constant. Thus we choose $p_t = k$ as our trial equation. Also, $p_{t+1} = k$. On substituting p_t and p_{t+1} in (xv), we get

$$k + \frac{d}{b}k = \frac{a+c}{b}$$

$$\Rightarrow \quad k = \frac{a+c}{b+d}$$

Thus, the general solution of equation (xv) is

$$p_t = A\left(-\frac{d}{b}\right)^t + \frac{a+c}{b+d} \quad \ldots\text{(xvi)}$$

If we are given some initial condition, say p_0 *i.e.*, the price at $t = 0$, the value of the constant A can be obtained. On substituting $t = 0$ in the general solution equation, we get

$$p_0 = A + \frac{a+c}{b+d} \Rightarrow A = p_0 - \frac{a+c}{b+d}$$

Now, equation (xvi) can be rewritten as

$$p_t = \left(p_0 - \frac{a+c}{b+d}\right)\left(-\frac{d}{b}\right)^t + \frac{a+c}{b+d} \quad \ldots\text{(xvii)}$$

Equation (xvii), which is a solution to the difference equation (xv), represents the time path for p_t. That is, given t we can find the corresponding equilibrium price p_t. This gives an oscillatory time path (a regular back and forth path) and the model is known as the cobweb model. There are three varieties of oscillation patterns in the model. The model is called **explosive** if $d>b$, **uniform** if $d = b$ and **damped** if $d<b$. The model is called the cobweb model because when we trace out the prices and the quantities in subsequent periods, a cobweb is spun around the demand and supply curves.

The cobweb model was an example of a situation in Economics which gives rise to a first order difference equation. We now consider a situation which gives rise to a second order difference equation.

15.5.2 Samuelson Multiplier-Accelerator Interaction Model

This model assumes that the national income y_t consists of three components

i) Consumption (C_t)

ii) Investment (I_t)

iii) Government Expenditure (G_t)

The Consumption (C_t) in the time period t is assumed to be proportional to the income of the previous period, y_{t-1}.

Investment (I_t) in period t is assumed to be proportional to the increase in the consumption in the time period t over the consumption in the time period $t-1$, *i.e.*, I_t is assumed to be proportional to $C_t - C_{t-1}$. These set of assumptions give rise to the following set of equations:

Income Function $\quad y_t = C_t + I_t + G_0$

Consumption Function $C_t = \gamma y_{t-1} \quad (0 < \gamma < 1)$

Investment Function $\quad I_t = \alpha(C_t - C_{t-1}) \quad (\alpha > 0)$

The government expenditure is assumed to be constant equal to G_0. The constant γ represents the marginal propensity to consume and α represents the acceleration coefficient.

Substituting the value C_t from consumption function into investment function, we get

$$I_t = \alpha\gamma(y_{t-1} - y_{t-2})$$

Now substituting the value of consumption function and the value of $I_t$obtained above in the income function (y_t), we get

$$y_t = \gamma y_{t-1} + \alpha\gamma(y_{t-1} - y_{t-2}) + G_0$$

$$\Rightarrow \quad y_t - \gamma y_{t-1} - \alpha\gamma(y_{t-1} - y_{t-2}) = G_0$$

$$\Rightarrow \quad y_t - \gamma\,(1 + \alpha)y_{t-1} + \alpha\gamma y_{t-2} = G_0$$

By replacing t by $t + 2$ in the above equation, this equation can be rewritten as

$$\Rightarrow \quad y_{t+2} - \gamma\,(1 + \alpha)\,y_{t+1} + \alpha\gamma y_t = G_0$$

Now we have a linear non-homogeneous difference equation of order 2 that can be easily solved.

Since $\phi(t) = G_0$, a constant, the particular integral ($P.I$) is easy to obtain. The calculation of complimentary function ($C.F$) may pose some problems because in this case the characteristic equation of the associated homogeneous difference equation (with $y_{t+2} = x^2$, $y_{t+1} = x$, and $y_t = 1$) is

$$x^2 - \gamma\,(1 + \alpha)\,x + \alpha\gamma = 0$$

This equation has roots

$$x = \frac{\gamma\,(1+\alpha) \pm \sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma}}{2}$$

$$\left[\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ for a quadratic equation, } ax^2 + bx + c = 0\right]$$

The nature of roots depends on the constants α and γ,and a number of cases may arise. We have no intention of going into the complicated details as we have already established our claim that there are situations in economic analysis which can be tackled using difference equations.

Check Your Progress 3

1) Obtain the difference equation in each of the following special cases of the cobweb model and solve it

   i) $Q_{dt} = 18 - 3P_t$, $Q_{st} = -3 + 4P_{t-1}$

   ii) $Q_{dt} = 5 - 2P_{t,}$ $Q_{st} = -2 + 5P_{t-1}$

   iii) $Q_{dt} = 19 - 7P_t$ $Q_{st} = 6P_{t-1} - 4$

   whereQ_{dt} stands for quantity demanded, Q_{st} stands for quantity supplied and P is the price.

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2) Obtain the difference equation in each of the following special cases of the Samuelson multiplier accelerator interaction model given by the income, consumption and investment function equations in section 15.5.2 and solve them.

i) *Take* $\alpha = \frac{2}{3}, \gamma = \frac{49}{50}$

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ii) *Take* $\alpha = \frac{2}{3}, \gamma = \frac{24}{25}$

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15.6 LET US SUM UP

This unit dealt with dynamic economic processes in discrete time. The next unit too will deal with discrete dynamic processes, but the processes studied will be non-linear. The present unit began by explaining the difference between continuous and discrete processes. After touching upon first-forward difference, the unit went on to explain the nature of a difference equation. You were familiarised with homogeneous and non-homogeneous difference equation.

Subsequently, the unit discussed first order and second order difference equations. The unit considered linear equations only. Non-linear difference equations would have taken us beyond the scope of this unit. The unit explained in detail the method of solving first order and second order difference equations. The iterative method as well as the general method of solving first order difference equations were explained. The method of solving second order difference equations was discussed, and several examples were provided. Finally, the unit discussed the use of difference equation in economic modeling, with applications to the Cobweb model and the Samuelson multiplier-accelerator interaction model.

15.7 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) A difference equation is **homogeneous** if its constant term, the term that does not contains *y* (that is the variable) is zero, otherwise it is **non-homogenous**.

2) i) $y_t = 6^t y_0$

ii) $y_t = y_0$

iii) $y_t = 10t + y_0$

i) $y_t = 3^t y_0$

Check Your Progress 2

1) A difference equation that does not contain successive values of y_t greater than $y_{t+2}$is said to be of second-order.

2) i) $y_t = A(3)^t + \frac{5}{3}$

ii) $y_t = A(-7)^t + \left[\frac{25}{256} + \frac{3}{32}t + \frac{1}{8}t^2\right]$

iii) $y_t = A(-1)^t - \frac{7}{4} + \frac{7}{2}t$

iv) $y_t = A(4)^t + \frac{5}{4}t(4)^t$

3) i) $y_t = A_1(2)^t + A_2(3)^t$

ii) $y_t = (A_1 + tA_2)(-2)^t$

iii) $y_t = A_1(4)^t + A_2(7)^t + \frac{1}{3}$

iv) $y_t = A_1(8)^t + A_2(-5)^t + \left[-\frac{41}{1764} - \frac{1}{42}t\right]$

v) $y_t = (A_1 + tA_2)(3)^t + \left[\frac{7}{8} + \frac{3}{4}t + \frac{1}{4}t^2\right]$

Check Your Progress 3

1) i) $P_t = A(-\frac{4}{3})^t + 3$

ii) $P_t = A(-\frac{5}{2})^t + 1$

iii) $P_t = A(-\frac{6}{7})^t + \frac{23}{13}$

2) i) $y_t = A_1\left(\frac{14}{15}\right)^t + A_2\left(\frac{7}{10}\right)^t + 50\,G_0$

ii) $y_t = A\left(\frac{4}{5}\right)^t + 25\,G_0$

UNIT 16 NON-LINEAR DIFFERENCE EQUATIONS*

Structure

16.0 OBJECTIVES

After going through this Unit, you will be able to:

- Differentiate between linear and non-linear difference equations;
- Explain the concept of a phase diagram and the important role that phase diagrams play in understanding non-linear difference equations;
- Discuss how qualitative analysis of non-linear difference equations is carried out;
- Describe the procedure for linearising non-linear difference equations; and
- Discuss the application of non-linear difference equation to the Solow growth model and to Cycles and Chaos theory.

