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Indira Gandhi National Open University
School of Social Sciences

BECC-104
MATHEMATICAL METHODS FOR ECONOMICS - II

BECC-104

Mathematical Methods in Economics-II

School of Social Sciences
Indira Gandhi National Open University

Course Coordinator & Editor: Shri Saugato Sen

COURSE PREPARATION TEAM

		Unit Writer	Editor (Content, format & Language)
Block 1	**Functions of Several Variables**		**Shri Saugato Sen and Ms Chetali Arora**
Unit 1	Multivariate Calculus-I	Shri Saugato Sen School of Social Sciences IGNOU	
Unit 2	Multivariate Calculus -II		
Block 2	**Differential Equations**		**Shri Saugato Sen**
Unit 3	First-Order Differential Equations	Shri Jagmohan Rai Associate Professor PGDAV College University of Delhi	
Unit 4	Second-Order Differential Equations		
Block 3	**Linear Algebra**		**Shri Saugato Sen**
Unit 5	Vectors and Vector Spaces	Shri Saugato Sen School of Social Sciences IGNOU	
Unit 6	Matrices and Determinants		
Unit 7	Linear Economic Models		
Block 4	**Multivariate Optimisation**		**Shri Saugato Sen and Ms Chetali Arora**
Unit 8	Unconstrained Optimisation	Dr. S.P. Sharma Associate Professor of Economics, Delhi	
Unit 9	Constrained Optimisation with equality Constraints	Shri Saugato Sen School of Social Sciences IGNOU	
Unit 10	Duality	Shri Saugato Sen School of Social Sciences IGNOU	

PRINT PRODUCTION

Mr. Tilak Raj	Mr. Yashpal
Assistant Registrar, (Pub.) MPDD, IGNOU New Delhi	Section Officer (Pub.) MPDD, IGNOU New Delhi

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COURSE INTRODUCTION

Welcome to the second course on mathematical methods in Economics. You are coming to this course having already done the first course on mathematical methods in Economics. This means that unlike when you met the course BECC 102, this time you already have some insight into how mathematics is applied in economic analysis. You are familiar with the use of mathematics as a language. You understand how various concepts in mathematics are utilized to explain economic processes; you also know how different areas of economic analysis invoke specific ideas and concepts in mathematics to aid in explanation and analysis.

This course proceeds further from where course BECC 102 left off. The present course takes up further topics in mathematical methods. It has ten units spread over four Blocks. The first Block, titled **Functions of Several Variables,** has two Units: unit 1 titled **Multivariate Calculus I** and unit 2 titled **Multivariate Calculus II**. This first block deals with functions where one dependent variable is a function of several independent variables. This marks a change from what you studied in course BECC 102. This block discusses how we can study the change in the dependent variable as a result of changes in one or all of the independent variables. For this the block discusses techniques of differential calculus, like partial derivatives, total differentials, and total derivatives. The first unit discusses the mathematics of multivariate calculus, and the second discusses a variety of applications to economics, both in microeconomics and macroeconomics.

The second block is titled **Differential Equations**. In the last Block of the course BECC 102, you had studied difference equations and their applications to economics. Like difference equations, differential equations, too, are useful to study dynamic processes, that is processes that take place over time. However, unlike difference equations which show discrete dynamic processes, differential equations are useful to study continuous dynamic processes, and are equations that involve derivatives. This block also has two units. In the block on difference equations in BECC 102, the units were organized as linear and non-linear equations. In the present course, the block on differential equations has two units organized as first-order and second-order differential equations. Unit 3 is titled **First-order Differential Equations**, and unit 4 is titled **Second-order Differential Equations**.

The title of Block 3 in this course is **Linear Algebra**. As the name suggests this deals with topics having to do with linear equations and equation systems and their manipulations. This block has three units. Unit 5 titled **Vectors and Vector Spaces**, discusses vectors, which are ordered n-tuples of numbers. The unit discusses geometric and algebraic properties of vectors. The next unit, unit 6 is titled **Matrices and Determinants**. Matrices are the basis of linear algebra. Determinants are formed from a particular type of matrix called square matrices, and are single numbers. The final unit in Block 3, unit 7, titled **Linear Economic Models**, brings together the concepts and techniques discussed in units 5 and 6, and gives a variety of economic applications of linear algebra in economics.

The final Block of the course is titled **Multivariate Optimisation.** This Block has three units. Unit 8, titled **Unconstrained Optimisation**, discusses optimization of multivariate functions that are not subject to any constraints. The next unit, unit 9 discusses constrained optimization, and has as the title **Constrained Optimization with Equality Constraints**. Thus the constraints studied here are linear. The final unit of this Block, and of the course, unit 10 titled **Duality** discusses certain specialized topics related to optimization, particularly multivariate optimization, both constrained and unconstrained.

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BLOCK 1

FUNCTIONS OF SEVERAL VARIABLES

BLOCK 1 INTRODUCTION

The first Block of this course in on calculus. But it is on differential calculus applied to multivariate functions. The title of this Block is **Functions of Several Variables.** The Block has two units, titled multivariate Calculus-I and multivariate Calculus-II. The first Unit is on the mathematics of multivariate Calculus and the second unit discusses economic applications.

The first Block, titled **Functions of Several Variables,** has two Units: unit 1 titled **Multivariate Calculus I** and unit 2 titled **Multivariate Calculus II**. This first block deals with functions where one dependent variable is a function of several independent variables. This marks a change from what you studied in course MECC 102. This block discusses how we can study the change in the dependent variable as a result of changes in one or all of the independent variables. For this the block discusses techniques of differential calculus, like partial derivatives, total differentials, and total derivatives. The first unit discusses the mathematics of multivariate calculus, and the second discusses a variety of applications to economics, both in microeconomics and macroeconomics.

UNIT 1 MULTIVARIATE CALCULUS-I

Structure

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  1.2.2 Level Curves

  1.2.3 General Multivariate Functions

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  1.4.1 Total Differentials

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1.7 Homogeneous and Homothetic functions

1.8 Let Us Sum Up

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1.0 OBJECTIVES

After reading this unit you will be able to:

- Describe functions where a dependent variable depends on more than one independent variable;
- Explain the concept of a partial derivative;
- Discuss the techniques of total differentiation, and obtain total derivatives;
- Explain what homogeneous and homothetic functions are; and
- Explain the chain rule with regard to functions of more than one variable.

1.1 INTRODUCTION

In the course on mathematical methods in economics (BECC 102) that you studied in the previous semester, you learnt about differentiation. However, there you studied about differentiation of univariate functions, that is, functions where the dependent variable is dependent on one independent variable. This Unit takes up the case of function of more than one independent variable. It also considers differential calculus pertaining to multivariate functions. The Unit first discusses the concept of function of several variables in section 1.2 Section 1.3 deals with partial derivatives, which means differentiating the function with respect to one independent variable, keeping others unchanged. Section 1.4 takes up the case of total differentiation, which means the change in the dependent variable as a result of changes in all the independent variables. This section also discusses total derivatives. The next section, section 1.5 discusses the chain rule that you studied in the previous unit, but also somewhat different and new. The subsequent section, section 1.6, discusses the differentiation of functions that are defined implicitly. Of course, the discussion of implicit function is with regard to functions, which when expressed explicitly, would be multivariate. This section also discusses the important implicit function theorem. Finally, in section 1.7 you are introduced to an important type of multivariate functions, called homogeneous functions. Other than the definition of a homogeneous function, some important properties are discussed, including a theorem called Euler's theorem. Homothetic functions are defined as well, and their relationship to homogeneous functions is mentioned.

Let us state explicitly that the present unit discusses the content of all the topics and concepts only in mathematical terms. That is, the mathematics of these concepts is discussed in the present unit. The next unit, unit 2 is entirely devoted to the applications of the ideas in this unit to economics, like the theory of the consumer, producer's theory, markets, macroeconomics, and so on. Just understand the content of this unit well, and you will appreciate better the matter in the next unit, as well as what you learn in your microeconomics and macroeconomics courses.

1.2 FUNCTION OF SEVERAL VARIABLES

In this section we introduce you to functions where the dependent variable is a function of more than one independent variable. We begin with the simplest case where the dependent variable is a function of two independent variables. Next we discuss the idea of level curves and finally we take up the general case of n- independent variables.

1.2.1 Functions of Two Independent Variables

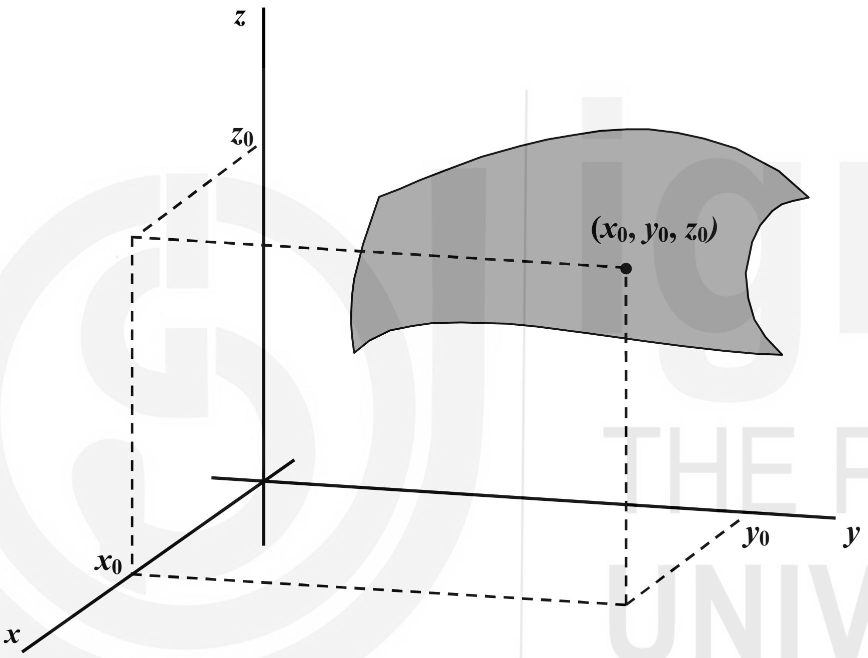
In the course on mathematical methods in economics that you studied in the last semester, we have discussed almost exclusively functions of one variable $[y = f(x)]$ that is, functions where a variable is dependent on one independent variable. But in real life one comes across cases where more than one independent variables influence one dependent variable. For simplicity, let us begin by considering a function which shows an independent variable, say z, being a function of two independent variables, x and y. In notation

$$z = f(x, y)$$

We can define the above function in the following way. A function *f* of two real variables *x* and *y* with domain D is a rule that assigns a specified number f(x, y) to each point (x,y) in domain D. In the above function, *x* and *y* are called the independent variables, or arguments of the function *f*. The variable *z* is called the dependent variable. The domain of the function *f* is the set of all possible ordered pairs of the independent variables, while the range is the set of corresponding values of the dependent variable. In some contexts, *z* is called the endogenous variable while *x* and *y* are called exogenous variables.

Apart from simplicity, one reason for beginning with functions of only two variables is that we are able to draw diagrams. We can depict $f(x, y)$ diagrammatically in three dimensional space by drawing three mutually perpendicular axes Ox, Oy and Oz. Figure 1.1 below shows such a diagram depicting a surface in three-dimensional space, and a point

(x_0, y_0, z_0) on that surface.



Now let us suppose that this surface is traced out by the function $z = f(x, y)$, and that z is traced out as (x,y) varies over the xy-plane. Then $z_0 = f(x_0, y_0)$.

1.2.2 Level Curves

If we have $z = f(x,y)$, the graph of this function in three-dimensional space can visualized as being cut by horizontal planes that are parallel to the $x\,y$ – plane. The intersections between the planes and the graph can be projected onto the $x\,y$ – plane. If the intersecting plane is at $z = k$ then the projection onto the $x\,y$ – plane is known as the level curve or contour at height *k* for the function *f*. the contour or level will consist of points that satisfy the equation

$$f(x, y) = k$$

Think of a map. It shows the geographical location of a place (in terms of latitudes and longitudes, for example). To show altitude or height, what we can do (in maps showing physical properties of places), is to draw a set of contours or level curves connecting those points on the map that lie at the same altitude, or elevation above sea level. So on a two-dimensional map (basically representing direction, distance, area, and so on) we can show the third dimension (altitude) by drawing a set of level curves.

