Indira Gandhi National Open University
School of Sciences

BMTC - 131
CALCULUS

Block

2

LIMIT AND CONTINUITY

Course Design Committee*

Prof. Rashmi Bhardwaj
G.G.S. Indraprastha University, Delhi

Dr. Sunita Gupta
L.S.R. College, University of Delhi

Prof. Amber Habib
Shiv Nadar University
Gautam Buddha Nagar, U.P

Prof. S. A. Katre
University of Pune, Pune

Prof. V. Krishna Kumar
NISER, Bhubaneswar

Dr. Amit Kulshreshtha
IISER, Mohali

Dr. Aparna Mehra
I.I.T. Delhi

Prof. Rahul Roy
Indian Statistical Institute, Delhi

Prof. Meena Sahai
University of Lucknow

Dr. Sachi Srivastava
University of Delhi

Faculty Members
School of Sciences, IGNOU

Prof. M. S. Nathawat (Director)
Dr. Deepika
Mr. Pawan Kumar
Prof. Poornima Mital
Prof. Parvin Sinclair
Prof. Sujatha Varma
Dr. S. Venkataraman

* The course design is based on the recommendations of the Programme Expert Committee and the UGC-CBCS template

Block Preparation Team

Prof. Amber Habib (*Editor*)
Shiv Nadar University
Gautam Buddha Nagar, U.P

Dr. Malathy A. (Language Editor)
School of Humanities
IGNOU, New Delhi

Dr. Deepika
School of Sciences
IGNOU, New Delhi

Course Coordinators: Prof. Parvin Sinclair and Dr. Deepika

Acknowledgement: To Prof. Parvin Sinclair for comments on the manuscript. Also, to Sh. Santosh Kumar Pal for the word processing and to Sh. S. S. Chauhan for preparing CRC of this block. Parts of this block are based on the course material of the previous course Calculus (MTE-01).

July, 2019

ISBN-978-93-89200-39-3

Further information on the Indira Gandhi National Open University courses, may be obtained from the University's office at Maidan Garhi, New Delhi-110 068 and IGNOU website www.ignou.ac.in.

Printed and published on behalf of the Indira Gandhi National Open University, New Delhi by Prof. M. S. Nathawat, Director, School of Sciences.

BLOCK 2 LIMIT AND CONTINUITY

This is the second of the five blocks which you will be studying for the course calculus. We assume that you are familiar with the real number system and real functions. But, just to refresh your memory, we have given a brief account of real numbers and their properties, as well as some types of functions in Unit 6. It is also possible that some of you have not studied certain aspects of the real number system and functions earlier. In that case, Unit 6 will help you prepare a firm ground for the imposing structure of calculus which follows.

Calculus has two fundamental procedures, differentiation and integration, which can be formulated in terms of a concept called the 'limit'. In Unit 7, we begin with helping you acquire an intuitive sense of this concept. The word 'intuitive' can mean several things. Its use here means "experience based, without proof". Following this intuitive presentation, we present the formal definition of 'limit'. In addition, we will introduce you to functions that require the use of limits, namely, the exponential, logarithmic functions and hyperbolic functions.

In Unit 8, we will continue to use 'limit' to explore a new concept, namely, that of continuity. We will also discuss the types of discontinuity, and end the unit by stating the intermediate value theorem for continuous functions, and giving some of its applications.

In Unit 6 to Unit 8, we have included a number of examples. Please go through them carefully. They will help you in a better understanding of the concepts discussed and will also serve as a guide in solving the exercises.

At the end of the block, you will find miscellaneous examples and exercises, covering the concepts you have studied across the units. Please solve the exercises on your own. At the end of each unit, and after the miscellaneous exercises, we do provide some solutions/answers to the exercises concerned. These are only as a support for you to be able to check whether you have been able to solve the problem correctly or not. Please do not look at these solutions till you have spend enough time on studying the unit and trying all the exercises.

A word about some signs used in the unit! Throughout each unit, you will find theorems, examples and exercises. To signify the end of the proof of a theorem, we use the sign ■. To show the end of an example, we use ***. Further, equations that need to be referred to are numbered sequentially within a unit, as are exercises and figures. E1, E2 etc. denote the exercises and Fig. 1, Fig. 2, etc. denote the figures.

NOTATIONS AND SYMBOLS (used in Block 2)

$\in$ $(\notin)$	belongs to (does not belong to)
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}(\mathbb{Z}^+)(\mathbb{Z}^-)$	the set of integers(the set of positive integers)(the set of negative integers)
$\mathbb{Q}(\mathbb{Q}^*)$	the set of rational numbers (non-zero rational numbers)
$\mathbb{R}(\mathbb{R}^+)(\mathbb{R}^-)$	the set of real numbers(the set of positive real numbers)(the set of negative real numbers)
$\Rightarrow (\Leftrightarrow)$	implies (implies and is implied by)
iff	if and only if
$\therefore$	therefore
w.r.t.	with respect to
s.t.	such that
$< (\leq)$	is less than (is less than or equal to)
$> (\geq)$	is greater than (is greater than or equal to)
$\exists$	there exists
$\forall$	for all
$f : X \to Y$	f is a function from the set X to the set Y
$\{x \mid x \text{ satisfies } P\}$	the set of all x such that x satisfies the property P
$\lvert x \rvert$	modulus of the real x
$\lim_{x \to a} f(x)$	limit of $f(x)$ as x tends to a
$x \to f(x)$	a functions f taking x to $f(x)$
${}^nC_r$	the number of combinations of r things taken out of n, ${}^nC_r = \frac{n!}{r!(n-r)!}$
$\approx$	is approximately equal to
$\max\{x, y\}$	the maximum of x and y
$\min\{x, y\}$	the minimum of x and y

Please see the notations and symbols used in Block 1.

UNIT 6

REAL NUMBERS

Structure   Page No.

6.1 INTRODUCTION

In this unit, we shall provide you with a review of the basic facts about the system of real numbers. Perhaps, you are already familiar with some of these from your studies in school. But, a quick look through this unit will help you in refreshing your memory. And some of the concepts, like infimum and supremum, are likely to be new to you.

In Sec. 6.2 of this unit, we shall present the arithmetic and order properties of the real number system. In Sec. 6.3, we shall introduce the concept of supremum and infimum. We shall discuss absolute value and the intervals on the real line in Sec. 6.4 and Sec. 6.5 respectively. You have already studied functions and their graphs in Unit 2 and Unit 3. In Sec. 6.6, you will find more examples of functions and their graphs. We end the unit with even and odd functions, monotonic functions and periodic functions in Sec.6.7.

And now, we will list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

Objectives

After reading this unit, you should be able to:

- list the basic properties of real numbers;
- derive other properties of real numbers with the help of the basic ones;
- define the supremum and infimum of a given set;
- obtain the absolute value of a real number; and
- determine whether a given function is even, odd, monotonic or periodic.

6.2 PROPERTIES OF REAL NUMBERS

The real number system (which we often call simply "the reals") is first of all a set on which the operations are defined. Real number system is the foundation on which a large part of mathematics, including calculus, rests. Thus, before you actually start learning calculus, it is necessary to understand the real number system.
Here we shall quickly recall some properties of real numbers.

Operation of Addition:

$A1$ **$\mathbb{R}$ is closed under addition.**
If x and y are real numbers, then $x+y$ is a unique real number.
For example: $3+5=8$ is real.

$A2$ **Addition in $\mathbb{R}$ is associative.**
$x+(y+z)=(x+y)+z$ holds for all x, y, z in $\mathbb{R}$.
For example: $(0.2+0.5)+0.3=0.2+(0.5+0.3)$

$A3$ **Existence of Additive identity** (z**ero).**
There is a unique real number, 0, such that $x+0=0+x=x$ for all x in $\mathbb{R}$. 0 is called the **additive identity** in $\mathbb{R}$.
For example: $\frac{1}{4}+0=0+\frac{1}{4}=\frac{1}{4}$.

$A4$ **Exstence of Additive inverse.**
For each real number x , there exists a real number y (called the **additive inverse** of x , and denoted by $-x$) such that $x+y=y+x=0$.
For example: $\frac{1}{2}+\left(-\frac{1}{2}\right)=\left(-\frac{1}{2}\right)+\frac{1}{2}=0.$

$A5$ **Addition is commutative.**
$x+y=y+x$ holds for all x, y in $\mathbb{R}$.
For example: $32.2+42.8=42.8+32.2$

Operation of Multiplication:

$M1$ **$\mathbb{R}$ is closed under multiplication.**
If x and y are real numbers, then xy is a real number.
For example: $3\times 4=12$ is real.

$M2$ **Multiplication is associative.**
$x\ (y z)=(xy)\ z$ holds for all x, y, z in $\mathbb{R}$.

For example: $\frac{2}{3}\times\left(\frac{3}{4}\times\frac{4}{5}\right)=\left(\frac{2}{3}\times\frac{3}{4}\right)\times\frac{4}{5}$

$M3$ **Existence of Multiplicative Identity (Unity)**
There exists a unique real number 1, such that $x1 = 1x = x$ for every x in $\mathbb{R}$. The number 1 is called the multiplicative identity (unit element or unity) in $\mathbb{R}$.
For example: $8.5\times1 = 1\times8.5 = 8.5$

$M4$ **Existence of Multiplicative Inverse.**
For each non-zero real number x, there exists a real number y (called the **multiplicative inverse** of x, and denoted by x^{-1}, or by $1/x$) such that $xy = yx = 1$.

For example: $6\times\left(\frac{1}{6}\right)=\left(\frac{1}{6}\right)\times6=1$

$M5$ **Multiplication is commutative.**
$xy = yx$ holds for all x, y in $\mathbb{R}$.
For example: $7\times8 = 8\times7$
The next property involves addition as well as multiplication.

D **Multiplication is distributive over addition.**
$x(y+z) = xy + xz$ holds for all x, y, z in $\mathbb{R}$.
$(x+y)z = xz + yz$ holds for all x, y, z in $\mathbb{R}$.

For example: $\frac{1}{2}\times(2+4)=\frac{1}{2}\times2+\frac{1}{2}\times4$, and
$(2+3)\times4 = 2\times4+3\times4$

It may be noted that, for any two real numbers x and y, the result of subtraction of y from x is denoted by $x - y$ and is defined as $x + (-y)$. Similarly, the division $x \div y$ (also denoted by x / y) is defined as xy^{-1}, provided $y \neq 0$.

Now we are ready to list a few more properties of real numbers.

1. $0x = 0, -(-x) = x$ and $(-1)x = -x$, for all x in $\mathbb{R}$.
2. $-(x+y) = (-x)+(-y)$ for all x, y in $\mathbb{R}$.
   For example: $-(4+5) = (-4)+(-5) = -9$
3. If $xy = 0$, then either $x = 0$ or $y = 0$
   For example: $(x-1)(x-2) = 0$ gives $x-1 = 0$ or $x-2 = 0$.
4. $(x^{-1})^{-1} = x$ for all $x \neq 0$ in $\mathbb{R}$.
   For example: $\left((5)^{-1}\right)^{-1} = (1/5)^{-1} = 5$
5. If x and y are non- zero numbers such that $x^{-1} = y^{-1}$, then $x = y$.

Example 1: A person calculates $\frac{80}{9}$ using the following steps:

$\frac{80}{9}=\frac{80}{4+5}$ (Line 1)

$=\frac{80}{4}+\frac{80}{5}$ (Line 2)

$= 20+16$ (Line 3)

$=36$ (Line 4)

which line of the calculation is wrong and why?

Solution: Line 2 is wrong because distributive law does not work for division.

That is $a \div (b+c) \neq (a \div b) + (a \div c)$ or $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$.

***

Try the following exercises now.

E1) Is the following subset of $\mathbb{R}$ closed w.r.t. multiplication? Give reasons.

$$\mathbf{S} = \{-3, -2, -1, -\frac{1}{2}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{2}, 1, 2, 3\}.$$

E2) Find the mistake done in the following lines of some calculation. Explain the nature of the error.

$1573 - 697$	$= 1573 - (700 - 3)$	Line 1
	$= (1573 - 700) - 3$	Line 2
	$= 873 - 3$	Line 3
	$= 870$	Line 4

Now, we shall discuss the order relation on $\mathbb{R}$.

Order Relation on $\mathbb{R}$: The order relation $'>'$, which you have been already using, has the following properties: For example, all positive numbers are greater than zero.

We write $x < y$ (and read x is less than y) to mean $y > x$. We write $x \leq y$ (and read x is less than or equal to y) to mean either $x < y$ or $x = y$. We write $x \geq y$ (and read x is greater than or equal to y) if either $x > y$ or $x = y$.

Symbol $'\forall'$ denotes 'for all' or 'for each'.

***O*1 Law of Trichotomy holds.**
For any two real numbers x, y, one and only one of the following holds: $x > y, x = y, x < y$.
For example: Since $9 > 5$, therefore $9 \neq 5$ and $9 \not< 5$.

***O*2 '>' is transitive.**
If $x > y$ and $y > z$, then, $x > z, \forall x, y, z \in \mathbb{R}$.
For example: Since $\frac{1}{5} > \frac{1}{7}$ and $\frac{1}{7} > \frac{1}{9}$, therefore, $\frac{1}{5} > \frac{1}{9}$

***O*3 Addition is monotone.**
If x, y, z, in $\mathbb{R}$ are such that $x > y$, then, $x + z > y + z$.
For example: Since $9.5 > 7.5$, therefore, $9.5 + 4 > 7.5 + 4$, i.e. $13.5 > 11.5$.

***O*4 Multiplication is monotone only under a specific condition.**
If x, y, z in $\mathbb{R}$ are such that $x > y$ and $z > 0$, then, $xz > yz$.
For example: Since $6 > 3$ therefore, $6 \times 2 > 3 \times 2$, i.e. $12 > 6$.
Caution: $x > y$ and $z < 0 \Rightarrow xz < yz$.
For example: Since $6 > 3$, therefore, $6 \times (-2) < 3 \times (-2)$, i.e. $-12 < -6$.
Some more properties are:

1. If $a < b$ and $c < d$, then, $a + c < b + d$.
   For example: Since $5 < 7$ and $4 < 6$, therefore, $5 + 4 < 7 + 6$

2. a^2 is non-negative for all a in $\mathbb{R}$.
   For example: $(-5)^2 = (-5)(-5) = 25$ is non-negative.
   The only $a \in \mathbb{R}$ for which $a^2 = 0$ is $a = 0$.

3. If a and b are positive real numbers, then
   i) $a^2 = b^2 \Leftrightarrow a = b$.
   ii) $a^2 > b^2 \Leftrightarrow a > b$
   iii) $a^2 < b^2 \Leftrightarrow a < b$
   For example: "If x is positive then $x^2 > 1 \Leftrightarrow x > 1$."

The symbol $\Leftrightarrow$ denotes 'implies and is implied by' which is equivalent to 'if and only if'.

4. If $b > 0$, then $a^2 < b^2 \Leftrightarrow -b < a < b$.
   If the square of a real number is less than the square of a positive real number, then the number must lie between the positive and negative value of the positive real number.
   For example: $x^2 < 25 \Leftrightarrow -5 < x < 5$.

5. For any reals x and y, if $x \neq y$, then $\frac{x+y}{2}$ is between x and y. That is
   $x < \frac{x+y}{2} < y$ or $y < \frac{x+y}{2} < x$.
   For example: Take $x = 4$ and $y = 10$, $4 < \frac{4+10}{2} = 7 < 10$.

Now we shall discuss supremum and infimum in the following section.

6.3 SUPREMUM AND INFIMUM

Now consider a real number such as $\sqrt{2}$, which we wish to plot on the real number line. How do we determine its location? First we see that 1 is smaller than $\sqrt{2}$ (since $1^2 = 1 < 2$), while 2 is larger (since $2^2 = 4 > 2$). So $\sqrt{2}$ lies between 1 and 2. By considering one-tenths we can get a better idea of the location. We see that 1.4 is too small $(1.4^2 = 1.96 < 2)$, while 1.5 is too large $1.5^2 = 2.25 > 2)$. So $\sqrt{2}$ lies between 1.4 and 1.5. If we need more accurancy we can zoom into the one-hundredths level and see that $\sqrt{2}$ lies between 1.41 and 1.42 $(1.41^2 = 1.988 < 2$ and $1.42^2 = 2.016 > 2)$. Thus, to plot a number which is not precisely known, we seek known numbers which are just above and below it.

Instead of a single number, we may be worried about the location of an entire set and it may not be easy to figure out each element of the set. For example, can you precisely list the elements of $A = \{x \in \mathbb{R} : x^4 + x - 1 = 0\}$? Probably not. But you can at least see that no $x > 1$ satisfies $x^4 + x - 1 = 0$, since in this case $x^4 + x - 1 > 1^4 + 1 - 1 = 1$. Thus, the set A lies entirely to the left of 1. Similarly, we see that no $x < -2$ is a solution, since in this case $x^4 + x - 1 = x(x^3 + 1) - 1 > (-2)(-7) - 1 = 13$. Thus, the set A is entirely to the right of -2. We have learned that all the elements of A lie between -2 and 1. Often, such knowledge is enough to reach useful conclusions about A.

This leads to the following definition:

Definition: Let S be a subset of $\mathbb{R}$. An element u in $\mathbb{R}$ is said to be an **upper bound** of S if $u \geq x$ holds for every x in S. In other words, for a given set, a number which is greater than or equal to all the elements is known as upper bound of that set. We say that S is **bounded above**, if there is an upper bound of S.

Working on similar lines, we define the lower bound in the following definition.

Definition: A **lower bound** for a given set S is a real number v such that $v \le x$ for all $x \in S$. We shall say that a set is **bounded below**, if we can find a lower bound for it.

Let us find upper and lower bounds in following example:

Example 2: Find the upper and lower bounds of $A = \left\{ \frac{n+2}{n} : n \in \mathbb{N} \right\}$.

Solution: For an upper bound, we need to find a number u such that $\frac{n+2}{n} \le u$ for all $n \in \mathbb{N}$.

We have $\frac{n+2}{n} = 1 + \frac{2}{n} \le 3 \ \forall n \in \mathbb{N}$. Thus, 3 is an upper bound of the set A.

Also, $\frac{n+2}{n} = 1 + \frac{2}{n} \ge 1 \ \forall n \in \mathbb{N}$. Thus, 1 is a lower bound of the set A.

***

You may notice that if u is an upper bound of a set S, then so is $u+1$. In fact, $u+2, u+3, \ldots, u+r$, where r is any positive real number are all upper bounds of S. In general, from among all the upper bounds of a set S, can we always choose an upper bound u such that u is less than or equal to every upper bound of S? We can, and this upper bound has a special name, we call this u the **least upper bound (l.u.b.)** or **the supremum of S.** For example, -1 is the supremum of $\mathbb{Z}^-$, where $\mathbb{Z}^- = \{\ldots -4, -3, -2, -1\}$.

This leads to the following definition:

Definition: The upper bound of any subset S of $\mathbb{R}$, which is less than any other upper bound of S is called its **supremum** or **least upper bound (l.u.b.)**. Further, the lower bound of S which is greater than any other lower bound of S will be called its **infimum** or **greatest lower bound (g.l.b.)**.
Sometimes, we write supremum of a set S as Sup(S) and infimum of a set as Inf(S).

Example 3: Consider the set $T = \{x \in \mathbb{R} : x^2 \le 4\}$, find its supremum and infimum.

Solution: Here $x^2 \le 4 \Rightarrow -2 \le x \le 2$. Here all members of the set T are less than or equal to 2. Thus, 2 is an upper bound of T and all the reals greater than 2 are the upper bounds of T, since 2 is the least from all the upper bounds. Thus 2 is the supremum of T.
Similarly, all the members of T are greater than or equal to -2. Thus -2 is a lower bound of T. All the real numbers less than -2 are also lower bounds of T. Here -2 is the greatest of all the lower bounds of T. Thus -2 is infimum of T.

