Block 2
Production and Cost

UNIT 5 PRODUCTION FUNCTION WITH ONE AND MORE VARIABLE INPUTS

Structure

5.0 OBJECTIVES

After going through this unit, you should be able to:

- understand the concept of production function and its types;
- mathematically comprehend various concepts of production theory introduced in Introductory Microeconomics of Semester 1;
- explain the concepts of homogeneous and homothetic functions along with their properties;
- analyse different types of production functions, *viz.* Linear, Leontief, Cobb-Douglas and CES production function; and
- discuss the impact of technical progress on the production function or an isoquant.

5.1 INTRODUCTION

Production in Economics means creation or addition of value. In production process, economic resources or inputs in the form of raw materials, labour, capital, land, entrepreneur, etc. are combined and transformed into output. In other words, firm uses various inputs/factors, combines them with available technology and transforms them into commodities suitable for satisfying human wants. For example, for making a wooden chair or table, raw materials like wood, iron, rubber, labour time, machine time, etc. are combined in the production process. Similarly, cotton growing in nature needs to be separated from seeds, carded, woven, finished, printed and tailored to give us a dress. All the activities involved in transforming raw cotton into a dress involve existence of some technical relationship between inputs and output.

The present unit is an attempt to build up on the foundation of the Theory of Production you learnt in your Introductory Microeconomics course of Semester 1. Units 6 and 7 of the Introductory Microeconomics course comprehensively discussed Production function with one variable input and with two or more variable inputs, respectively. This theoretical base shall be combined with the mathematical tools you have already learnt in your Mathematical Economics course of Semester 1. Section 5.2 will give a brief review along with the Mathematical comprehension of what we already know about the production theory. Section 5.3 shall explain the concepts of Homogeneous and Homothetic functions along with their properties. Further, in Section 5.4 we will elaborate upon the types of production functions, *viz.* Linear, Leontief, Conn-Douglas and CES production functions. This Unit ends with representation of the impact of technological progress on the production function, along with the Hick's classification of technical progress.

5.2 PRODUCTION FUNCTION

A firm produces output with the help of various combinations of inputs by harnessing available technology. The production function is a technological relationship between physical inputs or factors and physical output of a firm. It is a mathematical relationship between maximum possible amounts of output that can be obtained from given amount of inputs or factors of production, given the state of technology. It expresses flow of inputs resulting in flow of output in a specific period of time. It is also determined by the state of technology. Algebraically, production function can be written as:

$$Q = f\ (A, B, C, D, \dots)$$

where Q stands for the maximum quantity of output, which can be produced by the inputs represented by A, B, C, D,…, *etc*. where f (.) *represents the technological constraint of the firm*.

5.2.1 Short-run Production Function

A Short run production function is a technical relationship between the maximum amount of output produced and the factors of production, with at least one factor of production kept constant among all the variable factors. A two factor short run production function can be written as:

$$Q = f(L, \bar{K})$$

where, Q stands for output, L for Labour which is a variable factor here, K for Capital, and f (.) represents functional relationship. A bar over letter K indicates that use of capital is kept constant, that is, it is a fixed factor of production. Supply of capital is usually assumed to be inelastic in the short run, but elastic in the long run. This inelasticity of the factor is one of the reasons for it to be considered fixed in the short run. Hence, in the short run, all changes in output come from altering the use of variable factor of production, which is labour here.

Total Product (TP)

Total Product (TP) of a factor is the maximum amount of output (Q) produced at different levels of employment of that factor keeping constant all the other factors of production. Total product of Labour (TP_L) is given by:

$$TP_L = Q = f(L)$$

Average Product (AP)

Average product is the output produced per unit of factor of production, given by:

Average Product of Labour, $AP_L = \frac{Q}{L}$ and Average Product of Capital, $AP_K = \frac{Q}{K}$.

Marginal Product (MP)

Marginal Product (MP) of a factor of production is the change in the total output from a unit change in that factor of production keeping constant all the other factors of production. It is given by: Marginal Product of Labour, $MP_L = \frac{\Delta Q}{\Delta L}$ or $\frac{\partial Q}{\partial L}$ and Marginal Product of Capital, $MP_K = \frac{\Delta Q}{\Delta K}$ or $\frac{\partial Q}{\partial K}$, where Δ stands for "change in" and ∂ denotes partial derivation in case of a function with more than one variable [here we are considering a production function with two factors of production, Q = f (L,K)].

Law of Diminishing Marginal Product

The law of diminishing marginal product says that in the production process as the quantity employed of a variable input increases, keeping constant all the other factors of production, the marginal product of that variable factor may at first rise, but eventually a point will be reached after which the marginal product of that variable input will start falling.

5.2.2 Law of Variable Proportions

Also called the law of non-proportional returns, law of variable proportions is associated with the short-run production function where some factors of production are fixed and some are variable. According to this law, when a variable factor is added more and more to a given quantity of fixed factors in the production process, the total product may initially increase at an increasing rate to reach a maximum point after which the resulting increase in output become smaller and smaller.

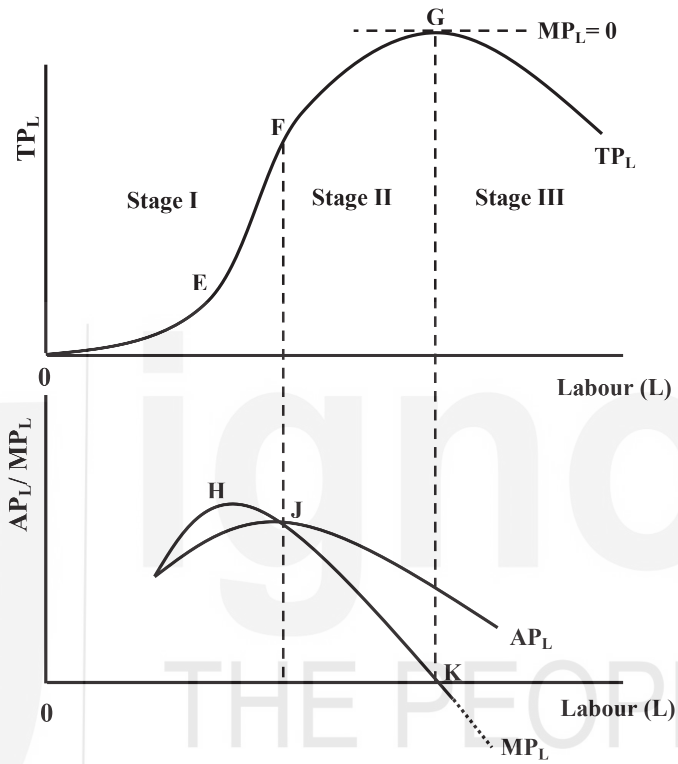

Fig. 5.1: Law of Variable Proportion

Stage 1: This stage begins from origin and ends at point F (in part (a) of the Fig. 5.1). Corresponding to the point F, you may see the AP_L reaches maximum and $AP_L = MP_L$ represented by point J in part (b) of Fig. 5.1. Point E where the total product stops increasing at an increasing rate and starts increasing at diminishing rate is called *point of inflexion*. At point E, TP_L changes its curvature from being convex to concave.

Stage 2: This stage begins from point F and ends at point G (in part (a) of the Fig. 5.1).

Corresponding to the point F, you may see the AP curve reaches its maximum (point J) and both AP and MP curves are having falling segments along with MP reaching 0 i.e., MP curve touches the horizontal axis (at point K). From point F to point G, the total product increases at a diminishing rate, marginal product falls but remains positive. At point K marginal product of the variable factor reduces to zero. Since both the average and marginal products of the variable factor fall continuously, this stage is known as stage of *diminishing returns*.

Stage 3: Beginning from point G, the total product declines and slopes downward. Marginal product of variable factor is negative. Given the fixed factor, the variable factor is too much in proportion and hence this stage is called stage of *negative returns*.

Remember: *

At point H, slope of $MP_L = 0$, *i.e.*, $\frac{dMP_L}{dL} = 0$

Up to point E (the inflection point) $\frac{d^2TP_L}{dL^2} > 0$ (denoted by the convexity of the TP_L), and from point E onwards till point G, $\frac{d^2TP_L}{dL^2} < 0$ (denoted by the concavity of the TP_L).

At point F and J, $MP_L = AP_L$, *i.e.*, $\frac{dQ}{dL} = \frac{Q}{L}$

At point J, slope of $AP_L = 0$, *i.e.*, $\frac{dAP_L}{dL} = 0$

At point G, slope of TP_L, i.e. MP_L or $\frac{dQ}{dL} = 0$

Point K onwards, $MP_L < 0$

Relationship between Average Product and Marginal Product

1) So long as MP curve lies above AP curve, the AP curve is sloping upwards. That is, when MP > AP, AP is rising.
2) When MP curve intersects AP curve, this is the maximum point on the AP curve. That is, when MP = AP, AP reaches its maximum.
3) When MP curve lies below the AP curve, the AP curve slopes downwards. That is, when MP < AP, AP is falling.

Proof: Consider total product of Labour $TP_L = f(L)$, then $AP_L = \frac{f(L)}{L}$.

For maximisation of AP_L differentiating it w.r.t. L and putting it equal to 0,

First order condition (FOC): $\frac{d}{dL}\left(AP_L\right) = 0 \Rightarrow \frac{d}{dL}\left[\frac{f(L)}{L}\right] = 0$

$$\frac{df(L)}{dL} \cdot \frac{1}{L} - \frac{f(L)}{L^2} = 0$$

$$\frac{df(L)}{dL} = \frac{f(L)}{L} \qquad \text{...(1)}$$

$\Rightarrow MP_L = AP_L$ when AP_L reaches its maximum.

Second order condition (SOC): $\frac{d^2(AP_L)}{dL^2} \leq 0 \Rightarrow \frac{d}{dL}\left(\frac{d}{dL}AP_L\right) \leq 0 \Rightarrow \frac{d}{dL}\left(\frac{df(L)}{dL}\right) \leq 0$

Using Equation 1, we get, $\frac{d}{dL}\left(\frac{df(L)}{dL}\right) = \frac{d}{dL}\left(\frac{f(L)}{L}\right) = \frac{f'(L)L - f(L).1}{L^2}$

Thus, $\frac{d^2(AP_L)}{dL^2} = \frac{1}{L}\left[f'(L) - \frac{f(L)}{L}\right] \leq 0 \Rightarrow f'(L) \leq \frac{f(L)}{L} \Rightarrow MP_L \leq AP_L$

* Refer Unit 10 of your course on Mathematical Methods in Economics during first semester (BECC-102) for the derivative criterion for finding maxima, minima, point of inflexion.

5.2.3 Long-run Production Function

In long run, all factors can be varied, thus, for a long-run production function all inputs vary proportionally. Consider a long-run two factor production function:

$$Q = f(L, K)$$

where, Q stands for output, L for Labour and K for Capital (here, K is without bar, that is, it represents a variable factor like L).

The basic assumption we make about the production function is monotonicity, which means as the factor labour (L) increases, given the factor capital (K), the production Q also increases. Similarly as the factor capital (K) increases, given the factor labour (L), the production Q increases. Thus, the first derivative of the production function is positive w.r.t. L and K i.e., $f'(L) > 0$, $f'(K) > 0$. In other words the marginal product of labour and capital are positive. The second assumption we usually make about the production function is with the curvature. The assumption is the concavity i.e., $f''(L) < 0$, $f''(K) < 0$; diminishing returns to the marginal product of L and K. But the second order cross partial derivative i.e.,

$\frac{\partial}{\partial L}\left(\frac{\partial f(K,L)}{\partial K}\right) = \frac{\partial}{\partial K}\left(\frac{\partial f(K,L)}{\partial L}\right) = \frac{\partial^2 f(K,L)}{\partial K \partial L} > 0$. The two second order cross partial derivatives are equal by Young's theorem.