16.1 INTRODUCTION

In the previous unit, we had discussed linear difference equations. In practice, several economic models give rise to non-linear dynamic relations. The introduction of non-linearity does not alter the essence of a difference equation. Let us consider a simple economic system. Suppose that the economic system under discussion can be described by a single variable $x \in X$, where $X \subseteq \mathbb{R}$ is a non-empty interval of the real number line. You have studied about real number line and subsets in the unit on sets. The variable x is referred to as the system variable and X is referred to as the system domain. The system domain consists of all possible values of the system variable. In our context, we think of the economic system as a dynamic one, that is, that the values of x change over time. We assume that time is measured in discrete periods $t \in \{0,1,2,3..\}$. The variable x changes only once in one period. The set Z^+ (*i.e.* a set of positive integers)is the set of all time periods and is called the *time domain* or *time horizon*. Time is

* Contributed by Shri Saugato Sen & Chetali Arora

indicated by a subscript on the variable x. Thus x_t is the value of x in time period t. All difference equations describe the evolution of a variable over time. The dynamic evolution of the system is described by a function f, which shows how x_t is dependent on past values of x. We write

$$x_{t+1} = f(x_t)$$

Here the function f is called the law of motion or the system dynamics. In the previous unit, we were concerned with difference equations that could be described by a linear equation like

$$x_{t+1} = a + bx_t$$

The non-linear form includes the linear form as a special case, but has the advantage of allowing a much broader range of varieties of time paths to emerge.

Actually the non-linear difference equation of the above type can be expressed as $x_{t+1} = f(x_t, t)$. Here t itself appears as an argument, that is, an independent variable. However, we consider only autonomous difference equation where time does not appear as an independent variable. Hence we restrict ourselves to the study of equations of the type

$$x_{t+1} = f(x_t)$$

Notice that x_{t+1} depends only on the value of x in the period immediately preceding to it, that is, on x_t and not on $x_{t-1}$and other past values of x. Thus the value of x in any period depends only on the value of x in the immediately preceding period and not on the values of x in distant past periods. This type of difference equation is called a ***first-order difference equation***. In this unit, we will only consider first-order difference equation.

A general form of linear difference equation would be

$$x_{t+1} = f(x_t, x_{t-1}, x_{t-2}, \dots, x_{t-k+1}, t)$$

where$k$ is a positive integer called the *order* of the difference equation. The above equation also considers time as an independent variable. As we saw above, if t does not enter as a variable itself, then the difference equation is autonomous.

The solution to the autonomous difference equation of degree one, $x_{t+1} = f(x_t)$, is a sequence$\{x_t\}_{t=0}^{\infty}$, such that $x_t \in X$ and the equation $x_{t+1} = f(x_t)$holds. Such a sequence is referred to as the *trajectory* of the equation $x_{t+1} = f(x_t)$.

Let us look again at our difference equation, $x_{t+1} = f(x_t)$. Notice that we can shift it by a period and write $x_t = f(x_{t-1})$.

Given an initial condition x_0 at $t = 0$, we can work out a solution sequence as follows:

$$x_1 = f(x_0)$$

$$x_2 = f(x_1) = f\left[f(x_0)\right] = f^2(x_0)$$

$$x_3 = f(x_2) = f\left[f^2(x_0)\right] = f^3(x_0)$$

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$$x_t = f\left(x_{t-1}\right) = f\left[f^{t-1}\left(x_0\right)\right] = f^t\left(x_0\right)$$

Where, f^t denotes the t th iterate of f, not its power. The function $x\left(t, x_0\right) = f^t\left(x_0\right)$ is called the *flow* of the system $x_t = f\left(x_{t-1}\right)$

There is very simple but powerful method of analysing the dynamics of the above equation. To use this technique, we must draw the graph of the function f, as well as a 45° line into a $\{x_t, x_{t+1}\}$ plane. We take up the discussion of trying to arrive at a solution of first-order difference equation by making use of such diagrams (called phase diagrams) in the next section.

After that, the unit will discuss how non-linear difference equations can be approximated linearly in small neighbourhoods around specific points. Following this the unit will discuss a few applications of non-linear difference equations in section 16.4. First, the unit will discuss the growth model propounded by Solow. Finally the unit will discuss periodic cycles as well as aperiodic behavior of economic systems. This aperiodic behavior is called chaos.

16.2 PHASE DIAGRAMS AND QUALITATIVE ANALYSIS

We draw phase diagrams in the case of non-linear equations to help us get a qualitative 'solution' of a difference equation. To understand phase diagrams, consider a difference equation

$x_{t+1} = f(x_t)^*$, where $t = 0, 1, 2, 3\ldots$

Case I: $0 < f'(x) < 1$ and $f''(x) < 0$

We first consider the phase diagrams of the non-linear difference equation by assuming that the function is upward sloping, that is $f'(x) > 0$ and that the function flattens as x increases, that is, $f''(x) < 0$.

*A general non-linear, first order difference equation is given by $x_{t+1} = f(x_t, t)$, $t = 0, 1, 2\ldots$ Here, we are considering an equation which does not explicitly depends on time, that is $x_{t+1} = f(x_t)$, where $t = 0, 1, 2, 3\ldots$ This is called *nonlinear, first-order, autonomous difference equation.*

If we plot this curve on a graph which has x_{t+1} on the vertical axis and x_t on the horizontal axis, we get a picture as in Figure16.1. Let us denote this curve (also called the phase line) equation by

$$x_{t+1} = f_a(x_t) \qquad \ldots(1)$$

We have also plotted a 45° line in diagram 16.1. The intersection of the curve f_a with the 45°^{0} line marks an equilibrium point where $x_{t+1} = x_t$. Given an initial value x_0 on the horizontal axis, from our equation 1, we get the vertical-coordinate value as x_1 [where $x_1 = f_a(x_0)$], this is what is mapped by the Phase line f_a in figure 16.1; considering the other way round, x_0 on the horizontal axis, marked straight up to the phase line at point A gives the value of x_1. Next we map another pair given by x_1 and x_2 [where, $x_2 = f_a(x_1)$]. Note that $x_1$is transplotted from vertical axis to the horizontal axis with the help of 45° line (having slope = 1). Steps are repeated in similar fashion to plot subsequent pairs. This process is what we learnt in previous unit, called the *Iteration* method.