It is the same idea that we apply to the diagrammatical representation of functions of two variables. You can think of the graph of a function in three-dimensional space (as shown on a two-dimensional diagram) as being represented by horizontal planes that are parallel to the x-y plane. These intersections between the planes and the graph, we project onto the x-y plane.

At this point, think back to the unit on coordinate geometry that you studied in course BECC 102. There you studied that the equation of a *line* parallel to the y-axis is $x = c$, while the equation of a *line* parallel to the x-axis is $y = c$

In the present context, we are talking of the equation of a plane (because it is in three dimensions x-y-z). hence the equation of the plane parallel to the xy-plane will be, in an analogous fashion, $z = c$. This shows the projection of the intersection of this plane with the graph at height c for *z*. since $z = f(x, y)$, this projection is called the level curve for *f* at height *c*. This level curve then consists of all points that satisfy the equation

$$f(x, y) = c$$

The level curve thus connects points whose functional values are equal. In the above, the level curve connects all the points for which the value of *f* is *c*. it is the locus of all points, that is, the combination of *x* and *y*, for which the value of *z* is equal to *c*.

1.2.3 General Multivariate Functions

In subsection 1.2.1 above, we had discussed about functions of two variables. We can extend the discussion to functions in which the dependent variable is a function of several independent variables. Let us denote a list of n variables by $(x_1, x_2, \ldots x_n)$ We say that the variable *x* is indexed by *i*, where *i* = 1,2,…,*n*. This collection of n-variables (each x_i is a real number) is called a vector. A vector is an ordered n-tuple. This was mentioned in Units 1 and 2 of course BECC 102 that you studied in the last semester. You will study about vectors in much greater detail in Block 3 of the present course.

Coming back to the vector $x = (x_1, x_2, \ldots x_n)$, suppose a variable z is a function of all these n variables. We denote this function as $z = f(x_1, x_2, \ldots, x_n)$.

One point that presents itself immediately is that now we are in n+ 1 dimension. Suppose n =2 where n is the number of independent variable. This is what we discussed in subsection 1.2.1. There n + 1 was = 3 and we could depict f on a diagram. When n is greater than 2, we end up with more than 3 dimensions, and the function cannot be depicted diagrammatically. So we have to think of the function conceptually and in an abstract manner. When a line is generalised to two-dimensions, it is called a plane. Above that it is called a hyperplane. A general surface in higher dimensions is called a hypersurface. Just remember that the function $z = f(x, y)$, which can as well be depicted $z = f(x_1, x_2)$ is simply a special case of $z = f(x_1, x_2, ..., x_n)$, where n = 2.

1.3 PARTIAL DERIVATIVES

In your study of derivatives in the course BECC 102, you came to realise that differential calculus studies how the dependent variable changes due to a change in the independent variable, when the change in the independent variable is infinitesimally small. In that course you studied the derivatives of functions of a single independent variable. In this unit we have discussed in the section above about functions where the dependent variable is a function of several independent variables. So intuitively you can think of exploring the idea about how to study changes in the dependent variable due to changes in the independent variables. Consider the simplest multivariate function $z = f(x_1, x_2)$. We want to see how changes due to changes in x_1 and x_2. Now we can study the change in z when both the independent variables change together or when one of them changes and the other does not. Extend the idea to a general multivariate function $z = f(x_1, x_2, ..., x_n)$. The rest of the unit is concerned with a study of such changes. In this section we deal with the study of derivatives of the function due to a change in only one of the independent variables at a time. Subsequent sections will deal with situations of all independent variables changing. When we take derivatives of the function with respect to an independent variable, keeping other independent variables constant, it is called a partial derivative. Let us begin the study of partial derivatives

1.3.1 First- Order Partial Derivatives

Consider the function $z = f(x_1, x_2, ..., x_n)$. Here let us assume that the variables x_i $(i = 1,2,\dots,n)$ are all independent of one another so that each variable can individually vary without influencing the other independent variables. Suppose there is a change in x_1 by Δx_1 while $x_2, \dots x_n$ all remain unchanged (fixed), there will be a corresponding change in z, namely Δz. The quotient $\frac{\Delta z}{\Delta x_1}$ in this case can be expressed as

$$\frac{\Delta z}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, ..., x_n) - f(x_1, x_2, ... x_n)}{\Delta x_1}$$

If we take the limit of $\frac{\Delta z}{\Delta x_1}$ as $\Delta x_1 \to 0$, the limit we obtain is called the partial derivative of with respect to x_1. The term *partial* derivative is used to indicate that the other independent variables are held constant. We can derive similar partial derivatives of the function with respect to each of the other independent variables. The process of taking partial derivatives is known as partial differentiation.

We saw in course BECC 102 that derivatives are denoted by the symbol *d*. Thus if we have y = f(x) then the derivative of y with respect to x is denoted $\frac{dy}{dx}$. Partial derivatives are denoted by the symbol ∂. This is a variation of the lower-case Greek letter 'delta' δ. Hence we write the above partial derivative as $\frac{\partial z}{\partial x_1}$ We call this "partial derivative of with respect to x_1. For the generic variable x_i, we write the partial derivative of with respect to x_i as $\frac{\partial z}{\partial x_i}$

We sometimes write the partial derivative as $\frac{\partial}{\partial x_i} z$. With such a symbol, the $\frac{\partial}{\partial x_i}$ part can be taken as a mathematical operator symbol showing "taking partial derivative (of some function)". Since z is a function of x_i (i = 1,2,…,n), we can also denote the partial derivative by $\frac{\partial f}{\partial x_i}$. We have seen earlier that when we have y = f(x), then $\frac{dy}{dx}$ is also sometimes denoted $f'(x)$. In the case of partial derivatives, we sometimes use subscripts to denote the partial derivatives. Thus $\frac{\partial f}{\partial x_i}$ is sometimes denoted f_i

If we denote a function as , say z =f(x, y, w, v), then we can denote the partial derivatives as f_x, f_y, f_w, f_v rather than f_1, f_2, f_3, f_4

To sum up, suppose we have $z = f(x_1, x_2, ..., x_n)$.

Then we can depict partial derivative of z with respect to, say x_3, by

$$\frac{\partial z}{\partial x_3} \text{ or } \frac{\partial f}{\partial x_3} \text{ or } \frac{\partial}{\partial x_3} f \text{ or } f_3$$

Let us now take a few examples to see how partial derivatives are computed. For simplicity let us consider functions of two independent variables. To compute partial derivatives, remember two things: first, when you take partial derivative with respect to one variable, treat the other variable(s) as constant; and second, remember that the derivative of a constant equals zero. Also remember that if you have a variable multiplied by a constant, then the derivative of this product with respect to the variable is the product of the constant and the derivative of this variable. For example, (d [cx]/dx =cdx/dx; and d[cx^2]/dx =2cx). Moreover, the usual rules of differentiation, like sum and difference rules, product and quotient rules, and the composite function rules hold in the case of partial derivatives also.

Example 1 Let $f(x, y) = x^3 y + y^4$

Partially differentiating with respect to x,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial}\left(x^3 y\right) + \frac{\partial}{\partial x}\left(y^4\right)$$

The first term on the right-hand side is $3x^2 y$ and the second term is zero (because y is a constant, and so is y raised to the power 4.

Thus $\frac{\partial f}{\partial x} = 3x^2 y$

Similarly

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^3 y\right) + \frac{\partial}{\partial y}\left(y^4\right) = x^3 y + 4y^3$$

Example 2: Given $z = f\left(x_1, x_2\right) = 4x_1^2 + x_1 x_2 + 3x_2^2$

To find the partial derivatives, we have to remember that when we are computing partial derivative with respect to x1, we must treat x2 as constant. The variable x2 will drop out if it is an additive constant, such as the third term in the above example, but will be retained if it is a multiplicative constant, like in the second term in the above example.

Thus we have:

$$\frac{\partial z}{\partial x_1} = f_1 = 8x_1 + x_2$$

Similarly for computing the partial derivative with respect to x_2, we treat x_1 as constant. We obtain

$$\frac{\partial z}{\partial x_2} = f_2 = x_1 + 6x_2$$

Example 3 Given $z = f(w, v) = (w+4)(3w+2v)$

We can obtain the partial derivatives by using the product rule. For partial derivative with respect to w, we hold v constant. We obtain

$$f_w = (w+4)(3) + (3w+2v)(1) = 2(3w+v+6)$$

For partial derivative with respect to v, we hold w constant, and get:

$$f_v = (w+4)(2) + 0(3w+2v) = 2(w+4)$$

Example 4 Let us see in this example how the quotient rule is used

Suppose $z = (3w-2v)/\left(w^2+3v\right)$

We have $\frac{\partial z}{\partial w} = f_w = \frac{3\left(w^2+3v\right) - 2w(3w-2v)}{\left(w^2+3v\right)^2} = \frac{-3w^2+4wv+9v}{\left(w^2+3v\right)^2}$

$$\frac{\partial z}{\partial v} = f_v = \frac{-2\left(w^2+3v\right) - 3(w-2v)}{\left(w^2+3v\right)^2} = \frac{-w(2w+9)}{\left(w^2+3v\right)^2}$$

In the above we considered first-order partial derivatives. These are partial derivatives of the given function with respect to the arguments. A basic point about partial derivatives may be made. The first-order partial derivatives are themselves functions of the arguments of the original function, that is the first-order partial derivatives are themselves the functions of the independent variables in the original function.

Suppose we have $z = f(x_1, x_2)$

Then $f_1 = \dfrac{\partial f}{\partial x_1} = g(x_1, x_2)$ and

$f_2 = \dfrac{\partial f}{\partial x_2} = h(x_1, x_2)$

In the above g and h denote functions.

We may make a final point about first-order partial derivatives. Suppose we have $z = f(x_1, x_2, ..., x_n)$. Let us compute $\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, ..., \dfrac{\partial f}{\partial x_n}$. The collection of these partial derivatives in a collection of n-real numbers for specific values of $x_1, x_2, ..., x_n$. This collection is an n-tuple or vector. This vector is called a gradient vector denoted by ∇ or grad *f*.

1.3.2 Second-Order Partial Derivatives

In the above sub-section we discussed first-order partial derivatives. Let us now turn to second-order partial derivatives. Let us consider the function

$z = f(x, y)$. This function can yield two first-order partial derivatives $f_x = \dfrac{\partial f}{\partial x}$ and $f_y = \dfrac{\partial f}{\partial y}$

We mentioned towards the end of the previous sub-section that the first-order partial derivatives are themselves functions of x and y, that is,

$f_x = g(x, y)$ and

$f_y = h(x, y)$.

This means that the functions g and h can themselves be partially differentiated with respect to x and with respect to y. Now notice one thing: f_x is itself a partial derivative and it can be partially differentiated with respect to x, and with respect to y. The same is true of f_y

Just as we have $\dfrac{\partial}{\partial x} g(x, y)$,we can differentiate *g(x,y)* with respect to *y*. Similarly

$f_y = h(x,y)$ can be differentiated with respected to *x* and with respect to *y*

Let us differentiate $f_x = g(x, y)$

We get $\frac{\partial f_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

The right hand side is written $\frac{\partial^2 z}{\partial x^2}$

The second order partial derivative of *z* (or the function *f*) with respect to x is thus written:

f_{xx} or $\frac{\partial}{\partial x} f(x)$. It's also written $\frac{\partial f_x}{\partial x}$ or $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

Similarly, we can have the second-order partial derivative of z with respect to y. We thus have : $f_{yy} \equiv \frac{\partial}{\partial y} f_y$ or $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

This denotes the rate of change of f_y with respect to y, while x is held constant

Please remember that we saw that f_x is a function of y, and that f_y is a function of x as well.