***

Remark 1: Note that for the set T considered in Example 3 the l.u.b. belongs to the set. This may not be true in general. Consider the set of all negative real numbers $\mathbb{R}^- = \{x : x < 0\}$. What is the l.u.b. of this set? Wouldn't it be 0. Here $0 \notin \mathbb{R}^-$ but it is the l.u.b. of $\mathbb{R}^-$.

As in the case of l.u.b., remember that the g.l.b. of a set may or may not belong to the set. For example, consider $S = \{x \in \mathbb{R}, 1 < x < 2\}$. The infimum $(S) = 1$ does not belong to S. Now, does every set have an upper and a lower bound? Consider, $S_1 = \{x \in \mathbb{R} : x^2 < 5\}$. Here, $\text{Sup}\ (S_1) = \sqrt{5}$, and $\text{Inf}\ (S_1) = -\sqrt{5}$, therefore, S_1 is bounded. Now, consider $S_2 = \{x \in \mathbb{R} : x > \sqrt{8}\}$. S_2 has no upper bound but has a lower bound. If a set has an upper bound, it is called bounded above and if a set has a lower bound, it is called bounded below, and is given in the following definitions:

Definition: A set $S \subset \mathbb{R}$ is **bounded** if it has both an upper bound and a lower bound.

Would you agree, that the set T given in Example 3 is bounded?
Based on this discussion you would be able to solve the following exercises.

E3) Find the supremum and infimum of the following subsets of $\mathbb{R}$.

i) $S_1 = \{x \in \mathbb{R} : 1 < x < 2\}$

ii) $S_2 = \{x \in \mathbb{R} : x^2 < 5\}$

iii) $S_3 = \{x \in \mathbb{R} : x^2 > 7\}$

iv) $S_4 = \{-\frac{1}{x} : x \in \mathbb{N}\}$

E4) Give an example, with justification, of each of the following:

i) a set of real numbers having a lower bound, but no upper bound.

ii) a set of real numbers without any lower bound, but with a supremum.

iii) a set of real numbers whose g.l.b. does not belong to it.

iv) a bounded set of real numbers, whose inf and sup do not lie in it.

Now we are ready to state an important property of $\mathbb{R}$.

Archimedean Property: If a and b are any real numbers such that $b > 0$, then there is a positive integer n such that $nb > a$, that is $\underbrace{b + b + b + ... + b}_{n \text{ times}} > a$. It is shown in Fig. 1.

Archimedean property is named after the ancient Greek Mathematician Archimedes.

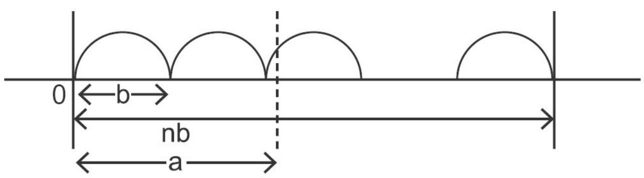


Fig. 1: For any reals a and b, where $b > 0$, $\exists\ n \in \mathbb{N}$ such that $a < nb$

For example, if a = 1 and b = ½ > 0, then there is $n = 3$, say, then $nb > a$. Note that n is not unique. If $nb > a$, then $(n+1)b > a$ and so on.

If a is any real number, there is a positive integer n such that $n > a$ (Archimedean property applied to a and 1).

Try the following exercises now.

E5) Let $A = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$. Is it bounded? If yes, find supremum and infimum of A. If it is not bounded, find a subset of it that is bounded.

E6) Prove that supremum and infimum of any set are unique, if they exist.

Now, let us discuss absolute value of a real number.

6.4 ABSOLUTE VALUE

You may recall the modulus function and its graph discussed in Unit 3. In this section we shall define the absolute value. You will realise the importance of this simple concept as you study the later units.
The absolute value of a real number is its magnitude, or the distance between the origin and the point representing the real number on the real number line. For this, consider two points 2.5 and -2.5 on the real number line. The distance of both the points from O (origin or 0) is 2.5. We denote this distance by $|2.5|$ and $|-2.5|$ and call it the absolute value.

The notation $|x|$, with a vertical bar on each side was introduced by Karl Weierstrass in 1841.

Definition: If x is a real number, its **absolute value**, denoted by $|x|$ (read as **modulus of** x, or **mod** x), is defined by the following rules:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

For instance,

$$|4| = 4, |-4| = -(-4) = 4,$$

$$|2.5| = 2.5, |-3| = 3.$$

Note that the absolute value of a real number is **never negative**, as $-x$ is positive if x is negative.
The absolute value of a real number is either positive or zero. Moreover, 0 is the only real number whose absolute value is 0. Therefore, $|0| = 0$.

Example 4: Find the Absolute value of the following numbers:

i) $|\pi - 2|$, ii) $|2 - \pi|$.

Solution: i) $|\pi - 2| = \pi - 2$ [since $\pi - 2 > 0$]

ii) $|2 - \pi| = -(2 - \pi) = \pi - 2.$ [since $2 - \pi < 0$]

***

Example 5: Solve $|x - 2| = 5$

Solution: Now $|x-2| = \begin{cases} x - 2 \ ; & x - 2 \geq 0 \\ -(x-2) \ ; & x - 2 < 0 \end{cases}$

If $|x - 2| = 5$, then the two possibilities are $x - 2 = 5$ or $-(x-2) = 5$. We get $x = 7, -3$.

***

You may now try the following exercises.

E7) Evaluate the following:

i) $|-15|$ ii) $\left|\frac{2}{3}\right|$

iii) $|-4.3|$ iv) $-|-6|$ v) $\left|\sqrt{2}-2\right|$

E8) Evaluate $\frac{|x|}{x}$ for i) $x > 0$ and ii) $x < 0$.

E9) Place appropriate symbol $(<, >$ or $=)$ between the following pairs of real numbers.

i) $|5|$ and $|-7|$

ii) $\left|\frac{1}{-8}\right|$ and $\left|\frac{1}{8}\right|$

iii) $|2^x|$ and $|(-2)^x|$ for any natural number x.

iv) $-|-\sqrt{10}|$ and $|-\sqrt{10}|$

We shall discuss some of the important properties of $|x|$ in the following theorems, which we shall be using in calculus:

Theorem1: If a and b are any real numbers, then

i) $|a| \geq 0$ [Non-negativity]

ii) $|a| = \max\{-a, a\}$

Proof: (i) By the law of trichotomy (O1 in Sec. 6.2), applied to the real numbers a and 0 we have $a > 0$, $a = 0$ or $a < 0$. So, we get the following three cases:

i) If $a > 0$, then $|a| = a > 0$

ii) If $a = 0$, then $|a| = 0$

iii) If $a < 0$, then $|a| = -a > 0$

Thus, from (i), (ii) and (iii), we get $|a| \geq 0$ for all $a \in \mathbb{R}$.

(ii) Again using these cases for any $a \in \mathbb{R}, a > 0$ or $a = 0$ or $a < 0$.

i) Since $a > 0$, therefore, $|a| = a$ and $a > -a$. So that,

$\max\{-a, a\} = a$. Hence, $|a| = \max\{-a, a\}$

ii) Since $a = 0$, therefore, $-a = 0$. Also $|a| = 0$. So that,

$\max\{-a, a\} = 0$. Hence, $\max\{-a, a\} = |a|$

iii) Since $a < 0$, therefore, $|a| = -a$ and $-a > a$. So that,

$\max\{-a, a\} = -a$. Hence, $|a| = \max\{-a, a\}$

Thus, from (i), (ii) and (iii), we get

$|a| = \max\{-a, a\}$ ■

The identity given in the following theorem is sometimes used as a definition of absolute value of a real number.

Theorem 2: For any real number $a, \sqrt{a^2} = |a|$.

Proof: Since $a^2 = (-a)^2$, a and $-a$ are the two square roots of a^2.

If $a \geq 0, \sqrt{a^2} = a$

If $a < 0, \sqrt{a^2} = -a$

It follows that $\sqrt{a^2} = |a|$. ■

Some additional useful properties of the absolute value are given in the following theorem:

Theorem 3: If x and y are real numbers, then

i) $|-x| = |x|$ [Evenness (reflection symmetry of the graph)]

ii) $|xy| = |x||y|$ [Multiplicativity of absolute value]

iii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, if $y \neq 0$. [Preservation of division]

Proof: i) Using Theorem 2, $|-x| = \sqrt{(-x)^2} = \sqrt{x^2} = |x|$.

ii) Again using Theorem 2,
$|xy| = \sqrt{(xy)^2} = \sqrt{x^2y^2} = \sqrt{x^2}\sqrt{y^2} = |x||y|$ ■

iii) You may like to prove it yourself.

Remark 2: The result of Theorem 3 (ii) can be generalised as given below.
If $x_1, x_2, x_3, ...x_n$ are real numbers, then $|x_1 x_2....x_n| = |x_1||x_2|....|x_n|$.

If $x_1 = x_2 = ... = x_n = x$, then $|x^n| = |x|^n$.

In the following theorem, we shall prove the triangle inequality.

Theorem 4 (The triangle inequality): If x and y are two real numbers, then

$$|x + y| \leq |x| + |y|$$

Proof: Let us consider i) $x + y \geq 0$ and ii) $x + y < 0$

i) If $x + y \geq 0$, $|x + y| = x + y$

$\leq |x| + |y|$ [since $x \leq |x|$ and $y \leq |y|$, using Theorem 1, $|x| = \max\{x, -x\}$]

ii) If $x + y < 0, |x + y| = -(x + y)$

$= (-x) + (-y)$

$\leq |x| + |y|$ [since $-x \leq |x|$ and $-y \leq |y|$, using Theorem1].

Therefore, $|x + y| \leq |x| + |y|$. ■

Two other useful properties concerning inequalities are:

i) $|a| \le b \Leftrightarrow -b \le a \le b$

ii) $|a| \ge b \Leftrightarrow a \le -b \text{ or } b \le a.$

These relations may be used to solve inequalities involving absolute values.
For example:

$$|x-5| \le 11 \Leftrightarrow -11 \le (x-5) \le 11$$
$$\Leftrightarrow -6 \le x \le 16$$

Example 6: Calculate $|\pi-2|+|\pi-3|+|2\pi-7|$.

Solution: The quantities $\pi-2$ and $\pi-3$ are positive, so they remain unchanged when the absolute bars are dropped. However, because $2\pi-7$ is negative, we get

$$|\pi-2|+|\pi-3|+|2\pi-7| = \pi-2+\pi-3-(2\pi-7)$$
$$=2$$

***

Example 7: Find the value of $\left|2\times\left(\frac{2}{3}-0.5\right)\right|$.

Solution: $\left|2\times\left(\frac{2}{3}-0.5\right)\right| = \left|2\times\frac{1}{6}\right|$

$$=\left|\frac{1}{3}\right| = \frac{1}{3}.$$

***

Example 8: Solve the equation

$$|x-3|+|x-4|=1.$$

Solution: We need to discuss the following three cases:

i) $x \le 3$

ii) $3 < x \le 4$

iii) $x \ge 4$

i) When $x \le 3$.

$$|x-3|+|x-4|=1 \Rightarrow -(x-3)-(x-4)=1$$
$$\Rightarrow x=3$$

ii) When $3 < x \le 4$

$$|x-3|+|x-4|=1 \Rightarrow x-3-(x-4)=1$$
$$\Rightarrow 3 < x \le 4 \text{ is a solution.}$$

iii) When $x > 4$

$$|x-3|+|x-4|=1 \Rightarrow x-3+x-4=1$$
$$\Rightarrow x=4.$$

Since we assumed $x > 4$, there is no solution for the case $x > 4$.

In conclusion, the solution of the equation $|x-3|+|x-4|=1$ are $3 \le x \le 4$.

***

Now, try the following exercises.

E10) Which of the following is the graph of $y = \frac{x}{|x|}$ for $x \neq 0$?

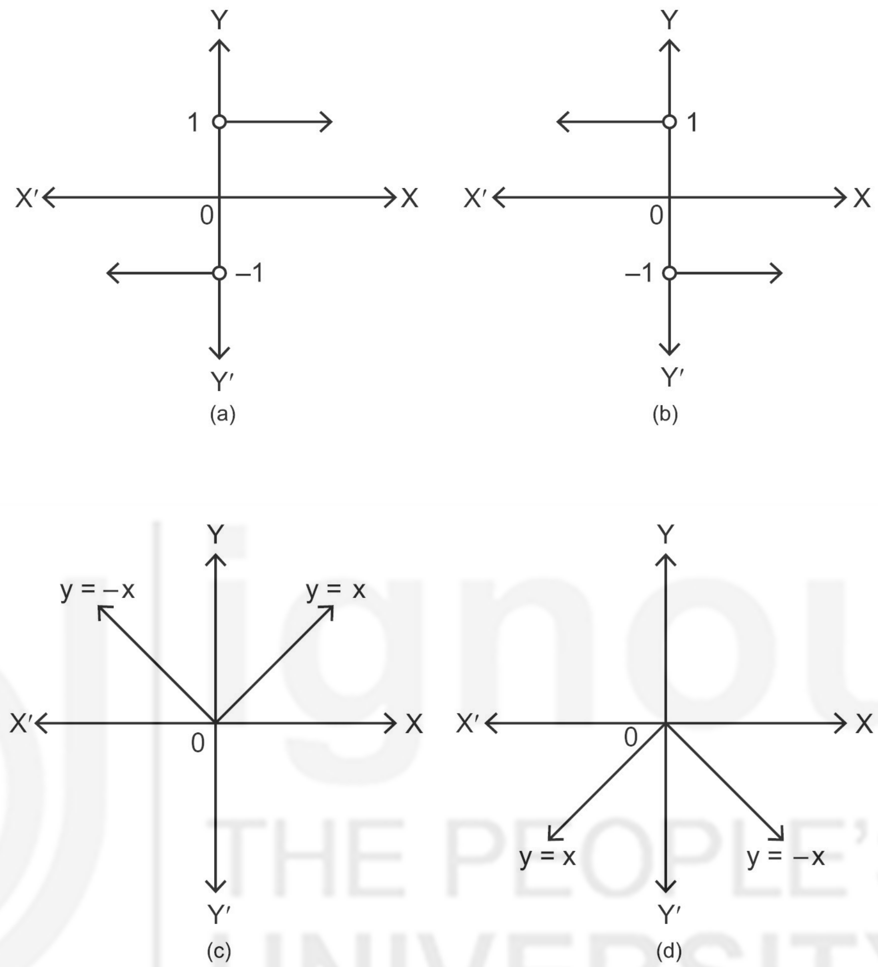


Fig. 2

E11) Prove the following for any real numbers x, y:

i) $x = 0 \Leftrightarrow |x| = 0$

ii) $|1/x| = 1/|x|$, if $x \neq 0$

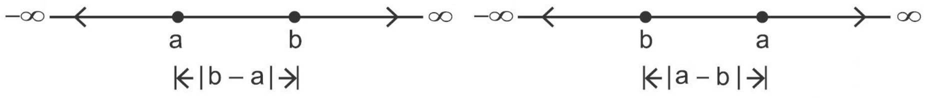
iii) $|x - y| \leq |x| + |y|$.

Inequalities involving absolute values arise at several places in calculus. In the next section, we shall see how the set $\{x : |x - a| < k\}$ can be represented geometrically.

6.5 INTERVALS ON THE REAL NUMBER LINE

We have discussed that the absolute value or the modulus of any number a is nothing but the distance between the point denoting it from the point 0 on the real number line. In the same way, $|a - b|$ denotes the distance between the points corresponding to the two numbers a and b. It may be noted that $a < b$ if and only if a lies to the left of b on the real number line, as shown in

Fig.3(a), and $a > b$ if and only if a lies to the right of b on the real number line as shown in Fig. 3(b).



(a) $\mathbf{b > a}$ **(b)** $\mathbf{a > b}$

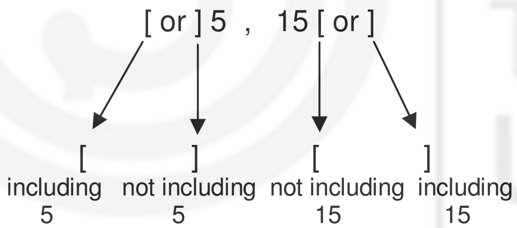
Fig. 3: Representation of $\mathbf{b > a}$ **and** $\mathbf{a > b}$

Now, let us consider the set S of all real numbers which lie between two given real numbers say 0 and 2.3. So $S = \{x \in \mathbb{R} | 0 \le x \le 2.3\}$. Here we say that x attains the values from 0 to 2.3.

For example, the interval 2 to 4 include all real numbers such as: $2.1,\ 2.1111,\ 2.5, 2.75,\ 2.80001,\ \pi, 7/2,\ 3.7937$ and a lot more.

A convenient notation for representing intervals on a number line is called interval notation. Given two real numbers a and b with $a < b$, the closed interval $[a, b]$ is defined as the set of all real numbers x such that $a \le x$ and $x \le b$ or more concisely, $a \le x \le b$. The open interval $]a, b[$ is defined as the set of all real numbers x such that $a < x < b$.

For example, consider the interval $[5,15]$



It may be noted that a mix of brackets can also be used. For example, $]4,13]$ means from 4 to 13, do not include 4 but do include 13.

On real number line, we draw a thick line to show the values we are including and a filled-in circle, when we want to include the endpoints or a hollow open circle when we do not want to include the endpoints.
For example,

means all the numbers between 5 and 15, but do not include 5, and do include 15.

The inclusion of the endpoints in the interval gives the four different sets as given in Table 1. In the representation of this closed interval, we put thick black dots at a and b to indicate that they are included in the set. The endpoints of a **closed interval** are included in the interval.

The endpoints of an **open interval** are not included in the interval. Note that in this case we draw a hollow circle around a and b to indicate that they are not included in the number line.

The sets [a, b[and]a, b] are called half-open (or half-closed) intervals or semi-open (or semi-closed) intervals, as they contain only one endpoint.

Table 1: Intervals on the Real Number line.

S.No	Notation	Interval Type	Sets	Geometrical Representation
1.	$]a, b[$	Open interval	$\{x : a < x < b\}$ not including a and b.	
2.	$[a, b]$	Closed interval	$\{x : a \le x \le b\}$ including both a and b.	
3.	$]a, b]$	Half open or half closed or open on left and closed on right	$\{x : a < x \le b\}$ including b, but not a	
4.	$[a, b[$	Closed on left and open on right	$\{x : a \le x < b\}$ Including a but not b.	

In particular, if $a = b,]a, a[\, = \,]a, a] = [a, a[= \phi$ and $[a, a] = a$.

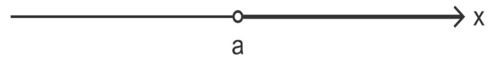
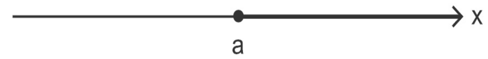
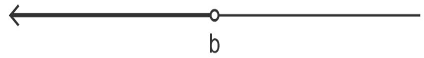
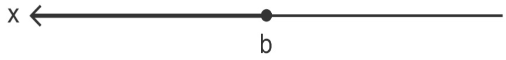
You may check that the four types of intervals given in Table 1 are bounded. If both the endpoints in an interval are reals, then that interval is called a **finite interval** (**bounded interval) .** A bounded interval is open if it includes neither of its endpoints, half-open if it includes only one endpoint, and closed if it includes both endpoints. The symbol "∞" (pronounced infinity) is used for intervals that are not limited in one direction or another.

Note that ∞ does not denote a real number, it merely indicates that an interval extends without end. We use open bracket with infinity, because we do not reach it. We can say that an interval having ∞ or $-\infty$ as one of its end-points is

- left-bounded, if the left endpoint **is a real**.
- right-bounded, if the right endpoint **is a real**.
- unbounded, if none of the endpoints are reals, both of them are infinite endpoints.

Apart from the four types of intervals listed in Table 1, there are a few more types. These are given in Table 2.

Table 2: Unbounded Intervals on Real Number line.