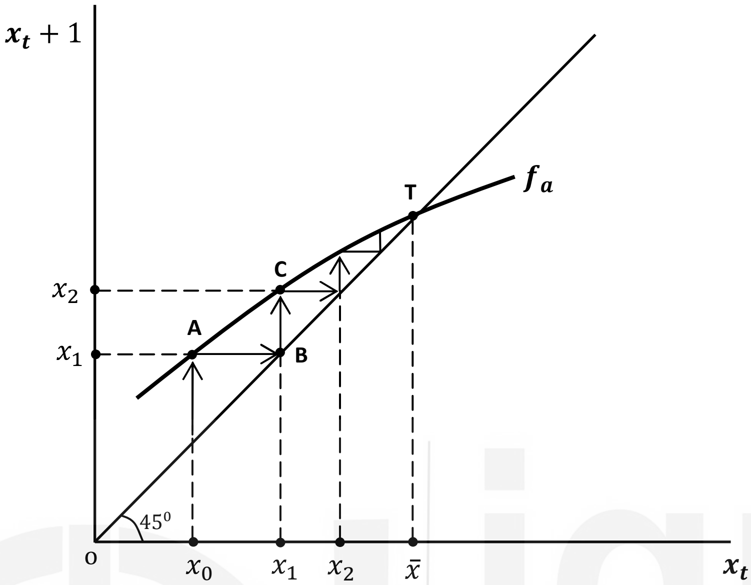

Figure 16.1: *Phase Diagram for* $0 < f'(x) < 1$ *and*$f''(x) < 0$

On similar lines, the Graphic iterations discussed above will be applicable to the following three cases as well:

Case II: $f'(x) > 1$ and$f''(x) < 0$

Let Phase line equation be give by $\qquad x_{t+1} = f_b(x_t)$

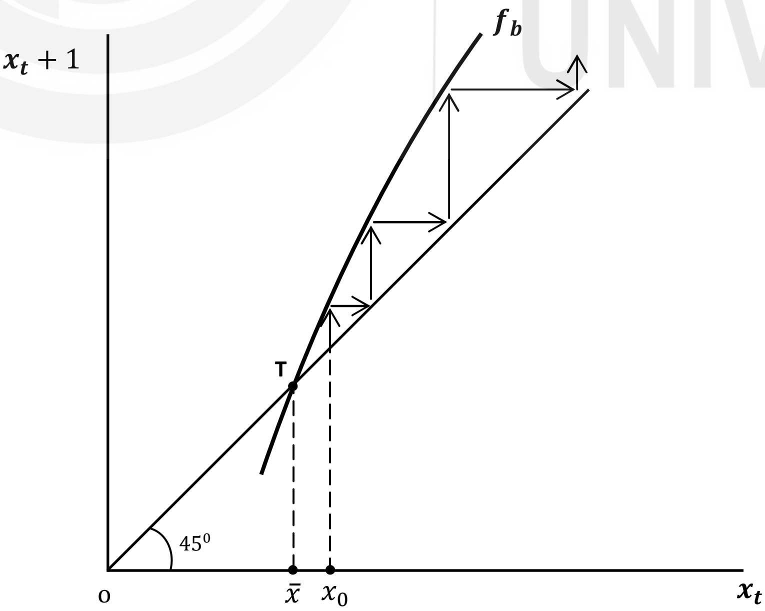

Figure 16.2: *Phase Diagram for* $f'(x) > 1$ *and*$f''(x) < 0$

Case III: $-1 < f'(x) < 0$

Let Phase line equation be give by $x_{t+1} = f_c(x_t)$

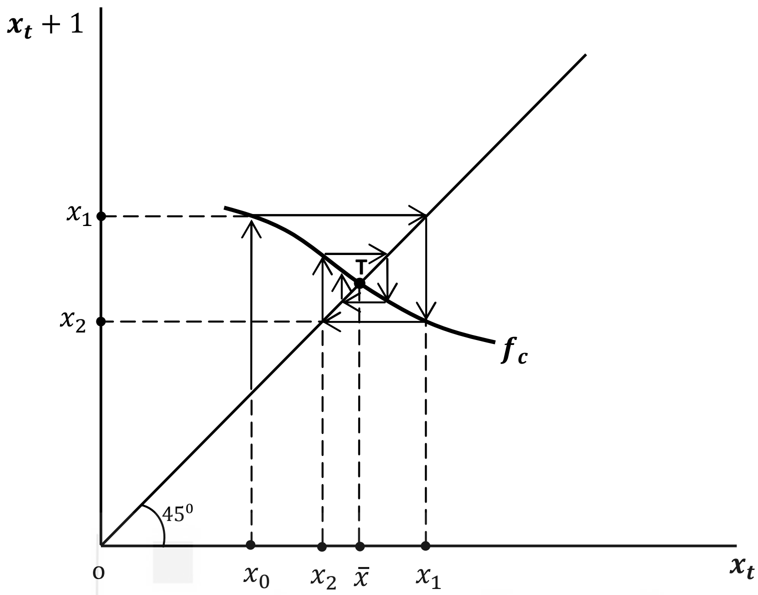

Figure 16.3: *Phase Diagram for* $-1 < f'(x) < 0$

Case IV: $f'(x) < -1$

Let Phase line equation be give by $x_{t+1} = f_d(x_t)$

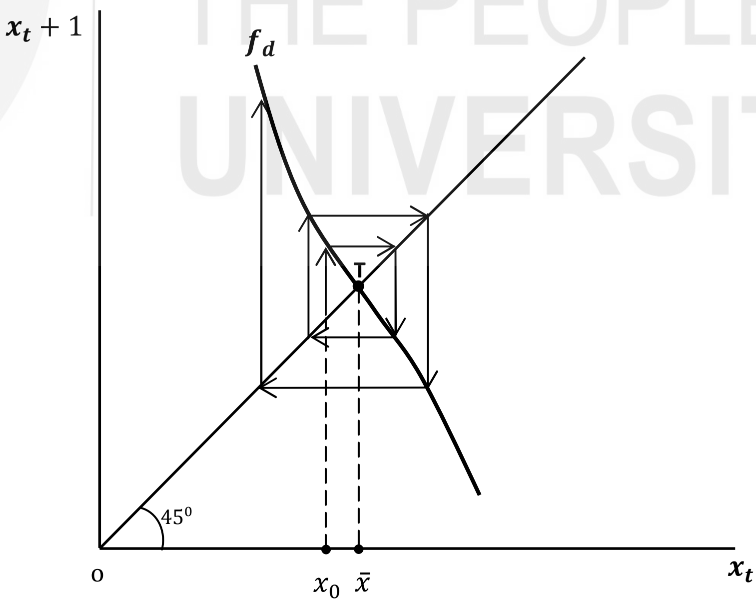

Figure 16.4: *Phase Diagram for* $f'(x) < -1$

The above four cases illustrate four basic types of Phase lines, each indicating a different time path. The intertemporal (where 'intertemporal' refers to 'across the time') equilibrium value of x denoted by $\bar{x}$ is given by the intersection of the phase lines in each diagram with the respective 45° line, labeled as point T. At point T, $x_{t+1} = x_t$

Our next task is to see whether given an initial value $x_0 \neq \bar{x}$, the pattern of change as indicated by the phase line will lead consistently towards $\bar{x}$— which is the case of *Convergence*, or away from it — the case of *Divergence*. In Case I above, phase line f_a leads us from x_0 to $\bar{x}$ in a steady path, without oscillation. Even when x_0 is placed to the right of $\bar{x}$, the movement towards $\bar{x}$ will be steady and in the left direction. This is referred to as convergence to the equilibrium. Under Case II, phase line f_b with a slope exceeding 1, showcases a divergent path. Here beginning from an initial value x_0 greater than $\bar{x}$, we are lead steadily away from the equilibrium value to higher and higher values of x. An initial value, lower than $\bar{x}$, will give rise to similar steady divergent movements in the opposite direction.

In Cases III and IV, we encounter steady oscillatory movements, along with the phenomenon of overshooting the equilibrium mark. Phase line f_c with absolute slope less than 1, will result in convergence, along with the extent of overshooting (given by x_0 leading to x_1 which exceeds$\bar{x}$ to be followed by x_2 falling short of$\bar{x}$) diminishing in successive periods. In case of phase line f_d whose absolute slope exceeds 1, there is an opposite scenario of a divergent time path.