$f_x = g(x, y)$, and $f_y = h(x, y)$

Hence we can obtain two more second-order partial derivatives:

1) $fxy = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

   This is the partial derivative of z first with respect to y and then with respect to x.

2) $fyx = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$

   This is the partial derivative of z first with respect to x and then with respect to y.

Two points may be made here:

1) f_{xy} is also denoted by $\frac{\partial^2 z}{\partial x \partial y}$ and f_{yx} by $\frac{\partial^2 z}{\partial y \partial x}$

2) the order of differentiation is from right to left (←). For example f_{xy} means first differentiate *z* with respect to. *y* and then with respect to *x*. Similarly f_{yx} means first differentiate *z* with respect to x and then with respect to *y*. The partial derivatives f_{xy} and f_{yx} are called cross-partial derivatives.

Young's Theorem

In the case of continuous functions with continuous partial derivatives, $f_{xy} = f_{yx}$. This is called Young's Theorem and can be stated as under:

The order of cross partial derivatives does not matter. That is, it does not matter whether a function is differentiated first with respect to x and then again differentiated with respect of y or vice versa. Symbolically:

$$f_{xy} = f_{yx}$$

The above was for the case of second-order derivatives

Check Your Progress 1

1) Explain the concepts of (a) A multivariate function (b) Partial derivative.

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2) Find the partial derivatives f_x and f_y of the functions

i) $f(x, y) = 5x^2 + 6y^2$

ii) $f(x, y) = -10xy^2$

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3) Determine all first-order and second-order derivatives for the function

$f(x, y) = 8x^3 - 4x^2y + 10y^2$

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1.4 TOTAL DIFFERENTIAL AND TOTAL DERIVATIVES

Let us consider again the function $z = f(x, y)$

The partial derivatives gave us a small change in *z* when there was a small change in *x* holding *y* constant, or a small change in *y* holding *x* constant. Let us now study what happens to *z* if both *x* and *y* were to change. For that we turn to the study of differentials

1.4.1 Total Differential

Let us begin by considering a function of a single variable, *z =f(x)*. for a change in *x, Δx*, the change in *z* will be *Δz*. In the course BECC 102, we have seen *dz/dx* to be the change in *z* when there is a small unit change in x. so, when there is change of Δx in x the change in z will be:

$$\Delta z = \frac{dz}{dx}\Delta x$$

Writing dz for Δz and dx for Δx, we get

$$dz = \frac{dz}{dx}dz$$

This dz on the left-hand side is called the differential of z.

Now let us consider

If $z = f(x, y)$ is a function, then the total differential dz can be expressed as:

$$dz = f_x d_x + f_y d_y = \frac{\partial z}{\partial x} \cdot d_x + \frac{\partial z}{\partial y} d_y \text{ approximately}$$

Let us try to understand it in the following manner. In the above, $\frac{\partial z}{\partial x}$ shows the incremental change in z when x changes by a small amount. So the change in z due to change of dx in x is $\frac{\partial z}{\partial x}dx$

Similarly, the change in z due to a change of dy in y is $\frac{\partial z}{\partial y}dy$. Thus the total change in z due to change in both x and y of dx and dy respectively is

$$dz = f_x d_x + f_y d_y = \frac{\partial z}{\partial x} \cdot d_x + \frac{\partial z}{\partial y} d_y$$

This is known as the total differential of z.

The expression dz shows the increment in the function $z = f(x, y)$ when there is an infinitesimal increments in x and well as y. For example if $z = x^3 + y^3$, then total differential can be expressed as $dz = f_x d_x + f_y d_y = 3x^2 d_x + 3y^2 d_y$

If we take a general multivariate function

$z = f(x_1, x_2, ..., x_n)$ then

$$dz = \frac{\partial z}{\partial x_1} dx_1 + \frac{\partial z}{\partial x_2} dx_2 + ... + \frac{\partial z}{\partial x_n} dx_n \text{, or}$$

$dz = f_1 dx_1 + f_2 dx_2 + ... + f_n dx_n = \sum_{i=1}^{n} f_i dx_i$ (recall the rules and properties of summation as discussed in course BECC 102)

Let us now discuss some rules of differentials.

Rules of differentials

The following rules on total differential will be found useful. Let z and w represent two functions of x and y, then

1) $dk = 0$ (constant-function rule)

2) $d(w \pm z) = dw \pm dz$
$= (f_x dx + f_y dy) \pm (g_x dx + g_y dy)$ (Sum-difference rule)

3) $d(wz) = w.dz + z\,dw$
$w(g_x dx + g_y dy) + z(f_x dx + f_y dy)$ (Product rule)

4) $d\left(\frac{w}{z}\right) = \frac{z.dw - wdz}{z^2}$
$= \frac{z(f_x dx + f_y dy) - w(g_x dx + g_y dy)}{z^2}$ (Quotient Rule)

5) $d(kz^n) = knz^{n-1} dz$ (power-function rule)

6) The chain rule:

If z = z(u) and u = u(x) then

$$dz = d(z(u)) = \frac{d}{du}(z)du$$

Here du is not an arbitrary increase in u but happens to be a differential of u. we have

$du = d(u(x)) = \frac{d}{dx}(u)dx$. Hence

$$dz = \left[\frac{d}{du}(z)\right]\left[\frac{d}{dx}(u)dx\right]$$

Example.

If $z = u^n$, where $u = f(x, y)$, then

$$dz = \frac{d}{dx}(u^n) \cdot du = nu^{n-1} \cdot du$$

Let us now solve some problems on total differentials.

1) Find du when $u = 3x^3 + 2y^2 + y^3$

Answer:. Total differential du is given by:

$$du = f_x d_x + f_y d_y = 9x^2 d_x + (uy + 3y^2) d_y$$
$$= 9x^2 d_x + y(u + 3y) d_y$$

2) Find total differentials of the following functions

a) $u = \frac{x^2 - y^2}{x^2 + y^2}$ b) $w = e^{x^2 - y^2}$ c) $u = \log(x^2 + y^2)$

Answer: a) $u = \frac{x^2 - y^2}{x^2 + y^2}$, apply quotient rule

$$= \frac{z(f_x dx + f_y dy) - w(g_x dx + g_y dy)}{z^2}$$

$$\frac{(x^2 + y^2)d(x^2 - y^2) - (x^2 - y^2)d(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)(2xdx - 2ydy) - (x^2 - y^2)(2xdx + 2ydy)}{(x^2 + y^2)^2}$$

$$= \frac{4xy^2 dx - 4x^2 ydy}{(x^2 + y^2)^2}$$

b) $w = e^{x^2 - y^2}$

Put $u = x^2 - y^2$ so that $w = e^u$ and $dw = e^u du$ (1)

Also $du = d(x^2) - d(y^2) = 2xdx - 2ydy$ (2)

From (11) and (12), we get

$$dw = e^{x^2 - y^2} \cdot (2xdx = 2ydy) = 2xe^{x^2 - y^2} dx - 2ye^{x^2 - y^2} dy$$

c) $u = \log(x^2 + y^2)$

Let us try it by using the formula:

$$du = f_x dx + f_y dy = \frac{1 \times 2x}{(x^2 + y^2)} dx + \frac{1 \times 2y}{(x^2 + y^2)} dy$$

$$= \frac{2x}{x^2 + y^2} dx + \frac{2y}{(x^2 + y^2)} dy = \frac{2xdx + 2ydy}{(x^2 + y^2)} = \frac{2(xdu + ydy)}{(x^2 + y^2)}$$

1.4.2 Total Derivatives

Let us go back to our familiar function of two variables

$z = f(x, y)$

Here we assume that x and y are independent variables, and also independent of each other.

Consider the case where x and y are not independent variables but are dependent variables of other functions, for example,

$x = g(t), y = h(t)$, where t is the independent variable. Thus, z is a function of x and y; x and y are functions of t. We want to investigate as to what would be the derivative of z with respect to t. What would be dz/dt?

We know $\frac{\partial z}{\partial x}$ is the change in z due to a small unit change in x holding y constant hence $\frac{\partial z}{\partial x}\frac{dx}{dt}$ will be the amount of change in u due to a small unit change in t, that is transmitted through x. similarly $\frac{\partial z}{\partial y}\frac{dx}{dt}$ is the amount of change in z due to a small unit change in t transmitted through y.

Hence, the total change in z due to a small unit change in t will be the sum of these two effects. We write

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dx}{dt}$$

This we can also express as

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dx}{dt}$$

This $\frac{dz}{dt}$ is called the total derivative of z with respect to t.

If we have z = (x, y, u, …) where x = x(t), y = y(t), u = u(t), …

Then $\frac{dz}{dt} + \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dx}{dt} + \frac{\partial z}{\partial u}\frac{du}{dt} + \ldots.$

Now let us again go back to the function

$z = f(x, y)$

Now instead of thinking of x and y being dependent variables that are functions of an independent variable, say, t, let us suppose x is a function of y. so our given function is

$z = f(x, y)$

, where $x = g(y)$

We can combine the two functions f and g into a composite function:

$$z = f[g(y), y]$$

In the above we see that z, x, and y are related as follows: the variable y affects z through two channels. It can affect z directly through the function f, and indirectly through the function g. thus the variable y is the ultimate source of variation in z. The indirect effect of y on z can be represented as $\frac{\partial z}{\partial x}\frac{dx}{dy}$. The direct effect is simply f_y

Hence the total derivative of z with respect to y is obtained by combining the direct and the indirect effects;

$$\frac{dz}{dy} = \frac{\partial z}{\partial x}\frac{dx}{dy} + f_y$$

It can also be obtained alternatively by simply taking total derivative in the usual manner:

$$\frac{dz}{dy} = \frac{\partial z}{\partial x}\frac{dx}{dy} + \frac{\partial z}{\partial y}\frac{dy}{dy}$$

Thus $\frac{dz}{dy} = \frac{\partial z}{\partial x}\frac{dx}{dy} + \frac{\partial z}{\partial y}$

Check Your Progress 2

1) a) Explain what you understand by total differential.

   b) How are partial derivatives related to total derivatives?

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2) Find total differential of the following functions.

   a) $u = 3x^3 - 2y^2$ b) $u = \frac{x}{x+y}$ c) $x = AL^{\frac{1}{2}}K^{\frac{1}{2}}$

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1.5 THE CHAIN RULE FOR MULTIVARIATE FUNCTIONS

You are aware of the chain rule for function of one variable. It is based on the function-of-a-function rule. Let us suppose z is a function of y, and y in turn is a function of x. We can depict this as $z = f\big(y(x)\big)$. we know in this case how to take the derivative of z with respect x: we first take the derivative of z with respect to y and then multiply this with the derivative of y with respect to x. We can write this as follows:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

As similar process works in the case of partial derivatives. There is a chain rule applicable to the case of partial derivatives as well. We only have to take care in following the rules of partial differentiation. We shall try to make this clear with the help of some examples.

Just a matter of notation for the examples: unlike earlier, we denote one argument in the function by z. Earlier, we denoted the dependent variable by z. Please do not get confused.

Example 1

Suppose we have $u = f(x, y, z)$and here x, y, z are themselves each a function of some variable, say t. then we can find $\frac{du}{dt}$ as follows:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Notice that this is just the total derivative that we used in the previous section. This gives us occasion to mention that the total derivative is used when the variables inside the parentheses (the argument of the function) are themselves functions of only one independent variable. In the example above, x, y, z are functions of the single variable t. In the next example we consider a case where variables that are arguments in the given function, are themselves functions of more than one variable. This is where we use the chain rule, and have to use partial derivatives for composite function. Just study the examples below, and it will become clear.

Example 2

If we have $w = f(x, y, z)$ and x, y, z are themselves each a function of variable r and s then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

The next example could also have been depicted using total derivatives as we did in the previous section.