Notation	Interval Type	Sets	Geometrical Representation
$]a, \infty[$	open right ray	$\{x : a < x\}$ greater than a	
$[a, \infty[$	closed right ray	$\{x : a \le x\}$ greater than or equal to a	
$]-\infty, b[$	open left ray	$\{x : x < b\}$ less than b	
$]-\infty, b]$	closed left ray	$\{x : x \le b\}$ less than or equal to b	
$]-\infty, \infty[$	open interval	$\mathbb{R}$	

Based on this here is an example.

Example 9: Describe the subset of real numbers that the following inequalities represent. Also, show them on real number line.

a) $x \le 4$ b) $-3 \le x < 2$.

Solution: a) The inequality $x \le 4$ denotes all the real numbers less than or equal to 4 as shown in Fig.4

Fig. 4: Representation of $x \le 4$

b) The inequality $-3 \le x < 2$ denotes that $x \ge -3$ and $x < 2$. This “double inequality” denotes all the real numbers between -3 and 2, including -3 but not including 2 as shown in Fig. 5.

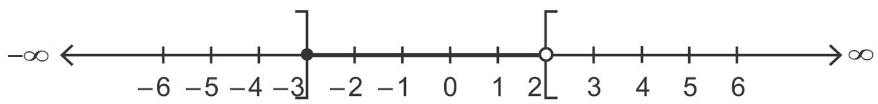


Fig. 5: Representation of $-3 \le x < 2$.

***

You may now try the following exercises.

E12) State whether the following are true or false. Give reasons for your answers.

i) $0 \in [1, 8]$ ii) $-1 \in\ ]-\infty, 2[$

iii) $1 \in [1, 2]$ iv) $5 \in\ ]5, \infty[$.

E13) Write the inequalities for the intervals given in E12. Also, show these on the real line.

You have studied functions in Unit 2. Now, in the following section, we shall discuss various functions with their graphs.

6.6 FUNCTIONS AND THEIR GRAPHS

You have already studied the notion of a function along with some special functions like constant function, identity function, linear, x^2 and $|x|$ etc. along with their graphs in Unit 2 and Unit 3. Addition, subtraction, multiplication, division, composition of two functions, etc. have also been studied in Unit 2. As the concept of function is of paramount importance in Mathematics and in other disciplines as well, we would like to extend our study about functions from where we finished earlier.

In this course throughout, we shall consider functions whose domain and co-domain are both subsets of $\mathbb{R}$. Such functions are often called **real functions** or **real-valued functions** of a real variable. We shall, however, simply use the word 'function' to mean a real function.

In functions, the variable x used in describing a function is often called a **dummy variable** because it can be replaced by any other letter. Thus, for example, the rule $f(x) = -x,\ x \in \mathbb{N}$ can as well be written in the form $f(t) = -t,\ t \in \mathbb{N}$. The variable x (or t or u) is also called an **independent variable**, and $f(x)$ is **dependent** on the independent variable. Sometimes, we write $f(x) = y,$ where y is the dependent variable.

Let us discuss more examples based on functions.

Example 10: Suppose $f(x) = 2x^3 - x$. Find $f(-1), f(0), f(2), f(\pi), f(x+h)$ and $\dfrac{f(x+h) - f(x)}{h}$, where x and h are real numbers and $h \neq 0$.

Solution: In this case, the rule defined for the function f tells us to subtract the independent variable x from twice its cube. Thus, we have

$$f(-1) = 2(-1)^3 - (-1) = -2 + 1 = -1$$

$$f(0) = 2(0)^3 - (0) = 0$$

$$f(2) = 2(2)^2 - (2) = 6$$

$$f(\pi) = 2\pi^3 - \pi$$

To find $f(x+h),$ we begin by writing the formula for f in more neutral terms, say as

$$f(\square) = 2(\square)^2 - (\square)$$

Then, we insert the expression $x+h$ inside each box, we obtain

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - (x+h) \\ &= 2(x^3 + 3x^2h + 3xh^2 + h^3) - (x+h) \\ &= 2x^3 + 6x^2h + 6xh^2 + 2h^3 - x - h \end{aligned}$$

Finally, if $h \neq 0$

$$\frac{f(x+h)-f(x)}{h}=\frac{[2x^3+6x^2h+6xh^2+2h^3-x-h]-(2x^3-x)}{h}$$
$$=6x^2+6xh+2h^2-1.$$

***

You may note that the expression $\frac{f(x+h)-f(x)}{h}$ is called a difference quotient and will be used in later units to compute the derivatives.

Example 11: It is known that an object dropped from a height in a vaccum will fall a distance of d metre in t seconds according to the formula $d(t)=9t^2,\ t\geq 0$.

i) How far will the object fall in the first second? In the next 2 seconds?

ii) How far will it fall during the time interval $t=1$ to $t=1+h$ seconds?

iii) What is the average rate of change of distance (in m/sec.) during the time $t=1$ sec. to $t=3$ sec.?

iv) What is the average rate of change of distance during the time $t=x$ sec. to $t=x+h$ sec.?

Solution: i) $d(1)=9$

In the first second, the object will fall $9m$.

In the next two seconds, the object will fall $d(3)-d(1)=81-9=72m$.

Therefore, the object will fall 72 m in next 2 sec.

ii) The required distance $=d(1+h)-d(1)=9(1+h)^2-9(1)^2=9h^2+18h$.

iii) Average rate of change of distance

$$=\frac{\text{change in distance}}{\text{change in time}}=\frac{d(3)-d(1)}{3-1}=\frac{72}{2}=36\ \text{m/sec.}$$

iv) Average rate of change of distance from $t=x$ to $t=x+h$ is

$$=\frac{d(x+h)-d(x)}{(x+h)-x}=\frac{d(x+h)-d(x)}{h}=\frac{9(x+h)^2-9x^2}{h}=9h+18.$$

***

Certain functions are defined differently on different parts of their domain and are expressed in terms of more than one formula. We refer to such functions as **piecewise-defined functions**. The following example is of piecewise-defined function.

Example 12: Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x)=\begin{cases} x\cos x & \text{if } x<2 \\ 3x^2-1 & \text{if } x\geq 2\end{cases}$, find $f(-0.5), f(\pi/2)$ and $f(2)$. Here, x is in radian.

Solution: To find (-0.5), we use the first line of the formula because $-0.5<2$.

$f(-0.5)=-0.5\cos(-0.5)=-0.5\cos 0.5\approx -0.5\times 0.27=-1.35$

To find $f(\pi/2)$, we use the first line of the formula because

$\frac{\pi}{2}\approx 1.57<2\,; f(\pi/2)=\frac{\pi}{2}\cos\frac{\pi}{2}=\frac{\pi}{2}(0)=0$

Finally, because $2 \ge 2$, we use the second line of the formula to find $f(2)$.

$f(2) = 3(2)^2 - 1 = 11$.

***

Now, you may try the following exercises.

E14) Find the domain and range of the following functions defined as $y = f(x)$:

i) $y = 3 - |x|$

ii) $y = \dfrac{12}{x}$

iii) $y = x^4 - 2x^2 - 3$.

E15) If $f(t) = \sqrt{t^2 - 16}$, find all values of t, for which, $f(t)$ is a real. Also, find t, for which, $f(t) = 3$.

E16) If $g(x) = \dfrac{4 - x^2}{x^2 + x}$, find the domain of $g(x)$. Also, solve $g(x) = 0$.

E17) Let $f(x) = \dfrac{|x|}{x}, \ x \ne 0$.

i) Find $f(-7)$ and $f(3)$.

ii) For what value of x, the function is defined.

iii) Find the range of f.

iv) Does $f(2+8)$ equal $f(2) + f(8)$? Justify.

v) Does $f(-1+6)$ equal $f(-1) + f(6)$? Justify.

E18) Let $f(x) = 3x + 2$. Does $f(a^2)$ ever equal $(f(a))^2$? Justify.

E19) A charter bus has 50 seats and will not run unless at least 30 of those seats are filled. When there are 30 passengers, a ticket costs Rs.30, but each ticket is reduced by Rs.5 for every passenger over 30. Express the total amount collected by the charter bus as a function of the number of passengers p.

In addition to various types of functions, you have studied in Unit 2, we extend our study to more functions such as greatest integer function, rational function, polynomial function, trigonometric functions, etc.

1. **The Greatest Integer Function:** Take a real number x. Either it is an integer, say n (so that $x = n$) or it is not an integer. If it is not an integer, using l.u.b. property it can be shown that there is an $n \in \mathbb{Z}$ such that $n \le x < n+1$.[The proof of this is not included in this course.] Further, for a given real number x, we can find only one such integer n. We say that n is the greatest integer not exceeding x, and denote it by $[x]$. For example, $[3] = 3$ and $[3.5] = 3, [-3.5] = -4$.

   Let us consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x], \ x \in \mathbb{R}$, which assumes the value of the greatest integer, that is less than or equal to x. Such a function is called the **greatest integer function**. The graph of the function is as shown in Fig. 6. (It resembles the steps of an infinite staircase).

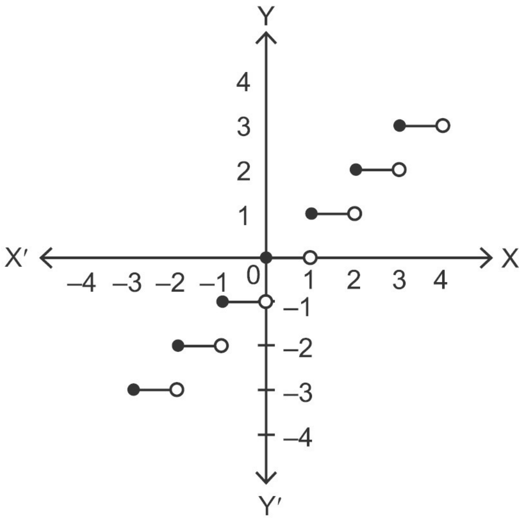


Fig. 6: $f(x) = [x]$

Notice that the graph consists of infinite many line segments of unit length, all parallel to the x-axis. Also, from the definition of $[x]$, we can see that

$[x] = -2$ for $-2 \le x < -1$

$[x] = -1$ for $-1 \le x < 0$

$[x] = 0$ for $0 \le x < 1$

$[x] = 1$ for $1 \le x < 2$

$[x] = 2$ for $2 \le x < 3$ and so on.

2. **Polynomial Functions:** A polynomial function is a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_n$, where $a_0, a_1, ..., a_n \in \mathbb{R}$ and n is a non-negative integer. If $a_0 \neq 0$, the integer n is called the degree of the polynomial. The functions defined by $f(x) = 2x^3 + 5x - \frac{1}{2}$, $f(x) = \sqrt{2}x^4 - \pi x$, etc, are some examples of polynomial functions, whereas the function g defined by $g(x) = 2x^{\frac{1}{2}} + 3$ is not a polynomial function. (why?)

3. **Rational Functions:** A rational function is the quotient of two polynomial functions. It can be written in the form $f(x) = g(x)/k(x)$, where $g(x)$ and $k(x)$ are polynomial functions of degree n and m. This is defined for all real x, for which $k(x) \neq 0$. For example,

$$f(x) = \frac{2x-3}{x^4 + 2x - 5},\ f(x) = \frac{1}{x^2}, \text{etc.}$$

4. **Signum Function:** The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is called the **signum function**. The domain of the signum function is $\mathbb{R}$ and the range is $\{-1, 0, 1\}$. The graph of the signum function is given in Fig. 7.

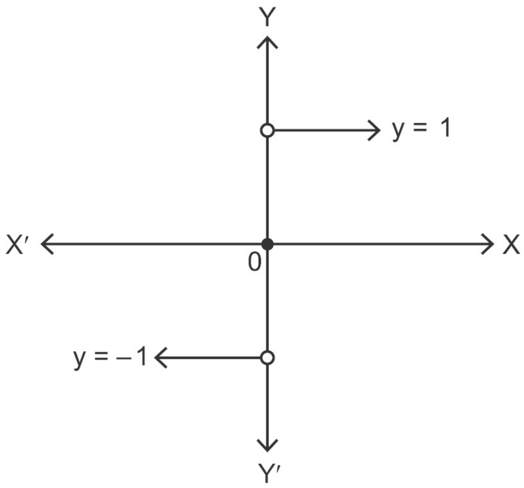
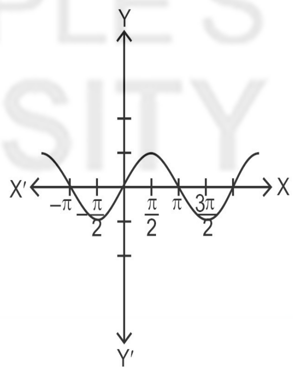
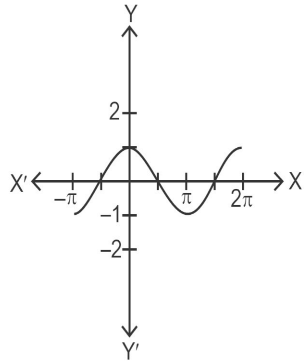
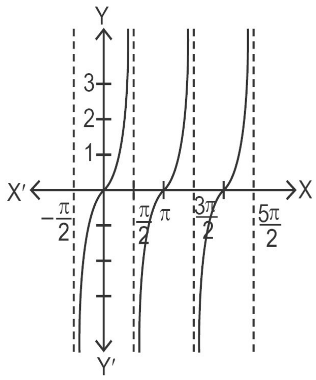
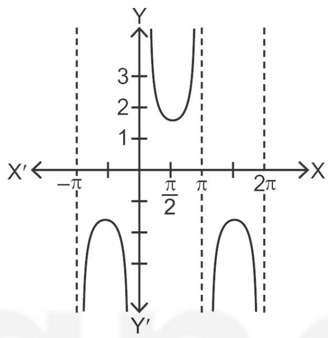
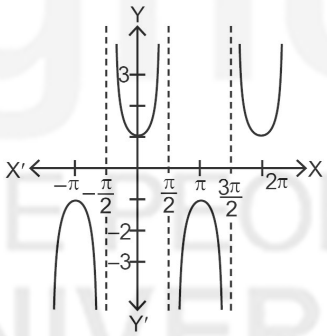
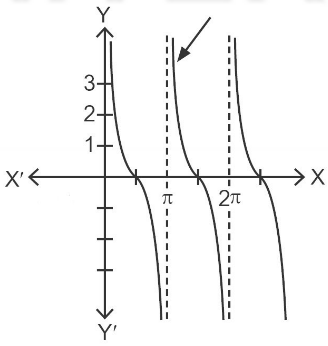


Fig. 7: The Signum Function

5. **Trigonometric (or Circular) Functions:** Trigonometric functions are the sine, cosine, tangent, secant, cosecant, and cotangent functions. The forms of trigonometric functions with their domain, range and graph are given in Table 3.

Table 3: Trigonometric functions.

Functions	Domain	Range	Graph
$f(x) = \sin x$	$\mathbb{R}$	$[-1, 1]$	
$f(x) = \cos x$	$\mathbb{R}$	$[-1, 1]$	

$f(x) = \tan x$	$\mathbb{R} - \left\{(2n+1)\dfrac{\pi}{2} : n \in \mathbb{Z}\right\}$	$\mathbb{R}$	
$f(x) = \operatorname{cosec} x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$	$\mathbb{R} -]-1,1[$	
$f(x) = \sec x$	$\mathbb{R} - \left\{(2n+1)\dfrac{\pi}{2} : n \in \mathbb{Z}\right\}$	$\mathbb{R} -]-1,1[$	
$f(x) = \cot x$	$\mathbb{R} = \{n\pi : n \in \mathbb{Z}\}$	$\mathbb{R}$	

Let us draw the graph of functions in the following examples.

Example 13: Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $y = f(x) = x^2, x \in \mathbb{R}$. Complete the Table given below. What is the domain and range of the function? Draw the graph of f.

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$									

Solution: The completed table is given below:

x	-4	-3	-2	-1	0	1	2	3	4
$y=f(x)=x^2$	16	9	4	1	0	1	4	9	16

Domain of $f=\{x : x\in \mathbb{R}\}$. Range of $f=\{x^2 : x\in \mathbb{R}\}$. The graph of f is given by Fig. 8.

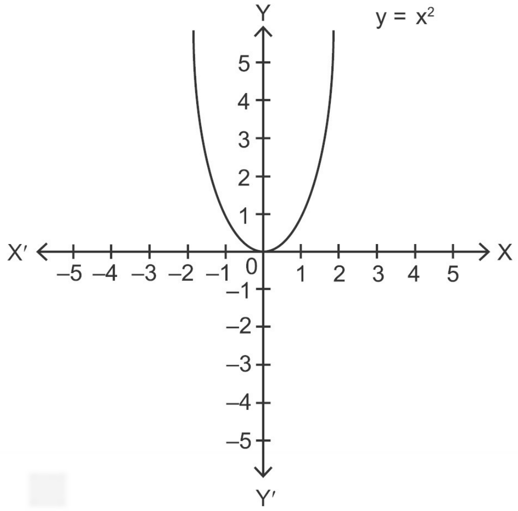


Fig. 8: Graph of $f(x)=x^2$

***

Example 14: Draw the graph of the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^3, x\in\mathbb{R}$.

Solution: We have
$f(0)=0, f(1)=1, f(-1)=-1, f(2)=8, f(-2)=-8, f(3)=27 : f(-3)=-27,$ etc.
The graph of f is given in Fig. 9.

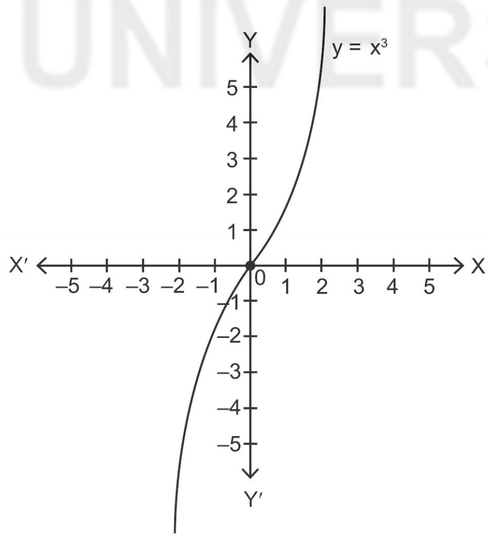


Fig. 9: Graph of $f(x)=x^3$

***

Example 15: Consider the real valued function $f:\mathbb{R}-\{0\}\to\mathbb{R}$ defined by $f(x)=\dfrac{1}{x}, x\in\mathbb{R}-\{0\}$. Complete the table given below using this definition. What is the domain and range of the function?

x	-2	-1.5	-1	-0.5	0.25	0.5	1	1.5	2
$y=\frac{1}{x}$	...	...	...	...	...	...	...	...	...

Solution: The completed table is given by

x	-2	-1.5	-1	-0.5	0.25	0.5	1	1.5	2
$y=\frac{1}{x}$	-0.5	-0.67	-1	-2	4	2	1	0.67	0.5

The domain is the set of all real numbers except 0 and its range is also the set all real numbers except 0. The graph of f is given in Fig. 10.

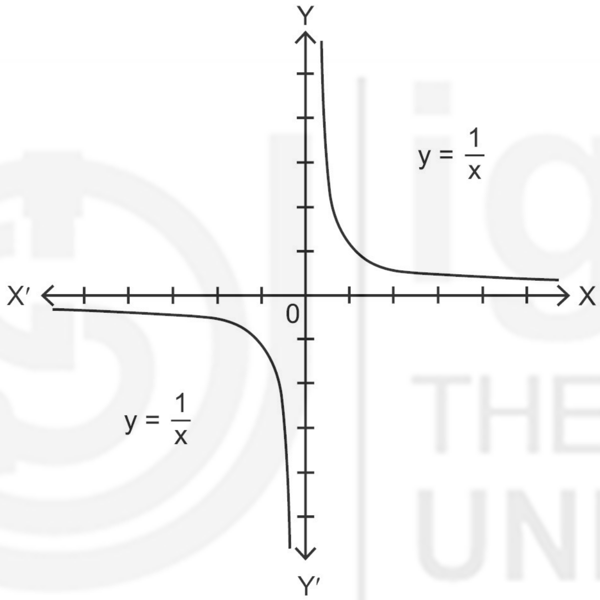


Fig. 10: Graph of $f(x)=\frac{1}{x}$

***

Example 16: Draw the graph of the equation $x^2+y^2=16$.