Thus, two basic rules can be drawn from above discussion:

i) A first-order, autonomous, non-linear difference equation will have a locally stable steady-state equilibrium point ($\bar{x}$) if the absolute value of the derivative (which is nothing but the slope of the phase line), given by $f'(\bar{x})$ is less than 1, whereas will be unstable if the absolute value of the derivative is greater than 1 at that point (i.e. at$\bar{x}$).

ii) A first-order, autonomous, non-linear difference equation will lead to oscillations in x_t if $f'(x)$ is negative for all $x_t > 0$, whereas there will be monotonic movement in x_t if $f'(x)$ is positive for all $x_t > 0$.

Check Your Progress 1

1) What is a non-linear difference equation?

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2) What do you understand by a phase diagram? What is it used for?

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3) What is the basic difference in the equilibrium of a linear difference equation and that of a non-linear one?

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16.3 LINEARISING NON-LINEAR DIFFERENCE EQUATIONS

After obtaining equilibrium point(s), that is, the fixed point(s) of a difference equation, an important task left is to explore the stability of that equilibrium. For non-linear equations, we can only explore stability of equilibrium point(s) in a small neighbourhood. This is done by linearising the non-linear function at equilibrium and then testing the slope of the linear approximation. In fact, this is just an extension of what we did when we studied phase diagrams in the previous section: we looked at the slope of $f(x)$ function in the phase diagram to judge about the stability of the equilibrium. We investigated the slope of $f(x)$ in the region of the equilibrium.

You have seen in an earlier unit how we use Taylor's series expansion to linearise a non-linear function. Here too, we shall use Taylor's series expansion. We attempt to linearise$f(x)$, *i.e.* our Non-linear difference function by finding a first-order Taylor series expansion of this function, with equilibrium as the point of expansion. Although, a first-order Taylor series expansion produces a linear approximation to a non-linear function, the problem is that the approximation is good only for a limited range around the point of expansion. Also, the greater the degree of curvature of the original function, the smaller that range will be. Of course, our expansion need not be limited to first-order only, and we can take the expansion to as high an order as we like. Moreover, if the original curve is very greatly curved, we may require higher orders of expansion to approximate the original curve, but the higher orders of expansion introduce non-linear elements into the expansion, and since the idea is to take a linear expansion, we stop at the first-order, that is, at a linear expansion.

To take a first-order Taylor series approximation to a general function $f(x)$, we begin by choosing the value of x which would determine the point around which we will construct a linear approximation to the non-linear function. Let us denote this value of x by x^*,which means that the value of the function $f(x)$ at the point of approximation is $f(x^*)$. Then we can write, as the approximation to the function $f(x)$ at some arbitrary point x^*:

$$f(x) \approx f\left(x^*\right) + \frac{df(x^*)}{dx}(x - x^*)$$

Note that the derivative on the right-hand side of the equation is also evaluated at x^*. The closer x is to x^*, the closer the value of the approximation.

Now we apply this approximation to a first-order difference equation. Remember we are using the following equation as our first-order difference equation:

$$x_{t+1} = f(x_t)$$

and using x^* or $\bar{x}$to denote an equilibrium of the system. Approximating the function close to the equilibrium gives

$$x_{t+1} = f(x_t) = f(x^*) + \frac{df(x^*)}{dx}(x_t - x^*)$$

Now note that since $x^*$is an equilibrium point, $f(x^*) = x^*$. Hence we can write the above equation as $x_{t+1} = x^* + \frac{df(x^*)}{dx}(x_t - x^*)$ …(2)

Let us define a new variable x^D, as the deviation of the current value of x from its equilibrium value x^*. Thus,$x^D{}_t = x_t - x^*$ and $x^D{}_{t+1} = x_{t+1} - x^*$. Hence, we can write the equation (2) as

$$x^D_{t+1} = \frac{df(x^*)}{dx} x^D_t$$

…(3)

In interpreting equation 3, we must remember that we have evaluated the first derivative $\frac{df(x^*)}{dx}$ at a single point (here the equilibrium point), thus it is a constant. Given this, equation 3 becomes a homogeneous equation in x^D, with constant coefficient. This makes it a *linear first-order homogeneous difference equation*. Since it is a homogeneous equation, it means that its equilibrium is at $x^D = 0$, but since $x^D$is the deviation of the original untransformed variable from its equilibrium, it means when $x^D = 0$, $x = x^*$. So if equation 2 is stable with x^D converging to its equilibrium, then x too must converge to its own equilibrium.

Check Your Progress 2

1) Why do we need to linearise non-linear difference equations?

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2) Suggest a method by which we can approximate a first-order non-linear difference equation near a small neighbourhood of its equilibrium point.

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16.4 APPLICATIONS OF NON-LINEAR DIFFERENCE EQUATIONS

In this section we discuss some applications of non-linear difference equations. Probably the most familiar among the various applications are some models of economic growth. Even several consumption models display non-linear relations, especially when the utility function is not very restrictive. However we are not discussing consumption functions here, but taking up other non-linear dynamic processes. Specifically we shall take up for discussion a non-linear growth model suggested by Robert Solow, and Cycles and Chaos.

16.4.1 The Solow Model

Let us see what we understand by a growth model. Since we are discussing dynamics, we would want to see how a variable evolves over time. Here time is expected to pass in discrete levels. In the theory of economic growth, we see how the aggregate economic output or income grows over time. Your "Principles of Microeconomics" course in this semester will introduce you to the concept of a production function, where output is a function of labour and capital, in your. Here, we will think of an aggregate production function for the economy as a whole. Your "Principles of Macroeconomics" course will introduce you to the study of the economy in aggregate, that is, the Macroeconomics aspect of an economy. By combining our understanding from both the courses, here we are basically examining how the aggregate output or what we call the Gross Domestic Product (GDP) of an economy grows as a result of growth of total labour and capital (machines and equipment) over time.

Professor Robert Solow in 1956 propounded a model of economic growth. This model has become the central growth model in economic analysis, and has served as the baseline growth model upon which further work on growth has been carried out and new models suggested. Prof. Solow went on to receive the Nobel Prize in Economics in 1987 for this model. The economic growth model of Robert Solow which we consider here is also known as the Neoclassical growth model. This model contains difference equations for two variables, but by a trick common to growth models we are able to reduce it to a single difference equation model.

We begin with an aggregate production function:

$$Y_t = f(K_t, L_t)$$

Where *Y* is aggregate output, *K* is aggregate capital and *L* is aggregate labour. The time subscripts on each variable indicate that there are no lags in the production process.

For simplicity, labour (which is here assumed to be identical to population; that is, the labour force participation rate is100 per cent) is assumed to grow at an exogenous proportional rate *n*. In other words, we get a difference equation:

$$L_t = (1 + n)L_{t-1}$$

Capital grows as a result of net investment, which is given by gross investment minus an allowance for depreciation. Thus, Net investment is defined as the change in the stock of capital in two successive time periods.

This is a neoclassical model, so all savings are invested in productive physical capital, hence investments are considered to be equal to the savings. Consider the following equation that gives the value of capital at period t (K_t) as a function of capital at period $t-1$ (K_{t-1}):

$$K_t = sF(K_{t-1}, L_{t-1}) + (1-\delta)K_{t-1}$$

Here, s is average total saving per output, $\frac{S}{Y}$. So $S = sY = sF(K, L)$. We are also considering saving S equals investment I. So we can write, $I = sF(K, L)$.

It must be clear that,

$I_{t-1} = K_t - K_{t-1}$. This means $K_t = I_{t-1} + K_{t-1}$. If we subtract depreciation at rate δ, we get

$K_t = I_{t-1} + (1-\delta)K_{t-1}$. Substituting for I_{t-1} as $sF(K_{t-1}, L_{t-1})$, we can write

$K_t = sF(K_{t-1}, L_{t-1}) + (1-\delta)K_{t-1}$

Note that saving is done in period $t-1$ for new capital to emerge in period t. The above equation tells us that the current period's capital (K_t) is equal to the part of last period's capital left after deducting for depreciation $[(1-\delta)K_{t-1}]$ plus saving (equaling investment) done out of last year's income (output) which shows up in new capital goods and equipment in the current period, *i.e.*, $[sF(K_{t-1}, L_{t-1})]$.