Example 3

Suppose $u = f(x, y, z)$ and if y and z themselves depend on x, that is, $y = y(x), z = z(x)$. Then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}$$

Example 4

Let $u = f(x, y, z)$ and if x depend on t, y depends on x, and z depends on y, that is, $x = x(t), y = y(x), z = z(y)$. Then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\frac{\partial y}{\partial x}\frac{\partial x}{\partial t}$$

Notice that in the above example, z depends on *y,* which depends on *x,* which depends on *t*. Thus *z* and *y* ultimately depend on *t.*

Example 5

Suppose $w = f(x, y, z)$ where $x = x(r,s), y = (r), z = z(y)$. Then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}$$

So now you should be able to grasp the technique of the chain rule in the case of partial derivatives, and should be able to carefully observe which variable depends on which other(s), and carry out the required differentiation. You should be able to see from the way the variables are related to each other how to use the chain rule, and when to take the total derivatives.

Let us take one final case and see how the chain rule and total derivatives could be combined to obtain the relevant results.

Example 6

Suppose $y = f(x_1, x_2)$ where

$x_1 = g(w_1, w_2)$ and

$x_2 = h(w_1, w_2)$

The partial derivatives of y with respect to w_1 and w_2 are calculated by making use of the composite-function rule (chain rule) that we show below. Let us begin by taking total differential of y:

$$dy = \frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2$$

Since we have x_1 and x_2 as functions of w_1 and w_2, we can compute the other total differentials:

$$dx_1 = \frac{\partial x_1}{\partial w_1} dw_1 + \frac{\partial x_1}{\partial w_2} dw_2 \text{ and}$$

$$dx_2 = \frac{\partial x_2}{\partial w_1} dw_1 + \frac{\partial x_2}{\partial w_2} dw_2$$

Substituting for dx_1 and dx_2 in the equation for the total differential of y, and collecting the terms for dw_1 and dw_2 ,we get

$$dy = \left(\frac{\partial y}{\partial x_1} \frac{\partial x_1}{\partial w_1} + \frac{\partial y}{\partial x_2} \frac{\partial x_2}{\partial w_1} \right) dw_1 + \left(\frac{\partial y}{\partial x_1} \frac{\partial x_1}{\partial w_2} + \frac{\partial y}{\partial x_2} \frac{\partial x_2}{\partial w_2} \right) dw_2$$

1.6 DERIVATIVES OF IMPLICIT FUNCTIONS

Now let us discuss a type of functions called implicit functions. Suppose we take a simple function z =f (x), for example $z = 9x^3$. Here z is called an explicit function of x, because the value of the variable z is explicitly dependent on the value of x. If on the other hand, we were to write the function as $f(z,x)=0$ it becomes an implicit function relating z with x. so the specific function mentioned above can be written in an implicit form as $z - 9x^3 = 0$.

We may write a general multivariate function in explicit form as $z = f(x_1, x_2, \ldots x_n)$. This may be implicitly written as $F(z, x_1, x_2, \ldots, x_n) = 0$. Sometimes we may encounter an equation of the form $f(x_1, x_2, \ldots x_n) = 0$. You must realize that it is implicitly defining a function which, when written explicitly may be of the form $x_1 = f(x_2, x_3, \ldots x_n)$.

An explicit function can always be expressed in an implicit manner by moving the f(.) to the left-hand-side of the equation, but it is not always possible to express an implicit function in an explicit form. When it can be so expressed will be the topic of study in Block 3 of this course, when we discuss the implicit-function theorem.

Instead, let us turn to the derivatives of implicit functions. To do this, let us recall level curves . Take a function $f(x_1, x_2) = k$. This, you would recall was the equation of a level curve. This is also an implicit function. We can solve this single equation in two unknowns and one of the unknowns can be expressed in terms of the other, say, $x_2 = x_2(x_1)$

We can substitute this back into the implicit function and write

$$f(x_1, x_2(x_1)) \equiv k$$

The slope of any level curve is the derivative $\frac{dx_2}{dx_1}$. But this is conceptually true only if in the implicit function we have defined x_2 as a function of x_1.

Here we have done so.

Here our function $x_2 = x_2(x_1)$ is well defined. We can get $\frac{dx_2}{dx_1}$ by differentiating the identity $f(x_1, x_2(x_1)) \equiv k$, with respect to x_1, using the chain rule. We get

$$\frac{\partial f}{\partial x_1}\frac{dx_1}{dx_1} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx_1} \equiv \frac{\partial k}{\partial x_1} \equiv 0$$

,or $f_1 + f_2 \frac{dx_2}{dx_1} \equiv 0$

If we assume $x_2 \neq 0$,

$$\frac{dx_2}{dx_1} \equiv \frac{-f_1}{f_2}$$

This shows that the slope of a level curve at any point is the ratio of the first-order partial derivatives evaluated at some particular point on the given level curve. Let us next consider a general multivariate function.

Given the implicit function $f(x_1, x_2, ..., x_n) = 0$, the partial derivative of the jth argument of the function with respect to the i^{th} argument of the function x_i , $\frac{\partial x_j}{\partial x_i}$

is obtained by first finding the total differential $f_1 dx_1 + f_2 dx_2 + ... + f_n dx_n = 0$.

Then we divide by dx_i:

$$f_1 \frac{dx_1}{dx_i} + f_2 \frac{dx_2}{dx_i} + ... + f_j \frac{dx_j}{dx_i} + ... + f_i + ... + f_n \frac{dx_n}{dx_i} = 0$$

We set all differentials other than dx_i and dx_j equal to zero. Then

$$f_j \frac{dx_j}{dx_i} + f_i = 0$$

$$\Rightarrow \frac{dx_j}{dx_i} = -\frac{f_i}{f_j}$$

1.7 HOMOGENEOUS AND HOMOTHETIC FUNCTIONS

Let us consider a function F of two variables x and y, that is, *F(x, y).* This function is said to be homogeneous of degree r, if for all x and y in the domain,

$$F(\lambda x, \lambda y) \equiv \lambda^r F(x, y).$$

We can extend this definition to a function of n-variables. Suppose $z = f(x_1, x_2, ..., x_n)$. Then f is said to be homogeneous of degree r if

$$f(\lambda x_1, \lambda x_2, ..., \lambda x_n) \equiv \lambda^r f(x_1, x_2, ..., x_n).$$

$F(x, y)$ is linearly homogeneous (homogeneous of degree 1) if and only if $F(x, y) = yf\left(\frac{x}{y}\right)$, where $f\left(\frac{x}{y}\right) = F\left(\frac{x}{y}, 1\right)$

How do we show this? Let us begin by showing the 'if' (sufficiency) part.

Given:

$F(x, y) = yf\left(\frac{x}{y}\right)$,we have

$$F = (\lambda x, \lambda y) = \lambda yf\left(\frac{\lambda x}{\lambda y}\right) = \lambda yf\left(\frac{x}{y}\right) = \lambda F(x, y)$$

Now let us show the only if (necessary condition) part:

Given F(x,y) is linearly homogeneous (homogeneous of degree 1). Then

$F(\lambda x, \lambda y) = \lambda F(x, y)$, for any λ

Put $\lambda = \frac{1}{y}$. The we have

$$F\left(\frac{x}{y}, 1\right) = \frac{1}{y} F(x, y)$$

Thus $yF\left(\frac{x}{y}, 1\right) = F(x, y)$

But $F\left(\frac{x}{y}, 1\right) = f\left(\frac{x}{y}\right)$.

Hence, $F(x, y) = yf\left(\frac{x}{y}\right)$

Differentiation of a Homogeneous Function

A very important property of homogeneous functions is with regard to the differentiation of homogeneous functions.

Let us have a homogeneous function *f(x, y)* of degree *r*. then we have:

$$f(\lambda x, \lambda y) \equiv \lambda^r f(x, y)$$

Differentiating with respect to x we have:

$\lambda f_x(\lambda x, \lambda y) = \lambda^r f_x(x, y)$, where $f_x = \frac{\partial f}{\partial x}$

If we divide by λ we get:

$$f_x(\lambda x, \lambda y) = \lambda^{r-1} f_x(x, y)$$

If you notice carefully, the above equation shows that the function f_x is homogeneous of degree r – 1. The above holds for the function f_y too, where f_y is the partial derivative with respect to y. the above holds in the case of any multivariate homogeneous function.

This result says that if a function is homogeneous of degree r, then each of its partial derivatives is homogeneous of degree $r - 1$.

Euler's equation

This is a very important property displayed by homogeneous equations. Let $z = f(x, y)$

be a homogeneous function of degree r. then the following relation holds as an identity:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = rz$$

We can prove this using implicit differentiation and the chain rule:

Let us have the homogeneous function

$$f(\lambda x, \lambda y) \equiv \lambda^r f(x, y) \quad \text{(a)}$$

Let us partially differentiate the left-hand side of equation (a) with respect to λ. We get:

$$\frac{\partial f(\lambda x, \lambda y)}{\partial \lambda x}\frac{\partial \lambda x}{\partial \lambda} + \frac{\partial f(\lambda x, \lambda y)}{\partial \lambda y}\frac{\partial \lambda y}{\partial \lambda}$$

$$= xf_{\lambda x} + yf_{\lambda y} \quad \text{(b)}$$

Now let us obtain the partial derivative of the right-hand side of equation (a) with respect to λ. We now get:

$$\frac{\partial \lambda^r}{\partial \lambda} f(x, y) + \lambda^r \frac{\partial f(x, y)}{\partial \lambda}$$

$$= \begin{matrix} r\lambda^{r-1} f(x, y) + 0 \\ = r\lambda^{r-1} f(x, y) \end{matrix} \quad \text{(c)}$$

Since for equation (a), left-hand side = right hand side, therefore equation (b) = equation (c). Hence we have

$$xf_{\lambda x} + yf_{\lambda y} = r\lambda^{r-1} f(x, y)$$

Now λ can be any number. Let λ be 1. Then, we get Euler's theorem:

$$xf_x + yf_y = rf(x, y), \text{ i.e. } x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = rz$$

The above can be generalised for functions of more than two variables.

Homothetic Functions

To understand homothetic function, recall the concept of function-of-function, or composite function. Suppose we have y = f(x) and x = g(w), then we can write y =f[g(w)] = h(w). We can extend the idea of composite function to the case of multivariate function. This we indeed did in this unit, specially when looking at partial derivatives of implicit functions, the chain rule etc. Using this concept of composite functions in the case of multivariate functions, let us try to explain what homothetic functions are.

If f(x,y) is a homogeneous function, then any function g[f(x,y)] is a homothetic function provided g' is positive. In other words, any positive monotonic function of a homogeneous function is a homothetic function.

Check Your Progress 3

1) Find the derivative dz/dt, given

   i) $Z = x^2 – 8xy – y^3$, where x = 3t and y = 1 – t

   ii) $Z = 7u + vt$, where $u = 2t^2$ and v = t + 1

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2) Consider the following implicit functions. Find dy/dz for each of these functions.

   i) $F(x,y) = y – 3x^4 = 0$

   ii) $F(x,y) = x^2 + y^2 - 19 = 0$

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3) Determine whether the following functions are homogeneous. If so, of what degree?

i) $f(x,y) = \sqrt{xy}$

ii) $f(x,y) = x^3 - xy + y^3$

iii) $f(x,y,w) = \dfrac{xy^2}{w} + 2xw$

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1.8 LET US SUM UP

You just finished reading the first unit of this course. This unit was on the mathematics of multivariate functions. The unit began by discussing the concept of functions where the dependent variable is a function of more than one independent variable. These independent variables are called arguments of the function. You were familiarized with the idea of level curves which are contours showing diagrammatically the third variable projected in a two-variable plane. The unit then discussed the important concept of partial derivatives, where the function is differentiated with respect to one variable, with the other variables held constant. For second-order cross partial derivatives, you learnt about Young's theorem. Moving on from there, the unit went on to discuss total differentials and total derivatives. We saw that when we want to investigate what happens to the dependent variable as a result of changes in all the independent variables we use total differentials and total derivatives.