Note that, $\frac{\partial^2 f(K,L)}{\partial L^2} = f''(L)$ and $\frac{\partial^2 f(K,L)}{\partial K^2} = f''(K)$ are the second order own partial derivative w.r.t L and K respectively.

Output Elasticity of a Factor

Given the production function, X = f (L) the elasticity of output with respect to factor (L) is given by the ratio of proportionate change in output (X) to proportionate change in use of factor (L). Output Elasticity of Factor L (e_L) is given by:

$$e_L = \frac{d\,(\log X)}{d\,(\log L)} = \frac{\%\Delta X}{\%\Delta L} = \frac{dX}{dL} \times \frac{L}{X} = \frac{MP_L}{AP_L}$$

Example 1

Consider a production function as follows: $Q = 6K^2L^2 - 0.10K^3L^3$, where Q is the total output produced, and K and L the two factors of production. With factor K fixed at 10 units, determine

a) The total product function for factor L (TP_L)

b) The marginal product function for factor L (MP_L)

c) The average product function for factor L (AP_L)

d) Number of units of input L that maximises TP_L

e) Number of units of input L that maximises MP_L

f) Number of units of input L that maximises AP_L

g) The boundaries for the three stages of production

Solution:

a) $TP_L = 6\,(10)^2\, L^2 - 0.10\,(10)^3\, L^3 = 600L^2 - 100L^3$

b) $MP_L = \frac{d(TP_L)}{dL} = 1200\, L - 300\, L^2$

c) $AP_L = \frac{TP_L}{L}$

$= \frac{600\, L^2 - 100\, L^3}{L} = 600\, L - 100\, L^2$

d) For maximisation of TP_L put $\frac{d(TP_L)}{dL} = 0$

$\Rightarrow 1200\, L - 300\, L^2 = 0$

$\Rightarrow L\,(1200 - 300\, L) = 0 \Rightarrow L = 0$ or $L = 4$

Checking for second order condition for maximisation, i.e., $\frac{d^2(TP_L)}{dL^2} < 0$, we get $L = 4$ that maximises TP_L.

e) Condition for maximisation of MP_L is given by, $\frac{d(MP_L)}{dL} = 0$

$1200 - 600\, L = 0 \Rightarrow L = 2$ maximises MP_L

f) Condition for maximisation of AP_L is given by, $\frac{d(AP_L)}{dL} = 0$

$600 - 200\, L = 0 \Rightarrow L = 3$ maximises AP_L

g) Stage I: Labour units 0 - 3, Begins from origin till the point where AP_L reaches its maximum.

Stage II: Labour units 3 - 4, begins at point where AP_L reaches its maximum till the point when MP_L reduces to 0 with TP_L reaching its maximum.

Stage III: Labour units 4 - ∞, begins at point where $MP_L = 0$ till the point where $MP_L < 0$.

Example 2

The production function of firm is given by $X = 8L + 0.5L^2 - 0.2L^3$ where X is the output produced and L denotes 100 workers.

a) Determine the point at which $MP_L = AP_L$

b) Find the range over which production function exhibits the property of diminishing marginal productivity of labour?

c) How many workers should be employed so that MP_L becomes zero?

d) Find TP_L, MP_L and AP_L when the firm employs 150 workers.

Solution:

a) Total product of labour $TP_L = X = 8L + 0.5L^2 - 0.2L^3$

$MP_L = \frac{dX}{dL} = 8 + L - 0.6L^2$

$AP_L = \frac{X}{L} = 8 + 0.5L - 0.2L^2$

The point at which $MP_L = AP_L$ is given by

$8 + L - 0.6L^2 = 8 + 0.5L - 0.2L^2$

$0.5L - 0.4L^2 = 0$

$L (0.5 - 0.4L) = 0$

Either $L = 0$ or $L = \frac{0.5}{0.4} = 1.25$ or 125 workers

Hence the point at which $MP_L = AP_L$ is L = 125 workers

b) The production function exhibits diminishing marginal productivity of labour over range where $\frac{dMP_L}{dL} < 0$

That is, $1 - 1.2L < 0$ or $L > \frac{1}{1.2}$ $\Rightarrow L > 0.83$ or 83 workers.

Hence the range over which production function exhibits the property of diminishing marginal product of labour is L > 0.83 or more than 83 workers.

c) The number of workers for MP_L to become zero is given by solution

$$8 + L - 0.6L^2 = 0$$

or $$0.6L^2 - L - 8 = 0$$

Therefore, $L = \frac{1 + \sqrt{1+19.2}}{2 \times 0.6}$ = 4.58 (approx). The other root being negative is neglected. Thus when $MP_L = 0$ about 458 workers are being employed.

d) When firm employs 150 workers, L = 1.5

Substituting this value in TP_L, MP_L and AP_L we get

$TP_L = 8L + 0.5L^2 - 0.2L^3$

$= 8 \times 1.5 + 0.5 \times (1.5)^2 - 0.2\,(1.5)^3 = 12.45$ units

$MP_L = 8 + L - 0.6L^2$

$= 8 + 1.5 - 0.6(1.5)^2 = 8.15$ units

$AP_L = 8 + 0.5L - 0.2L^2$

$= 8 + 0.5 \times 1.5 - 0.2(1.5)^2 = 8.3$ units

5.2.4 Isoquants

Isoquants is the locus of all possible input combinations which are capable of producing the same level of output (Q). In Fig. 5.2, all the possible combinations of Labour (L) and Capital (K), for instance (L_1,K_1), (L_2,K_2) and (L_3,K_3), produce a constant level of Output (Q).

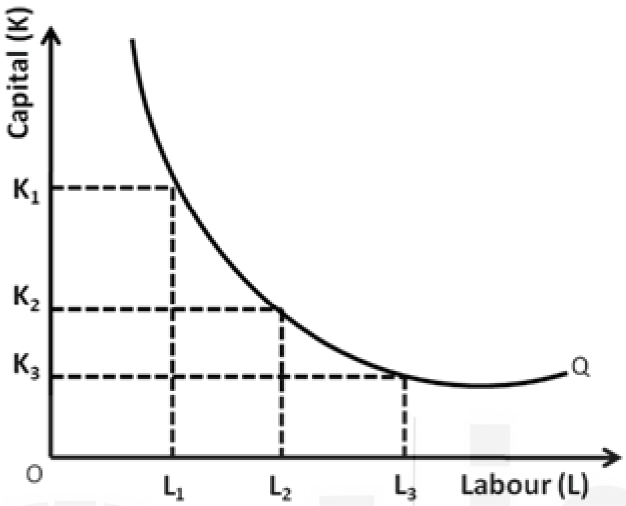

Fig. 5.2: Isoquant

Properties of Isoquants

1) Isoquants are negatively sloped.
2) A higher isoquant represents a higher output.
3) No two isoquants intersect each other.
4) Isoquants are convex to the origin. The convexity of isoquant curves implies diminishing returns to a variable factor.

Isoquant Map

An Isoquant map is a family of isoquant curves, where each curve represents a specified output level. Three such curves with different output levels (Q_1, Q_2 and Q_3) forming an Isoquant map is given in Fig. 5.3.

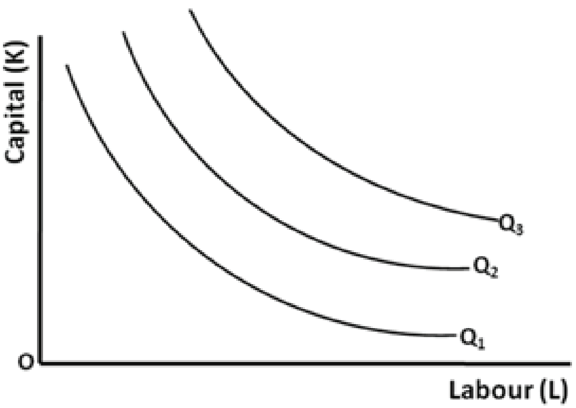

Fig. 5.3: Isoquant Map

Isocost Line

An Isocost line represents various combinations of two inputs that may be employed by a firm in the production process for a given amount of Budget and prices of the factors. Slope of an Isocost line is given by $\frac{w}{r}$, that is, the factor price ratio, where, w and r represent the prices paid to Labour and Capital factors, respectively. Refer Fig. 5.4, where we have three different isocost lines with different budget outlays represented by C_1, C_2 and C_3, such that $C_3 > C_2 > C_1$.

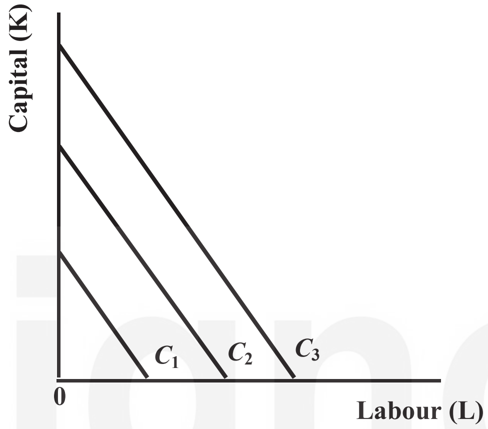

Fig. 5.4: Isocost Lines

5.2.5 Marginal Rate of Technical Substitution

Marginal rate of technical substitution (MRTS) is the rate at which one factor can be substituted for another along an Isoquant. Along an Isoquant output remains constant, *i.e.* (dQ = 0)

$$\Delta K.MP_K + \Delta L.MP_L = 0$$

$$\Delta K.MP_K = \Delta L.MP_L$$

$$\frac{\Delta K}{\Delta L} = \frac{MP_L}{MP_K}$$

$$MRTS_{LK} = \frac{MP_L}{MP_K}$$

As quantity of labour is increased and quantity of capital employed is reduced, the amount of capital that is required to be replaced by an additional unit of labour so as to keep the output constant will diminish.

5.2.6 Producer's Equilibrium

A rational producer attempts to maximise his profits either by maximising the production of output for a given level of cost of production or by minimising the cost of production of a given level of output. Either way the producer chooses, it will result in employment of an optimum combination of resources in the production process so that $MRTS_{LK}$ equals the price ratio of the factors. That is, producer's equilibrium is given by the condition:

$$MRTS_{LK} = \frac{w}{r} \Rightarrow \frac{MP_L}{MP_K} = \frac{w}{r}$$

The equilibrium condition represents the tangency between the isoquant and the isocost line. In Fig. 5.5, at point E, $MRTS_{LK} = \frac{w}{r}$.

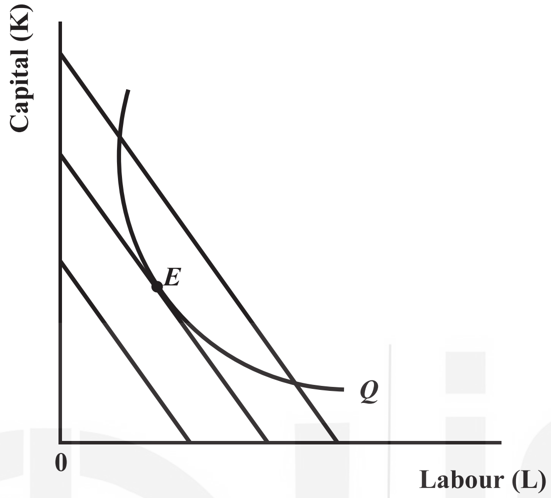

Fig. 5.5: Producer's Equilibrium

5.2.7 Elasticity of Technical Substitution

Elasticity of Technical substitution in production is a measure of how easy it is to shift between factors in the production process. It is given by :

$$\sigma = \frac{\text{Proportionate change in ratio of inputs (K \& L) used}}{\text{Proportionate change in marginal rate of technical substitution of L for K}}$$

$$= \frac{\text{Proportionate change in K/L}}{\text{Proportionate change in } MRTS_{LK}}$$

$$= \frac{\frac{\Delta K/L}{K/L}}{\Delta MRTS_{LK} / MRTS_{LK}}$$

At equilibrium, $MRTS_{LK} = \frac{w}{r}$, therefore we get, $\sigma = \frac{\frac{\Delta K/L}{K/L}}{\frac{\Delta w/r}{w/r}}$

5.2.8 Economic Region of Production

The economic theory focuses on only those combinations of factors which are technically efficient; i.e., where the marginal products of factors are diminishing but positive. These combinations, forming the efficient region of production, are represented by the downward sloping and convex to the origin isoquants. Refer to the following Fig. 5.6 where factors L and K are assumed to be substitutable but not perfectly. When firm goes on substituting L for K, a point like P is reached where $MRTS_{LK}$ given by $\frac{MP_L}{MP_K}$ diminishes to 0 (as $MP_L = 0$ at point P). This implies, at point P, no more K can be given up for having more of L. Beyond point P, as L rises, MP_L becomes negative. In order to produce the fixed output (Q_1), the mismanagement caused by the excessive L units needs to be corrected. This is done by increasing the employment of factor K (since $MP_K > 0$) as L increase beyond point P. This gives us the positively sloped isoquant below ridge line OB. Similarly, at point R, $MP_K = 0$. So, as K increases beyond point R, to make up for the negative MP_K, L would also have to be increased.