Solution: The graph of the equation $x^2+y^2=16$ is a circle of radius 4, centered at origin. Algebraically, by solving this equation for y in terms of x, we get

$$y=\pm\sqrt{16-x^2}$$

This equation does not define y as a function of x because the right side is "multiple valued" in the sense that values of x in the interval]-4, 4[produce two corresponding values of y. For example, if x = 2, then $y=\pm\sqrt{12}$, and hence (2, $\sqrt{12}$) and (2, $-\sqrt{12}$) are two points on the circle that lie on the same vertical line as shown in Fig.11 (a). However, we can regard the circle as the union of two semicircles $y=\sqrt{16-x^2}$ and $y=-\sqrt{16-x^2}$ as shown in Fig. 11 (b).

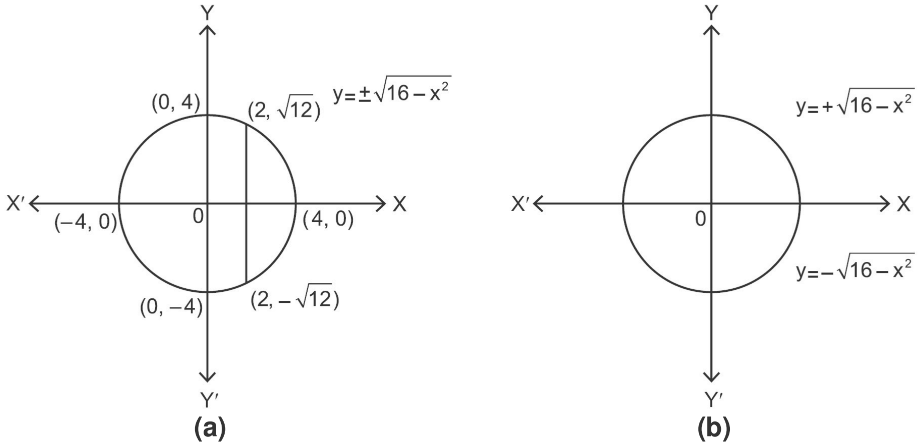


Fig. 11: Graph of equation $x^2 + y^2 = 16$

***

Now, try the following exercises:

E20) Sketch the graph of the function defined piecewise by the formula

$$f(x) = \begin{cases} 0, & x \le -1 \\ \sqrt{1-x^2}, & -1 < x < 1 \\ x, & x \ge 1 \end{cases}$$

E21) Given below are the graphs of four functions depending on the notion of absolute value. The functions are $x \to -|x|$, $x \to |x|+1$, $x \to |x+1|$, $x \to |x-1|$, though not necessarily in this order. (The domain in each case is $\mathbb{R}$). Can you identify them?

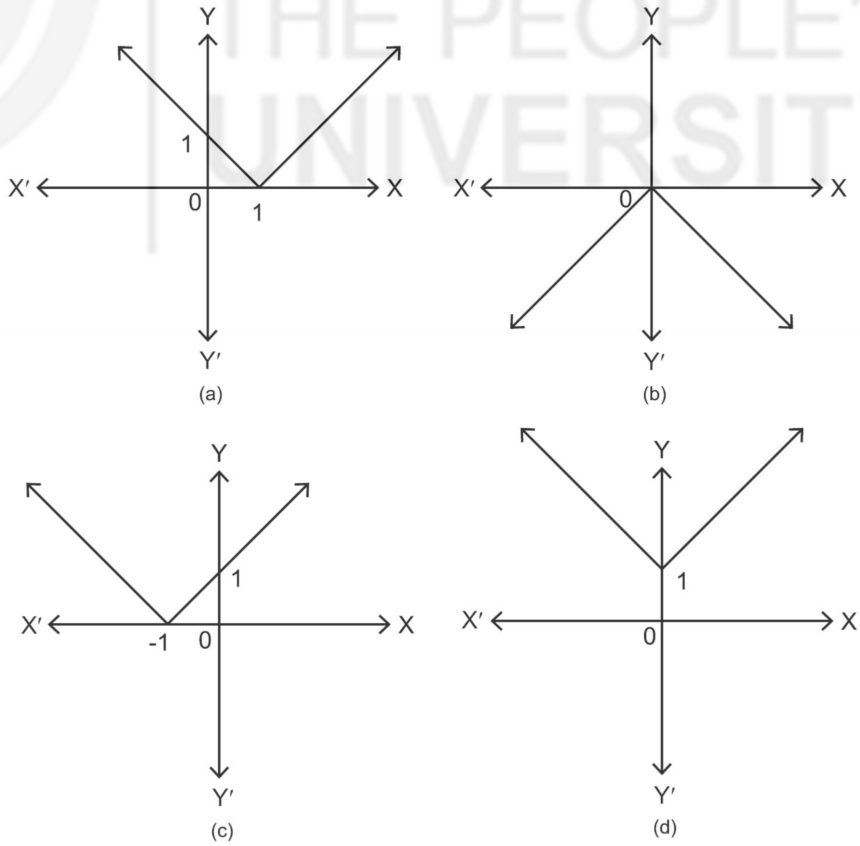


Fig. 12

So far, in this section, we have discussed functions and their graphs. You have studied that the inverse of a function f, which is denoted by f^{-1}, exists if f is one-one and onto in Unit 2. Here, we shall extend our study about the inverse functions.

Let us begin this with some examples.

Example 17: Consider a function f from $\mathbb{R}$ to $\mathbb{R}$, defined as $f(x) = \frac{x^5}{5} + 2$.

Find f^{-1}.

Solution: Before finding the inverse, let us first check whether the function is one-one and onto.

Let $x, y \in \mathbb{R}$ such that $f(x) = f(y)$

$\frac{x^5}{5} + 2 = \frac{y^5}{5} + 2 \Rightarrow x = y$.

Therefore, f is one-one.

Now, we solve $\frac{x^5}{5} + 2 = y$ for x in terms of y.

This gives us $x = \{5(y-2)\}^{\frac{1}{5}}$.

Thus, we have found the required pre-image for f and hence f is onto by definition.

Hence, f is one-one onto. Thus, f^{-1} exists, and f^{-1} is the function on $\mathbb{R}$ defined by $f^{-1}(y) = \{5(y-2)\}^{\frac{1}{5}}$.

***

Example 18: Consider the funciton $f : \{3,5\} \to \{2,4\}$. If $f(3) = 4$ and $f(5) = 2$, find if inverse is possible, $f^{-1}(3), f^{-1}(4), f^{-1}(5), f^{-1}(2)$.

Solution: The given function f is one-one and onto. Thus, f^{-1} exists. The domain of f is $\{3,5\}$ and range of f is $\{4,2\}$. Accordingly, domain of f^{-1} is $\{4,2\}$ and range of f^{-1} is $\{3,5\}$. Thus, $f^{-1}(3)$ does not exist.

Similarly, $f^{-1}(4) = 3$, $f^{-1}(5)$ does not exist and $f^{-1}(2) = 5$.

***

Now, let us take up the problem of drawing the graph of an inverse function.

There is an interesting relation between a pair of the graphs of a function and its inverse function because of which, if the graph of one of them is known, the graph of the other can be obtained easily.

Let $f : X \to Y$ be a one-one and onto function, and let $g : Y \to X$ be the inverse of f. A point (p, q) lies on the graph of $f \Leftrightarrow q = f(p) \Leftrightarrow p = g(q) \Leftrightarrow (q, p)$ lies on the graph of g. Now the points (p, q) and (q, p) are reflections of each other with respect to (w.r.t.) the line $y = x$. Therefore, we can say that the graphs of f and g are reflections (or mirror image) of each other w.r.t. the line $y = x$.

w.r.t stands for with respect to

Therefore, it follows that, if the graph of one of the functions f and g is given, that of the other can be obtained by reflecting it w.r.t. the line $y = x$. As an illustration, the graphs of the functions $y = x^3$ and $y = x^{1/3}$ are given in Fig. 13.

Do you agree that these two functions are inverses of each other? If the sheet of paper on which the graphs have been drawn is folded along the line $y = x$, the two graphs will exactly coincide.

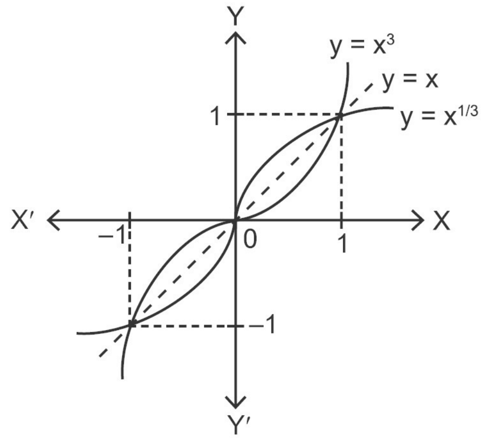


Fig. 13: Graph of two inverse functions $y = x^3$ and $y = x^{1/3}$

If a given function is not one-one and onto on its domain, we can choose a subset of the domain as well as codomain on which it is one-one and onto, and then define its inverse function in the domain in which the function is one-one and onto. For example, consider the function $f(x) = \sin x$. We know that $\sin(x + 2\pi) = \sin x$. The function is neither one-one nor onto on $\mathbb{R}$. But if we restrict the domain to the inverval $[-\pi/2, \pi/2]$ and codomain to $[-1,1]$, we find that it is one-one and onto. Thus, if $f(x) = \sin x \forall x \in [-\pi/2, \pi/2]$, then we can define

$$f^{-1}(x) = \sin^{-1}(x) = y \text{ if } \sin y = x.$$

Similarly, we can define $\cos^{-1}$ and $\tan^{-1}$ functions as inverse of cosine and tangent functions if we restrict the domains to $[0, \pi]$ and $]-\pi/2, \pi/2[$, respectively. The inverse trigonometric functions along with their domains and ranges are given in Table 4.

Table 4: Inverse Trigonometric Funcitons

Inverse Trigonometric Function	Domain	Range
$\sin^{-1}$	$[-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	$[-1,1]$	$[0, \pi]$
$\operatorname{cosec}^{-1}$	$\mathbb{R} -]-1,1[$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	$\mathbb{R} -]-1,1[$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\tan^{-1}$	$\mathbb{R}$	$\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$
$\cot^{-1}$	$\mathbb{R}$	$]0, \pi[$

Remark 3: The function $\sin^{-1} x$ should not be confused with $(\sin x)^{-1}$. In fact, $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.

Try the following exercise.

E22) Draw the graph of inverse trigonometric functions given in Table 4 and compare them with corresponding trigonometric functions.

Now let us discuss various types of functions in the following section.

6.7 TYPES OF FUNCTIONS

In this section, we shall talk about various types of functions, namely, even, odd, increasing, decreasing and periodic functions. In each case, we shall also try to explain the concept through graphs.

6.7.1 Even and Odd Functions

Consider the function f defined by $f(x) = x^2 \quad \forall x \in \mathbb{R}$. You will notice that $f(-x) = (-x)^2 = x^2 = f(x)\ \forall x \in \mathbb{R}$.

This is an example of an even function. Let's take a look at the graph (Fig.14) of this function. We find that the graph (a parabola) is symmetrical about the y-axis. If we fold the paper along the y-axis, we shall see that the parts of the graph on both sides of the y-axis completely coincide with each other. Such functions are called even functions.

Thus, a function f, defined on a domain D is **even function**, if for each $x \in D,\ f(-x) = f(x)$.

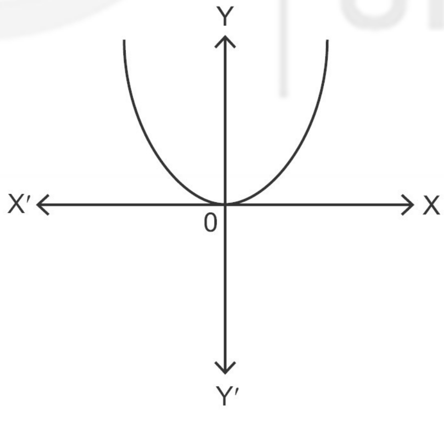


Fig. 14: Graph of even function

The graph of an even function is symmetric with respect to the y-axis. We also note that if graph of a function is symmetric with respect to the y-axis, the function must be an even function. Thus, if we are required to draw the graph of an even function, we can use this property to our advantage. We only need to draw that part of the graph which lies to the right of the y-axis and then just take its reflection w.r.t. the y-axis to obtain the part of the graph which lies to the left of the y-axis.

Now, try the following exercise.

E23) Given below are two examples of even functions, along with their graphs. Try to convince yourself, by calculations as well as by looking at the graphs, that both the functions are, indeed, even functions.

i) The absolute value function on $\mathbb{R}$ $f : x \to |x|$. The graph of f is shown in Fig. 15(a).

ii) The function g defined on the set of non-zero real numbers by setting $g(x) = 1/x^2,\ x \neq 0$. The graph of g is shown in Fig. 15(b).

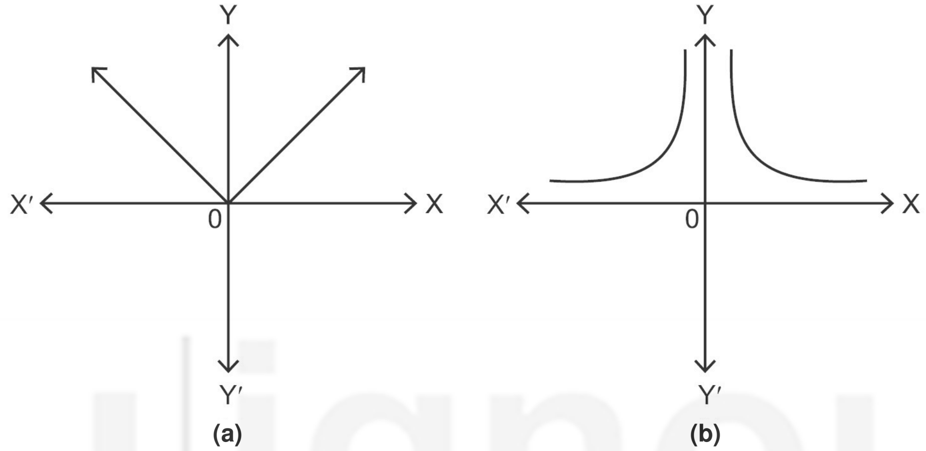


Fig. 15

Now, let us consider the function f **defined** by setting $f(x) = x^3\ \forall x \in \mathbb{R}$. We observe that $f(-x) = (-x)^3 = -x^3 = -f(x)\ \forall x \in \mathbb{R}$.

If we consider another function g, given by $g(x) = \sin x$, we shall be able to note again that $g(-x) = \sin(-x) = -\sin x = -g(x)$.

The functions f and g above are similar in one respect, that is, the image of $-x$ is the negative of the image of x. Such functions are called odd funtions. Thus, a function f defined on D is said to be an **odd function** if $f(-x) = -f(x)\ \ \forall x \in D$.

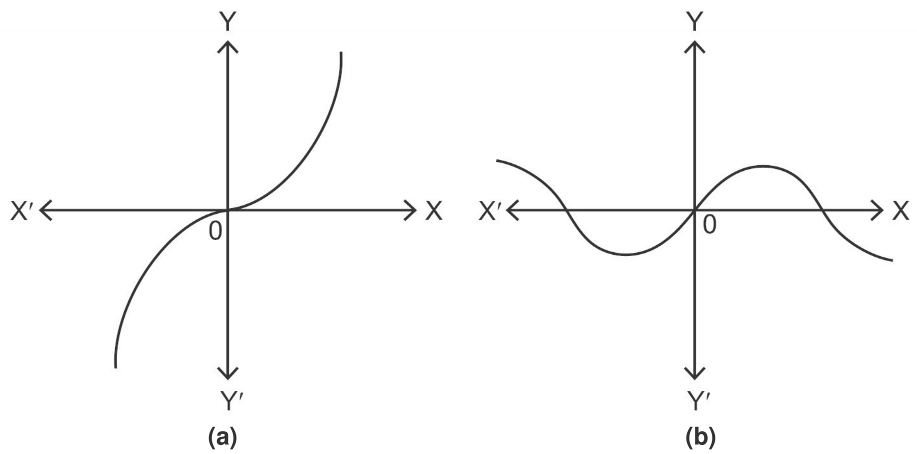


Fig. 16: Graph of odd function

If $(x, f(x))$ is a point on the graph of an odd function f, then $(-x, -f(x))$ is also a point on it. This can be expressed by saying that the graph of an odd function is symmetric with respect to the origin. In other words, if you rotate the graph of an odd function through 180° about the origin you will find that you get the original graph again as shown in Fig. 16. Conversely, if the graph of a function is symmetric with respect to the origin, the function must be an odd function. The above facts are often useful while handling odd functions.

Now, try the following exercise.

E24) We are giving below two functions along with their graphs in Fig. 17(a) and 17 (b) respectively. By calculations as well as by looking at the graphs, find out whether each is ever or odd.

i) The identity function f on $\mathbb{R}$ defined as $f(x) = x$ shown in Fig.17 (a).

ii) The function g defined on the set of non-zero real numbers by setting $g(x) = 1/x,\ x \neq 0$ shown in Fig. 17(b).

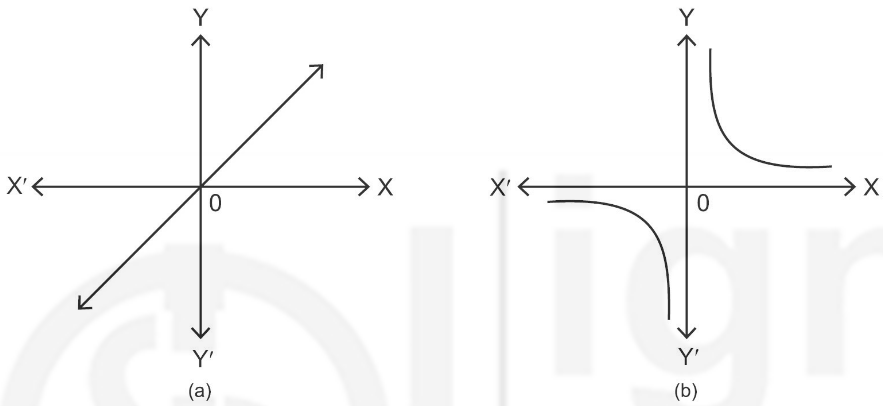


Fig. 17

While many of the functions that you will come across in this course will turn out to be either even or odd, there will be many more which will be neither even nor odd.

Consider, for example, the function f on $\mathbb{R}$, defined by

$f(x) = (x+1)^2$

Here $f(-x) = (-x+1)^2 = x^2 - 2x + 1$. Is $f(x) = f(-x)\ \forall x \in \mathbb{R}$?

The answer is 'no'. Therefore, f is not an even function.

Is $f(x) = -f(-x)\ \forall x \in \mathbb{R}$? Again, the answer is 'no'. Therefore f is not an odd function. The same conclusion could have been drawn by considering the graph of f which is given in Fig. 18.

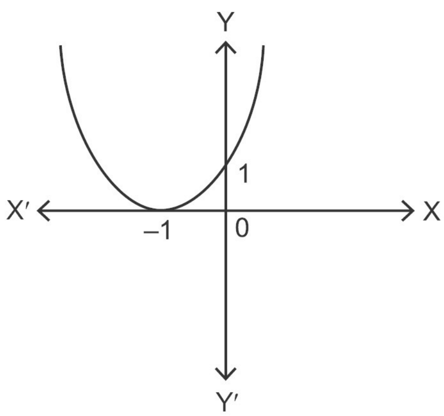


Fig. 18: Graph of a function neither ever nor odd.