Now let us make an assumption about the production function. Let us assume that the production function displays Constant Returns to Scale (CRS). This means that if labour and capital are scaled up by a factor λ, the output will increase by the same factor λ.

That is, CRS implies $\lambda Y_t = F(\lambda K_t, \lambda L_t)$

Let us assume $\lambda = \frac{1}{L_t}$. Then, $\frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}, 1\right)$

where $\frac{K_t}{L_t}$ is the current capital-labour ratio. Right-hand side of the equation shows the amount a single worker would produce if he or she had available a capital stock equal to the current capital-labour ratio. The left-hand side of this equation is output per worker and is denoted y_t. We can write the above equation as $y_t = f(k_t)$ where, $k_t = \frac{K_t}{L_t}$.Here, we have depicted output per capita as a function of capital per capita.

Recall the equation $K_t = sF(K_{t-1}, L_{t-1}) + (1-\delta)K_{t-1}$. Let us divide both sides of the above equation by L_t. We get $\frac{K_t}{L_t} = s\frac{F(K_{t-1}, L_{t-1})}{L_t} + \frac{(1-\delta)K_{t-1}}{L_t}$

In this equation, the right hand side has terms pertaining to both time periods $t-1$ and t. To resolve this, let us multiply and divide all terms on the right hand side by L_{t-1}. We get

$$\frac{K_t}{L_t} = \left(\frac{sF(K_{t-1}, L_{t-1})}{L_{t-1}}\right)\left(\frac{L_{t-1}}{L_t}\right) + \left(\frac{(1-\delta)K_{t-1}}{L_{t-1}}\right)\left(\frac{L_{t-1}}{L_t}\right) \qquad \ldots(4)$$

The term $\frac{F(K_{t-1}, L_{t-1})}{L_{t-1}}$ in above equation is output per worker in time period $t-1$ and the term $\frac{K_{t-1}}{L_{t-1}}$ is capital-labour ratio in period $t-1$. We hadseen above that $L_t = (1+n)L_{t-1}$. From this we get $\frac{L_{t-1}}{L_t} = \frac{1}{(1+n)}$. Using the notation for per capita quantities we had developed earlier, we can write equation (4) as,

$$k_t = \frac{sf(k_{t-1})}{1+n} + \left(\frac{1-\delta}{1+n}\right)k_{t-1} \qquad \ldots(5)$$

Here n and δ are exogenous (i.e., given and fixed from outside the model). Hence, we get a first-order difference equation given by equation 5, where k_t is a function of k_{t-1}. We have not specified a precise functional form for f (k_t), so we limit ourselves to a qualitative, phase diagram analysis of equation (5). We can see that per capita production function $y_t = f(k_t)$ has usual production function properties. Also $f'(k) > 0, f''(k) < 0.$ considering these assumptions, we can draw a phase diagram of the type given in figure 16.5.

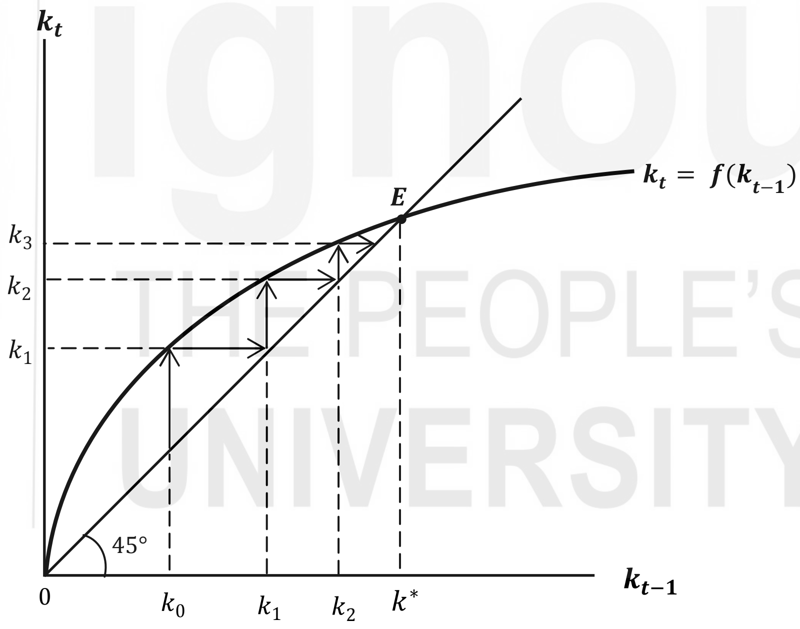

Figure 16.5: *Phase Diagram for a Neoclassical Growth Model*

At equilibrium (that is at point E), $k_{t-1} = k_t = k^*$, say. Then, even though we may not have a precise definition of $f(k)$, we can see that equation (B) must satisfy the relationship

$$\frac{f(k^*)}{k^*} = \frac{(n+\delta)}{s}$$

16.4.2 Cycles and Chaos

Till now we have considered only those non-linear difference equations for which the slope of the function f (.), does not change sign, that is, the graph of x_{t+1} against x_t, or x_t against x_{t-1}, does not change sign. In other words, the graph of x_t against x_{t-1} in phase diagram, is either monotonically increasing or monotonically decreasing, but never has the shape of a hill (inverted U), or valley (U-shaped). In this sub-section, we consider non-linear difference equations that generate hill-shaped curves in the phase diagram. This kind of difference equation generates interesting dynamic behavior, such as *Cycles* which repeat themselves every two or three periods, or even dynamic processes in which there is irregularity in the behavior of x_t. This type of an irregular process is called *Chaos*. It is beyond the scope of this unit to provide a detailed analysis of such cycles as well as chaotic behavior. In this sub-section, we give a simple exposition of the topic.

Consider the first-order, nonlinear, autonomous difference equation

$$x_{t+1} = A\, x_t(1 - x_t), \text{ where } t = 0, 1, 2, \ldots \qquad \ldots(6)$$

The equilibrium values $\bar{x}$ are obtained by solving

$$\bar{x} - A\bar{x}(1-\bar{x}) = 0 \Rightarrow A\bar{x}^2 - A\bar{x} + \bar{x} = 0$$

$$\Rightarrow A\bar{x}^2 + \bar{x}(1-A) = 0 \Rightarrow \bar{x}^2 + \bar{x}\frac{(1-A)}{A} = 0$$

This gives $$\bar{x}\left[\left(\frac{1-A}{A}\right) + \bar{x}\right] = 0$$

The two steady points are $\bar{x} = 0$and $\bar{x} = \left(\frac{A-1}{A}\right)$.

From the second steady-state point, we can say that a strictly positive steady-state equilibrium exists only if A> 1. If A ≤ 1, then the steady states are zero or negative. These we do not discuss as they do not have much relevance in Economics. Recall from our discussion of phase diagrams that a steady-state equilibrium point of any first-order autonomous, non-linear difference equation is locally stable, if the absolute value of its slope, that is its derivative is less than 1at that point. The value of the derivative of equation 6 above at the two steady points $\bar{x} = 0$ and $\bar{x} = \left(\frac{A-1}{A}\right)$ we found above is given as follows:

$$\frac{dx_{t+1}}{dx_t} = A - 2Ax_t$$

At$\bar{x} = 0$, $\frac{dx_{t+1}}{dx_t} = A$ and at $\bar{x} = \frac{A-1}{A}$, $\frac{dx_{t+1}}{dx_t} = 2 - A$

The above result implies that the steady point $\bar{x} = 0$ is unstable (as we assumed A> 1). The point $\bar{x} = \frac{A-1}{A}$ will be locally stable only if absolute value of $2 - A$, that is $|2 - A|$< 1⟹ 1 <A < 3. This was what concluded the basic rule (i) under section 16.2.