Ridge Lines

The ridge line OA is the locus of those points of isoquants where marginal product of capital is zero and ridge line OB is the locus of those points of isoquants where marginal product of labour is zero. See the following Fig. 5.6. A rational producer will operate in the region bound by the two ridge lines called the *economic region of production*. The regions outside the ridge lines are called *regions of economic nonsense* (technically inefficient region).

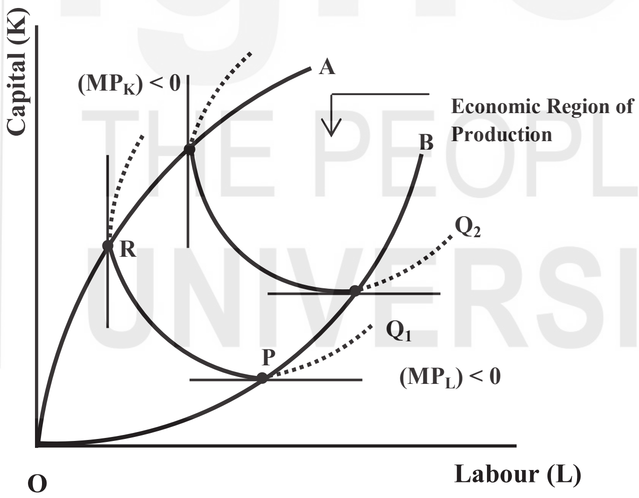

Fig. 5.6: Economic Region of Production

5.3 HOMOGENOUS AND HOMOTHETIC FUNCTIONS

5.3.1 Homogenous Function

A function $f(X_1, X_2,\ldots, X_n)$ is said to be homogenous of degree k if

$$f(mX_1, mX_2,\ldots, mX_n) = m^k f(X_1, X_2,\ldots, X_n)$$

where m is any positive number and k is constant.

A zero-degree homogeneous function is one for which

$$f(mX_1, mX_2, \ldots, mX_n) = m^0 f(X_1, X_2, \ldots, X_n)$$

In a similar way, Homogeneous production function of first degree can be expressed as

$$f(mX, mY) = m^1 f(X, Y)$$

Here X and Y are the two factors of production. It simply says if factors X and Y are increased m times, total production also increases m times.

In case of a Linear Homogeneous production function or Homogeneous production function of first degree with k = 1, if all factors of production are increased in a given proportion, output also increases in the same proportion. This represents the case of constant returns to scale (CRS). When k > 1, production function yields increasing returns to scale (IRS), whereas when k < 1, it yields decreasing returns to scale (DRS).

Euler's Theorem

For a homogenous of degree k function $f(X_1, X_2, \ldots, X_n)$, Euler's theorem gives the following relationship between a homogeneous function and its partial derivatives:

$$X_1 \frac{\partial f}{\partial X_1} + X_2 \frac{\partial f}{\partial X_2} + \ldots + X_n \frac{\partial f}{\partial X_n} = k\, f(X_1, X_2, \ldots, X_n)$$

Properties of Homogenous Functions

1) If $f(X_1, X_2, \ldots, X_n)$ is homogenous of degree k then it's first order partial derivatives will be homogenous of degree (k – 1).

2) For a homogenous of degree k function f (.), if f(X) = f(Y), then f(tX) = f(tY).

   Proof: We have $f(tX) = t^k f(X)$ and $f(tY) = t^k f(Y)$ (1)

   Given, f(X) = f(Y) (2)

   From (1) and (2), we get

   f(tX) = f(tY)

3) Level curves of a homogenous function f(X, Y) have constant slopes along each ray from the origin. That is, if f(X, Y) is a homogeneous production function of degree k, then the MRTS is constant along rays extending from the origin.

Returns to Scale

Returns to scale are a measure of technical property of a production function that examines how output changes subsequent to a proportional change in all the factors of production. If proportional change in output is equal to the proportional change in factors, then there are constant returns to scale (CRS). If proportional change in output is less than the proportional change in factors, there are decreasing returns to scale (DRS), whereas if proportional change in output exceeds the proportional change in factors, there are increasing returns to scale (IRS).

A homogenous function of degree k, exhibits

i) Constant Returns to Scale if $k = 1$

ii) Increasing Returns to Scale if $k > 1$

iii) Decreasing Returns to Scale if $k < 1$

5.3.2 Homothetic Function

A Homothetic function is a monotonic transformation* of a homogeneous function. It is given by the form:

$$H(X_1, X_2, \ldots, X_n) = F[f(X_1, X_2, \ldots, X_n)]$$

where, $f(X_1, X_2, \ldots, X_n)$ represents a homogeneous function of degree k, and F (.) is a monotonically increasing function. In case of such a function, $H(X_1) \geq H(X_2) \Leftrightarrow H(tX_1) \geq H(tX_2)$ for all $t > 0$, where symbol $\Leftrightarrow$ represents "if and only if."

Given a homogeneous function of degree 2, $f(X, Y) = XY$, a homothetic function $H(X, Y)$, which is the monotonic transformation of $f(X, Y)$ could be in the following forms:

$H_1(X, Y) = XY + 2$;

$H_2(X, Y) = (XY)^2$;

$H_3(X, Y) = X^3Y^3 + XY$;

$H_4(X, Y) = \ln X + \ln Y$ (where ln stands for "natural log")

$H_5(X, Y) = e^{XY}$

Important:

A Homogeneous production function implies that it is homothetic as well, but converse is not true, for instance, $f(X,Y) = XY + 1$ is homothetic, but not homogeneous [*proof*: $f(tX, tY) = t^2XY + 1$ and $t^2 f(X,Y) = t^2XY + t^2$, here $f(tX, tY) \neq t^2 f(X,Y)$, hence $f(X,Y) = XY + 1$ is not homogeneous].

Properties of Homothetic Functions

1) Level curves of a Homothetic function are radial expansion of one another, i.e., $H(X_1, Y_1) = H(X_2, Y_2) \Leftrightarrow H(mX_1, mY_1) = H(mX_2, mY_2)$ for all $t > 0$.

2) Level curves of a Homothetic function $H(X, Y)$ have constant slopes along each ray from the origin.

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* **Monotonic transformation—** A transformation of a set that preserves the order of that set. For instance, if the original function is f (X, Y), a monotonic transformation is represented by F [f(X, Y)] so that, if $f_1 > f_2$, then $F(f_1) > F(f_2)$, where F(.) is the strictly increasing function on f(.).

When we have a homothetic production function, the above property implies that Isoquants have a constant slope ($MRTS_{XY}$) along any ray from origin. $MRTS_{XY}$ only depends on factor proportion $\left(\frac{X}{Y}\right)$, that is, it is a homogenous function of degree 0.

Check Your Progress 1

1) As the quantity of a variable input increases, explain why the point where marginal product begins to decline is reached before the point where average product begins to decline. Also explain why the point where average product begins to decline is reached before the point where total output begins to decline?

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2) Given a firm's production function, $X = 50 + 30L - L^2$, where X is the output produced and L the amount of Labour employed, also the Average Revenue function is AR = 1200 – 3X, answer the following:

 a) Find MP_L and the value of L at which $MP_L = 0$

 b) Does the production function show diminishing marginal productivity of labour?

 c) Write down an expression of MRP_L as a function of L and find its value where L = 10.

 d) Is it profitable to employ more or less labourers? Explain.

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3) Which of the following functions is homogenous? Write their degrees of homogeneity.

 a) $\frac{X}{Y}$

 b) $X + Y^2$

 c) $X^{3/4}\,Y^{1/4} + 2X$

 d) $4X^5Y + 5X^4Y^2 - 2X^3Y^3$

 e) $X^3Y + 3X^2Y^2 - 2X^3Y^2$

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5.4 TYPES OF PRODUCTION FUNCTIONS

In this sub-section we are introducing some functions of more than one independent variable which have certain unique mathematical properties. These properties facilitate derivation of certain interesting results which are attractive for economic analysis.

5.4.1 Linear Production Function

A Linear production function is given by the following form:

$$Q = \alpha K + \beta L$$

Where, Q stands for output, K and L, the two inputs in production, α and β, the two constant terms. Production function of this form represents inputs which behave as perfect substitutes to each other in the production process. For this reason MRTS remains constant along an isoquant resulting in a straight line downward sloping Isoquant curves. Refer Fig. 5.7 for Isoquants of a linear production function.

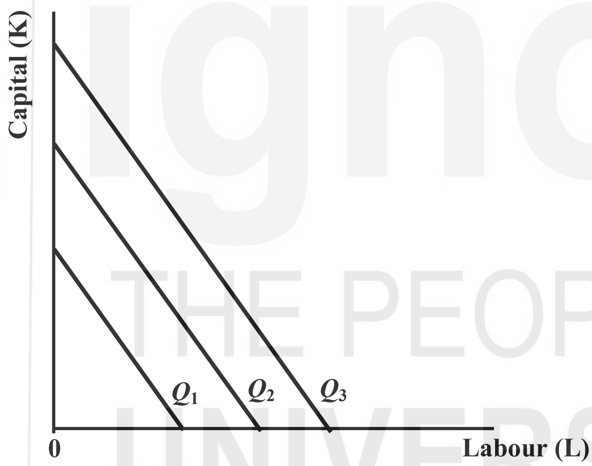

Fig. 5.7: Isoquants for a Linear Production function

$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\beta}{\alpha}$, which is a constant. , Now, Elasticity of Technical Substitution is given by, $\sigma = \frac{\Delta\frac{K}{L}/\frac{K}{L}}{\Delta\,MRTS/MRTS}$. Here along the Isoquant, MRTS remains constant, so that $\Delta\,MRTS = 0$. This implies that Elasticity of Technical Substitution$(\sigma) = \infty$ for a linear production function. That is, inputs are perfectly substitutable for each other in the production process.

5.4.2 Leontief Production Function

Production technology sometimes could be such that factors of production must be employed in a fixed proportion. For instance, to produce a unit of output, capital and labour must be employed in proportion of 2:1, so that no output increase could be possible by increasing the units of capital alone or of labour alone, or of both in a different ratio than 2:1. Leontief production technology represents the case where inputs must be combined in fixed proportions, for this reason it is also called a Fixed-proportions production function. The Leontief production function is given as the following form:

$$Q = \min\left(\frac{K}{\theta_K}, \frac{L}{\theta_L}\right)$$

Where Q is the output produced, K and L represent the factors of production, θ_K and θ_L are the unit input requirements. That is, to produce a single unit of output, θ_K unit of factor K and θ_L units of factor L are needed. Consequently for Q units of output, $\theta_K Q$ units of factor K and $\theta_L Q$ units of factor L will be needed. Thus, the fixed proportion of factors to produce output is given by $\frac{K}{L} = \frac{\theta_K}{\theta_L}$. Factors of such production function behave as Perfect compliments to each other in the production process. Refer Fig. 5.8 for L-shaped Isoquants of a Leontief production function.