You will observe that the graph is symmetric neither with respect to the y-axis, nor with respect to the origin.

Now, there should be no difficulty in solving the exercise below.

E25) Which of the following functions are even, which are odd, and which are neither even nor odd?

i) $f(x) \to x^2 + 1 \quad \forall x \in \mathbb{R}$

ii) $f(x) \to x^3 - 1 \quad \forall x \in \mathbb{R}$

iii) $f(x) \to \cos x, \quad \forall x \in \mathbb{R}$

iv) $f(x) \to x\,|x| \quad \forall x \in \mathbb{R}$

v) $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

6.7.2 Monotonic Functions

Does the profit of a company increase with production?
Does the volume of gas decrease with increase in pressure?
Problems like these require the use of increasing or decreasing function. Any function which is any one of these types is called a **monotone function**.
Now, let us see what we mean by an increasing function.

Example 19: Consider the functions g and h defined by

$g(x) = x^3$ and $h(x) = \begin{cases} -1 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$

Note that whenever $x_2 > x_1$, we get $x_2^3 > x_1^3$, that is, $g(x_2) > g(x_1)$.
In other words, as x increases, $g(x)$ also increases. This fact can also be seen from the graph of g shown in Fig. 19.

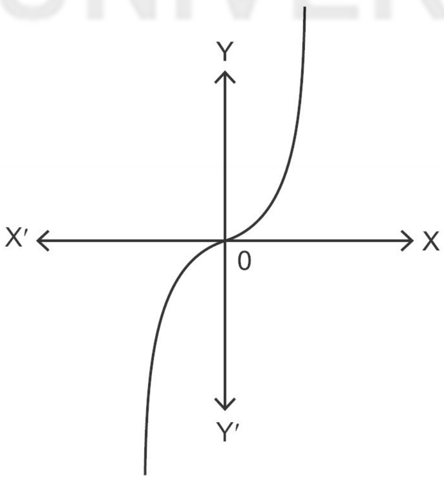


Fig. 19: Graph of g

Let us find out how $h(x)$ behaves as x increases. In this case, we see that if $x_2 > x_1$, then $h(x_2) \ge h(x_1)$. (You can verify this by choosing any values for x_1 and x_2). Equivalently, we can say that $h(x)$ increases (or does not decrease) as x increases. The same can be seen from the graph of h in Fig. 20.

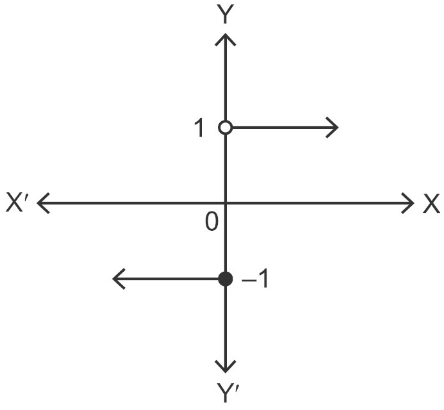


Fig. 20: Graph of h

Functions like g and h above are called **increasing or non-decreasing functions.** This leads to the following definition:

Definition: A function f defined on a domain D is said to be **increasing** (or **non-decreasing**) if, for every pair of elements $x_1\ x_2 \in D,\ x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$. Further, we say that f is **strictly increasing** if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$ (strict inequality).

Clearly, the function $g(x) = x^3$ discussed above, is a strictly increasing function while h is not a strictly increasing function.

***

We shall now study another concept which is, in some sense, complementary to that of an increasing function.

Example 20: Consider the function f defined on $\mathbb{R}$ by setting

$$f(x) = \begin{cases} 1, & \text{if } x \leq -1 \\ -x, & \text{if } -1 < x < 1 \\ -1, & \text{if } x \geq 1 \end{cases}$$

The graph of f is as shown in Fig. 21.

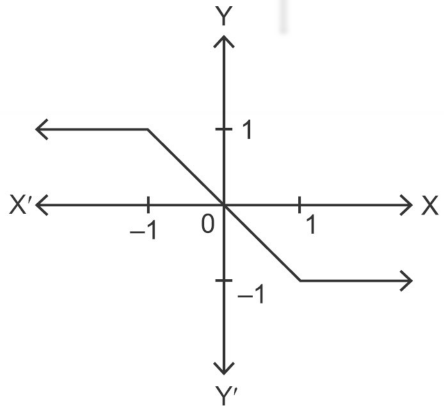


Fig. 21: The graph of f

From the graph we can easily see that as x increases f does not increase. That is, $x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1)$ or $f(x_2) \not> f(x_1)$.

Now consider the function $g : x \to -x^3 (x \in \mathbb{R})$.

The graph of g is shown in Fig. 22.

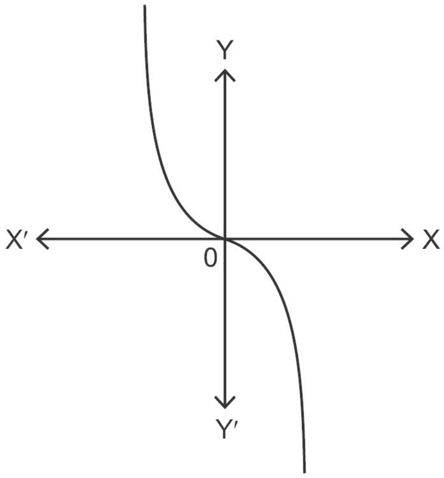


Fig. 22: The graph of g

Since $x_2 > x_1 \Rightarrow x_2^3 > x_1^3 \Rightarrow -x_2^3 < -x_1^3 \Rightarrow g(x_2) < g(x_1)$, we find that as x increases, $g(x)$ decreases. Functions like f and g are called **decreasing or non-increasing functions**.

***

The above example suggests the following definition:

Definition: A function f defined on a domain D is said to be **decreasing** (or non-increasing) if for every pair of elements $x_1, x_2 \in D$ with $x_2 > x_1$ we have $f(x_2) \le f(x_1)$. Further, f is said to be **strictly decreasing** if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$.

We have seen in Example 20 that, from the two decreasing functions f and g, the function g is strictly decreasing, while f is decreasing but not strictly decreasing.

Let us now give the definition of monotonic function.

Definition: A function f defined on a domain D is said to be a **monotonic function** if it is either increasing or decreasing on D.

The phrases 'monotonically increasing' and 'monotonically decreasing' are often used for 'increasing' and 'decreasing', respectively. While many functions are monotone, there are many others which are not monotone. Consider, for example, the function $f : x \to x^2 (x \in \mathbb{R})$. You have seen the graph of f in Fig. 8. This function is neither increasing nor decreasing.

If we find that the given function is not monotone, we can still determine some subsets of the domain on which the function is increasing or decreasing. For example, the function $f(x) = x^2$ is strictly decreasing in $]-\infty, 0]$ and is strictly increasing in $[0, \infty[$.

Now, try the following exercise.

E26) Fig. 23 shows the graphs of some functions. Classify them as non-decreasing, strictly decreasing, neither increasing nor decreasing:

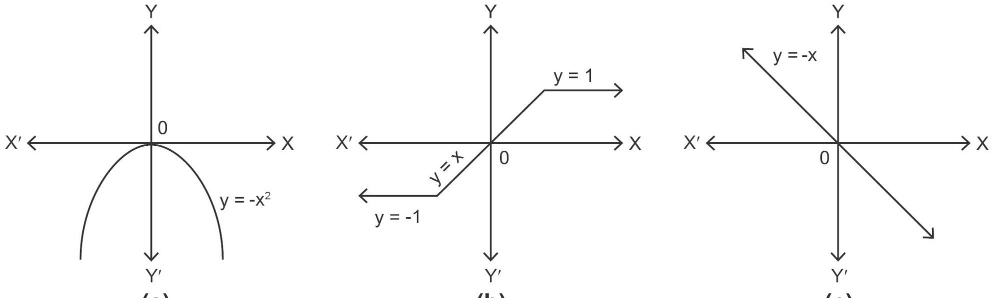


Fig. 23

Now, we shall discuss periodic functions.

6.7.3 Periodic Functions

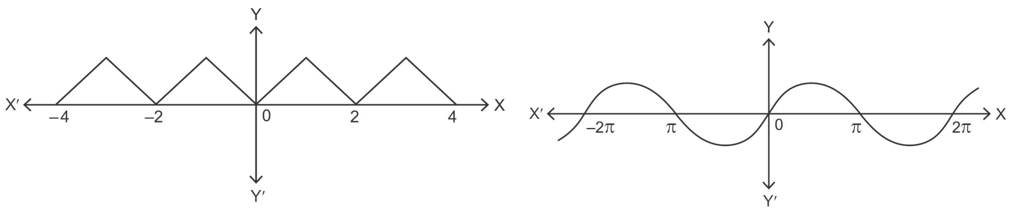
Periodic functions occur very frequently in applications of mathematics to various branches of science. Many phenomena in nature such as propogation of water waves, sound waves, light waves, electromagnetic waves, heartbeats, orbits, etc. are periodic and we need periodic functions to describe them. Similarly, seasons, monsoon etc. can also be described in terms of periodic functions.

Look at the following patterns:

(a) (b)

Fig. 24: Patterns

You must have come across patterns similar to the ones shown in Fig. 24 on the borders of sarees, wall papers etc. In each of these patterns a design keeps on repeating itself. A similar situation occurs in the graphs of periodic functions. Look at the graphs in Fig. 25.



(a) (b)

Fig. 25: Graph of a periodic function

In each of the figures shown above the graph consists of a certain pattern repeated infinitely. Both these graphs represent periodic functions. To understand the situation, let us examine these graphs closely.

Consider the graph in Fig. 25(a). The portion of the graph lying between $x = -1$ and $x = 1$ is the graph of the function $x \to |x|$ on the domain $-1 \le x \le 1$.

This portion is being repeated both to the left as well as to the right, by translating (pushing) the graph through two units along the x-axis. That is to say, if x is any point of $[-1, 1]$, then the ordinates at $x, x \pm 2, x \pm 4, x \pm 6, \ldots$ are all equal. The graph, therefore, represents the function f defined by $f(x) = |x|$, if $-1 \le x \le 1$ and $f(x+2) = f(x)$.

The graph in Fig. 25(b) is the graph of the sine function, $x \to \sin x, \forall x \in \mathbb{R}$. You will notice that the portion of the graph between 0 and 2π is repeated both to the right and to the left. You know already that $\sin(x + 2\pi) = \sin x$, $\forall x \in \mathbb{R}$. We now give a precise meaning to the term "a periodic function".

Definition: A function f defined on a domain D is said to be a **periodic function** if there exists a positive real number p such that $f(x+p) = f(x)$ for all $x \in D$. The number p is said to be a period of f.
If there exists a smallest positive p with the property described above, it is called the **period of f**.

As you know, $\tan(x + n\pi) = \tan x \ \forall n \in \mathbb{N}$. This means that $n\pi, n \in \mathbb{N}$ are all periods of the tangent function. The smallest of these, that is π, is the period of the tangent function.

See if you can do this exercise.

E27) i) What are the periods of the functions given in Fig. 25?

ii) Can you give one other period of each of these functions?

As another example of a periodic function, consider the function f defined on $\mathbb{R}$ by setting $f(x) = x - [x]$.

Let us recall that $[x]$ stands for the greatest integer not exceeding x.
The graph of this function is as shown in Fig. 26.

From the graph (as also by calculation) we can easily see that $f(x+n) = f(x) \ \forall x \in \mathbb{R}$, and for each positive integer n.

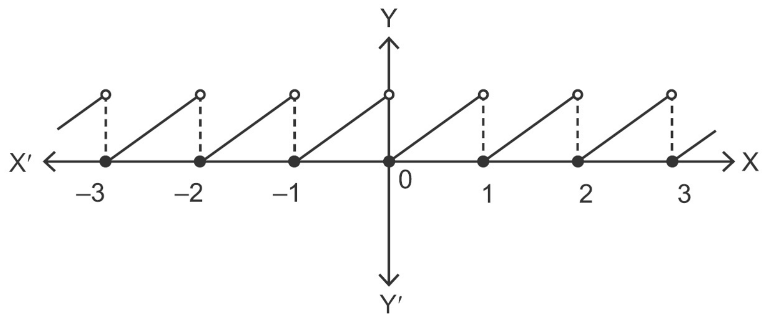


Fig. 26: Graph of f

The given function is therefore periodic, the numbers $1, 2, 3, 4$ being all the periods. The smallest of these, namely 1 , is the period.
Thus, the given function is periodic and has the period 1.

Remark 4: Monotonicity and periodicity are two properties of functions which cannot coexist except for a constant function. A monotone function can never be periodic, and a periodic function can never be monotone.
In general, it may not be easy to decide whether a given function is periodic or not. But sometimes it can be done in a straightforward manner.

Example 21: Find whether the function $f : x \to x^2 \; \forall x \in \mathbb{R}$ is periodic or not.

Solution: We start by assuming that it is periodic with period p.
Then, we must have $p > 0$ and $f(x+p) = f(x) \; \forall x$.

$\Rightarrow \quad (x+p)^2 = x^2 \;\; \forall x$

$\Rightarrow \quad 2xp + p^2 = 0 \;\; \forall x$

$\Rightarrow \quad p(2x+p) = 0 \;\; \forall x$

Considering $x \neq -p/2$, we find that $2x + p \neq 0$. Thus, $p = 0$. This is a contradiction.

Therefore, there does not exist any positive number p such that $f(x+p) = f(x), \; \forall x \in \mathbb{R}$ and, consequently, f is not periodic.

***

Now try the following exercises.

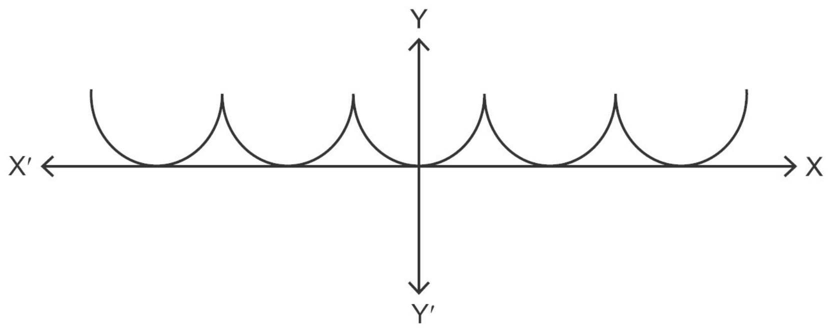
E28) Examine whether the following functions are periodic or not. Write the periods of the periodic functions.

i) $x \to \cos x$ ii) $x \to x + 2$

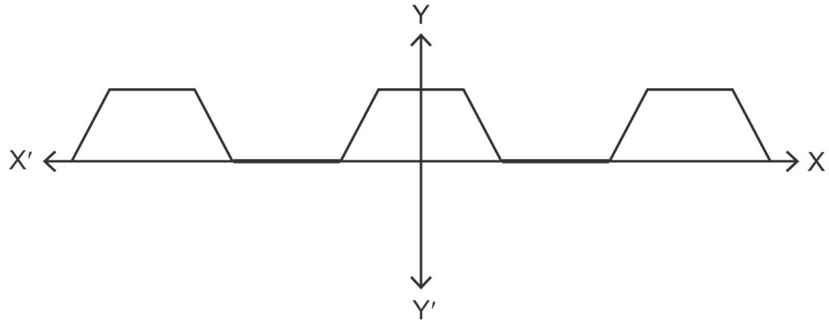
iii) $x \to \sin 2x$ iv) $x \to \tan 3x$

v) $x \to \cos(2x+5)$ vi) $x \to \sin x + \sin 2x$

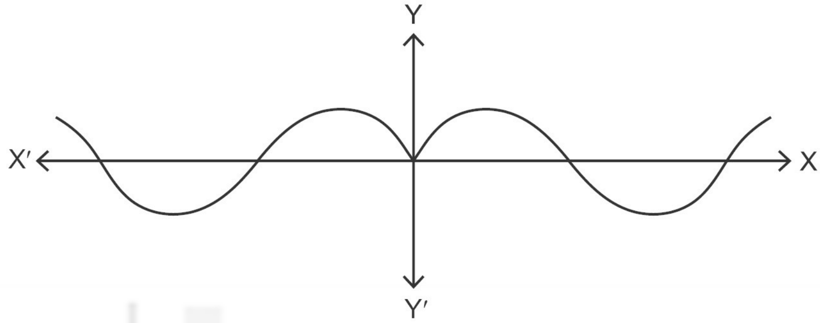
E29) The graphs of three functions are given below in Fig. 27. Classify the functions as periodic and non-periodic.



(a)



(b)



(c)

Fig. 27

E30) Is the sum of two periodic functions also periodic? Give reasons for your answer.

We end with summarising what we have discussed in this unit.

6.8 SUMMARY

In this unit we have covered the following points:

1. Briefly revised the basic properties of real numbers for addition, multiplication, archimedean property, etc.

2. Let S be a subset of $\mathbb{R}$. An element u in $\mathbb{R}$ is said to be an **upper bound** of S if $u \geq x$ holds for every x in S. In other words, for a given set, a number which is greater than or equal to all the elements is known as upper bound of that set. We say that S is **bounded above**, if there is an upper bound of S.

3. A **lower bound** for a given set S is a real number v such that $v \leq x$ for all $x \in S$. We shall say that a set is **bounded below**, if we can find a lower bound for it.

4. The upper bound of any subset S of $\mathbb{R}$, which is less than any other upper bound of S is called its **supremum** or **least upper bound (l.u.b.)**. Further, the lower bound of S which is greater than any other lower bound of S will be called its **infimum** or **greatest lower bound (g.l.b.)**.

5. A set $S \subset \mathbb{R}$ is **bounded** if it has both an upper bound and a lower bound.

6. The absolute value of a real number x is defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

7. The various types of intervals in $\mathbb{R}$, where $a, b \in \mathbb{R}$, are

Open: $]a, b[= \{x \in \mathbb{R} : a < x < b\}$

Closed: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Semi-open: $]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ or $[a, b[= \{x \in \mathbb{R} : a \leq x < b\}$

8. We discussed the graph of inverse function.

9. A function f, defined on a domain D, is even function, if for each $x \in D, f(-x) = f(x)$.

10. A function f, defined on a domain D, is odd function, if for each $x \in D$, $f(-x) = -f(x)$.

11. A function f defined on a domain D is said to be **increasing** (or **non-decreasing**) if, for every pair of elements $x_1\ x_2 \in D$, $x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$. Further, we say that f is **strictly increasing** if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$ (strict inequality).
Clearly, the function $g(x) = x^3$ discussed above, is a strictly increasing function while h is not a strictly increasing function.

12. A function f defined on a domain D is said to be **decreasing** (or non-increasing) if for every pair of elements $x_1, x_2 \in D$ with $x_2 > x_1$ we have $f(x_2) \leq f(x_1)$. Further, f is said to be **strictly decreasing** if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$.

13. A function f defined on a domain D is said to be a **monotonic function** if it is either increasing or decreasing on D.

14. A function f defined on a domain D is said to be a **periodic function** if there exists a positive real number p such that $f(x+p) = f(x)$ for all $x \in D$. The number p is said to be a period of f. If there exists a smallest positive p with the property described above, it is called the **period of f**.

6.9 SOLUTIONS/ANSWERS

E1) No, $(-3) \times (-2) = 6 \notin \mathbf{S}$

E2) Line 2, As associative law does not work for subtraction, i.e. $a - (b - c) \neq (a - b) - c$.