Following this, the unit went on to discuss the chain rule. We saw that the chain rule in the context of partial derivatives and total derivatives is similar to, and yet different from the chain rule in the context of single-variable differentiation. Next, the unit discussed implicit functions. You were familiarized with the difference between explicit and implicit functions. Partial derivatives for implicit functions were discussed. You learnt about the implicit function theorem. The unit then considered implicit functions in the context of level curves. You saw how the derivative of one independent variable with respect to the other is equal to the negative of the ratio of partial derivatives of the dependent variable to each of the independent ones.

In the end, the unit discussed homogeneous functions. The definition of homogeneous functions was given. Certain properties of homogeneous functions were also provided. You were introduced to Euler's theorem. Finally mention was made of homothetic functions, and their relationship to homogeneous functions was stated

1.9 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) (i) See section 1.2 (ii) see section 1.3 and answer.

2) (i) $f_x = 10x$; $f_y = 12y$

   iii) $f_x = -10y^2$; $f_y = -20x$

3) See section 1.3 and answer

Check Your Progress 2

1) See section 1.4 and answer.

2) See section 1.4 and answer.

Check Your Progress 3

1) See section 1.5 and answer.

2) See section 1.6 and answer.

3) See section 1.7 and answer.

UNIT 2 MULTIVARIATE CALCULUS-II

Structure

2.0 Aims and Objectives

2.1 Introduction

2.2 Applications of Level Curves and Partial Derivatives in Economics

2.3 Applications of Total Differentials, Total Derivatives and Chain Rule in Economics

2.4 Applications of Implicit Functions in Economics

2.5 Applications of Homogeneous Functions and Euler's Theorem in Economics

2.6 Let Us Sum Up

2.7 Answers to Check Your Progress Exercises

2.0 AIMS AND OBJECTIVES

In the previous Unit, you were introduced to the mathematics of multivariate calculus. In this unit, we shall discuss some applications of multivariate calculus in economics. After going through the unit, you will be able to:

- State some important curves in economic theory that are level curves;
- Apply partial derivatives to areas in microeconomic and macroeconomic theory;
- Describe some applications of total differentials and total derivatives in economics;
- Examine how implicit functions may be used in economic theory; and
- Discuss how homogeneous functions and Euler's theorem can be applied in economic analysis.

2.1 INTRODUCTION

In the previous Unit, you were introduced to multivariate functions, and also differential calculus applicable to multivariate functions. There you learnt about partial derivatives, about total differentials and total derivatives, about implicit functions and differentiation of implicit functions, and about homogeneous and homothetic functions and Euler's theorem. In unit 1, you learnt about the mathematics of these concepts. Here, we intend to discuss elements of economic theory to which these concepts are applied. While the applications presented here are certainly not exhaustive, we hope you will get a flavor of how multivariate calculus is applied in economics. Reading the present unit together with unit 1, you will gain an understanding about which mathematical concept to invoke to make clear specific economic ideas. For instance, from your microeconomics course, you learnt about the importance of marginal utility, marginal costs, marginal revenue etc. In this unit, you will learn that partial derivatives are the appropriate device for expressing the idea of marginal units in multivariate functions. For each concept that you learnt in the previous unit, we present some suitable economics examples. In this unit, you shall be getting acquainted with these progressively.

The unit is organized as follows. In the next section, section 2.2 we discuss applications in economics of partial derivatives and level curves.. We mainly take examples from microeconomics, specifically from consumer theory and theory of the firm; we discuss some macroeconomic examples as well. In the section after that, section 2.3 we discuss how total differentials, total derivatives and chain rule find applications in economics. Following this, the discussion moves to a discussion of applications of implicit functions in economics. In the final section, section 2.5, the unit discusses some homogeneous functions found in economics, and discusses an important application of Euler's theorem. As you study the unit, you will discover that many concepts that you have and will encounter in your study of microeconomics and macroeconomics are approached from a mathematical point of view. You will gain an insight into the structure of microeconomic and macroeconomic theory, a structure which is mathematical in nature. You will also realise that mathematical concepts are such as to lend themselves easily into a language for communicating ideas of economics.

We strongly urge you to have unit 1 open along with this, the present unit, so that you can immediately refresh your learning of the corresponding mathematical concept for each economic application we deal with in this unit.

2.2 APPLICATIONS OF LEVEL CURVES AND PARTIAL DERIVATIVES IN ECONOMICS

Recall your study of multivariate functions and level curves in unit 1. We saw multivariate functions can be written as $Z = f(X, Y)$

If the function has n arguments we can write it as

$$z = f(x_1, x_2, \dots, x_n)$$

To understand the applications of such functions in economics, what you have to do is think of a situation where a variable depends on more than one variable.

For example, demand of a good x depends upon a variety of factors like its own price (P_x), the prices of related goods (P_z), money income of the consumer (M), his tastes, habits and fashion (i.e. consumer preferences-I) and so on. We can represent a demand function as,

$$D_x = f(P_x, P_{z,} M, T \dots),$$

Similarly, we have production function: $P = f(L, K)$ where P stands for production/output, L for labour and K for capital.

Economic situations typically depicted using multivariate functions. For simplicity we consider only two-variable analysis. For example in a production function, $p = f(L, K)$, we assume and capital (K) used in the process of production. Similarly, in a utility function, $u = f(x, y)$, we assume utility (u) depending upon the quantity of two goods x and y consumed. With the use of partial derivatives we can identify whether two goods are competitive or complementary.

For example think of production, where output of a good depends on two factors labor and capital. Let labor be denoted by L and capital by K. let the output be denoted by Q. Then we have a production function

$$Q = f(L, K)$$

Now suppose we have several inputs, not just two. Let the number of inputs be n. Let us denote the collection of n inputs by an n-tuple $(x_1, x_2, \ldots x_n)$

We can then write the production function as

$$Q = f(x_1, x_2, \ldots, x_n)$$

Similarly, we can think of a utility function which shows the utility U that a consumer derives from the consumption of two goods X and Y. We can show this as

$$U = f(X, Y)$$

Again, suppose there are n goods $Y_1, Y_2 \ldots Y_n$ and the amount consumed of these goods are denoted by

$$(y_1, y_2, \ldots y_n)$$

In this case, we can depict the utility function as

$$U = f(y_1, y_2, \ldots, y_n)$$

As a final example, this time from macroeconomics, let the money demand M^d be a function of the interest rate r and income. We can show this as

$$M^d = g(r, Y)$$

So you understand that in all these functions a dependent variable depends on more than one variable. In the rest of the unit, we study how changes in the independent variable induce changes in the dependent variable. Sometimes we consider the change in one independent variable keeping the others constant, and sometimes we consider changes in all the independent variables together.

Now we turn to a discussion of level curves in economics. At this point you must remember the discussion in section 1.2.2 of unit 1. You know from that discussion that a level curve shows the curve where the dependent variable is fixed at a certain level. If we have a function $Z = f(X,Y)$, the graph of this function in three-dimensional space can be visualized as being cut by horizontal planes that are parallel to the $x\ y$ – plane. The intersections between the planes and the graph can be projected onto the $x\ y$ – plane. If the intersecting plane is at $z = k$ then the projection onto the $x\ y$ – plane is known as the level curve or contour at height k for the function f. The contour or level will consist of points that satisfy the equation

$$f(x, y) = k$$

To understand the economic application, think of the simple production function that we just considered

$$Q = f(L, K)$$

If we were to depict this graphically, we can show L and K in the x-y plane and the output Q vertically on the z –axis. Then the graph of the production function Q will be like a 'hill' or dome. To think of level curves suppose the value of Q is fixed at a level say k. Then the production function will be k = f (L, K). Here k is a *constant*. The graph of the production function will be a 'slice' on the hill or dome at a height k. If we were to take a different fixed value of k, then there will be a 'slice' at different height on the dome. To put this slightly differently, a level curve for the production function above is the locus of all points showing combinations of labour and capital that yield an output equal to k. All the combinations of labour and capital that result in the same specific output level will be on the same level curve. In microeconomics, in the theory, you have come across level curves under the name 'isoquant'. So for a production function (with two factors) each isoquant is a different level curve.

Similarly, consider the simple utility function $U = f(X, Y)$

If the level of utility were to be fixed at a given level say $\overline{U}$ then a level curve would be produced as the locus of all combinations of X and Y that give the same level of utility equal to $\overline{U}$. In the context of the theory of the consumer that you studied in microeconomic theory, you were familiarized with level curves under the name 'indifference curves'. So for a utility function (of two goods), each indifference curve is a different level curve, showing a different level of utility.

Let us now discuss applications of partial derivatives in economics. Recall the discussion on partial derivatives in section 1.3 in unit1. Now we take some functions in economics and see where partial derivatives can be applied. If we take the production function above and see by how much the output increases if we were to increase labour input by one unit while holding capital constant, we will get the marginal product of labour. So if we have $Q = f(L, K)$, the marginal product of labour will be $\frac{\partial Q}{\partial L}$

Similarly we can have $\frac{\partial Q}{\partial K}$

Now consider the utility function

$U = f(X, Y)$

Here, if we increase X by one unit while holding Y constant, the change in utility that would result is the marginal utility of X: $\frac{\partial U}{\partial X}$

In a similar manner, we can have the marginal utility of Y: $\frac{\partial U}{\partial Y}$

Let us consider simple solved example from basic microeconomic theory.

Given the demand function of two goods X_1 and $X_{2'}$, as $x_1 = -8p_1 + 3p_2 + 11$ and $x_2 = 2p_1 - 3p_2 + 17$, find the effect of change in prices of two goods on their demand. Interpret the results.

Alternatively, this example could be thought of being about finding, what is called, partial marginal demands (PMD). There are four possibilities.

1) PMD of X_1, when price of X_1 changes (holding other variables constant) $= \frac{\partial x_1}{\partial p_1} = -8$

2) PMD of X_1, when price of X_2 changes $\frac{\partial x_1}{\partial p_2} = 3$

3) PMD of X_2, when price of X_2 changes $= \frac{\partial x_2}{\partial p_2} = -3$

4) PMD of X_2, when price of X_1 changes $= \frac{\partial x_2}{\partial p_1} = 2$

Interpretation:

1) $\frac{\partial x_1}{\partial p_1} = -8$, shows that when the price of X_2 is held constant, a rise in price of X_1, by one unit decreases $X_1's$ demand by 8 units

2) $\frac{\partial x_1}{\partial p_2} = 3$, shows that when the price of X_1 is held constant, a rise in price of X_2, by one unit increases the demand of X_1 by 3 units

3) $\frac{\partial x_2}{\partial p_2} = -3$, shows that when the price of X_1 is held constant, a rise in price of X_2, by one unit decreases $X_2's$ demand by 3 units

4) $\frac{\partial x_2}{\partial p_1} = 2$, shows that when the price of X_1 is held constant, a rise in price of X_1, increases demand for X_2 by 2 units.

[Note that Price of X_1 is p_1 and price of X_2 is p_2 Demand of X_1 is x_1 and demand of X_2 is x_1]

Two goods x and y are complementary when, other things being held constant, rise in the price of one (say petrol), decreases the demand for the other (say cars). In this case both the cross partial derivatives should be negative i.e.

$$\frac{\partial x_1}{\partial p_2} < 0; \qquad \frac{\partial x_2}{\partial p_1} < 0$$

On the other hand, two goods X and Y are competitive (or substitute goods), when other things being held constant, if the increase in the price of one (say coffee) increases the demand for the other (say tea). In this case both the order partial derivatives should be positive i.e.

$$\frac{\partial x_1}{\partial p_2} > 0; \qquad \frac{\partial x_2}{\partial p_1} > 0$$

Example.