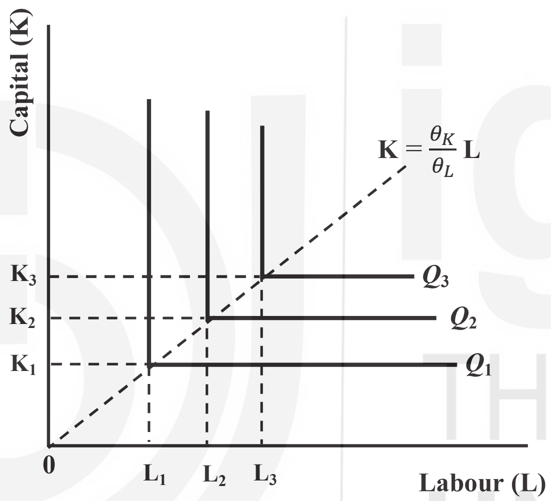

Fig. 5.8: Isoquants for a Leontief Production function

In Fig. 5.8 you may notice, if K = K_1 and L = L_2, then we have $\frac{K_1}{\theta_K} < \frac{L_2}{\theta_L}$, thus ,

$Q = \frac{K_1}{\theta_K}$. In this case, the technically efficient level of L factor would be given by, $\frac{K_1}{\theta_K} = \frac{L}{\theta_L} \implies L = \frac{\theta_L}{\theta_K} K_1$, which is, as you may notice from the figure is given by L_1. The equation for the line from the origin, at which factor proportion equals $\frac{\theta_K}{\theta_L}$, is given by $K = \frac{\theta_K}{\theta_L} L$.

Since there exist no possibility for altering the factor proportion, any change in MRTS, does not result in change in factor proportion which remains fixed. That is $\Delta\left(\frac{K}{L}\right) = 0$. This, implies that Elasticity of Technical Substitution(σ) = 0 for a Leontief production function.

5.4.3 Cobb-Douglas Production Function

A widely used form of production function is the Cobb-Douglas production function which takes the following form:

$$Q = AL^{\alpha}K^{\beta}$$

where Q is the output, L and K the factors of production, and A, α, β are all positive constants. The Isoquants of this production function are hyperbolic, asymptotic to both the axis (i.e. it never touches any axis). Refer Fig. 5.9.

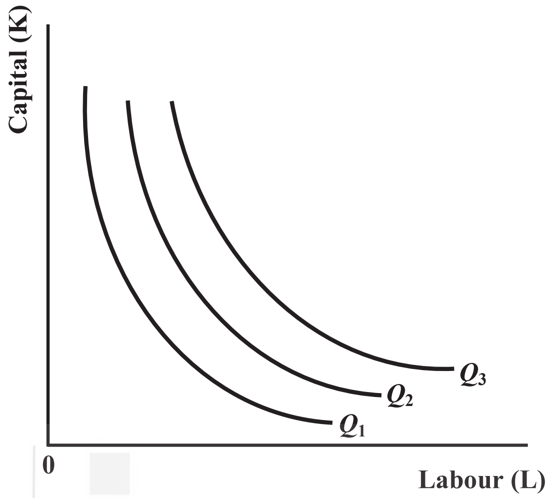

Fig. 5.9: Isoquants for a Cobb-Douglas production function

Some properties of a Cobb-Douglas production function are as follows:

Returns to Scale

For the Cobb-Douglas production function $Q = AL^{\alpha}K^{\beta}$, when

$\alpha + \beta = 1$ there are constants returns to scale (CRS)

$\alpha + \beta > 1$ there are increasing returns to scale (IRS)

$\alpha + \beta < 1$ there are decreasing returns to scale (DRS)

Average and Marginal Product of Factors

For a Cobb-Douglas Production Function $Q = AL^{\alpha}K^{\beta}$

Average Product of Labour, $AP_L = \dfrac{AL^{\alpha}K^{\beta}}{L}$

$$= AL^{\alpha-1}K^{\beta}$$

Average Product of Capital, $AP_K = \dfrac{AL^{\alpha}K^{\beta}}{K}$

$$= AL^{\alpha}K^{\beta-1}$$

Marginal Product of Capital, $MP_k = \dfrac{\partial Q}{\partial K}$

$$= \beta AL^{\alpha}K^{\beta-1}$$

Marginal Product of Labour, $MP_L = \dfrac{\partial Q}{\partial L}$

$$= \alpha A L^{\alpha-1} K^{\beta}$$

Marginal Rate of Technical Substitution (MRTS)

Now, we have $MP_L = \alpha A L^{\alpha-1} K^{\beta}$ and $MP_K = \beta A L^{\alpha} K^{\beta-1}$

$$\text{MRTS} = \frac{MP_L}{MP_K} = \frac{\alpha A L^{\alpha-1} K^{\beta}}{\beta A L^{\alpha} K^{\beta-1}} = \frac{\alpha}{\beta}\frac{K}{L}$$

Product Exhaustion Theorem

In case of a homogeneous production function of degree one, if each unit of every factor of production is given a reward equal to the marginal product of that factor, the total output will be exactly divided among those factors. This is what is referred to as the Product Exhaustion theorem.

Consider a CRS Cobb Douglas production function, $Q = A.L^{\alpha}K^{\beta}$, where $\alpha + \beta = 1$.

Now, $MP_L = A.\alpha\left(\frac{K}{L}\right)^{\beta}$ and $MP_K = A.\beta\left(\frac{L}{K}\right)^{\alpha}$

According to Euler Theorem, if production function is homogeneous of first degree, then

Total Output (Q) = L. MP_L + K.MP_K

$$Q = L.A\alpha\left(\frac{K}{L}\right)^{\beta} + K.A.\beta\left(\frac{L}{K}\right)^{\alpha}$$

$$Q = A.\alpha L^{1-\beta}K^{\beta} + A.\beta L^{\alpha}K^{1-\alpha}$$

$$= A.(1-\beta)L^{1-\beta}K^{\beta} + A.\beta L^{\alpha}K^{1-\alpha}$$

$$= A.L^{1-\beta}K^{\beta}$$

$$= A.L^{\alpha}K^{\beta}$$

$$= Q$$

Thus in Cobb Douglas production with $\alpha + \beta = 1$ if wage rate = MP_L and rate of return on capital (K) = MP_K, then total output will be exhausted.

Elasticity of Substitution

$$\mathbf{e_s} \text{ or } \boldsymbol{\sigma} = \frac{\text{Proportinate change in } \frac{K}{L} \text{ ratio}}{\text{Proportinate change in } MRTS_{LK}}$$

$$= \frac{d\left(\frac{K}{L}\right)/\frac{K}{L}}{dMRTS_{LK}/MRTS_{LK}}$$

$$= \frac{d\left(\frac{K}{L}\right)/\frac{K}{L}}{d\left(\frac{\alpha K}{\beta L}\right)/\left(\frac{\alpha K}{\beta L}\right)}$$

$$= \frac{d\left(\frac{K}{L}\right)/\frac{K}{L}}{\frac{\alpha}{\beta}.d\left(\frac{K}{L}\right)/\frac{\alpha}{\beta}\left(\frac{K}{L}\right)} = 1$$

Output Elasticity of Factors

Elasticity of Output of Labour = $e_L = \frac{\partial Q}{\partial L}.\frac{L}{Q} = \frac{\frac{\partial Q}{\partial L}}{\frac{Q}{L}}$

$$= \frac{MP_L}{AP_L} = \frac{\alpha A L^{\alpha-1}K^{\beta}}{AL^{\alpha-1}K^{\beta}}$$

$$e_L = \alpha$$

Elasticity of Output of Capital = $e_K = \frac{\partial Q}{\partial K}.\frac{K}{Q} = \frac{MP_K}{AP_K}$

$$e_K = \frac{\beta A L^{\alpha}K^{\beta-1}}{AL^{\alpha}K^{\beta-1}}$$

$$e_K = \beta$$

Example 3

Consider the Cobb-Douglas production function below:

$Q = 10L^{0.45}K^{0.30}$

Where Q is the output produced using factors L (Labour) and K (Capital). Calculate

a) Output Elasticities for Labour and Capital.

b) Change in Output, when Labour increases by 15%

c) Change in output, when both Labour and Capital increase by 15%

Solution:

a) Elasticity of output with respect to Labour, $e_L = \frac{\partial Q}{\partial L}.\frac{L}{Q} = \frac{MP_L}{AP_L}$

$$= \frac{10\times 0.45L^{-0.55}K^{0.30}}{10\times L^{-0.55}K^{0.30}}$$

$= 0.45$

Elasticity of output with respect to Capital, $e_K = \frac{\partial Q}{\partial K} . \frac{K}{Q} = \frac{MP_K}{AP_K}$

$$= \frac{10 \times 0.30 L^{0.45} K^{-0.70}}{10 \times L^{0.45} K^{-0.70}}$$

$$= 0.30$$

b) $e_L = \frac{\%\Delta Q}{\%\Delta L}$

We have $e_L = 0.45$ and $\%\Delta L = 15$, therefore $\%\Delta Q = e_L \times \%\Delta L = 6.75$. Hence, output will increase by 6.75%.

c) Change in Output when Labour increase by 15% = 6.75% (calculated in part b)

Similarly, change in Output when Capital increases by 15% = $e_K \times \%\Delta K$ = 4.5%

Therefore, Change in Output when both Capital and Labour increase by 15% = (6.75 + 4.5)% = 11.25%

Example 4

Suppose a commodity (Q) is produced with two inputs, labour (L) and capital (K) and production function is given by $Q = 10\sqrt{LK}$. What type of returns to scale does it exhibit?

Solution:

The above production function can be rewritten as $Q = 10.L^{1/2}K^{1/2}$

Now, increase both the factors by a positive constant λ.

$$Q' = 10(\lambda L)^{1/2}(\lambda K)^{1/2}$$

$$= \lambda^{1/2}\lambda^{1/2}10L^{1/2}K^{1/2}$$

$$Q' = \lambda Q$$

Increasing L and K by λ results in increase in output (Q) by λ. Hence this shows constant returns to scale.

5.4.4 The CES Production Function

Linear, Leontief and Cobb-Douglas production functions are a special case of the Constant Elasticity of Substitution (CES) production function, which has been jointly developed by Arrow, Chenery, Minhas and Solow. CES production function is a general production function wherein elasticity of factor substitution can take any positive constant value. The function is given by the following equation:

$$Q = C\,[\alpha K^{\rho} + (1 - \alpha)L^{\rho}]^{1/\rho}$$

Where, Q stands for output.

'C' is an efficiency parameter, a measure of technical progress. The value of C > 0 and any change in it resulting from technological or organisational change causes shift in the production function.

'α' is a distribution parameter, determining factor shares and $0 \leq \alpha \leq 1$. It indicates relative importance of capital (K) and labour (L) in various production processes.

ρ is a substitution parameter, used to derive elasticity of substitution (σ) between factors K and L, given by $\sigma = \frac{1}{1-\rho}$. The value of ρ is less than or equal to 1 and can be $-\infty$. The two extreme cases are when $\rho \rightarrow 1$ or $\rho \rightarrow -\infty$.

i) When $\rho \rightarrow 1$, the elasticity of substitution tends towards ∞, the case representing Linear Production function where factors are perfect substitutes to each other in the production process giving straight line Isoquants.

ii) When $\rho \rightarrow -\infty$, the elasticity of substitution tends towards 0, the case representing Leontief Production function where factors are perfect compliments to each other in the production process giving L-shaped Isoquants.

iii) When $\rho = 0$, the elasticity of substitution = 1, then CES production function becomes a Cobb-Douglas production function giving convex Isoquants.

CES Production function are extensively used by economists in the empirical studies of production processes because it permits the determination of the value of elasticity of factor-substitution from the data itself rather than prior fixing of the value of substitution elasticity (σ).