E3) i) Clearly 2 is an upper bound of S_1, and all other upper bounds are greater than 2. Therefore, Sup $(S_1) = 2$.
Similarly, 1 is a lower bound of S_1 and all other lower bounds are less than 1. Thus, $\text{Inf}\,(S_1) = 1$.

ii) Note that $S_2 = \{x \in \mathbb{R} : -\sqrt{5} < x < \sqrt{5}\}$ So, $\sqrt{5}$ is the least upper bound of S_2. Sup $(S_2) = \sqrt{5}$. $-\sqrt{5}$ is the greatest lower bound of S_2. Thus $\text{Inf}\,(S_2) = -\sqrt{5}.$

iii) Sup (S_3) does not exists and $\text{Inf}\,(S_3)$ does not exist.

iv) Sup $(S_4) = 0$, $\text{Inf}\,(S_4) = -1$

E4) Any example satisfying the condition may be given. One example of each set is given below:

i) The set $\{1, 2, 3, \ldots\}$ has a lower bound, e.g., 0.

ii) The set $\{\ldots -3, -2, -1, 0, 1, 2, \ldots\}$ does not have a lower bound.

iii) The g.l.b. of the set $S = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$ is 0 and $0 \notin S$.

iv) $\{x : x \in \mathbb{R}$ and $1 \le x \le 2\}$ is a bounded set as it is bounded above by 2 and below by 1.

E5) Yes, it is bounded, as supremum of A = 2 and Infimum of A = 1. Since the set A is bounded above as well as bounded below, it is bounded.

E6) Let us prove that the supremum of a set, if it exists, is unique. Suppose that $S \subseteq \mathbb{R}$ is bounded above and that $a, b \in \mathbb{R}$ are supremum of S. You may note that both a and b are upper bounds of S. Since a is the least upper bound of S and b is an upper bound of S, therefore $a \le b$. Similarly, b is a least upper bound and a is an upper bound of S, therefore $b \le a$. Thus $a = b$, shows that the supremum of a set is unique.
You may like to prove the uniqueness of infimum yourself.

E7) i) 15, ii) $2/3$, iii) 4.3, iv) -6, v) $2-\sqrt{2}$

E8) i) If $x > 0$, then $|x| = x$ and $\frac{|x|}{x} = \frac{x}{x} = 1$

ii) If $x < 0$, then $|x| = -x$ and $\frac{|x|}{x} = \frac{-x}{x} = -1$.

E9) i) $|-5| < |7|$, ii) $\left|\frac{1}{-8}\right| = \left|\frac{1}{8}\right|$, iii) $\left|2^x\right| = \left|(-2)^x\right|$ for all natural number x,

iv) $-|-10| < |-10|$

E10) If $x > 0$, then $|x| = x$, which imples $y = \frac{x}{|x|} = \frac{x}{x} = 1.$

If $x < 0$, then $|x| = -x$, which implies $y = \frac{x}{|x|} = \frac{x}{-x} = -1.$

Therefore, answer (a) is correct.

E11) i) $|x| = \max\{x, -x\}$. Hence $x = 0 \Rightarrow x = 0$
$|x| = 0 \Rightarrow \max\{x, -x\} = 0 \Rightarrow x = 0$

ii) If $x > 0, |x| = x$ and $|1/x| = 1/x = 1/|x|$

If $x < 0, |x| = -x$ and $|1/x| = -1/x = 1/|x|$

iii) $|x - y| = |x + (-y)| \le |x| + |-y| = |x| + |y|$

E12) i) False, ii) True, iii) True, iv) False

E13) i) $1 \le x \le 8$, ii) $-\infty < x < 2$, iii) $1 \le x \le 2$, iv) $5 < x < \infty$

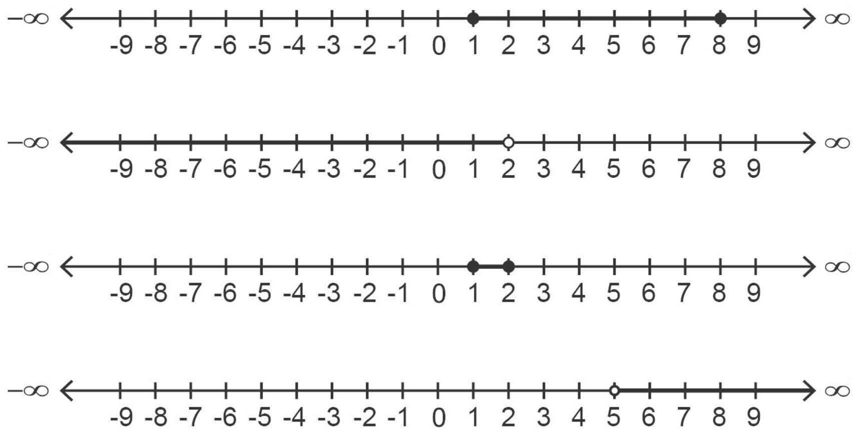


Fig. 28

E14) i) Domain = $\mathbb{R}$
Range = $]-\infty, 3]$

ii) Domain = $\mathbb{R} - \{0\}$
Range = $\mathbb{R}$

iii) Domain = $\mathbb{R}$
Range = $\mathbb{R}$

E15) $t^2 \ge 16, -4 \ge t \ge 4$

$f(t) = 3 \Rightarrow \sqrt{t^2 - 16} = 3 \Rightarrow t^2 - 16 = 9 \Rightarrow t^2 = 25 \Rightarrow t = 5 \text{ or} -5$.

E16) Domain of g = $\mathbb{R} - \{0, -1\}$
$g(x) = 0 \Rightarrow x = 2, -2$.

E17) i) $f(-7) = \frac{|-7|}{-7} = -1, \; f(3) = \frac{|3|}{3} = 1.$

ii) The function f is defined for $\mathbb{R} - \{0\}$

iii) Range of f is $\{1, -1\}$

iv) $f(2+8) = \frac{|2+8|}{2+8} = 1$

$f(2) + f(8) = \frac{|2|}{2} + \frac{|8|}{8} = 1 + 1 = 2$

Thus, $f(2+8) \ne f(2) + f(8)$

v) $f(-1+6) = \frac{|-1+6|}{-1+6} = 1$

$f(-1) + f(6) = \frac{|-1|}{1} + \frac{|6|}{6} = 1 + 1 = 2$

Thus, $f(-1+6) \neq f(-1)+f(6)$

E18) $f(a^2) = 3a^2 + 2$

$(f(a))^2 = (3a^2+2)^2 = 9a^4 + 12a^2 + 4$

Thus, $f(a^2) \neq (f(a))^2$

E19) Let the number of passengers be p and $T(p)$ denote the total amount collected. Then, $T(p) = \begin{cases} 900, & \text{if } p = 30 \\ 900 + (p-30).25, & \text{if } 50 \geq p > 30 \end{cases}$

E20)

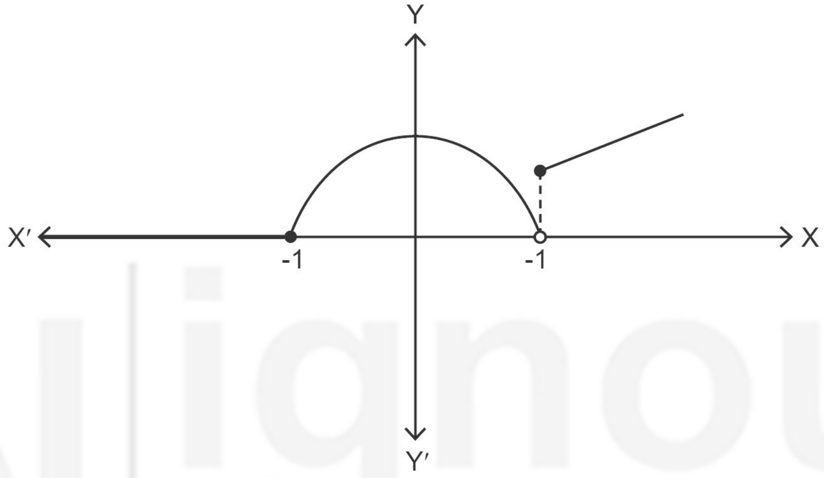


Fig. 29

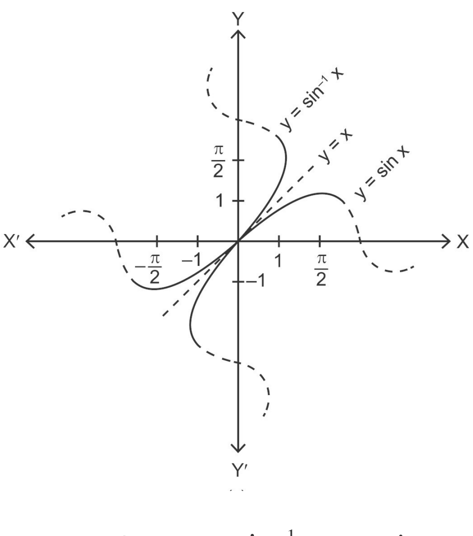
E21) i) $x \rightarrow |x-1|$

ii) $x \rightarrow -|x|$

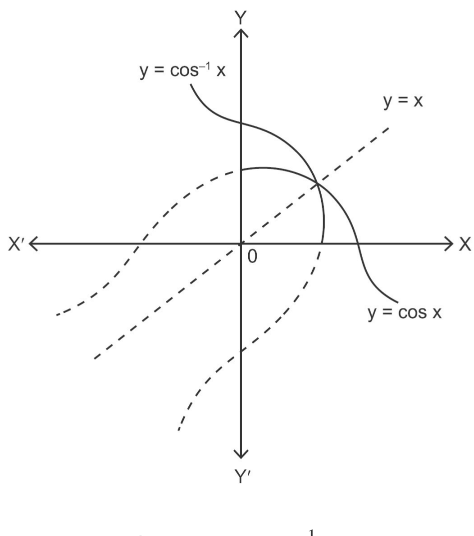
iii) $x \rightarrow |x+1|$

iv) $x \rightarrow |x|+1$

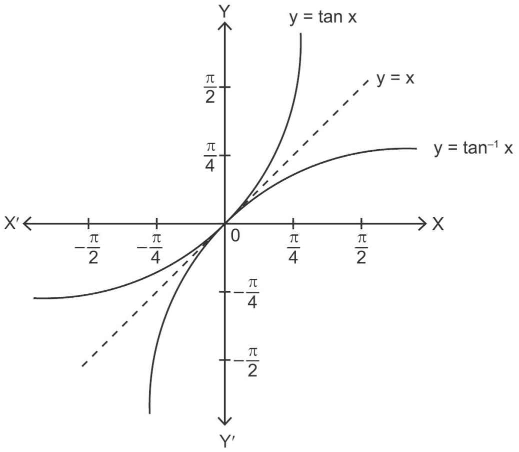
E22)



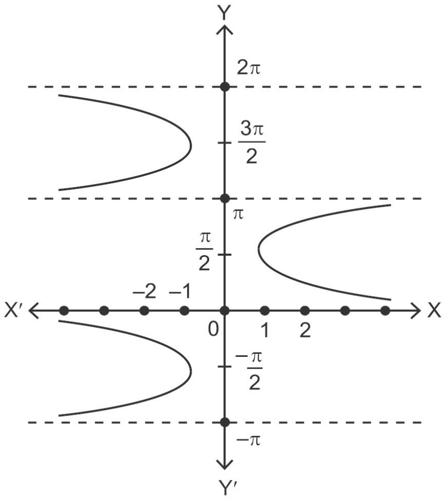
(a) Graph of $\sin^{-1} x$ **and** $\sin x$



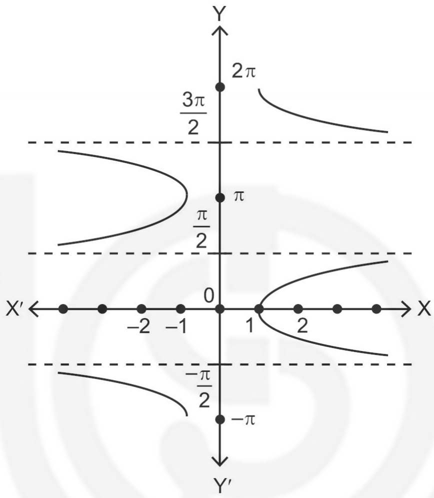
(b) Graph of $\cos^{-1} x$ **and** $\cos x$



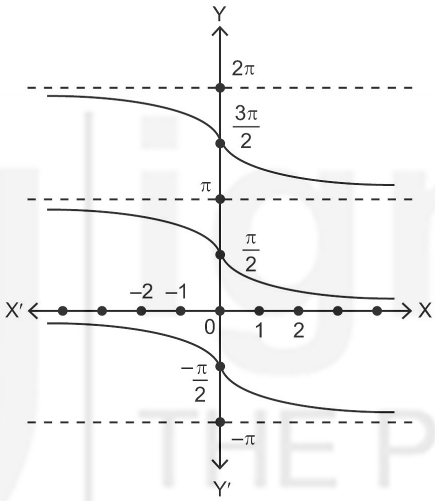
(c) Graph of $\tan^{-1} x$ **and** $\tan x$



(d) Graph of $\operatorname{cosec}^{-1} x$



(e) Graph of $\sec^{-1} x$



(f) Graph of $\cot^{-1} x$

Fig. 30

E23) i) $f(x) = |x| \Rightarrow f(-x) = |-x| = |x| = f(x)$. Hence, f is even.

ii) $g(x) = 1/x^2 \Rightarrow g(-x) = 1/(-x)^2 = g(x)$. Hence, g is even.

E24) i) $f(x) = x \Rightarrow f(-x) = -x = -f(x)$. Hence, f is odd.

ii) $g(x) = 1/x \Rightarrow g(-x) = -1/x = -g(x)$. Hence, g is odd.

E25) i), iii) and v) are even

iv) is odd

ii) is neither even nor odd

E26) i) neither increasing, nor decreasing

ii) non-decreasing

iii) strictly decreasing

E27) i) The period of the function in Fig. 25(a) is 2. Other periods are $4, 6, 8, \ldots$

ii) The period of the function in Fig. 25(b) is 2π. Other periods are $4\pi, 6\pi, \ldots$

E28) i) Periodic with period 2π
Since $\cos(x+2\pi) = \cos x$ for all x.

ii) not periodic.

iii) Periodic with period π.

iv) Periodic with period $\pi/3$.

v) Periodic with period π.

vi) Periodic with period 2π.

E29) i) and ii) are periodic, iii) is not.

E30) No. For example, $x-[x]$ and $|\sin x|$ are periodic, but their sum is not.

UNIT 7

LIMIT

Structure | Page No.

7.1 INTRODUCTION

We now begin the study of calculus, starting with a concept of fundamental importance to it, namely, 'limit'. As you read the later units, you will realise that the seeds of calculus were sown as early as the third century B.C. But it was only in the nineteenth century that a rigorous definition of a limit was given by Weierstrass. Before him, Newton and Cauchy had clear ideas about limit, but none of them had given a formal and precise definition. They had depended, more or less, on intuition (without actually defining it) or on geometry.

The introduction of limit revolutionised the study of calculus. The cumbersome proofs which were used by the Greek mathematicians have given way to neat, simpler ones.

You may already have an intuitive idea of limit, which we shall discuss in Sec. 7.2. In Sec. 7.3, we shall give you a precise definition of this concept. The limit at infinity has been discussed in Sec. 7.4. We extend our study of limit by discussing theorems based on limits in Sec. 7.5. In Sec. 7.6, we shall introduce exponential and logarithmic functions. In Sec. 7.7, we shall discuss hyperbolic functions and their inverse functions.

The functions that you will come across in this course will be real valued as we discussed in Unit 6.

And now we shall list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

Objectives

After reading this unit, you should be able to:

- find the limits of functions whenever they exist;
- define limit formally using $\varepsilon-\delta$;
- apply operations like addition, subtraction, etc. on limit;
- define exponential function and logarithmic function; and
- define hyperbolic functions and their inverses.

7.2 LIMIT (AN INTUITIVE APPROACH)

The concept of limit is the fundamental building block on which all other concepts of calculus are based. In this section, we shall study limits informally with the objective of developing an "intuitive feel" for the basic ideas.

You must have heard limits in various situations in daily life such as

- A vehicle stops at red-light by reducing its speed from say 40 km/hr to 0 by getting closer and closer to 0.
- Any object reaches terminal velocity when falling.

In the examples given above, you find that there is some target fixed. Sometimes, we cannot work something out directly but we can see what it should be as we get closer and closer!

Suppose, we have a real function and we want to know how the function behaves at the values close to a given value $x = a$. We choose values of x that get closer and closer to $x = a$ and we evaluate the function at these values.

Here is an example that illustrates the numerical and graphical approaches of the concept of limit.

Let us investigate the behaviour of the function f defined by $f(x) = \dfrac{x^2-1}{x-1}, x \neq 1,$ at the values of x near 1.

We see that the function f is not defined at $x = 1$ as $x-1$ is in the denominator. Taking the values of x which are near 1, but not equal to 1, we can write $f(x) = \dfrac{x^2-1}{x-1} = \dfrac{(x-1)(x+1)}{(x-1)} = x+1$ as $x-1 \neq 0$ and, so, division by $(x-1)$ is possible. The Table 1 gives the values of $f(x)$ for the values of x close to 1 from the left $(x < 1)$, but not equal to 1.

Table 1

x	-1	0	0.5	0.8	0.9	0.94	0.99	0.999	0.9999	0.99999
f(x)	0	1	1.5	1.8	1.9	1.94	1.99	1.999	1.9999	1.99999

When we examine the bottom row of Table 1, we see that $f(x)$ has the values 1.9 at $x = 0.9$, 1.99 at $x = 0.99$, 1.999 at $x = 0.999$, and so on. Therefore, as x moves closer to 1, $f(x)$ moves closer to 2. In fact, it appears that we can get the value of $f(x)$ as close as we like to 2 by making x sufficiently close to 1. Table 2 gives the values of $f(x)$ for values of x close to 1 from the right $(x > 1)$, but not equal to 1.

Table 2

x	x	1.5	1.2	1.1	1.05	1.01	1.001	1.0001	1.000001
f(x)	$f(x)$	2.5	2.2	2.1	2.05	2.01	2.001	2.0001	2.000001

Again, from Table 2, we observe that $f(x)$ moves closer to 2, when x moves sufficiently closer to 1. This is somewhat strengthened by considering the graph of the function f given by $y = f(x)$ in Fig. 1.

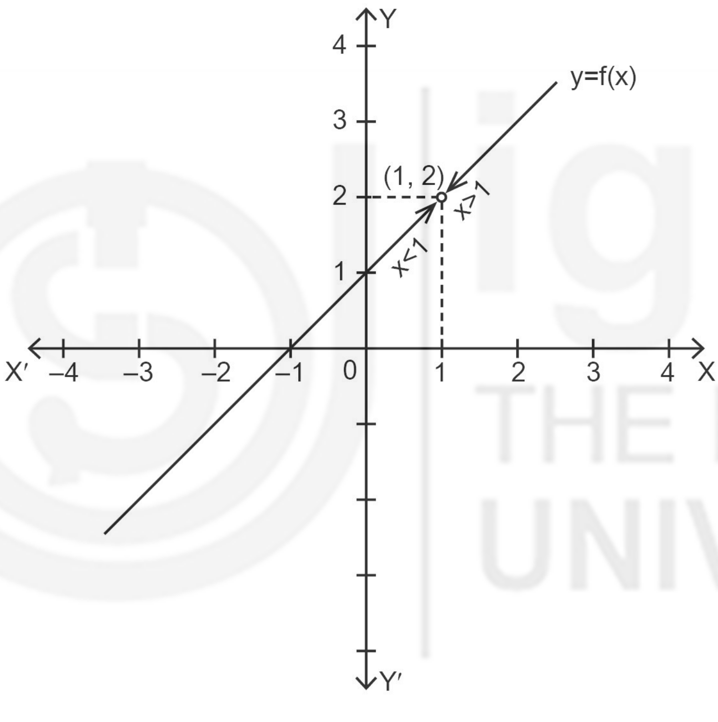


Fig. 1: Graph of $\mathbf{f(x) = \dfrac{x^2 - 1}{x - 1}}$**, near** $\mathbf{x = 1}$

Fig. 1 shows that as x gets closer to 1 (on either side of 1), $f(x)$ gets close to 2.