The demand for functions of two goods x_1 and x_2, find whether the goods are complementary or competitive? The functions are:

$x_1 = \frac{10}{p_1^2 p_2}$; $x_2 = \frac{150}{p_1 p_2^2}$ where p_1 and p_2 are the prices of x_1 and x_2 respectively

From the first function: $\frac{\partial x_1}{\partial p_2} = \frac{10}{p_1^2} \cdot \frac{\partial}{\partial p_2}\left(\frac{1}{p_2}\right) = \frac{-10}{p_1^2 p_2^2} < 0$

From the second function: $\frac{\partial x_2}{\partial p_1} = \frac{150}{p_2^2} \cdot \frac{\partial}{\partial p_1}\left(\frac{1}{p_1}\right) = \frac{-15}{p_1^2 p_2^2} < 0$

Since both $\frac{\partial x_1}{\partial p_2}$ and $\frac{\partial x_2}{\partial p_1}$ are negative, therefore, the goods are complementary.

Example.

Find are nature of goods x_1 and x_2 when

$x_1 = p_1^{-0.4} e^{0.2 p_2}$ and $x_2 = p_2^{-0.6} e^{0.5 p_1}$

where p_1 and p_2 represent their respective prices.

$$\frac{\partial x_1}{\partial p_2} = p_1^{-0.4} \cdot e^{0.2 p_2} \times 0.2 = 0.2 x_1 > 0$$

$$\frac{\partial x_2}{\partial p_1} = p_2^{-0.6} \cdot e^{0.5p_1} \times 0.5 = 0.5 p_2^{-0.6} e^{0.5p_1} = .5x_2 > 0$$

Since both the partial derivatives are positive, therefore, the goods x_1 and x_2 are competitive.

Application to Partial Elasticities (PE)

Keeping in mind the formula for point elasticity of demand, we can calculate four partial elasticities, for the two demand functions,

$x_1 = f_1(p_1, p_2)$ and $x_2 = f_2(p_1, p_2)$ where x_1 and x_2 represent demand for two goods X_1 & X_2 and p_1 & p_2 are their prices. These elasticities are as follows.

a) PE of demand for x_1 $w.r.t\ p_1 = e_{11}{}^{*} \dfrac{\partial x_1}{\partial p_1} \times \dfrac{p_1}{x_1}$

b) PE of demand for x_1 $w.r.t\ p_2 = e_{12} \dfrac{\partial x_1}{\partial p_2} \times \dfrac{p_2}{x_1}$

c) PE of demand for x_2 $w.r.t\ p_2 = e_{22} \dfrac{\partial x_2}{\partial p_2} \times \dfrac{p_2}{x_2}$

d) PE of demand for x_2 $w.r.t\ p_1 = e_{21} \dfrac{\partial x_2}{\partial p_1} \times \dfrac{p_1}{x_2}$

Note: Where as e_{11} and e_{22} are direct PEs of demand, but e_{12} and e_{21} are cross PEs of demand.

Let us take some examples to illustrate

Example 1. Given the demand functions $x_1 = p_1^{-1.5} p_2^{.4}$ and $x_2 = p_1^{.6} p_2^{-.7}$, find the direct as well as cross partial elasticities. (x_1 and x_2 are demands p_1, p_2 are the prices of two goods X_1 and X_2 respectively)

Sol. a) **Direct PEs**

$$e_{11} = \frac{\partial x_1}{\partial p_1} \times \frac{p_1}{x_1} = -1.5 p_1^{-2.5} p_2^{.4} \times \frac{p_1}{p_1^{-1.5} p_2^{.4}} = \frac{-1.5 p^{-1.5} p_2^{.4}}{p^{-1.5} \cdot p_2^{.4}} = -1.5$$

$$e_{22} = \frac{\partial x_2}{\partial p_2} \times \frac{p_2}{x_2} = -.7 p_1^{.6} \cdot p_2^{-1.7} \times \frac{p_2}{p_1^{.6} p_2^{0.7}} = \frac{-.7 p_1^{.6} \times p_2^{.7}}{p_1^{.6} \times p_2^{.7}} = -.7$$

b) **Cross PEs**

$$e_{12} = \frac{\partial x_1}{\partial p_2} \times \frac{p_2}{x_1} = \frac{.4 \times p_1^{-1.5} \times p_2^{-.6} \times p_2}{p_1^{-1.5} p_2^{.4}} = \frac{.4 p_1^{-1.5} \times p_2^{.4}}{p_1^{-1.5} p_2^{.4}} = .4$$

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* 1) In e_{11}, the LHS 1 stands for x_1 and the RHS 1 stands for p_1

2) First write PD to avoid mistakes (only suggestive)

Example 2. Demand for a goods x is given as a function of money income (M) and price p_x as $x = .9M^{1.1}p^{-0.7}$. Find price elasticity and income elasticity of demand.

Sol. Given $x = .9M^{1.1}p_x^{-0.7}$ $\qquad [x = f(M, p_x)]$

a) Price elasticity of demand $= \dfrac{\partial x}{\partial p_x} \times \dfrac{p_x}{x}$

$$= -.7 \times .9M^{1.1} \times p_x^{-1.7} \times \frac{p_x}{.9M^{1.1}p_x^{-0.7}} = -.7$$

b) Income elasticity of demand $= \dfrac{\partial x}{\partial M} \cdot \dfrac{M}{x}$

$$.9M^{.1} \times 1.1 p_x^{-0.7} \times \frac{M}{.9M^{1.1}p_x^{-0.7}} = 1.1$$

Example 3. From the following demand function, find income and cross elasticity of demand when consumer income (m) = Rs.500, price of $x(p_x) = 10$ and price of $z(p_z) = 15$.

$$x = 800 - \frac{p_x^2}{5} + \frac{p_z}{60} + \frac{m}{10}.$$

a) Cross elasticity of demand $= \dfrac{\partial x}{\partial p_z} \times \dfrac{p_z}{x}$

when $\dfrac{\partial x}{\partial p_z} = 0 - 0 + \dfrac{1}{60} + 0 = \dfrac{1}{60}$.

$$x = 800 - \frac{10}{5} + \frac{15}{60} + \frac{500}{10} \qquad (\text{For } m = 500, p_x = 10, p_z = 15)$$

$$= 800 - 2 + \frac{1}{4} + 50 = 848.25$$

is cross $\quad = \dfrac{1}{{}_4\cancel{60}} \times \dfrac{\cancel{15}^{1}}{848.25} = \dfrac{1}{3393}$

b) Income elasticity of demand $\dfrac{\partial x}{\partial m} \cdot \dfrac{m}{x}$, where

$\dfrac{\partial x}{\partial m} = -0 + 0 + \dfrac{1}{10} = \dfrac{1}{10}$, $m = 500$ and

$x = 848.25$ (already calculated)

is $e_m = \dfrac{1}{1\cancel{0}} \times \dfrac{50\cancel{0}}{848.25} = \dfrac{50}{848.25} = .0589$

Application to Utility Function

We know any marginal function is first derivative of total function. Thus Marginal = $\frac{d}{d_x}$ (Total). Since it is a care of partial derivative dealing with functions of two variables, therefore, we will find partial derivatives like $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial j}$ etc. Let us take some examples.

Example 1. Let the utility function be $u=(x+7)(y+2)$. Find the marginal utilities with respect to the two goods x and y when 5 units of x and 3 units of y are consumed.

Sol. Given utility function $u=(x+7)(y+2)$ or

a) MU with respect to $x=\frac{\partial u}{\partial x}=y+2$.

For $x=5, y=3, Mu_x = y+2=3+2=5$

b) MU with respect to $y=\frac{\partial u}{\partial y}=x+7$.

For $x=5, y=3, Mu_y = x+7=5+7=12$

Example 2. If the utility function is $u=ax+by+c\sqrt{xy}$, find the ratio of marginal utilities of the two goods x and y.

Sol. Given: $u=ax+by+c(xy)^{1/2}$

$$Mu_x = \frac{\partial u}{\partial x} = a+\frac{cy}{2\sqrt{xy}} = \frac{2a\sqrt{xy}+cy}{2\sqrt{xy}}$$

$$Mu_y = \frac{\partial u}{\partial y} = b+\frac{cx}{2\sqrt{xy}} = \frac{2b\sqrt{xy}+cx}{2\sqrt{xy}}$$

Therefore, ratio of $Mu_s = \frac{Mu_x}{Mu_y} = \frac{2a\sqrt{xy}+cx}{2\sqrt{xy}} \div \frac{2b\sqrt{xy}+cx}{2\sqrt{xy}}$

$$= \frac{2a\sqrt{xy}+cy}{2b\sqrt{xy}+cx}$$

Application to Production Function

A production function represent total product (TP). Therefore, marginal product will be the 1st partial derivative of TP with respect to one of the factors, keeping other factor constant. Therefore,

a) Marginal Product of Labour (L) $=\frac{\partial}{\partial L}(TP)$

b) Marginal Product of Capital (K) $= \frac{\partial}{\partial K}(TP)$

Euler Theorem can also be stated in terms of partial derivatives. If $P = f(L, K)$ is the production function of two factors, labour (L) and capital (K), then the Euler Theorem states that

$$L \cdot \frac{\partial P}{\partial L} + K \cdot \frac{\partial P}{\partial K} = P$$

$$L \cdot MP_L + K \cdot MP_K = P$$

This is also called product exhaustion theorem or the adding-up problem. It states that:

If all factors are paid according to their marginal product then the total product is exhausted.

That is $P - (L \cdot MP_L + KMP_K) = 0$

Let us now take some illustrations

Example 1.

Given the production function $P = 2(LK)^{1/2}$, where P is total product, L is labour and K is capital as factors of production

1) Find the marginal product of the two factors
2) Show that Euler Theorem is satisfied
3) What shall be the payment to labour is 5 units of labour are used, when capit remains fired at 20

Sol. Given TP : $P = 2.L^{1/2}K^{1/2}$

i) $\therefore MP_L = \frac{\partial P}{\partial L} = 2 \times \frac{1}{2} L^{-\frac{1}{2}} \times K^{\frac{1}{2}} = \left(\frac{K}{L}\right)^{\frac{1}{2}}$

and $MP_L = \frac{\partial P}{\partial K} = 2 \times \frac{1}{2} L^{\frac{1}{2}} K^{-\frac{1}{2}} = \left(\frac{L}{K}\right)^{\frac{1}{2}}$

ii) Euler Theorem states that: $L \cdot MP_L + KMP_K = P$

or $L \cdot \left(\frac{K}{L}\right)^{\frac{1}{2}} + K \cdot \left(\frac{L}{K}\right)^{\frac{1}{2}} = L^{\frac{1}{2}} K^{\frac{1}{2}} + K^{\frac{1}{2}} \cdot L^{\frac{1}{2}} = 2L^{\frac{1}{2}} \cdot K^{\frac{1}{2}} = P$

The Euler Theorem is satisfied.

iii) When $L = 5$ and $K = 20$, then $MP_L = \left(\frac{K}{L}\right)^{\frac{1}{2}} = \left(\frac{20}{5}\right)^{\frac{1}{2}} = (4)^{\frac{1}{2}} = 2$

Example 2.

A production function is given by $Ax^{1/3}y^{1/3}$ where x stands for labour and y stands for capital. Answer the following questions

a) What is the behaviour of each factor?

b) What is the nature of returns to scale?

c) Find whether total product is exhausted or not?