Since the above equation is a first-order nonlinear difference equation, we can draw a phase diagram for the equation. The intersection of the phase line with the 45° line will occur at our steady points, that is at 0and $\left(\frac{A-1}{A}\right)$. The graph will peak at x= $\frac{1}{2}$, where the slope given by $\frac{dx_{t+1}}{dx_t}$= 0. Second derivative of the function, given by $\frac{d^2x_{t+1}}{dx_t^2} = -2A$, which is negative, implying that the graph of the equation will be inverted U shape. (refer Figure 16.6)

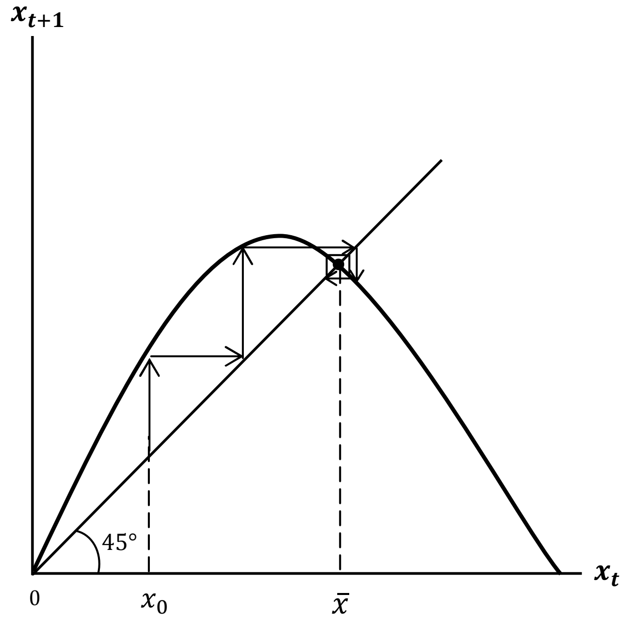

Figure 16.6: *Phase Diagram for the first-order, nonlinear, autonomous difference equation* $x_{t+1} = Ax_t(1-x_t)$.

Note yourself from the graph that the intersection of the phase line with the 45° line at point will happen to the left of the peak, if $1 < A < 2$. This implies that the slope is positive at the stable-steady point $\bar{x} = \frac{A-1}{A}$. Whereas, when $2 < A < 3$, the intersection will be on the right of the steady point. This satisfies the condition of local stability, as the slope of the phase line will be negative at the stable steady-state point.

A negative slope with less than 1 absolute value will mean that x_t will converge to $\bar{x}$ from either direction within a neighbourhood, but the path of approaching it will be oscillatory. Refer the figure. Starting from x_0, the slope is positive, with x_t increasing monotonically in the initial few periods. However as x_t approaches the neighbourhoodof$\bar{x}$, the slope becomes negative with x_t becoming oscillating in the neighbourhood before it converges to the steady state.

What will be the behavior of the phase line when $A \geq 3$? Firstly, $\bar{x} = \frac{A-1}{A}$ will no longer be a stable steady state, rather will be unstable. Secondly, the hill-shaped phase line possesses a peculiar characteristic which does not hold for a monotonically phase diagram—that is, x_t will not diverge endlessly to 0 or infinity, but will be oscillating within a bound range, though it will not be converging to the steady state, but could converge to regular periodic behavior.

When a non-linear difference equation throws up this kind of a inverted U-shape phase diagram, there arise threshold in the behavior of the function in the sense that small changes in the value of A can lead to dramatic changes in the behavior of x and in its trajectory. For instance, in our above example, if the value of A lies between 2 and 3, x follows a simple convergent alternations, that is, x takes on alternate values but moves towards convergence. As the value of A becomes 3 or above, the trajectories become very complicated. For some values between 3 and 4, x_t settles into periodic alternations, or what is called a limit cycle. This basically means that x takes values only within a certain range, and alternates values within this range. It can be shown that when the value of A is 3.2, if the dynamic system runs long enough, the system will settle down to a pattern that is called a period 2 cycle, getting back and forth between a value of 0.513 and 0.799. This is an example of the alternation version of the limit cycle. Limit cycles produce oscillations but these appear mainly in the case of higher order difference equations. The alternations, however, give the essence of a limit cycle. This is an example of a stable limit cycle.

However, there can be cases when the limit cycles are unstable. These cycles have the same basic properties as an unstable equilibrium. The basic feature is that if we start from a value in the limit cycle, we will stay in that same cyclical path always, neither converging nor diverging; if we start from a value on either side of the cycle, we will diverge from it.

If we increase the value of A steadily, different results emerge. At A = 3.4, the range of the interval within which the value of x lies is bigger, but it is still between two values. If A = 3.5, the system follows a period 4 cycle, that is going from $x = 0.382$ to $x = 0.827$ to $x = 0.501$ to $x = 0.875$. but if A is slightly higher at A = 3.84, the system is back to a period three cycle.

The interesting thing about values of A, and this is where chaos enters, is that the periodicity of alternations is not always smooth. If we set A = 3.58, we will find that the system alternates around an upper equilibrium, but never reapeats itself. It does not display any pattern which repeats over and over. It becomes aperiodic. In other words, the system is chaotic.

In what way is the study of chaos useful in economics? First of all, chaos is very hard to distinguish from a random process. And what is a random process? You will study about uncertainty and probability in the course on Statistics in the next semester. Some of you may already be familiar with probability. Random processes are associated with probability. A random variable is a variable whose value is determined by chance. Non-random or non-probability-based processes are called deterministic. The dynamic processes that we have been studying using non-linear (or even linear in the previous unit) difference equation is a deterministic process. The significance of chaos is that chaos depicts a situation where a deterministic process mimics the pattern of a random process. It is very difficult to distinguish a random process from a chaotic process. A fundamental difference between a random process and a chaotic process is that a random process cannot be predicted. However in a chaotic system, given the parameters, future values can fairly easily be predicted.

Check Your Progress 3

1) Explain the fundamental difference equation in the Solow growth model.

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2) a) What is the basic shape of the phase diagram of a chaotic system.

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b) What do you understand by a limit cycle?

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3) How is chaos important in economics? In what way is a chaotic process different from a random process?

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16. 5 LET US SUM UP

This unit followed unit 15 which was on linear difference equation. Like the previous unit, this unit dealt with discrete dynamic processes, that is, processes where the value of a variable depends on the past values of the same variable. The variables change values in discrete steps rather than continuously. The unit was concerned with general difference equations, of which linear difference equations were of one special type. Hence, the term 'non-linear' in the title is to be interpreted as general difference equations, rather than being restricted to the case of difference equations that are not linear.

The Unit began with explaining the general nature of difference equations. We saw that linear difference equations are a specific type of difference equations. The unit restricted itself to first-order difference equations, that is, to equations in which the value of the variable in period t depends only on value of the variable in period $t-1$, and not on the values in time periods $t-2, t-3$, and so on. The value of the variable thus depends on the immediate past value of the variable and not on the values of the variable in the distant past. We saw that we can shift the time period of the equation and still maintain the first order of the equation by considering that the value of the variable in period $t+1$ depends only on the value of the variable in period t.

In the next section, the Unit introduced you to phase diagrams (or phase line) as a way to depict first-order, nonlinear, autonomous difference equations, and say something about its possible solutions. The phase diagram is constructed by showing the current value of the variable on the horizontal axis and the future value of the variable on the vertical axis; or depicting the past value of the variable on the horizontal axis and the current value on the vertical axis. We saw that unlike linear difference equations of the first degree, non-linear difference equations of the first degree can intersect the 45° line at more than one place. If the curve depicting the difference equation is upward sloping then the equilibrium point (fixed point) can be converging (attractor) or diverging (repellor). A downward sloping curve leads to oscillations. Since the unit restricted itself to first-order difference equations, the unit relied mainly on graphic solution methods and hence phase diagrams became a very important tool of analysis.