In this illustration, the value which the function should assume at a given value $x = 1$, did not really depend on how x is tending to 1. Note that there are essentially two ways x could approach 1 either from left or from right, i.e. all the values of x near 1 could be less than 1 or could be greater than 1. Therefore, it is clear that in both the cases (from left and from right) as x gets closer and closer to 1, $f(x)$ gets closer and closer to 2.

The symbol $\rightarrow$ denotes "tends to".

We express this by saying that as x tends to 1 or x approaches 1, $f(x) = x + 1$ tends to 2 or $f(x) = x + 1$ approaches 2. We abbreviate the statement with the notation $f(x) \rightarrow 2$ as $x \rightarrow 1$.

Another way of expressing this is to say that the limit of $f(x)$ at $x = 1$ is 2, which we write as $\lim_{x \to 1} f(x) = 2.$

In general, if the limit of a function f at a given point a is L, we write the limit as $\lim_{x \to a} f(x) = L$ and read "the limit of $f(x)$, as x approaches a, equals L". The function f has a limit L at $x = a,$ as $f(x)$ moves closer and closer to the number L whenever x gets closer and closer to a number a. An alternate notation is $f(x) \to L$ as $x \to a,$ which is generally read as "$f(x)$ approaches L as x approaches a". As you see, we are interested in the behaviour of the function near a, $x \neq a$ that is x is near a. This means that we never consider $x = a$ in finding the limit of a function as x tends to a. In fact, $f(x)$ need not be even defined at $x = a.$

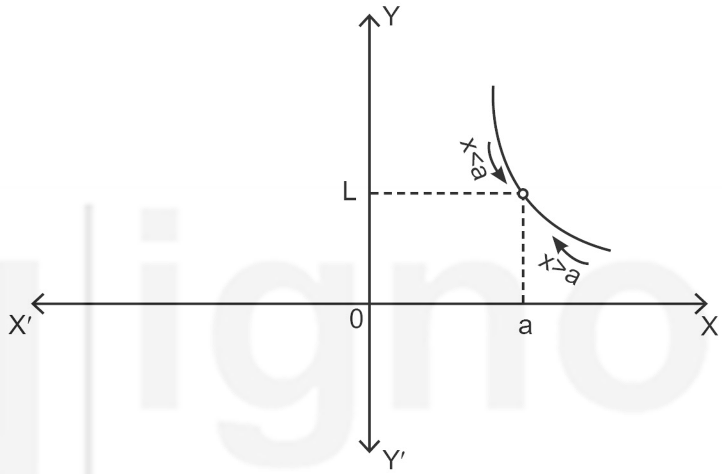


Fig. 2: f(x) when x takes values close to a

Fig. 2 shows that $f(a) \neq L,$ but $\lim_{x \to a} f(x) = L.$

Let us consider more examples to understand this.

Example 1: Investigate the limit of the function f defined by $f(x) = \sin x$ as x gets closer to $\pi/2$. The angle x is measured in radians.

Solution: Here, we tabulate the approximate values of $f(x)$ near $\pi/2$. Table 3 gives the values of $f(x)$ for values of x that approach $\pi/2$. From the values of Table 3, we observe that $\lim_{x \to \pi/2} \sin x = 1$.

Table 3: f(x) = sinx

$x > \pi/2$	$f(x) = \sin x$	$x < \pi/2$	$f(x) = \sin x$
2.0	0.909297	1.0	0.841471
1.8	0.973848	1.2	0.932039
1.6	0.999574	1.5	0.997495
1.59	0.99816	1.55	0.999784
1.58	0.999958	1.57	0.999999

Fig. 3 illustrates that $\lim_{x \to \pi/2} \sin x = 1$ from either side, that is left as well as right.

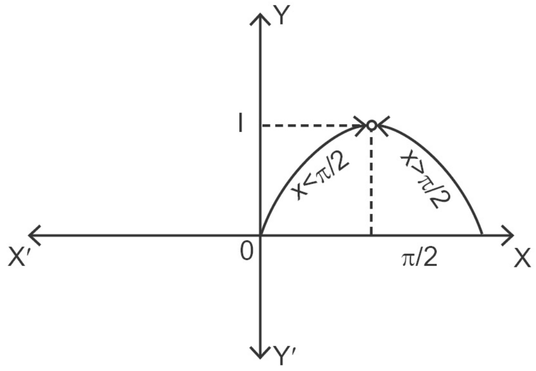


Fig. 3: $f(x) = \sin x$ **when** x **takes values close to** $\pi/2$

***

Example 2: Guess $\lim_{x \to 1} f(x)$, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 + 1$.

Solution: First, we tabulate values of $f(x)$ near 1 from the left side and the right side. Table 4 gives the values of $f(x)$ as x takes values nearer and nearer to 1. We see that as the values of x approach 1, $f(x)$ tends to 2.

Table 4: $f(x) = x^2 + 1$

$(x > 1)$	1.2	1.1	1.05	1.01	1.005	1.001
$f(x)$	2.44	2.42	2.103	2.02	2.01	2.002
$(x < 1)$	0.8	0.9	0.95	0.99	0.999	0.995
$f(x)$	1.64	1.81	1.903	1.9801	1.9989	1.990

We find that, as x gets closer and closer to 1, $f(x)$ gets closer and closer to 2. Alternatively, we express this by saying that as x approaches 1, $f(x)$ approaches 2. That is, the limit of $f(x)$ is 2, at $x = 1$. We write $\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 + 1) = 2.$

***

Now the question arises, whether the limit of a function is useful? It is indeed useful. Often we find functions that are undefined at certain values. This means that the function equals something like 0/0, or infinity over infinity for specific values of "x". We do not know what those expressions mean! We shall study these expressions in Unit 12. But using limits, we can know what the function is approaching when the variable "x" approaches that value. We don't need to care whether or not the function is defined at that point.

Example 3: Consider the function $\dfrac{\sin x}{x}$, and investigate its limit as $x \to 0$.

Solution: This function is not defined at $x = 0$, but it has no bearing on finding the limit. Table 5 and Table 6 show samples of x-values approaching 0 from the left side and from the right side respectively. In both cases, the values of $\dfrac{\sin x}{x}$ are calculated to four decimal places.

Table 5: Values of $\frac{\sin x}{x}$ as $x \to 0$ from the left.

x (in radian)	-1.0	-0.8	-0.6	-0.4	-0.2	-0.01
$f(x)=\frac{\sin x}{x}$	0.8415	0.8967	0.9411	0.9736	0.9934	0.9999

Left side

Table 6: Values of $\frac{\sin x}{x}$ as $x \to 0$ from the right.

x (in radian)	0.01	0.2	0.4	0.6	0.8	1.0
$f(x)=\frac{\sin x}{x}$	0.9999	0.9934	0.9736	0.9411	0.8967	0.8415

Right side

The graph of this is shown in Fig. 4, it has a missing point at (0,1).

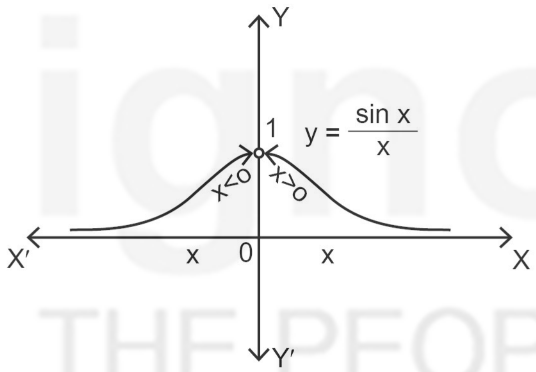


Fig. 4: $f(x) \to 1$ as $x \to 0$ from left as well as right side.

From Table 5, Table 6 and Fig. 4, we see that as x approaches 0, the values of the function appear as 0.9999.... and so we guess that

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

***

Have these examples helped you to reach the meaning of 'limit'? Would you agree with the following definition.

Limit (An Informal View): Given a function $f:\mathbb{R}\to\mathbb{R}$, and $x\in\mathbb{R}$, if the values of $f(x)$ can be made as close as we like to $L\in\mathbb{R}$ by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x\to a} f(x)=L$$

This is read as "the limit of $f(x)$ as x approaches a is L".

Here the limit is commonly called a **two-sided limit** because it requires the value of $f(x)$ to get closer and closer to L as the values of x are taken from both sides (left and right) of $x=a$.

However, some functions show different values on the two sides of x-values.

Now try the following exercise.

E1) Comment on the following limits at the given value of x.

i) $f(x)=\dfrac{x^2-9}{x-3}$ as $x \to 3.$

ii) $f(x)=\dfrac{x^2-2x}{x^2-4}$ as $x \to 2.$

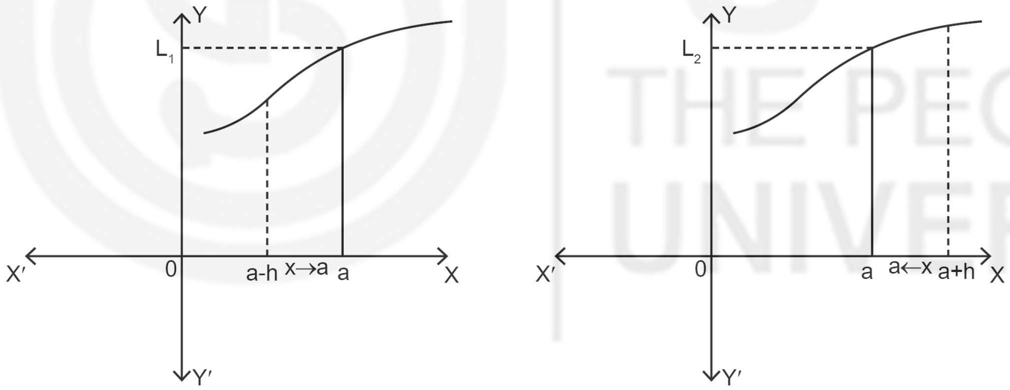
iii) $f(x)=4x-1$ as $x \to 0.$

If the values of the function f approach a limit L_1 as x approaches 'a' from the left, we say that the **left hand limit (LHL)** of $f(x)$ is L_1 as $x \to a$. We denote it by writing $\lim\limits_{x\to a^-} f(x)=L_1$ or $\lim\limits_{h\to 0} f(a-h)=L_1,\ h>0.$

Similarly, if $f(x)$ approaches a limit L_2 as x approaches 'a' from the right we say, that a **right hand limit (RHL)** of $f(x)$ as $x \to a$ is L_2. We write it as $\lim\limits_{x\to a^+} f(x)=L_2$ or $\lim\limits_{h\to 0} f(a+h)=L_2,\ h>0.$

The superscript "$-$" indicates a limit from the left and the superscript "$+$" indicates a limit

One-sided limits L_1 and L_2 are illustrated in Fig. 5.



a) $\lim\limits_{x\to a^-} f(x)=L_1$ b) $\lim\limits_{x\to a^+} f(x)=L_2$

Fig. 5

Let us illustrate a function in the following example, where the left hand limit and the right hand limit are not equal.

Example 4: Consider the function f defined by $f(x)=\begin{cases}2, & x\le 0\\ 4, & x>0\end{cases}$, comment on its limit at $x=0$.

Solution: Graph of this function is shown in the Fig. 6. It is clear that the left hand limit of f at 0 is $\lim\limits_{x\to 0^-} f(x)=2$.

Similarly, the right hand limit of f at 0 is $\lim\limits_{x\to 0^+} f(x)=4$.

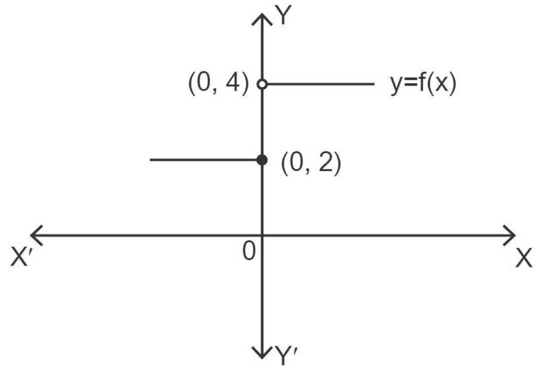


Fig. 6

In this case, the right and left hand limits are different, and hence we say that the limit of $f(x)$ does not exist as x tends to zero (even though the function is defined at 0).

***

In this discussion, you would have noted that there are situations where $LHL = RHL$ and where $LHL \neq RHL$. In such cases, we say that the limit of f as x approaches a **does not exist**, as the values of $f(x)$ do not get closer and closer to some single number L as $x \to a$. In general, the following condition must be satisfied for the two-sided limit of a function to exist.

The relationship between one-sided and two-sided limits: The two-sided limit of a function f exists at a if and only if both of the one-sided limits exist at $x = a$ and have the same value, i.e.,
$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.

Remark 1: If one or both of the one-sided limits fail to exist, the two-sided limit does not exist. Sometimes, we will be interested in the limit of a function from only one side.

Let us now discuss more examples.

Example 5: Consider the function f defined by $f(x) = x + 10$. Find the limit of the function f at $x = 2$.

Solution: Let us compute the value of the function f at x very near to 2. Some of the points near and to the left of 2 are 1.9, 1.95, 1.99, 1.995..., etc. Values of the function at these points are tabulated below. Similarly, the real numbers 2.001, 2.01, 2.1 are also points near and to the right of 2. Values of the function at these points are also given in Table 7.

Table 7

x	1.9	1.95	1.99	1.995	2.001	2.01	2.1
$f(x)$	11.9	11.95	11.99	11.995	12.001	12.01	12.1

From the Table 7, we deduce that the value of $f(x)$ at $x = 2$ should be greater than 11.995 and less than 12.001. It is reasonable to assume that the limit of $f(x)$ at $x = 2$ from the left of 2 is 12, i.e., $\lim_{x\to 2^-} f(x) = 12$.

Similarly, when x approaches 2 from the right, $f(x)$ should be taking value 12, i.e., $\lim_{x\to 2^+} f(x) = 12.$

Hence, it is likely that the left hand limit of $f(x)$ and the right hand limit of $f(x)$ are both equal to 12. Thus,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2} f(x) = 12.$$

Therefore the limit of the function f at $x = 2$ is equal to 12. Also, we observe that th e value of the function f at $x = 2$ also happens to be equal to 12.

***

Example 6: Consider the function f defined by $f(x) = x^3$. Find the limit of the function f at $x = 1$.

Solution: Proceeding as in the previous example, we tabulate the value of $f(x)$ at x near to 1. These values are given in the Table 8.

Table 8

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.729	0.970	0.997	1.003	1.030	1.331

From Table 8, we deduce that the limit of $f(x)$ at $x = 1$ should be greater than 0.997 and less than 1.003. It is reasonable to assume that the limit of the $f(x)$ at the left of 1 is 1, i.e., $\lim_{x\to 1^-} f(x) = 1.$

Similarly, when x approaches 1 from the right, $f(x)$ should be taking value 1, i.e., $\lim_{x\to 1^+} f(x) = 1.$

Hence, it is likely that the left hand limit of $f(x)$ and the right hand limit of $f(x)$ are both equal to 1. Thus,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 1.$$

Therefore, the limit of the function f at $x = 1$ is equal to 1.
We observe, again, that the value of the function f at $x = 1$ also happens to be equal to 1.

***

Example 7: Consider the function $f(x) = 2x$. Let us try to find the limit of this function at $x = 1$.

Solution: Table 9 is now self-explanatory.

Table 9

x	0.9	0.95	0.99	0.999	1.001	1.01	1.1
$f(x)$	1.8	1.90	1.98	1.998	2.002	2.02	2.2

We observe that as x approaches 1 from either left or right, the value of $f(x)$ seem to approach 2. We get, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 2$.

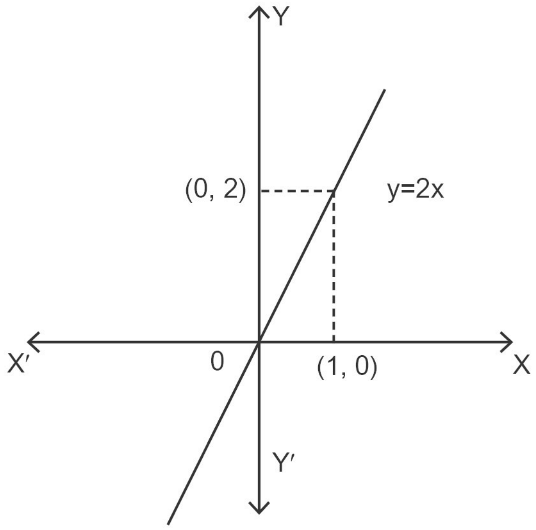


Fig. 7

Its graph is shown in Fig. 7, which strengthens this fact.

Here, again we note that the value of the function at $x = 2$ coincides with the limit at $x = 2$.

***

Example 8: Check the existence of $\lim_{x\to 2}[x]$, where [ ] is the greatest integer function. [You may refer Unit 6 for greatest integer function]

Solution: The value of $[x]$ is the largest integer which is less than or equal to x.

Now, let us draw its graph.

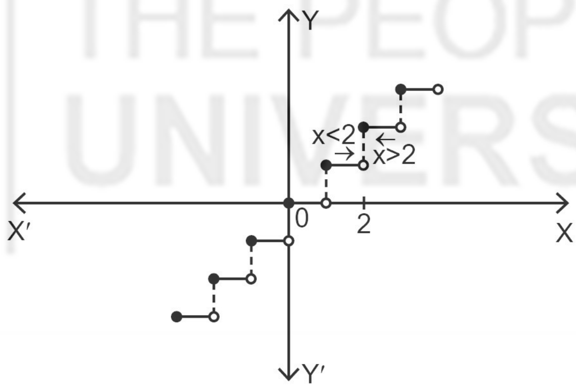


Fig. 8: Graph of $[x]$

If we consider the graph of the function $f(x) = [x]$, shown in Fig. 8, we see that if x approaches 2 from the left then $f(x)$ seems to tend to 1. At the same time, if x approaches 2 from the right, then $f(x)$ seems to tend to 2. This means that the limit of f exists if x approaches 2 from only one side (left or right) at a time, and thus $LHL \neq RHL$.

Thus, $\lim_{x\to 2}[x]$ does not exist.

***

Now try the following exercises.

E2) Find the following limits.

i) Consider the constant function f defined by $f(x) = 3$. Find its limit at $x = 2$.

ii) Consider the function f defined by $f(x) = x^2 + x$. Find $\lim_{x \to 1} f(x)$.

iii) Consider the function f defined by $f(x) = x + \cos x$. Find the $\lim_{x \to 0} f(x)$.

E3) Investigate $\lim_{x \to 2} (x^2 - x + 1)$.

E4) Evaluate $\lim_{x \to 2} \frac{1}{x^2}$.

E5) Find $\lim_{x \to 0} f(x)$, where

$$f(x) = \begin{cases} x - 2, & x < 0 \\ 0 \quad, & x = 0 \\ x + 2, & x > 0 \end{cases}$$

E6) Check whether the following limits exist or not.

i) $\lim_{x \to 1} \frac{|x-1|}{x-1}$

ii) $\lim_{x \to n} [x]$ where $n \in \mathbb{N}$

E7) Find $\lim_{x \to 1/3} \frac{9x^2 - 1}{3x - 1}$, if it exists.

So far, we have discussed limits informally. In the following section, we shall discuss, and apply, the formal definition of a limit.