Sol. Given $PF: P = Ax^{1/3} \cdot y^{1/3}$

a) Behaviour of factor's MP means nature of rate of change of MP of that factor i.e.

$$\frac{\partial}{\partial x}(MP_x) \text{ and } \frac{\partial}{\partial y}(MP_y)$$

i) $MP_x = \frac{\partial P}{\partial x} = A \cdot \frac{1}{3} x^{-\frac{2}{3}} y^{\frac{1}{3}} = \frac{1}{3} \cdot Ax^{-\frac{2}{3}} y^{\frac{1}{3}}$

Rate of change of $MP_x = \frac{\partial}{\partial x}(MP_x) = \frac{\partial}{\partial x}\left(\frac{1}{3} Ax^{-\frac{2}{3}} y^{\frac{1}{3}}\right)$

$$= \frac{1}{3} \times \frac{-2}{3} \times A \times x^{\frac{-5}{3}} \times y^{\frac{1}{3}} = \frac{-2}{9} Ax^{\frac{-5}{3}} \cdot y^{\frac{1}{3}} < 0$$

This shows that as amount of factor x in creases, its MP decreases, when y is held constant

ii) $MP_y = \frac{\partial P}{\partial y} = A \cdot \frac{1}{3} x^{\frac{1}{3}} y^{-\frac{2}{3}} = \frac{1}{3} \cdot Ax^{\frac{1}{3}} y^{-\frac{2}{3}}$

Rate of change of $MP_y = \frac{\partial}{\partial y}(MP_y) = \frac{\partial}{\partial y}\left(\frac{1}{3} Ax^{\frac{1}{3}} y^{\frac{-2}{3}}\right)$

$$= \frac{1}{3} \times \frac{-2}{3} \times A \times x^{\frac{1}{3}} y^{\frac{-5}{3}} = \frac{-2}{9} Ax^{\frac{1}{3}} y^{\frac{-5}{3}} < 0$$

This shows that as amount of factor y increase, its MP decreases, when x is held constant.

b) To show the nature of returns to scale, we increase both the factors by a fixed proportion say λ*. Then the PF becomes $\hat{P} = \lambda^{\frac{2}{3}} \cdot Ax^{\frac{1}{3}} \cdot y^{\frac{1}{3}} = \lambda^{\frac{2}{3}} \cdot P$

Since power of λ is $\frac{2}{3} < 1$, therefore, a 20% increase in both the factors bring about less than proportionate (i.e. less then 20%) increase in total

* λ read as Lamda is a proportion. If we increase every factor by, say, 20%, than $\lambda = \frac{20}{100} = \frac{1}{5} = .2$

product P. That is the production function shows decreasing returns to scale.

c) Let us see if Euler theorem holds good not if x and y factors are paid according to their MP, then TP should be exhausted i.e.

or TP – (Payment to factor x + Payment to factor) $= zero$

Now payment to $y = y \cdot MP_y = y \cdot \frac{1}{3} Ax^{\frac{1}{3}} \cdot y^{\frac{-2}{3}}$

Similarly payment to $y = y \cdot MP_y = y \cdot \frac{1}{3} Ax^{\frac{1}{3}} \cdot y^{\frac{-2}{3}}$

$$= \frac{1}{3} Ax^{\frac{1}{3}} \cdot y^{\frac{1}{3}} = \frac{1}{3} P .$$

is total payment $= \frac{1}{3} P + \frac{1}{3} P = \frac{2}{3} P < P$

Hence total product is not exhausted. Euler Theorem is not exhausted.

Example 3.

Let the production function by $Q = AL^a K^b$. Find the elasticity of production with respect to labour (L) and capital (K).

Sol. Given: Production function: $Q = AL^a K^b$

i) Labour elasticity of Production $= e_L = \frac{\partial Q}{\partial L} \cdot \frac{L}{Q}$

$$= A \cdot a\, L^{a-1} K^6 \times \frac{L}{AL^a \times K^b} = \frac{a \cdot A \cdot L^a K^b}{AL^a K^b} = a .$$

(which is the exponent of the factor L)

ii) Capital elasticity of Production $= e_K = \frac{\partial Q}{\partial K} \cdot \frac{K}{Q}$

$$= A \cdot bL^a K^{b-1} \times \frac{K}{AL^a K^b} = \frac{b \cdot A \cdot L^a K^b}{AL^a K^b} = b .$$

Check Your Progress 1

1) Given the utility function $U = x^2y^3$, find the marginal utilities. Determine if the marginal functions are increasing or decreasing.

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2) Given the production function $Q = 10L^{0.7}K^{0.3}$, find the marginal products. Determine if the marginal functions are increasing or decreasing.

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2.3 APPLICATIONS OF TOTAL DIFFERENTIALS, TOTAL DERIVATIVES AND CHAIN RULE IN ECONOMICS

In the previous section, you got a lengthy discussion of partial derivatives in economics. In this section we shall look at economic applications of total differentials as well as total derivatives in economics. Before we begin, let us introduce two notational points. The first one is that about functions. Suppose we have a function z = f (x, y). We can denote it by z =z (x, y). So the function itself we can denote by z instead of by f. This says that there is a variable which depends on x and y. We can sometimes talk of the relation between z, and x and y without using the equality sign. If we have z (x, y) you can think of it immediately as being about some variable z which depends on x and y. Thus we can talk about money demand M^d as M^d (r, Y) where r is interest rate and Y is income. This says money demand depends on the interest rate and income. We shall be using this notation intermittently in the rest of the unit.

The second point about notation concerns partial derivatives. Take a function U = f (x, y). The partial derivative of U with respect to x, $\frac{\partial U}{\partial x}$ can also be written U_x. Similarly the partial derivative of U with respect to y can be written U_y. The partial derivatives are also sometimes denoted f_x and f_y. Another notation is sometimes used. Suppose we have a function denoted U = (x_1, x_2). Then the partial derivative of U with respect to x_1 and x_2 is sometimes denoted U_1 and U_2 respectively, or even as f_1 and f_2 respectively.

So, having made these basic comments about notation, let us discuss some applications of differentials and derivatives to economics. For this section too, as for all sections of this unit, you just have to recall the basic concept that you learned in Unit 1.

Consider a saving function

$S = S(Y, i)$ where S is saving, Y is income, and i is interest rate. We assume that the function is continuous and has partial derivatives. In this saving function, $\frac{\partial S}{\partial Y}$ is the marginal propensity to save. If there is a change in *Y*, *dY*, the resulting change in S will be approximately $\frac{\partial S}{\partial Y}dY$. Similarly, given a change in i, di, we can say that the resultant change in S will be $\frac{\partial S}{\partial i}di$. Thus the total change in S due to the change in Y and in *i* is

$dS = \frac{\partial S}{\partial Y}dY + \frac{\partial S}{\partial i}di$. In the notation we had mentioned above, we can write this as

$$dS = S_Y dY + S_i di$$

Here *dS* is the total differential of the saving function. The process of obtaining the total differential of a function is called total differentiation.

In the saving function mentioned above, it is possible that only income changes; the interest rate stays constant. In that case, di = 0. So the total differential of the saving function becomes $dS = \frac{\partial S}{\partial Y}dY + 0 = \frac{\partial S}{\partial Y}dY$

Dividing both sides by dY we get

$$\left(\frac{dS}{dY}\right)_{i=\bar{i}} = \frac{\partial S}{\partial Y}$$

We can interpret the partial derivative $\frac{\partial S}{\partial Y}$ as the ratio of two differentials dS and dY, with the condition that the other variable in the saving function, I, remains constant. We could have also had a situation where Y did not change, only I did, and obtained

$$\left(\frac{dS}{di}\right)_{Y=\bar{Y}} = \frac{\partial S}{\partial i}$$

Now let us consider a function with n arguments. Think of the following utility function

$U = f(x_1, x_2, ..., x_n)$. Note that we can also write this also as

$U = U(x_1, x_2, ..., x_n)$ in the notation that we mentioned at the beginning of this section.

Differentiating U totally, we will get the total differential of U as:

$$dU = \frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 + \cdots + \frac{\partial U}{\partial x_n}dx_n$$

We can also write this as

$$dU = U_1 dx_1 + U_2 dx_2 + \ldots U_n dx_n = \sum_{i=1}^{n} U_i dx_i$$

As a final application of total differential, we will make use of one of the rules involving total differentials that you had studied in the previous unit which says

$$d(u \pm v) = du \pm dv$$

As an illustration of this, consider the basic macroeconomic national income equation for a closed economy: $Y = C + I + G$, where Y is income, C is aggregate private consumption, I is investment and G is government expenditure. Then the total differential of Y will be $dY = dC + dI + dG$

Now suppose private investment I was a function of the interest rate r. Then we would have

$$dY = dC + \frac{\partial I}{\partial r} dr + dG$$

After this discussion of application of total differentials in economics, let us discuss the application of total derivatives. To do this, think of the saving function $S = S(Y, i)$

We had considered this function above. Earlier, in this function we had assumed that income Y and interest rate i are independent variables. But suppose that both these variables somehow depend on the money supply M. So now we can write the saving function as $S = S(Y(M), i(M))$.

Now suppose we would like to see how savings change as a result of the money supply. Then we take the total derivative of S with respect to M. We get:

$$\frac{dS}{dM} = \frac{\partial S}{\partial Y}\frac{dY}{dM} + \frac{\partial S}{\partial i}\frac{di}{dM}$$

In the above, we assume that the only channels through which M affects S are through its influence on Y and i, and that M does not independently influence S. However, there may be cases where in a function the arguments themselves depend on a variable, and that this variable independently influences the dependent variable. For an example, we consider a production function, but in a dynamic long-run sense:

$$Q = Q(L, K, t)$$

This says that output depends not only on inputs capital and labour but on a third argument in the function, time, denoted by t. The presence of time in the production function suggests that production of the good is affected over time by technology, increase in skills, etc. This factor shifts the production function. Thus the production function is dynamic and not static. Now capital and labour can also change over time, and so we have $K = K(t)$ and $L = L(t)$. So the production function can be written

$$Q = Q(L(t), K(t), t)$$

We can use total derivative of Q with respect to t to show the rate of change of output over time. This will be

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial L}\frac{dL}{dt} + \frac{\partial Q}{\partial K}\frac{dK}{dt} + \frac{\partial Q}{\partial t}$$

Using an alternative notation, we can depict this as

$$\frac{dQ}{dt} = Q_L L'(t) + Q_K K'(t) + Q_t$$

The above can be considered as an application of chain rule also, although we had mentioned in unit 1 that for chain rule to apply, each of the arguments in the function should itself be a function of two variables. Say you have a function

Z = f (x, y) and x and y each is a function of r and s. So we have

$$z = f(x(r,s), y(r,s))$$

We can construct examples from economics. Say a person gets utility by being happy when his two children A and B get utility from consumption of food, denoted by F, and clothes denoted by C. Let U^A be the utility function of A and U^B be the utility function of B. Let U be the utility of this person.

Then we can write:

$$U = f\left[U^A(F,C), U^B(F,C)\right]$$

Notice that we have not made a distinction between the quantities of food and clothes consumed by the two children.

Suppose we want to see how the utility of this person changes when the consumption of clothes changes. This we can find by getting $\frac{dU}{dC}$. This will be equal to

$$\frac{dU}{dC} = \frac{\partial U}{\partial U^A}\frac{dU^A}{dC} + \frac{\partial U}{\partial U^B}\frac{dU^B}{dC}$$

We can similarly obtain $\frac{dU}{dF}$ to see the change in utility of this person when the consumption of food by his children.

2.4 APPLICATIONS OF IMPLICIT FUNCTIONS IN ECONOMICS

We have till now seen the application of partial derivatives, total differentials and total derivatives in economics. In this section, we look at the differentiation of implicit functions. For this we would again urge you to read the corresponding section on implicit functions in Unit 1, which is section 1.6.3.

If we have a function f (x, y), we find that x and y are related implicitly. To proceed further with this implicit function, we can introduce a third variable z, which is related as a single valued function of x and y: z = g (x, y). For z = k (a constant) we get a level curve. So level curves (like indifference curves and isoquants in economics) are implicit functions.

As x and y vary (not necessarily independent of each other), the change in z is obtained from the total differential:

$dz = g_x dx + g_y dy$, where g_x is the partial derivative of z with respect to x; you understand what g_y is

If the values of x and y are such that the value of z is 0, then $dz = g_x dx + g_y dy = 0$

Hence, $\frac{dy}{dx} = -\frac{g_x}{g_y}$

Let us apply this idea of differentiating implicit functions to economics. Consider a production function $Q = f(L, K)$. This gives rise to a family of isoquants. A specific isoquant will have the equation $f(L, K) = c$, where c is a constant. Differentiating this with totally we get

$f_L dL + f_K = 0$.