In the next section, we took a look at linearising non-linear difference equations. To this end, we utilised first-order Taylor Series expansion of the function $f(x)$ at the relevant point x^*. In the subsequent section, the unit took up for discussion two applications of non-linear difference equations: the Solow growth model, and a simple look at periodic *Cycles* as well as periodic *Chaos*. The Solow growth model showed the dynamics of how output per worker evolves over time as a function of capital per worker. A suitable phase diagram was provided with capital-worker ratio in one period shown on the vertical axis, and output-worker ratio shown on the horizontal axis. We showed the dynamics with the help of a non-linear difference equation relating capital per worker of the current period to the capital per worker of the previous period.

Finally, the unit discussed dynamic processes where the phase diagram of the non-linear function is inverted U-shaped. Important features and properties of such phase diagrams were discussed depending on whether the maximum point of the function lay to the left of the 45^0 line, or to its right. From such a quadratic difference equation, we studied the properties of limit cycles, as well as aperiodic cycles. The latter situation is called chaos, and at the end we saw that chaotic behavior mimics the behavior of random processes.

16.6 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) A non-linear difference equation is an equation that relates the value of a variable in one period to the value(s) of the same variable in past periods, where the shape of the function depicting the equation is not limited to certain specific shape. A linear difference equation is a special case of a non-linear difference equation.

2) A phase diagram plots the values of a variable on the vertical axis and the values of the same variable in the previous period on the horizontal axis. It is used to depict first-order difference equations. There is a 45^0 line, and the intersection of the function with the 45^0 line helps us know about equilibrium points.

3) (a) A linear difference equation has one equilibrium point; a non-linear equation has multiple equilibria. (b) an attractor is a stable equilibrium whereas a repellor is an unstable equilibrium.

Check Your Progress 2

1) We linearise a non-linear difference equation to be able to make statements about the stability of equations.

2) We can use first-order Taylor expansion to linearise a non-linear difference equation around its equilibrium value

Check Your Progress 3

1) The fundamental non-linear difference equation in the Solow model relates the capital-labour ratio to its value in the previous period. This is the basic function whose phase diagram is drawn.

2) (a) The basic shape of a chaotic dynamic system is quadratic with a hill shaped curve, though not all quadratic functions have chaotic behavior. (b) A limit cycle shows a periodic regular cyclical behavior with the interval within which the function can take its values being limited.

Chaos is important in Economics because it shows how deterministic equations can give rise to irregular aperiodic cyclical behavior. Chaos is different from a random process in that a random process' future values cannot be predicted (that is the nature of the variable being random!) while a chaotic dynamic process' future values can be predicted.

GLOSSARY

Adjoint of a matrix: The adjoint of a square matrix A is defined as the transpose of the matrix of the cofactors of the determinant of A.

Anti derivation: Reverse of derivation

Anti derivative: Integral

Cartesian coordinates: A system in which points on a plane are identified by an ordered pair of numbers, representing the distances to two or three perpendicular axes.

Continuity: A function is continuous if its graph can be drawn on a p without lifting the pen or the pencil with which it is being drawn.

Constant of Integration: Arbitrary constant C which gets added to F(x)

Constraint: it is a side-condition that has to be fulfilled in some optimisation exercises. It changes the value of the global optimum that would have prevailed in the absence of such a constraint.

Cofactor: The cofactor of any element a_{ij} of the determinant of A denoted by C_{ij}, is in fact a signed (+ or -) minor. The sign of the minor is determined by the rule: C_{ij} = (-1)i + jM_{ij}.

Constant: A quantity that remains fixed in the context of a given problem or situation.

Cobweb Model: A model of quantity demanded and supplied where demand depends on current price while quantity supplied depends on price prevailing one period earlier.

Calculus: A branch of mathematics which deals with change. It was described as the mathematics of change, motion and growth.

Coordinate plane: The plane determined by a horizontal number line, called the x-axis, and a vertical number line, called the y-axis, intersecting at a point called the origin. Each point in the coordinate plane can be specified by an ordered pair of numbers

Closed input-output model: In such a model, the household sector is treated as one of the industries and thus the entire output of each producing sector is absorbed by other producing sectors as secondary inputs or intermediate products. Here, no portion of the output is sold in the market as final product.

Constrained optimisation is best possible position under some conditions called constraints. For example a consumer maximizes his utility/ satisfaction subject to his money income called income or budget constraint.

Continuity: A function f(x) is **continuous** provided its graph is continuous, i.e., a **continuous** function does not have any break at any point of its graph.

Compounding: the process of accumulating the time-value of money or a financial asset forward in time

Continuity of a Function: Function f is continuous at some point c when the following two requirements are satisfied:

- $f(c)$ must be defined (i.e., c must be an element of the domain of f).

- The limit of $f(x)$ as x approaches c must exist and be equal to $f(c)$. (If the point c in the domain of f is not a limit point of the domain, then this condition is vacuously true, since x cannot approach c .
- **Critical Point:** It is also called stationary point where the first derivative is zero. It can be a point of:
  a) Maxima when slope changes from 0 to less than zero.
  b) Minima when slope changes from 0 to more than zero.
  c) Where the curvature of the curve changes either from converse to concave or from concave to converse. This point is called the point of inflexion.

Definite Integral: Here the numerical value of integral is a definite number, arbitrary constant is eliminated

Dependent Variable: in an equation or function, the variable whose value depends on that of other variable(s)

Discounting: calculating the present value of a future amount.

Dynamic Analysis: a type of analysis in which the objective is to either trace out the time paths of the variables, or to determine whether, given sufficient time, these variables will tend to converge to some equilibrium values.

Determinant: The determinant is a number (scalar) defined for a square matrix only. It is conventionally represented by enclosing the elements of the corresponding matrix within two vertical straight lines.

Discontinuity: If there is a jump or jerk in the graph at a point a over a range, the function is discontinuous

For finding the derivative of a function, the function must be continuous.

Diagonal matrix: A diagonal matrix is a square matrix, where there is at least one non-zero element on the principal diagonal and all the off-diagonal elements are zeros.

Dimension or order of a matrix: The number of rows and columns of a matrix constitutes its dimension or order.

Domain of a Function: In a function $f(x)=y$, the set X of input values is called the domain of f, and the set Y where f takes output values is called the codomain.

Differential Calculus is a branch of calculus which tries to study the phenomenon of finding a derived function – a function derived from a given function.

Derivative expresses the idea of change and is derived function on the basis of a condition that the function is continuous. It is also called differential coefficient $\frac{dy}{dx}$ if the function is $y=f(x)$.

Everywhere Continuous Function: Continuous at every point of its domain.

Endogenous Variable: in a model, the variables, whole values have to be determined as solutions to the model

Element of a matrix: Each member of a matrix is called an element of the matrix.

Endogenous variable: It is a variable whose value is given from within a given model.

Exogenous variable: It is a variable whose value is determined from outside a given model.

Hawkins-Simon Condition: This is the condition for positive levels of output, when a_{ij}'s are expressed in physical units. The condition requires that all principal minors of the technology matrix must be positive.

Homogeneous System of Equations: A system of linear equations is homogeneous if each constant to the right of the equality equals zero.

Homogeneous difference equation: A difference equation is **homogeneous** if its constant term (the term containing no y) is zero, otherwise it is **non-homogenous**.

Function : It is the rule of correspondence between dependent variable and independent variable (s) so that for every assigned value to the independent variable, the corresponding unique value for the dependent variable is determined.

Indefinite Integral: Integral involving an arbitrary constant. It cannot have a definite numerical value

Integration by substitution: When you make substitution for given substitution integrand and use function of a function rule to evaluate integral it is called integration by substitution

Integration-by-parts: a technique of working out integral of a product of two function

Input coefficient matrix: It is a matrix of various secondary inputs required by different producing sectors per unit of their output.