Check Your Progress 2

1) Consider the following production function:

$$Q = L^{0.75}K^{0.25}$$

a) Find the marginal product of labour, and marginal product of capital.

b) Show that the law of diminishing returns to the variable factor holds.

c) Show that if labour and capital are paid rewards equal to their marginal products, total product would be exhausted.

d) Calculate the marginal rate of technical substitution of capital for labour.

e) Find out the elasticity of substitution.

f) Show that the function observes constant returns to scale.

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2) Consider the following CES production function:

$$Q = [\alpha L^{\rho} + (1 - \alpha)K^{\rho}]^{1/\rho}$$

Where $0 \leq \alpha \leq 1$, $\rho \leq 1$, Q is output, L and K, the two factors of production.

a) Find marginal productivities of both the factors L and K.

b) Also give the expression for $MRTS_{LK}$.

c) Is this function Homothetic?

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5.5 TECHNOLOGICAL PROGRESS AND THE PRODUCTION FUNCTION

Technological progress has been one of the major forces behind Economic growth overtime. It enables output to rise even when the factors of production remain at a constant level. Technical progress could be shown with an upward shift of the production function (Refer Fig. 5.10 A, where with same level of Labour L_1, more of output could be produced, $X_1 > X$) or a downward movement of the production isoquant (Refer Fig. 5.10 B, where the same level of output X could be produced by fewer quantities of factors of production K_1 and L_1, with $K_1 < K_0$ and $L_1 < L_0$).

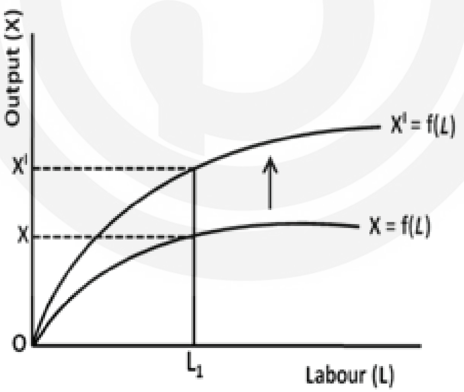

Fig. 5.10A: Technological change and production function

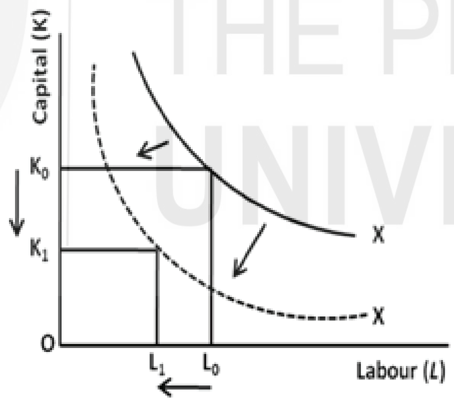

Fig. 5.10B: Technological change and Isoquant

5.5.1 Hicks Classification of Technological Progress

Hicks had distinguished three types of technical progress depending on its effect on rate of substitution of factors of production. They are as follows:

Capital Deepening Technical Progress

Technical progress is said to be capital-deepening (or Labour saving) when shifted Isoquant due to technical progress has lower $MRTS_{LK}$ at the equilibrium points. This results as MP_K increases more than MP_L. It simply means that the technical progress has resulted in increasing capital per worker or capital intensity in the economy. Refer Fig. 5.11, where as we move closer to the origin, $MRTS_{LK}$ falls along the equilibrium points.

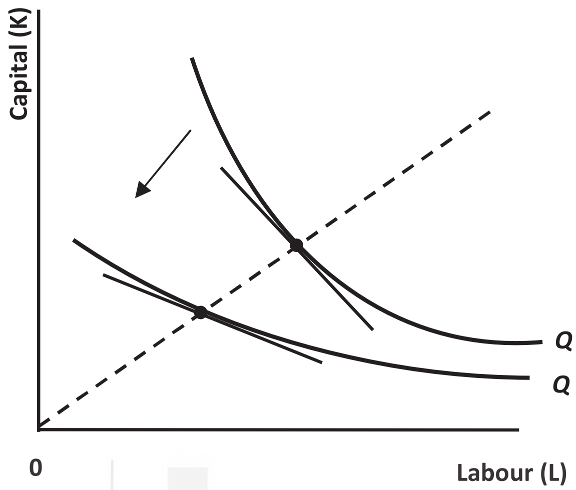

Fig. 5.11: Capital Deepening Technical Progress

Labour Deepening Technical Progress

Technical progress is said to be Labour-deepening (or Capital saving) when shifted Isoquant due to technical progress has higher $MRTS_{LK}$ at the equilibrium points. This results as MP_L increases more than MP_K. This results when technical progress decreases capital per worker or capital intensity in the economy. Refer Fig. 5.12, where as we move closer to the origin, $MRTS_{LK}$ rises along the equilibrium points.

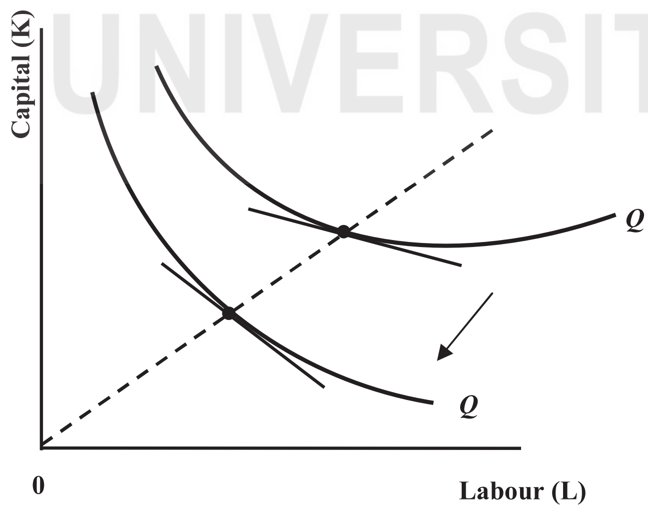

Fig. 5.12: Labour Deepening Technical Progress

Neutral Technical Progress

Technical progress is said to be neutral when for a shifted Isoquant due to technical progress $MRTS_{LK}$ does not change at the equilibrium points. Here, MP_K and MP_L both increase at same proportion. This is represented in

Fig. 5.13, where as we move closer to the origin, $MRTS_{LK}$ remains constant along the equilibrium points.

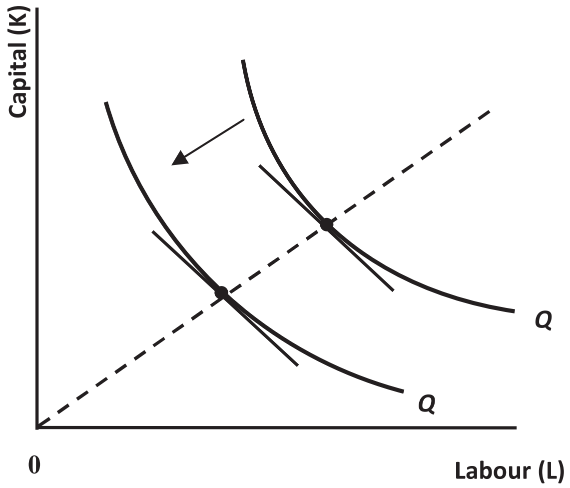

Fig. 5.13: Neutral Technical Progress

5.6 LET US SUM UP

A production function is a technological relationship between inputs and the output in a production process. The unit began with defining a production function, along with its two types— the short run and the long run production function. A short run production function is a technical relationship between output and inputs with at least one fixed input, whereas a long run function is a relationship between output and inputs with all inputs being variable. Unit proceeded with giving a brief review of the concepts comprehensively covered in Introductory Microeconomics course of Semester 1. In connection with the theory, the present unit discussed Mathematical treatment of the concepts like Total Product, Marginal Product, Average Product, Law of Variable Proportion, Output Elasticity of a factor, Marginal rate of technical substitution, Elasticity of Substitution, Producer's equilibrium, etc. Followed by this, the concepts of Homogeneous and Homothetic functions were touched upon. For a homogenous function, we learnt that the value of the function when all its arguments are multiplied by positive number m equals m^k times the value of the function with its original arguments. Monotonic transformation of this function is the Homothetic function. Thus, all the Homogeneous functions are Homothetic but converse is not true.

Various types of production functions, *viz.* Linear (Perfect Substitutes), Leontief (Perfect Compliments), Cobb Douglas, and CES production function were subsequently discussed. Text explained how a CES production function as a general function approaches a Leontief or a Linear production function for different values of ρ, which is referred to as the substitution parameter with the relation given by, $\sigma = \frac{1}{1-\rho}$, where σ represents elasticity of substitution between two factors of production. Unit concluded with a brief discussion about the impact of technological improvement on production function or an isoquant.

5.7 REFERENCES

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5.8 ANSWERS OR HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) Due to the operation of diminishing marginal returns, marginal product begins to decline at some point, but for some range though diminishing it remains greater than average product. Only when in its diminishing phase marginal product becomes less than average product, average product starts declining. That is why marginal product curve cuts the average product curve at the latter's highest point.

   Marginal product continues to diminish after it is equal to the maximum average product but remains positive which causes the total output to continue increasing. Only when marginal product becomes zero, the total product reaches its maximum level. As a result, total output continues to increase even after the maximum average product point and begins to decline only when marginal product becomes negative.

2) a) $MP_L = 30 - 2L$

   MP_L is 0 at $L = 15$

   b) $\frac{d\,(MP_L)}{dL} = -2 < 0$

   Yes, the production function shows diminishing marginal productivity of labour.

   c) $-12L^3 + 540L^2 - 7200L + 27000; -3000$

Hint: $TR = AR \times X = (1200 - 3X).\ X = 1200X - 3X^2$

Now, $MRP_L = MR\ .\ MP_L$

$$= \frac{d\,(TR)}{dL} \cdot \frac{dX}{dL}$$

$= (1200 - 6X) . (30 - 2L)$

$= [1200 - 6 \times (50 + 30L - L^2)]\ (30 - 2L)$

$= -12L^3 + 540L^2 - 7200L + 27000$

d) Since MRP_L is negative it is not profitable to employ more labourers rather it would be profitable to employ less labourers.

3. a) Yes, degree 0

b) No

c) Yes, degree 1

d) Yes, degree 6

e) No

Check Your Progress 2

1) a) $MP_L = 0.75\left(\frac{K}{L}\right)^{0.25}$; $MP_K = 0.25\left(\frac{L}{K}\right)^{0.75}$

b) Law of diminishing returns to variable factor would hold when $\frac{\partial MP_L}{\partial L} < 0$ and $\frac{\partial MP_K}{\partial K} < 0$, that is, when marginal product of a factor declines with increase in employment of that factor keeping constant the employment of other factor of production.