This is the approximate relation between changes dL and dK in the inputs along the isoquant through (L, K). This relation holds at all points on the isoquant. Hence the tangent gradient of the isoquant through (L, K) is

$$\frac{dK}{dL} = -\frac{f_L}{f_K}$$

The left-hand side $\frac{dK}{dL}$ is called the marginal rate of technical substitution. It shows the marginal rate of substitution of capital for labour.

On the right-hand side, f_L and f_K are the marginal products of labour and capital respectively.

Thus the marginal rate of technical substitution between capital and labour is equal to the negative of the ratio of the marginal product of labour and that of capital.

We can do a similar exercise in the case of consumer theory. Take an indifference curve $U = f(x, y)$. Now along each indifference curve, the utility remains constant. Thus, for a specific indifference curve, let utility be fixed at a level $\overline{U}$

Then, the total differential of this function would be $d\bar{U} = f_x dx + f_y dy = 0$

From this we get $\frac{dy}{dx} = -\frac{f_x}{f_y}$

The left-hand side is called the marginal rate of substitution of y for x. The right-hand side is the ratio of the marginal utility of x to that of y.

Let us now consider an application of implicit functions in macroeconomics. Consider a closed economy. Its basic accounting equation shows that total income is a sum of total private consumption, total private spending, and government expenditure:

$Y = C + I + G.$

Let consumption be an increasing function of income, and investment be an increasing function of income, and a decreasing function of the interest rate, *r*. Government spending is assumed to be exogenous. Then we can write the above equations as

$$Y = C(Y) + I(Y,r) + G$$
$$0 < C' < 1$$
$$I_Y > 0$$
$$I_r < 0$$

We can also consider a money market in which the exogenous supply of money, *M*, is equal to the demand for money, *L*, where *L* is an increasing function of income *Y*, and a decreasing function of the interest rate, *r*.

We can write for the money market the following equilibrium condition

$$M = L(Y,r)$$

We have now two equations: one for the goods market, and one for the money market. Leaving aside the government expenditure, we can consider the money supply as the only actual exogenous variable. We can take total differentials and get:

$$dY = C_Y dY + I_Y dY - I_r dr + dG$$

And for the money market

$$dM = L_y dY + L_r dr$$

If we take government spending and money supply to be fixed, then dG and dM will be zero, and the model can be solved.

Check Your Progress 2

1) Given a production function

$Y = f(K,L) = K^{0.3}L^{0.5}$, find the marginal rate of technical substitution.

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1) A firm finds that its production function is of the form

   $Q = 100K^{0.25}L^{0.75}$

   Where Q is output and K and L are capital and labour, respectively.

   i) Show that the equation of the isoquant for Q = 100 is given by $K = 1/L^3$

   ii) Show that this isoquant is negatively sloped.

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2) Consider the utility function

   U = XY, where X and Y are two goods consumed

   i) Show that the indifference curves are downward sloping and convex.

   ii) Give an economic interpretation to the slope and curvature of the indifference curve

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2.5 APPLICATIONS OF HOMOGENEOUS FUNCTIONS AND EULER'S THEOREM IN ECONOMICS

In this final section of the unit, we look at some applications of homogeneous functions in economics. For this section in this unit, you must read the corresponding section on homogeneous functions and their differentiation in unit 1.

Let us consider a production function F where output Q is a function of two inputs labour (L) and capital (K), that is, $Q = F(L, K)$. This production function is said to be homogeneous of degree r, if for all Land K in the domain,

$$F(\lambda L, \lambda K) \equiv \lambda^r F(L, K).$$

We can extend this definition to a production function of n-inputs x_1., x_2 ,…,x_n (Q is a function not just of labour and capital but of n inputs denoted as above. Suppose

$z = f(x_1, x_2, ..., x_n)$. Then f is said to be homogeneous of degree r if

$f(\lambda x_1, \lambda x_2, ..., \lambda x_n) \equiv \lambda^r f(x_1, x_2, ..., x_n)$. Here r determines the returns to scale of the production function. If

$F(x, y)$ is linearly homogeneous (homogeneous of degree 1) if and only if $F(x, y) = yf\left(\frac{x}{y}\right)$,where $f\left(\frac{x}{y}\right) = F\left(\frac{x}{y}, 1\right)$

How do we show this? Let us begin by showing the 'if' (sufficiency) part.

Given:

$F(x, y) = yf\left(\frac{x}{y}\right)$,we have

$$F = (\lambda x, \lambda y) = \lambda yf\left(\frac{\lambda x}{\lambda y}\right) = \lambda yf\left(\frac{x}{y}\right) = \lambda F(x, y)$$

Now let us show the only if (necessary condition) part:

Given F(x,y) is linearly homogeneous (homogeneous of degree 1). Then

$F(\lambda x, \lambda y) = \lambda F(x, y)$, for any λ

Put $\lambda = \frac{1}{y}$. The we have

$$F\left(\frac{x}{y}, 1\right) = \frac{1}{y} F(x, y)$$

Thus $yF\left(\frac{x}{y}, 1\right) = F(x, y)$

But $F\left(\frac{x}{y}, 1\right) = f\left(\frac{x}{y}\right)$.

Hence, $F(x, y) = yf\left(\frac{x}{y}\right)$

Differentiation of a Homogeneous Function

A very important property of homogeneous functions is with regard to the differentiation of homogeneous functions.

Let us have a homogeneous function *f(x, y)* of degree *r*. then we have:

$f(\lambda x, \lambda y) \equiv \lambda^r f(x, y)$

Differentiating with respect to x we have:

$\lambda f_x(\lambda x, \lambda y) = \lambda^r f_x(x, y)$, where $f_x = \dfrac{\partial f}{\partial x}$

If we divide by λ we get:

$f_x(\lambda x, \lambda y) = \lambda^{r-1} f_x(x, y)$

If you notice carefully, the above equation shows that the function f_x is homogeneous of degree r – 1. The above holds for the function f_y too, where f_y is the partial derivative with respect to y. the above holds in the case of any multivariate homogeneous function.

This result says that if a function is homogeneous of degree r, then each of its partial derivatives is homogeneous of degree $r - 1$.

Euler's equation

This is a very important property displayed by homogeneous equations. Let $z = f(x, y)$

be a homogeneous function of degree r. then the following relation holds as an identity:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = rz$$

We repeat the proof that that was provided in unit 1. We prove Euler's equation using implicit differentiation and the chain rule:

Let us have the homogeneous function

$$f(\lambda x, \lambda y) \equiv \lambda^r f(x, y) \quad \text{(a)}$$

Let us partially differentiate the left-hand side of equation (a) with respect to λ. We get:

$$\frac{\partial f(\lambda x, \lambda y)}{\partial \lambda x}\frac{\partial \lambda x}{\partial \lambda} + \frac{\partial f(\lambda x, \lambda y)}{\partial \lambda y}\frac{\partial \lambda y}{\partial \lambda}$$

$$= xf_{\lambda x} + yf_{\lambda y} \quad \text{(b)}$$

Now let us obtain the partial derivative of the right-hand side of equation (a) with respect to λ. We now get:

$$\frac{\partial \lambda^r}{\partial \lambda} f(x, y) + \lambda^r \frac{\partial f(x, y)}{\partial \lambda}$$

$$= r\lambda^{r-1} f(x, y) + 0$$
$$= r\lambda^{r-1} f(x, y) \quad \text{(c)}$$

Since for equation (a), left-hand side = right hand side, therefore equation (b) = equation (c). Hence we have

$$xf_{\lambda x} + yf_{\lambda y} = r\lambda^{r-1} f(x, y)$$

Now λ can be any number. Let λ be 1. Then, we get Euler's theorem:

$$xf_x + yf_y = rf(x, y), \text{ i.e. } x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = rz`$$

The above can be generalised for functions of more than two variables. Let us discuss an important application in economics of Euler's theorem.

Consider a production function that is homogeneous of degree 1. In other words, this production function displays constant returns to scale. Consider the following simple production function

$$Q = f(L, K)$$

If the Euler's theorem holds for the function $z = f(x, y)$

,then if z is homogeneous of degree 1, then Euler's theorem would imply $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

Now let us merely substitute z with Q, x with L, and y with K. Then we would have our production function above with constant returns to scale. Euler's theorem would imply:

$$L\frac{\partial Q}{\partial L} + K\frac{\partial Q}{\partial K} = Q$$

On the left hand side, $\frac{\partial Q}{\partial L}$ is the marginal product of labour and $\frac{\partial Q}{\partial K}$ is the marginal product of capital. If each worker gets paid a wage equal to his or her marginal product of labour, and each unit of capital gets paid a rental equl to its marginal product of capital, then $L\frac{\partial Q}{\partial L}$ is the total earnings of all the workers (that is, of labour as a whole) and $K\frac{\partial Q}{\partial K}$ is the total earnings of capital. Thus the left-hand side of the above equation shows the total earnings of all the factors of production (in this case, only labour and capital). This equals the right-hand side , which is simply Q, or the total output. So we see that if constant returns to scale prevail (which would, if the production function is homogeneous of degree one), then the total earnings of the factors of production is equal to the total output produced. Thus nothing is left over and there are no extra-normal profits. The product is totally exhausted. This is the famous product-exhaustion theorem which says that if there are constant returns to scale and if each factor is paid its marginal product, then the payments to the factors exhausts the entire output. Of course this holds only in the case of constant returns to scale.

Check Your Progress 3

1) Suppose you have a utility function $U = x^{\alpha} y^{\beta}$. Suppose U is fixed at the level U = k. Find the marginal rate of substitution of the utility function

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2) Consider the production function $Q = L^{\gamma} K^{\eta}$. Determine the degree of homogeneity of this production function.

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3) Show that if there are increasing returns to scale, the production exhaustion theorem will not hold

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2.6 LET US SUM UP

This unit was devoted to presenting some economic applications of the various concepts about multivariate functions and their differentiation. The unit began by presenting some familiar multivariate functions that you come across in economics, like utility function, production function, demand function, money demand function, investment function and so on. Also, the unit soon discussed a couple of examples of level curves in economics, namely, indifference curves and isoquants. The unit then went on to discuss partial derivatives in economics. We found that partial derivatives are ubiquitous in economics. In a multivariate function, whenever we speak of a marginal concept in economics, the corresponding mathematical concept is a partial derivative. Marginal utility, marginal product of a factor, marginal cost, marginal revenue, and marginal propensity to consume are all examples of partial derivatives,

The unit then went on to discuss the application of total differentials and total derivatives in economics. The applications were drawn from utility theory, production theory and macroeconomics. The application of the chain rule with respect to derivative of multivariate functions, where each argument in the original function is itself a function of two or more variables was also discussed with the aid of a constructed example about utility as a function of utility of others, with each utility in turn being functions of consumption of two goods.

After this you were presented with some applications of implicit functions and their differentiation. We saw that level curves are an important type of implicit functions. In economics, using the function for indifference curves and isoquants, we came across the concept of marginal rate of substitution, and the marginal rate of technical substitution. We also came across an example from macroeconomics.

Finally, the unit turned to a discussion of homogeneous functions and the Euler's theorem. We saw the homogeneity in utility functions, demand functions and production functions, where the degree of homogeneity being equal to, less than, or greater than one determines whether the production function displays constant, decreasing, or increasing returns to scale. At the end, the Euler's theorem was applied to demonstrate the product exhaustion theprem in the theory of production.

2.7 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) See section 2.3 and answer.

2) See section 2.3 and answer.

Check Your Progress 2

1) See section 2.4 and answer

2) See section 2.4 and answer

3) See section 2.4 and answer

Check Your Progress 3

1) See section 2.5 and answer.

2) See section 2.5 and answer.

3) See section 2.5 and answer.