Input-output model or input-output transactions matrix: It is a formulation that focuses on the interdependence of the producing sectors and, the interaction between the producing sectors on one hand and household sector of an economy on the other hand.

Intersection of sets : The intersection, written as A B, of two sets A and B is the set of all elements that belong to both A and B.

Inconsistent Equations: A system of equations that yields no solution

Independent Variable: in a function or model, the variable which is/are a causal variable(s), which influence the dependent variable.

Identity Matrix: It is a square matrix with 1s in the principal diagonal and zeros in the off-diagonal places.

Inverse of a matrix: Given a square matrix A, if there exists another square matrix B, such that, AB = BA = I, where I is an identity matrix, then B is said to be the inverse of the matrix A. For a matrix A, its inverse is generally denoted by A-1.

Integral Calculus is the other branch of calculus which finds the original function whose derivative is given. It is opposite of differential calculus and is called anti-derivative also.

Linearly Dependent Equations: These are a type of equations in a system of equations that may be derived from each mother by a series of linear operations

Limit of a Function at a Point: Suppose f is a complex-valued function, then we write $\lim_{x \to p} f(x) = L$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all real numbers x with $0 < |x - p| < \delta$, we have $|f(x) - L| < \varepsilon$.

Limit of a Sequence: provides a rigorous definition of the idea of a sequence converging towards a point called the limit. Suppose you have a **sequence of points** (i.e. an infinite set of points labelled using the natural numbers) in some sort of mathematical object (for example the real numbers) which has a concept of **nearness** (such as "all points within a given distance of a fixed point"). A point *L* is **the limit of the sequence** if for any prescribed nearness, all but a finite number of points in the sequence are that near to *L*.

Lagrangian Function: A function constructed to solve constrained optimization problems by combining the objective function and the constraint

Lagrange multiplier : A quantity in constrained optimization used in the Lagrangian function; it measure the shadow price of the variable in the objective function.

Limit: A concept used to describe the behavior of a function as its argument (independent variable) either "gets close" to some point, or as it becomes arbitrarily large; or the behavior of a sequence's elements as their index increases indefinitely. Suppose $f(x)$ is a real function and c is a real number. Then we write $\lim_{x \to c} f(x) = L$ to mean that $f(x)$ can be made to be as close to L as desired by making x sufficiently close to c. In that case, we say that "the limit of f of x, as x approaches c, is *L*".

Limit of a function: The idea if limit is closely connected with the idea of approximation and limit of a sequence. It is value of one variable approaching or approximating as another variable approaches or approximates a specific value. It can be a finite or infinite value. It a function approaches a finite number. The limit is finite; if it approaches $+\infty$ or $-\infty$ it is the case of infinite limit.

L' Hospital Rule: It is a rule for finding limits of function/s which lead to undefined/ indeterminate cases like $\infty - \infty, \frac{0}{0}, \frac{\infty}{\infty}$ etc. It states that if the original function is not differentiable, we can try finding limits of its derivative/s.

$$\underset{x \to a}{Lt} \frac{f(x)}{g(x)} = \underset{x \to a}{Lt} \frac{f'(x)}{g'(x)} = \underset{x \to a}{Lt} = \frac{f''(x)}{g''(x)} = \ldots\ldots$$

Market model: It is a formulation that seeks to explain the determination of equilibrium prices and quantities of different but related commodities.

Model: It is a set of equations, functional relationships and identities that seeks to explain some real world phenomenon.

Matrix: It is a rectangular array of numbers.

Matrix operations: The basic operations of addition and multiplication on matrices are called matrix operations.

Minor: The minor of any element a_{ij} of the determinant of some matrix A, denoted by M_{ij}, is the sub-determinant obtained by deleting ith row and jth column of the determinant of A.

National income model: It is a formulation that seeks to present and explain the relationships among national income and related aggregates and their determination.

Non-singular matrix: A square matrix A is said to be non-singular, if its determinant is not zero.

Null matrix: It is a matrix whose elements are all zero. It need not be a square matrix.

Overdetermined System of Equations: A system of m independent linear equations which is not inconsistent with n unknowns is overdetermined if $m > n$

Open input-output model: In such a model, the producing sectors interact with household sector of an economy through their purchase of primary inputs and sales of final products.

Optimisation: It is the best possible position under the given circumstances. It can be a point of either maximum/ maxima or Minimum/ minima e.g. obtaining maximum output with the minimum factor cost. A student wishes to maximise marks with minimum efforts in terms of study hours.

Order of a difference equation: The order of a difference equation is the order of the higher difference it contains

One-sided limit : Either of the two limits of a function $f(x)$ of a real variable x as x approaches a specified point either from below or from above. We write either $\lim_{x\to a^+} f(x)$ or $\lim_{x\downarrow a} f(x)$ for the limit as x approaches a from above (or "from the right"), and similarly, $\lim_{x\to a^-} f(x)$ or $\lim_{x\uparrow a} f(x)$ for the limit as x approaches a from below (or "from the left").

Parameter: A quantity that retains the same value throughout any particular problem but may assume different values in different problem.

Partial derivative: Ordinary derivatives apply to functions of one variable: $y = f(x)$. Partial derivative refer to finding differential coefficient of functions of two or more variables like $Z = f(x, y)$ or $P = f(L, K)$. Here we differentiate dependent variable (say z) with respect to either x or y by keeping/ assuming the other variable constant.

Primary inputs or factors of production: They are the basic participants of a production process. For their contribution to production, the net value of production is distributed among them as factor payments.

Quadratic Form: a form is a polynomial expression in which each component term has a uniform degree. In a quadratic form, each term is of the second degree.

Reduced Form: in a model depicted by a set of simultaneous equations, where as a solution to the system, the endogenous variables are expressed as functions only of the parameters

Range of a Function: Set of all output values of a function $f(x) = y$. The range of f is a subset of the codomain Y.

Relative Maxima: the set of points in whose neighbourhood only the function is maximised or minimised.

Right and Left Hand Limits: Take the limit of f as x approaches a is L. Note that x may approach a from above (right) or below (left), in which case the limits may be written as $\lim_{x \to a^+} f(x) = L$ or $\lim_{x \to a^-} f(x) = L$ respectively. If both of these limits are equal to L then this can be referred to as the limit of $f(x)$ at a. Conversely, if they are not both equal to L then *the* limit does not exist.

Set : A collection of objects or elements viewed as a single entity is called a set.

Scalar: An ordinary number is called a scalar. It is a matrix of d

Singular matrix: A square matrix A is said to be singular, if its determinant is zero.

Square matrix: A matrix, in which the number of rows and columns are equal, is called a square matrix.

Slope of a function: the ratio of the change in the dependent variable to a change in the independent variable.

Stationary Value: Point at which optimum is found

Sequence : An ordered list of objects (or events). For example, (X, Y, Z) is a sequence of letters that differs from (Y, Z, X), as the ordering matters.

Series: A series is a special type of sequence. For example, a series may be obtained by summing the first n terms of a sequence.

Symmetric matrix: It is a square matrix whose transpose is the matrix itself.

Secondary inputs or intermediate products: These are the products of other producing unit used by a given producing sector for its own production.

Transpose of a matrix: The transpose of a matrix, say A, is defined as a matrix that is obtained by interchanging the rows and columns of A.

Technology matrix: This is a matrix, that is obtained by subtracting a given input coefficient matrix from an identity matrix of appropriate dimension. This matrix is supposed to reflect the technology.

Union of sets : The union written as A B, of two sets A and B is the set whose elements belong to A or to B or to both.

Vector: A vector is a matrix consisting of only one row or one column.

Young's Theorem: States that the order of derivative does not matter that is $f_{xy} = f_{yx}$.

SUGGESTED READINGS

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