Since $\frac{\partial MP_L}{\partial L} = -0.25 \times 0.75\left(\frac{K^{0.25}}{L^{1.25}}\right) < 0$ and $\frac{\partial MP_K}{\partial K} = -0.75 \times 0.25\left(\frac{L^{0.75}}{K^{1.75}}\right) < 0$, therefore here law of diminishing returns to variable factor holds true.

c) To show $Q = MP_L . L + MP_K . K$ (the Product Exhaustion theorem)

Consider R.H.S, $MP_L . L + MP_K . K \Longrightarrow 0.75\left(\frac{K}{L}\right)^{0.25} . L + 0.25\left(\frac{L}{K}\right)^{0.75} . K$

$\Longrightarrow 0.75K^{0.25}L^{0.75} + 0.25K^{0.25}L^{0.75}$

$\Longrightarrow K^{0.25}L^{0.75}\ (0.75 + 0.25)$

$\Longrightarrow L^{0.75}K^{0.25}$, which equals Q (the L.H.S).

d) $MRTS_{LK} = 3\left(\frac{K}{L}\right)$

e) $\sigma = 1$

Hint: $\sigma = \dfrac{\Delta \frac{K}{L} \Big/ \frac{K}{L}}{\Delta MRTS / MRTS}$

Substituting value of $MRTS = 3\left(\frac{K}{L}\right)$

Elasticity of substitution = $\dfrac{\Delta K / L/K / L}{3\Delta\left(\frac{K}{L}\right)\Big/3\left(\frac{K}{L}\right)} = 1$

2) a) $MP_L = \alpha L^{\rho-1}\ [\alpha L^{\rho} + (1-\alpha)K^{\rho}\ ]^{\ 1/\rho-1}$

Hint: $MP_L = \left(\frac{1}{\rho}\right)\ [\alpha L^{\rho} + (1-\alpha)K^{\rho}\ ]^{\ 1/\rho-1}\ \rho\ \alpha L^{\rho-1} \Longrightarrow \alpha L^{\rho-1}\ [\alpha L^{\rho} + (1-\alpha)K^{\rho}\ ]^{\ 1/\rho-1}$

$MP_K = (1-\alpha)\ K^{\rho-1}\ [\alpha L^{\rho} + (1-\alpha)K^{\rho}\ ]^{\ 1/\rho-1}$

b) $MRTS_{LK} = \frac{\alpha}{1-\alpha}\left(\frac{L}{K}\right)^{\rho-1}$

c) Yes, the function is Homothetic since $MRTS_{LK}$ depends upon factor proportion $\left(\frac{L}{K}\right)$.

UNIT 6 COST FUNCTION

Structure

6.0 OBJECTIVES

After going through this unit, you should be able to:

- state the concept of cost minimisation;
- graphically and analytically approach the problem of cost minimisation;
- explain and derive conditional factor demand functions as a solution of the constrained optimisation problem of cost minimisation;
- subsequently derive the cost function as a function of factor prices and output;
- analyse average cost and marginal cost functions, along with the relationship between them; and
- discuss the concept of short-run and the long-run cost functions.

6.1 INTRODUCTION

A production activity is undertaken for earning profits, and the producer decides how much input to use to minimise its costs and maximise its profits. Profits are given by the difference between the revenue earnings from and the costs incurred during the production process. Costs, be it implicit or explicit, are the expenses incurred by the producer for undertaking the production of goods or services. Explicit costs are the out of pocket expenses which the producer makes payment for, like paying for raw materials, salaries and wages of staff employed, packaging and distribution expenses, etc. On the other hand, by implicit it simply means the implied or the opportunity cost of the self-owned inputs used by the producer in the production process, like opportunity cost of entrepreneurial skills of the entrepreneur, self-owned building used as office for business operations, etc. Economic profits are calculated using both, the explicit as well as the implicit costs. The optimal output of the firm is decided by maximisation of profits or by minimisation of costs incurred. The present unit is an attempt to analyse the approach of cost minimisation.

Unit begins with explaining the concept of cost minimisation. It proceeds with the sub-sections discussing the graphical and the analytical approach for cost minimisation. Subsequently, the concept of conditional input/factor demand functions will be introduced and plugging these optimal values, cost function will be derived. We will then derive the algebraic expression of average cost and the marginal cost functions from the cost functions. The unit also covers a mathematical proof of the relationship between the AC and the MC curve, which was already covered in Introductory Microeconomics course of Semester 1 (BECC-101). Towards the end, the concepts of variable and fixed factors of production, and consequently the short-run and the long-run cost functions have been discussed.

6.2 COST MINIMISATION

By cost minimisation it simply means to produce a specified output at the minimum cost. This in turn results from employment of a mix of factors or inputs so that the desired level of output is produced at the least cost. Consider a production function given by Q = f (K, L), where Q is the output produced by employing inputs K and L at given per unit factor prices r and w, respectively. Then total cost of producing a specified output Q will be given by:

$$C = Lw + Kr$$

Now cost minimisation would result in the following constrained optimisation problem:

$$\text{Min} \quad Lw + Kr$$
$$\text{s.t.} \quad Q = f(L, K)$$

Here, the problems entails finding out the cheapest way to produce a given level of output (Q) by a firm employing inputs K and L at the given inputs prices and a technology relationship f (L, K). The optimal solution of the above constrained minimisation problem will be given by (L*, K*) such that for all (L, K) satisfying Q = f (L, K), we will have $L^*w + K^*r \leq Lw + Kr$.

6.2.1 Graphical Approach for Cost Minimisation

Recall the concept of Producer's equilibrium we discussed in Unit 7 of your Introductory Microeconomics course of Semester 1 (BECC-101). A producer attains equilibrium by minimising the cost of producing output. This in turn involves employing a particular factor combination at the given factor prices and the input-output technological relationship. Fig. 6.1 represents such an optimal factor combination.

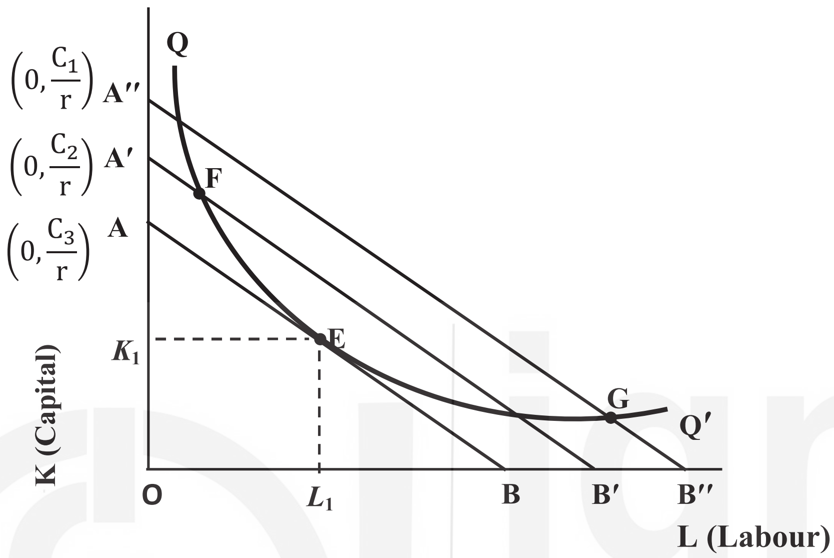

Fig. 6.1: Cost minimisation combination of factors

Lines AB, A'B' and A''B'' represent *Isocost lines*. An isocost line is a locus of various combinations of factor inputs (here K and L) that yield the same total cost (C) for the firm. The equation of the isocost line is given by,

$$C = Lw + Kr \Longrightarrow K = \frac{C}{r} - \frac{w}{r}L$$

This is a linear equation with slope $\frac{w}{r}$, a constant measuring the cost of one factor of production in terms of the other factor. For different values of C in the above equation, we get different isocost lines. In Fig. 6.1, A''B'' representing total cost C_1 is given by, $K = \frac{C_1}{r} - \frac{w}{r}L$. Similarly, outlays represented by A'B' and AB are C_2 and C_3, respectively. Given the factor prices, a higher outlay results in an outward parallel shift of the isocost line, thus, to the north-east, higher isocost lines correspond to higher levels of cost. In the above figure we have $C_1 > C_2 > C_3$.

Curve QQ' is the *isoquant* giving various combinations of factor inputs that yield the same level of output (Q). Fig. 6.1 shows an isoquant representing a given output level of output (let say Q*). Cost minimisation exercise involves minimising the total cost of producing a given level of output (here Q*). For instance, consider three possible factor combinations denoted by points E, F and G giving different cost of producing output level Q*. Point E provides factor combination producing output Q* at the cost of C_3, while F and G represents factor combinations producing output level Q* at the cost of C_2 and C_1, respectively. Among these possibilities, point E provides the least cost (= C_3) factor combination to the firm, as we know both C_1 and C_2 are higher than C_3 ($C_1 > C_2 > C_3$). Graphically at point E, slope of the isoquant given by the Marginal Rate of Technical Substitution ($MRTS_{LK}$— the rate at which two factors can be substituted with each other in the production of a constant level of output) equals the slope of the isocost line. Point E, the tangency point between isoquant and the isocost line gives us the optimal combination of factors of production, *i.e.* OL_1 amount of labour and OK_1 amount of capital. Symbolically, at E we have,

$$MRTS_{LK} = \frac{w}{r} \text{ or } \frac{MP_L}{MP_K} = \frac{w}{r} \qquad \left(\text{as we know } MRTS_{LK} = \frac{MP_L}{MP_K}\right)$$

6.2.2 Expansion Path

How does the cost-minimising factor combination changes as the output production increases, keeping constant the factor prices?— an expansion path answers such a question. Given factor prices, a firm can determine cost minimising combination of factors for every level of output following the rule $MRTS_{LK} = \frac{w}{r}$, that is the tangency between isoquants and isocost lines. The expansion path is nothing but a locus of such optimal factor combinations as the scale of production expands. In Fig. 6.2, expansion path OE is determining minimum cost combinations of labor (L) and capital (K) at each level of output. As output expands, (represented by higher isoquants $Q_3 > Q_2 > Q_1$) total cost of production rises as well, which is depicted in the figure by parallel rightward shifted isocost lines. The cost minimising factor combination occurs at point E_1, E_2, E_3, the locus of which, is called the expansion path.

Remember

The equation of the expansion path is determined by the cost minimisation rule, $MRTS_{LK} = \frac{w}{r}$.

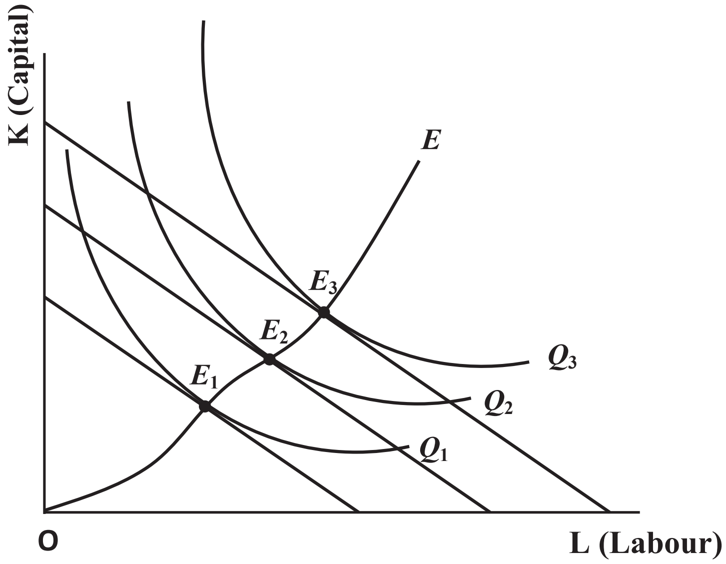

Fig. 6.2: Expansion Path

6.2.3 Analytical Approach for Cost Minimisation

Analytically, optimal combination of factors employed can be ascertained by finding the solution of the following constrained optimisation problem:

$$\text{Min} \quad Lw + Kr$$

$$\text{s.t.} \quad Q^* = f(L, K)$$

where Q* is the stipulated level of output produced.

We proceed solving the above problem by finding the Lagrangian function:

$$\mathcal{L} = Lw + Kr + \lambda\,[Q^* - f(L, K)]$$

The first-order optimisation conditions are:

$$\frac{\partial \mathcal{L}}{\partial L} = 0 \implies w - \lambda\left(\frac{\partial f}{\partial L}\right) = 0 \implies w = \lambda\, MP_L \qquad 1)$$

$$\frac{\partial \mathcal{L}}{\partial K} = 0 \implies r - \lambda\left(\frac{\partial f}{\partial K}\right) = 0 \implies r = \lambda\, MP_K \qquad 2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \implies Q^* - f(L, K) = 0 \implies Q^* = f(L, K) \qquad 3)$$

From (1) and (2) we get

$$\frac{MP_L}{MP_K} = \frac{w}{r} \qquad 4)$$

Same as is given by the tangency of the isocost and the isoquant. Thus, cost minimisation requires equality between the $MRTS_{LK}$ and factor price ratio. Equation (4) can also be written as

$$\frac{MP_L}{w} = \frac{MP_K}{r}$$

The above equation implies that a producer minimises cost of production when marginal output generated by the last monetary unit spent on each factor is equal.

Now, Equations (3) and (4) can be solved to arrive at the solution of the cost-minimisation problem. That is, we solve $\frac{MP_L}{MP_K} = \frac{w}{r}$ and $Q^* = f(L, K)$ to get optimal inputs (L*, K*). L* and K* represent the amount of labour and capital factors needed to be employed at the given prices of w and r respectively, so as to produce output level Q at the minimum cost. This minimum cost will then be given by

$$C = L^*w + K^*r$$

Example 1

Consider the production function Q = $L^{\frac{2}{3}}\ K^{\frac{2}{3}}$ where output Q is produced using factors L and K. Given per unit factor prices of L and K as Rs 10 and Rs. 5, respectively, find the expression for minimum cost of producing output Q.

Solution

We need to find solution (L*, K*) of the following constrained optimisation problem:

$$\text{Min} \quad 10L + 5K$$

$$\text{s.t.} \quad Q = L^{\frac{2}{3}} K^{\frac{2}{3}}$$

From the first order condition of the cost minimisation, we get the following equation:

$$\frac{MP_L}{MP_K} = \frac{w}{r} \Rightarrow \frac{\frac{2}{3}L^{\frac{-1}{3}} K^{\frac{2}{3}}}{\frac{2}{3}L^{\frac{2}{3}} K^{\frac{-1}{3}}} = \frac{10}{5}$$

$\Rightarrow \frac{K}{L} = 2 \Rightarrow K = 2L$, substituting this relation in our production function, we get the two conditional demand functions for input K and L as follows:

$$L^* = \frac{1}{\sqrt{2}} Q^{\frac{3}{4}}$$

$$K^* = \sqrt{2}\, Q^{\frac{3}{4}}$$

Therefore, the cost function is given by

$$C = 10\left(\frac{1}{\sqrt{2}} Q^{\frac{3}{4}}\right) + 5\left(\sqrt{2}\, Q^{\frac{3}{4}}\right)$$

$$= 10\sqrt{2}\, Q^{\frac{3}{4}}$$

Check Your Progress 1

1) Determine the equation for the expansion path of a production function given by $Q = LK^2$, where Q stands for output produced using factors K and L priced at Rs. 10 and Rs. 15, respectively.

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2) Production of a good is represented by the following technological relationship

$$Q = 10\sqrt{KL}$$

Where K and L are the factor inputs and Q is the output. Given per unit price of factor K as Rs. 3 and that of factor L as Rs. 12, what will be the minimum cost of producing 1000 units of this good?

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6.3 CONDITIONAL FACTOR DEMAND FUNCTION

As we saw in Sub-section 6.2.3, the solution of the cost minimisation problem

$$\text{Min} \quad Lw + Kr$$

$$\text{s.t.} \quad Q^* = f(L, K)$$

gives us L* and K* for the given values of w, r and Q*. That is, we arrive at the optimum level of factors employed in the production process for the given output level Q* and factor prices w and r. Any change in any one or all of these parameters will give us different values of L* and K*. Thus, solution to the cost minimisation problem involves finding L* and K* as a function of w, r and Q*. Symbolically, we get

$$L^*(w, r, Q^*)$$

$$K^*(w, r, Q^*)$$

which are known as the conditional factor demand functions as a solution of the constrained cost minimisation problem for the given production function, output level Q* and factor prices as w and r.

You might wonder why the word "conditional" before "factor demand function". The reason for this is— a factor demand function specifies profit maximising levels of factor employment at given unit factor price, when output level is free to be chosen, whereas a conditional factor demand function gives the cost minimising level of input employment at given unit factor prices, to produce a given level of output. That is, factor employment is conditional upon the output level to be produced. So along the conditional input demand function of L or K the output remains constant at Q*. Therefore any increase (or decrease) in Q* will be accompanied by a outward (or inward) shift of conditional demand function of L or K. Conditional input demand function for labour L (or capital K) is always negatively sloped with respect to its own price w (or r in case of capital K). Conditional input demand function captures only the substitution effect (similar concept like compensated or Hicksian demand functions which you have done in Consumer theory) and therefore is less elastic as compared to ordinary input demand function (a function of prices of inputs and output).

6.4 COST FUNCTION

A cost function is derived using production function and factor prices, assuming the rational of minimisation of the cost of production by the producer. On inserting conditional demand functions, i.e. L* (w, r, Q*) and K* (w, r, Q*) in our expression for total cost, C = Lw + Kr, we arrive at the cost function C (w, r, Q*).

$$C(w, r, Q^*) = L^*(w, r, Q^*)\, w + K^*(w, r, Q^*)\, r$$

A function of factor prices and output, the cost function C (w, r, Q*) gives the minimum cost of producing a specific level of output (Q*) for some given factor prices. You may note that optimisation has already been taken care of during construction of conditional factor demand functions, thus cost function gives the optimised solution to the producer regarding how much to employ of a factor in the production process.

Till now we have been considering a production function with two factor inputs (L and K), but in reality there can be more than two inputs in the production process. Let there be "n" factor inputs represented by vector X = (x_1, x_2, x_3,…. x_n) used in the production of output Q, such that our production function becomes, Q = f (x_1, x_2, x_3,…. x_n). Corresponding to factor vector X = (x_1, x_2, x_3,…. x_n), let factor price vector be W = (w_1, w_2, w_3,…., w_n). Then, the conditional factor demand function for factor i (where i = 1, 2,…., n) will be represented by x_i = (W, Q), total cost function by C (W, Q).

6.4.1 Properties of a Cost Function

Let us now discuss some properties of the cost function:

1) Cost function is non-decreasing in factor prices, that is, considering two factor price vectors W' and W, so that W' ≥ W, then C (W', Q) ≥ C (W, Q). Also, the function is strictly increasing in at least one factor price.

2) Cost function is non-decreasing in output. That is, Q' ≥ Q, then C (W, Q') ≥ C (W, Q) for W > 0. That is, increasing production increases the cost of production.

3) Cost function is homogeneous of degree 1 in factor prices. That is, a simultaneous change in all factor prices by a certain proportion (let say by λ, where λ > 0), changes the cost of production by the same proportion (λ). Symbolically,

$$C(\lambda W, Q) = \lambda C(W, Q) \qquad \text{for } W, Q, \lambda > 0$$

   Similarly it can be shown that the conditional factor demand functions (L and K) are homogeneous of degree zero.

4) Cost function is concave in factor prices. Symbolically,

$$C(tW + (1-t)W, Q) \geq tC(W, Q) + (1-t)C(W, Q) \qquad \text{for } t \in [0,1]$$

5) Shephard's Lemma: If a cost function C (W, Q) is differentiable at (W, Q) and $w_i > 0$ for i = 1,2,..,n, then a conditional factor demand function for factor i, that is, x_i (W, Q) is given by

$$x_i(W, Q) = \frac{\partial C(W,Q)}{\partial w_i}$$

   This lemma allows us to obtain conditional factor demand functions as partial derivatives of the cost function.

Example 2

For the production function $Q = L^{\frac{1}{3}} K^{\frac{1}{3}}$, where Q is the output, K and L the production factors with per unit prices as r and w, respectively, determine the cost function. Check the homogeneity condition and Shephard's Lemma.

Solution

Constrained optimisation problem is given by:

$$\text{Min } Lw + Kr$$

$$\text{s.t.} \quad Q^* = L^{\frac{1}{3}} K^{\frac{1}{3}}$$

We solve the above problem by Lagrangian method to arrive at the following condition pertaining to cost minimisation

$$\frac{MP_L}{MP_K} = \frac{w}{r} \Rightarrow \frac{\frac{1}{3}L^{\frac{-2}{3}} K^{\frac{1}{3}}}{\frac{1}{3}L^{\frac{1}{3}} K^{\frac{-2}{3}}} = \frac{w}{r}$$

$\Rightarrow \frac{K}{L} = \frac{w}{r} \Rightarrow K = \frac{w}{r} L$, substituting this relation in our production function, we get our conditional factor demand functions:

$$L^* (w, r, Q^*) = Q^{\frac{3}{2}} \left(\frac{r}{w}\right)^{\frac{1}{2}}$$

$$K^* (w, r, Q^*) = Q^{\frac{3}{2}} \left(\frac{w}{r}\right)^{\frac{1}{2}}$$

Therefore, cost function is given by

$$C (w, r, Q) = L^*(w, r, Q^*)\, w + K^* (w, r, Q^*)\, r$$

$$= w\left[Q^{*\frac{3}{2}} \left(\frac{r}{w}\right)^{\frac{1}{2}}\right] + r\left[Q^{*\frac{3}{2}} \left(\frac{w}{r}\right)^{\frac{1}{2}}\right]$$

$$= 2\sqrt{Q^{*3}wr}$$

In order to check the homogeneity condition, let input prices (w and r) increase by proportion λ. Therefore the cost function becomes:

$$C (\lambda w, \lambda r, Q^*) = 2\sqrt{Q^{*3}(\lambda w)(\lambda r)} = 2\lambda\sqrt{Q^{*3}wr} = \lambda\, C (w, r, Q^*)$$

Hence the cost function is homogeneous of degree one. You may also check the homogeneity condition for the conditional factor demand functions of L and K.

In order to check for Shephard's Lemma we differentiate the cost function with respect to the input price w (and r):

$$\frac{\partial C(w,r,Q^*)}{\partial w} = \frac{\partial 2\sqrt{Q^{*3}wr}}{\partial w} = 2 \times \frac{1}{2\sqrt{Q^{*3}wr}} \times Q^{*3}r = Q^{*\frac{3}{2}}\left(\frac{r}{w}\right)^{\frac{1}{2}} \Longrightarrow \text{Conditional input}$$

demand function for Labour L.

$\frac{\partial C(w,r,Q^*)}{\partial r} = \frac{\partial 2\sqrt{Q^{*3}wr}}{\partial r} = 2 \times \frac{1}{2\sqrt{Q^{*3}wr}} \times Q^{*3}w = Q^{*\frac{3}{2}}\left(\frac{w}{r}\right)^{\frac{1}{2}} \Longrightarrow$ Conditional input demand function for capital K.

6.4.2 Average and Marginal Cost Functions

Average Cost Function

Average cost is defined as the cost per unit of output produced. Average cost function, a function of input vector W and output Q, is derived from the total cost function as follows:

$$AC\ (W, Q) = \frac{C\ (W,Q)}{Q}$$

The function will determine the minimum per unit cost of producing a specific level of output, given the factor prices.

Marginal Cost Function

Marginal cost is the addition to total cost as an additional unit of output is produced. A function of input vector W and output Q, marginal cost function is derived from total cost function as a partial derivative of it with respect to the output:

$$MC\ (W, Q) = \frac{\partial C\ (W,Q)}{\partial Q}$$

It simply determines the minimum addition to the total cost from producing an additional unit of output, given the factor prices.

Example 3

Given a total cost function, $C = 100Q + wrQ^2$, find the average and marginal cost functions.

Solution

Average cost function (AC) = $\frac{C\ (W,Q)}{Q}$

$$= \frac{100Q + wrQ^2}{Q} = 100 + wrQ$$

Marginal cost function (MC) = $\frac{\partial C\ (W,Q)}{\partial Q}$

$$= \frac{\partial\left(100Q + wrQ^2\right)}{\partial Q} = 100 + 2wrQ$$

6.4.3 Relationship between AC and MC Function

We present below the relationship between the AC and the MC function.

We know, AC = $\frac{C\ (W,Q)}{Q}$

Rate of change of AC with respect to Q will be given by,

$$\frac{d}{dQ}\left(\frac{C\,(W,Q)}{Q}\right)$$

$$\Rightarrow \frac{QC\,'(W,Q) - C(W,Q)}{Q^2}$$

$$\Rightarrow \frac{C'(W,Q)}{Q} - \frac{C(W,Q)}{Q^2} \Rightarrow \frac{1}{Q}\left[C'(W,Q) - \frac{C(W,Q)}{Q}\right]$$

$$\Rightarrow \frac{1}{Q}[MC - AC]$$

From the above result it follows that, for Q > 0:

i) Slope of AC curve or rate of change of AC will be positive, that is $\frac{d}{dQ}(AC) > 0$, as long as MC > AC, or in other words, as long as MC curve lies above AC curve.

ii) Slope of AC curve or rate of change of AC will be zero, that is $\frac{d}{dQ}(AC) = 0$, when MC = AC, or in other words when MC curve intersects AC curve. This happens at the minimum point of the AC curve.

iii) Slope of AC curve or rate of change of AC will be negative, that is $\frac{d}{dQ}(AC) < 0$, when MC < AC, or in other words when MC curve lies below AC curve.

Fig. 6.3 represents such a relationship.

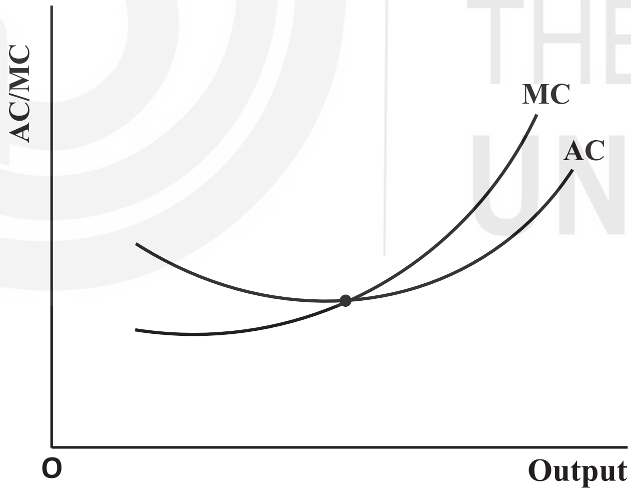

Fig. 6.3: Relationship between AC and MC curve

6.5 SHORT-RUN AND LONG-RUN COST FUNCTIONS

The distinction between the short-run and the long-run is done in terms of the fixed and variable factors of production. Fixed factors of productions are inputs in the production process that do not change with the level of output, generally comprising large capital assets, office building, machinery, etc. On the other hand, variable factors are inputs that change with the level of output. In the short-run firms may face some constraints in expanding or contracting their inputs so there exist some fixed factors at some predetermined levels along with some variable factors, whereas in the long-run all factors become variable. Let us now try to incorporate the distinction of short-run and long-run with the concept of cost function.

Let X_V represent a vector of variable inputs used in the production process, and X_F be the vector of fixed inputs. Similarly, vector W_V and W_F be the price vectors of variable and fixed factors, respectively.

6.5.1 Short-run Cost Function

In the short-run, we just discussed, exist some fixed factors (X_F) which are given, and some variable factors (X_V) which vary with the output level Q. Earlier, in Section 6.4, we derived cost function as a function of given parameters W and Q, now for the short-run cost function, X_F will be considered as another factor which is given in the short-run. Considering presence of both, the variable and the fixed factors in the production process, the short-run cost function will be given by:

$$C_S\,(W, Q, X_F) = W_V X_V\,(W, Q, X_F) + W_F X_F$$

Where, X_V (W, Q, X_F) is the conditional variable factor demand function, which in general depends upon the value of the given fixed factor X_F.

On the basis of the short-run cost function, we can further define the following:

Short-run average cost (SAC) function = $\frac{C_S(W,Q,X_F)}{Q}$

Short-run marginal cost (SMC) function = $\frac{\partial C_S(W,Q,X_F)}{\partial Q}$

Short-run variable cost (SVC) function = $W_V X_V$ (W, Q, X_F)

Short-run average variable cost (SAVC) function = $\frac{W_V X_V(W,Q,X_F)}{Q}$

Short-run fixed cost (SFC) function = $W_F X_F$

Short-run average fixed cost (SAFC) function = $\frac{W_F X_F}{Q}$

Remember:

If Q = 0, SVC = 0 as X_V = 0, but SFC = $W_F X_F$ i.e. when output production reduces to nil, short-run variable cost also reduces to nil as employment of variable inputs reduces to zero, but short-run fixed costs are still needed to be incurred regardless of the output level.

6.5.2 Long-run Cost Function

In the long-run, all factors become variable (X_V) that is they vary with output Q. Since there are no fixed factors (X_F), and all factors are variable (X_V), we can write vector X_V as vector X (our factor vector) with corresponding factor price vector given by W. Then, we get the following long-run cost function:

$$C_L (W, Q) = W X (W, Q)$$

which is the cost function we discussed in Section 6.4. From the above function, we can derive the following:

Long-run average cost (LAC) function = $\frac{C_L(W,Q)}{Q}$

Long-run marginal cost (LMC) function = $\frac{\partial C_L(W,Q)}{\partial Q}$

Check Your Progress 2

1) For the following total cost functions, find the AC, MC, AVC and AFC functions.

   a) $C = 5Q^3$

   b) $C = 10 + 7Q^2$

   c) $C = Q^3 - 4Q^2 + 10Q + 10$

   ....

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2) Given the technological relationship between output Q and inputs L and K priced at the per unit rate of w and r, respectively

$$Q = (L^\rho + K^\rho)^{\frac{1}{\rho}}$$

   a) Find the conditional factor demand functions L*(w, r, Q) and K*(w, r, Q)

   b) Derive the expression for the Cost function C (w, r, Q).

   c) Check for Shephard's Lemma, that is check whether $\frac{\partial C(w,r,Q)}{\partial w}$ = L*(w, r, Q) and $\frac{\partial C(w,r,Q)}{\partial r}$ = K*(w, r, Q)

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3) Use Shephard's lemma to derive factor demand functions from the following cost functions:

   a) $C = Q^2 (7w^3 + 5r^3)$

   b) $C = 2\, Q^2 \sqrt{wr}$

   where K and L are the two factors of production with per unit prices r and w, respectively.

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6.6 LET US SUM UP

The objective of profit maximisation explains the rationale behind cost minimisation by a producer of a good or a service. Costs incurred to carry out production include both, the explicit as well as the implicit costs. The present unit dealt with minimisation of such costs. For this, two approaches were explained— the graphical and the analytical. Graphical approach requires tangency between the isoquant curve and the isocost line. Analytical approach on the other hand involved solving the constrained optimisation problem of cost minimisation. We adopted the Lagrangian method for solving our optimisation problem, given the technological relationship between the factors and the output (*i.e.* the production function) and the factor prices.

On the basis of the optimisation exercise, the conditional factor demand functions— giving the optimum level of factors employed in the production process for the given output level Q and factor prices *w* and *r*, were derived. These functions were then employed to derive the cost function— a function of factor prices and output. By linking the production function with the factor prices, a cost function gives the minimum cost of producing a specific level of output, given the factor prices. Certain properties of a cost function were also touched upon. An important one being the Shephard's lemma, as per which a differentiable cost function can be used to derive conditional factor demand functions.

Subsequently, average cost and marginal cost functions were derived from the cost function. Average cost is the per unit cost of production, while marginal cost is the addition to cost from producing an additional unit. Further, the relationship between the AC and MC curve, which you have already studied in your Introductory Microeconomics course of Semester 1 (BECC-101), was mathematically established. Towards the end, the criterion distinguishing the short-run cost function from the long-run cost function was set in place by bringing in the picture the fixed and variable factors of production. Short-run cost function is composed of the cost incurred on both, the variable and the fixed factors. Long-run cost function, on the other hand is composed of the cost incurred on only the variable factors, as in the long-run all factors become variable. You must be clear with this by now— short-run and long-run are differentiated on the basis of the presence and absence of fixed factors of production, respectively.

6.7 REFERENCES

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6.9 ANSWERS OR HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) Equation for expansion path is given by, K = 3L

   Hint: Use cost minimisation condition $\frac{MP_L}{MP_K} = \frac{w}{r}$, where $MP_L = K^2$, $MP_K = 2LK$, w = 15 and r = 10.

2) Minimum cost = Rs. 1200

   Hint: Production function $Q = 10\sqrt{KL}$ can be written as $10K^{\frac{1}{2}}L^{\frac{1}{2}}$

   Use cost minimisation condition $\frac{MP_L}{MP_K} = \frac{w}{r}$, where $MP_L = 5K^{\frac{1}{2}}L^{\frac{-1}{2}}$, $MP_K = 5K^{\frac{-1}{2}}L^{\frac{1}{2}}$, w = 12 and r = 3 to arrive at the relation K = 4L. Then use this relation in production function $Q = 10\sqrt{KL}$, where Q = 1000 to get L = 50 and K = 200. Minimum cost thus equals 12(50) + 3(200) = 1200.

Check Your Progress 2

1) a) $AC = 5Q^2$, $MC = 15Q^2$, $AVC = 5Q^2$, AFC = 0

   b) $AC = \frac{10}{Q} + 7Q$, MC = 14Q, AVC = 7Q, $AFC = \frac{10}{Q}$

   c) $AC = Q^2 - 4Q + 10 + \frac{10}{Q}$, $MC = 3Q^2 - 8Q + 10$, $AVC = Q^2 - 4Q + 10$, $AFC = \frac{10}{Q}$

2) a) $L^*(w, r, Q) = \dfrac{Qw^{\frac{1}{\rho-1}}}{\left(w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}$

   $K^*(w, r, Q) = \dfrac{Qr^{\frac{1}{\rho-1}}}{\left(w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}$

   Hint: Use the cost-minimisation condition,

$$\frac{MP_L}{MP_K} = \frac{w}{r} \Rightarrow \frac{\frac{1}{\rho}(L^\rho + K^\rho)^{\frac{1}{\rho}-1}\rho L^{\rho-1}}{\frac{1}{\rho}(L^\rho + K^\rho)^{\frac{1}{\rho}-1}\rho K^{\rho-1}} = \frac{w}{r}$$

$$\Rightarrow \left(\frac{L}{K}\right)^{\rho-1} = \frac{w}{r}$$

$$\Rightarrow L = K\left(\frac{w}{r}\right)^{\frac{1}{\rho-1}} \qquad 1)$$

From our production function and Equation (1), we get

$$Q = (L^{\rho} + K^{\rho})^{\frac{1}{\rho}} \Rightarrow Q^{\rho} = \left[K\left(\frac{w}{r}\right)^{\frac{1}{\rho-1}}\right]^{\rho} + K^{\rho} \Rightarrow Q^{\rho} = K^{\rho}\left[\left(\frac{w}{r}\right)^{\frac{\rho}{\rho-1}} + 1\right]$$

$$\Rightarrow Q^{\rho} = K^{\rho}\left[\frac{w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}}{r^{\frac{\rho}{\rho-1}}}\right] \Rightarrow K^* = \frac{Q\, r^{\frac{1}{\rho-1}}}{\left(w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}$$

Substituting value of K*in Equation (1), we get

$$L^* = \frac{Q\, w^{\frac{1}{\rho-1}}}{\left(w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}$$

b) $C(w, r, Q) = Q\left(w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$

Hint: Insert values of L* and K* in the cost function given by,

$C(w, r, Q) = L^*(w, r, Q)\, w + K^*(w, r, Q)\, r$

3) a) $L^*(w, r, Q) = 21\, Q^2w^2$ and $K^*(w, r, Q) = 15\, Q^2r^2$

Hint: $\frac{\partial C(w,r,Q)}{\partial w} = L^*(w, r, Q) = 21\, Q^2w^2$ and $\frac{\partial C(w,r,Q)}{\partial r} = K^*(w, r, Q) = 15\, Q^2r^2$

b) $L^*(w, r, Q) = Q^2\sqrt{\frac{r}{w}}$ and $K^*(w, r, Q) = Q^2\sqrt{\frac{w}{r}}$