7.3 LIMIT (A FORMAL APPROACH)

After studying Sec. 7.2, you would have developed some understanding of what a limit is. It is not always convenient to guess the limit intuitively, thus we must use the precise definition of limit. Let us consider the function f defined by $f(x) = \begin{cases} 3x - 1 \text{ ; when } x \neq 1 \\ 4 \quad \text{ ; when } x = 1. \end{cases}$

Intuitively, we say that $\lim_{x \to 1} f(x) = 2$, since, when x is close to 1 but not equal to 1, $f(x)$ is close to 2. Can you find the closeness of x to 1, so that the difference between $f(x)$ and 2 is less than a very little amount, say 0.01? Mathematically we can write this as $|f(x) - 2| < 0.01$, if $|x - 1| < \delta$ (say), where δ is a very small number. $|f(x) - 2| = |(3x - 1) - 2| = |3x - 3| = |3(x - 1)|$

$= 3|x - 1| < 0.01$ that is $|f(x) - 2| < 0.01$ if $0 < |x - 1| < \frac{0.01}{3}$ or $0 < |x - 1| < 0.0033$.

We get an answer to our question, that the little amount δ is 0.0033. This means that if x is within a distance of 0.0033 from 1, then $f(x)$ will be within a distance of 0.01 from 2.

In the same way, if we change 0.01 to 0.001, we get $|f(x)-2| < 0.001$ if $0 < |x-1| < 0.00033.$

We can take 0.01, 0.001 or any other small positive number. For example, take it to be ε, which is any arbitrary positive number, then, in the same way you should check that

$|f(x)-2| < \varepsilon$ if $0 < |x-1| < \frac{\varepsilon}{3}$. So, $\frac{\varepsilon}{3}$ is a real number such that

$0 < |x-1| < \varepsilon/3 \Rightarrow |f(x)-2| < \varepsilon.$

This leads us to the following definition:

ε (epsilon) and δ (delta) are Greek letters .

Definition: Let f be a function defined at all points near a (except possibly at a). Let L be a real number. We say that f approaches the limit L as x approaches a if, for each real number $\varepsilon > 0$, we can find a real number $\delta > 0$, depending on ε, such that

$$0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon.$$

$|x-a| < \delta$ means that $-\delta < (x-a) < \delta$ i.e. $a-\delta < x < a+\delta$ i.e. $x \in]a-\delta, a+\delta[$ and $0 < |x-a|$ means that $x \neq a$. That is, $0 < |x-a| < \delta$ means that x can take any value **lying between** $a-\delta$ and $a+\delta$ except a.

The limit L is denoted by $\lim\limits_{x\to a} f(x)$. We also write $f(x) \to L$ as $x \to a$.

The $\varepsilon-\delta$ definition does not give us the value of L. It just helps us check whether a given number L is the limit of $f(x)$.

Note that, in the definition above, we take any real number $\varepsilon > 0$ and then choose some $\delta > 0$, so that $L-\varepsilon < f(x) < L+\varepsilon$, whenever $|x-a| < \delta$, that is, $a-\delta < x < a+\delta$.

In Unit 6, we have mentioned that $|x-a|$ can be thought of as the distance between x and a. In the light of this the definition of the limit of a function can also be interpreted as:

Given $\varepsilon > 0$, we can choose $\delta > 0$ such that if we choose x whose distance from a is less than δ, then the distance of its image from L must be less than ε. The picture in Fig. 9 may help you absorb the definition. Here, we first

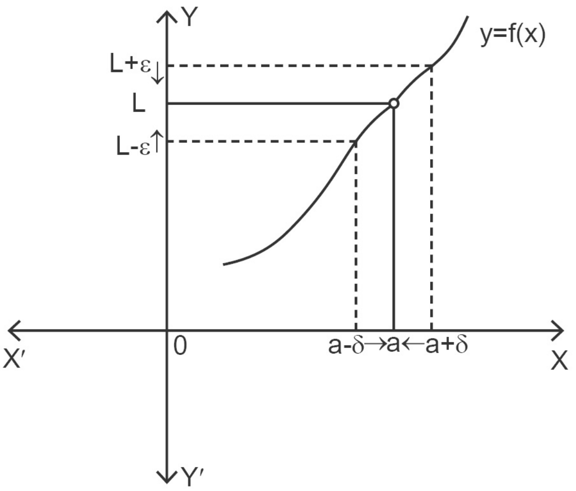


Fig. 9: $\varepsilon-\delta$ **definition of limit**

pick a number, $\varepsilon > 0$ and consider two horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$. Now, we take the band of width 2ε around the number L on the y-axis, and find a band around the number a on the x-axis, corresponding to the point of intersection of these lines with $f(x)$, so that for all x-values (excluding $x = a$) inside the band, the corresponding y-values lie inside the band.

We can do this in the following steps:

1. We first pick a given closeness, $]L-\varepsilon, L+\varepsilon[$ to L.

2. Then, we get close enough to a, $]a-\delta, a+\delta[$, so that all the corresponding y-values fall inside it. If a $\delta > 0$ can be found for each value of ε, then, we can say that L is the correct limit.

   If the process fails, then the limit L has been incorrectly computed, or the limit does not exist.

Remember, the number ε is given first and the number δ is to be produced, depending on ε.

Now, let us take up the following examples.

Example 8: Consider the function $f:\mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. How can we find $\lim_{x \to 0} f(x)$ using $\varepsilon - \delta$ definition of limit?

Solution:

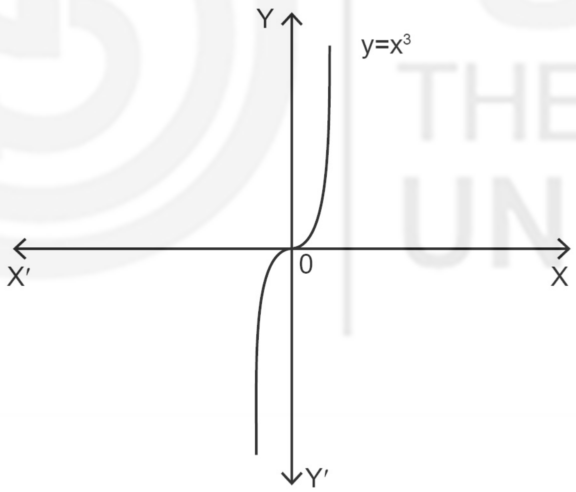


Fig. 10: Graph of x^3

Look at the graph of f in Fig. 10. You will see that when x is small, x^3 is also small. As x comes closer and closer to 0, x^3 also comes closer and closer to zero. It is reasonable to expect that $\lim f(x) = 0$ as $x \to 0$.
Let us prove that this is what happens.
Take any real number $\varepsilon > 0$. Then,
$| f(x) - 0 | < \varepsilon \Leftrightarrow \left| x^3 - 0 \right| < \in \Leftrightarrow | x^3 | < \varepsilon \Leftrightarrow | x | < \varepsilon^{1/3}$.

Therefore, if we choose $\delta = \varepsilon^{1/3}$ we get $| f(x) - 0 | < \varepsilon$ whenever $0 < | x - 0 | < \delta$. This gives us $\lim_{x \to 0} f(x) = 0$.

***

General Rule: A useful general rule to prove $\lim_{x \to a} f(x) = L$ is to write down $|f(x) - L|$ and then express it in terms of $|x - a|$ and relate ε and δ as much as possible.

Let us now see how to use this rule to calculate the limit in the following examples.

Example 9: Let us calculate $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ using $\varepsilon - \delta$ definition.

Solution: We know that division by zero is not defined. Thus, the function $f(x) = \frac{x^2 - 1}{x - 1}$ is not defined at $x = 1$. But, as we have mentioned earlier, when we calculate the limit as x approaches 1, we do not take the value of the function at $x = 1$. Now, to obtain $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$, we first note that $x^2 - 1 = (x - 1)\ (x + 1)$, so that, $\frac{x^2 - 1}{x - 1} = x + 1$ for $x \neq 1$. Therefore

$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1}\ (x + 1)$.

As x approaches 1, we can intuitively see that this limit approaches 2, as shown in the example considered in Sec 7.2.

To prove that the limit is 2, we first write $|f(x) - L| = |x + 1 - 2| = |x - 1|$, which is itself in the form $|x - a|$, since $a = 1$ in this case.

Let us take any number $\varepsilon > 0$. Now,

$$|(x + 1) - 2| < \varepsilon \Leftrightarrow |x - 1| < \varepsilon$$

Thus, if we choose $\delta = \varepsilon$, in our definition of limit, we see that $|x - 1| < \delta = \varepsilon \Rightarrow |f(x) - L| = |x - 1| < \varepsilon$. This shows that $\lim_{x \to 1} (x + 1) = 2$.

Hence, $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$.

***

Example 10: Prove that $\lim_{x \to 2} (x^2 + 1) = 5$, using $\varepsilon - \delta$ definition.

Solution: We shall prove that $\forall\ \varepsilon > 0, \exists \delta > 0$ such that $|x^2 + 1 - 5| < \varepsilon$ whenever $|x - 2| < \delta$.

Here, $f(x) - L = (x^2 + 1) - 5 = x^2 - 4$, and $x - a = x - 2$.

Now we write $|x^2 - 4|$ in terms of $|x - 2|$:

$$|x^2 - 4| = |x + 2||x - 2|$$

Thus, apart from $|x - 2|$, we have a factor, namely $|x + 2|$. To decide the limits of $|x + 2|$, let us put a restriction on δ. Remember, we have to choose δ. So let us say we choose a $\delta \leq 1$. What does this imply?

$$|x - 2| < \delta \Rightarrow |x - 2| < 1 \Rightarrow 2 - 1 < x < 2 + 1$$

$$\Rightarrow 1 < x < 3 \Rightarrow 3 < x + 2 < 5. \text{ [Recall Unit 6]}$$

Thus, we have $|x^2 - 4| < 5|x - 2|$. But our aim is to prove $|x^2 - 4| < \varepsilon$. For this we shall try to make $5|x - 2| < \varepsilon$. Now when will this be true? It will be true when $|x - 2| < \varepsilon/5$. So this $\varepsilon/5$ is the value of δ we were looking

for. But we have already chosen $\delta \le 1$. This means that given $\varepsilon > 0$, the δ we choose should satisfy $\delta \le 1$ and also $\delta \le \varepsilon/5$.
In other words, $\delta = \min\{1, \varepsilon/5\}$, should serve our purpose. Let us verify this:
$|x-2| < \delta \Rightarrow |x-2| < 1$ and $|x-2| < \varepsilon/5 \Rightarrow |x^2-4| = |x+2|.|x-2| < 5.\varepsilon/5 = \varepsilon$.

***

Remark 2: If f is a constant function on $\mathbb{R}$, that is, if $f(x) = k \forall x \in \mathbb{R}$, where k is some fixed real number, then $\lim_{x \to a} f(x) = k$.

Now, please try the following exercises.

E8) Using $\in -\delta$ definition of limit, show that

i) $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$

ii) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$

E9) Show that the function f, defined by $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right); & x \ne 0 \\ 0 & ; x = 0 \end{cases}$ does not approach 0 as $x \to 0$.

E10) Check, whether $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ is correct or not.

So far, in this section, you have studied formal definition of limit. Let us now see if analogous definitions hold for one-sided limits.

Definition: Let f be a function defined for all x in the interval $]a, b[, f$ is said to approach a **limit L as x approaches a from right** if, given any $\varepsilon > 0$, there exists a number $\delta > 0$ such that $a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$.
In symbols, we denote this limit by $\lim_{x \to a^+} f(x) = L$.

Similarly, the function $f :]a, b[\to \mathbb{R}$ is said to approach a limit L as **x approaches b from the left** if, given any $\varepsilon > 0, \exists \delta > 0$ such that $b - \delta < x < b \Rightarrow |f(x) - L| < \varepsilon$.
This limit is denoted by $\lim_{x \to b^-} f(x)$.
Note that in computing these limits, the values of $f(x)$ for x lying on only **one side** of the interval are taken into account.

Let us apply this definition to the function $f(x) = [x]$. We know that for $x \in [1, 2[, [x] = 1$. That is, $[x]$ is a constant function on $[1, 2[$. Hence $\lim_{x \to 2^-} [x] = 1$. Arguing similarly, we find that since $[x] = 2$ for all $x \in [2, 3[, [x]$ is, again, a constant function on $[2, 3[$, and $\lim_{x \to 2^+} [x] = 2$.

Let us improve our understanding of the definition of one-sided limits by looking at some more examples.

Example 11: Let f be defined on $\mathbb{R}$ by setting

$$f(x)=\begin{cases}\frac{|x|}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

We shall show that $\lim\limits_{x\to 0^-} f(x)$ equals -1.

Solution: When $x<0, |x|=-x$, and therefore, $f(x)=(-x)/x=-1$.

In order to show that $\lim\limits_{x\to 0^-} f(x)$ exists and equals -1, we have to start with any $\varepsilon>0$ and then find a $\delta>0$ such that, if $-\delta<x<0$, then $|f(x)-(-1)|<\varepsilon$.

Since $f(x)=-1$ for all $x<0, |f(x)-(-1)|=0$ and, hence, any number $\delta>0$ will work.

Therefore, whatever $\delta>0$ we may choose, if $-\delta<x<0$, then $|f(x)-(-1)|=0<\varepsilon$. Hence, $\lim\limits_{x\to 0^-} f(x)=-1$.

***

Example 12: f is a function defined on $\mathbb{R}$ by setting

$f(x)=x-[x]$, for all $x\in\mathbb{R}$.

Let us examine whether $\lim\limits_{x\to 1^-} f(x)$ exists.

Solution: This function can be written as $f(x)=x$, if $0\leq x<1$.

$f(x)=x-1$ if $1\leq x<2$, and, in general

$f(x)=x-n$ if $n\leq x<n+1$ (see Fig. 11)

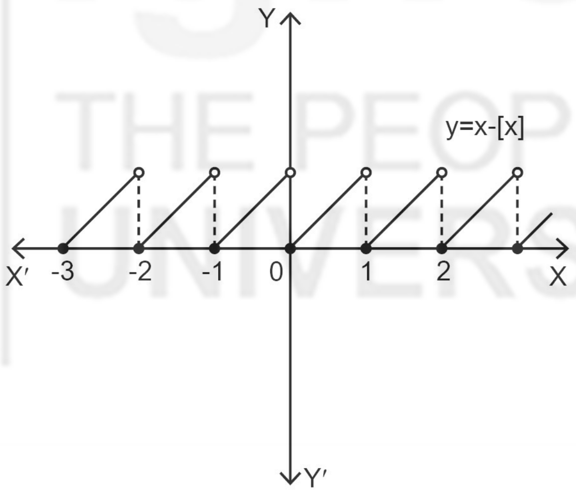


Fig. 11: Graph of $x-[x]$

Since $f(x)=x$ for the values of x less than 1 but close to 1, it is reasonable to expect that $\lim\limits_{x\to 1^-} f(x)=1$. Let us prove this by taking any $\varepsilon>0$ and choosing $\delta=\min\{1,\varepsilon\}$. We find that $1-\delta<x<1\Rightarrow f(x)=x$ and $|f(x)-1|=|x-1|<\delta\leq\varepsilon$.

Therefore, $\lim\limits_{x\to 1^-} f(x)=1$.

Proceeding exactly as above, we note that $f(x)=x-1$ if $1\leq x<2$, we can similarly prove that $\lim\limits_{x\to 1^+} f(x)=0$.

***

Now try the following exercise.

E11) Prove that

i) $\lim_{x \to 3^-} x - [x] = 1$.

ii) $\lim_{x \to 0^+} \frac{|x|}{x} = 1$.

iii) $\lim_{x \to 0^-} \frac{(x^2 + 2)|x|}{x} = -2$.

So far, we have been discussing limit of any real valued function at a finite value of x. Now, let us discuss the behaviour of function at ∞ or $-\infty$ in the following section.

7.4 LIMITS AT INFINITY

Take a look at the graph of the function $f(x) = \frac{1}{x}$, $x > 0$ in Fig. 12. We see from Fig. 12 that $f(x)$ comes closer and closer to zero as x gets larger and larger. This situation is similar to the one where we have a function $g(x)$ getting closer and closer to a value L as x comes nearer and nearer to some number a, that is when $\lim_{x \to a} g(x) = L$.

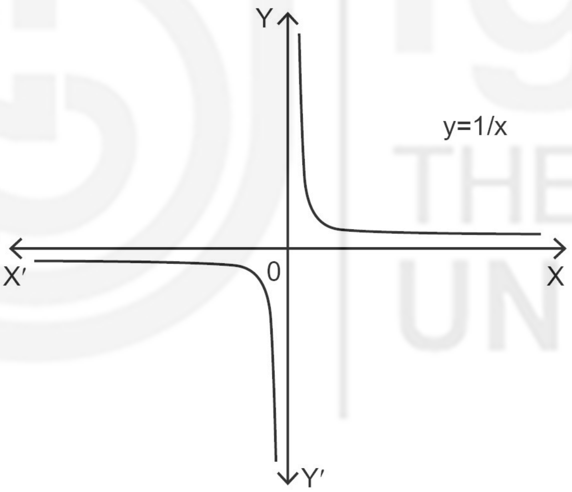


Fig. 12: Graph of $1/x$

The only difference is that in the case of $f(x)$, x is not approaching any finite value, and is just becoming larger and larger. We express this by saying that $f(x) \to 0$ as $x \to \infty$, or $\lim_{x \to \infty} f(x) = 0$.

Note that, ∞ is **not** a real number. We write $x \to \infty$ merely to indicate that x becomes larger and larger.

We now formalise this discussion in the following definition.

Definition: A function f is said to tend to a limit L as x tends to ∞ if, for each $\varepsilon > 0$ it is possible to choose K such that $|f(x) - L| < \varepsilon$ whenever $x > K$.

In this case, as x gets larger and larger, $f(x)$ gets nearer and nearer to L. We now give another example of this situation.

Example 13: Let f be defined by setting $f(x) = 1/x^2$ for all $x \in \mathbb{R} \setminus \{0\}$. Comment on $\lim_{x \to \infty} f(x)$.

Solution: Here f is defined for all real values of x other than zero. Let us substitute larger and larger values of x in $f(x) = 1/x^2$ and see what happens (see Table 10).

Table 10

x	100	1000	100,000
$f(x) = 1/x^2$	0.0001	0.000001	0.0000000001

We see that as x becomes larger and larger, $f(x)$ comes closer and closer to zero. Now, let us choose any $\varepsilon > 0$. If $x > 1/\sqrt{\varepsilon}$, then $1/x^2 < \varepsilon$. Therefore, by choosing $K = 1/\sqrt{\varepsilon}$, we find that $x > K \Rightarrow |f(x)| < \varepsilon$. Thus, $\lim_{x \to \infty} f(x) = 0$. Fig. 13 gives us a graphical idea of how this function behaves as $x \to \infty$.

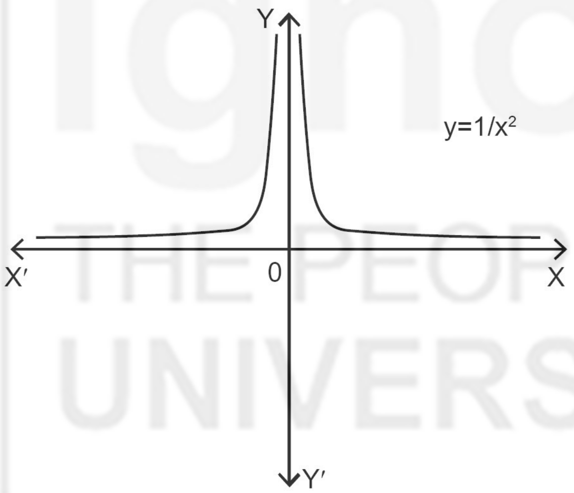


Fig. 13: $f(x) = \frac{1}{x^2}$ at $x \to \infty$

***

Sometimes we also need to study the behaviour of a function $f(x)$, as x takes smaller and smaller negative values. This leads to the following definition.

Definition: A function f is said to tend to a limit L as $x \to -\infty$ if, for each $\varepsilon > 0$, it is possible to choose K, such that $|f(x) - L| < \varepsilon$ whenever $x < -K$. The following example will help you in understanding this idea.

Example 14: Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{1 + x^2}$.

Comment on $\lim_{x \to -\infty} f(x)$.

Solution: The graph of f is as shown in Fig. 14.

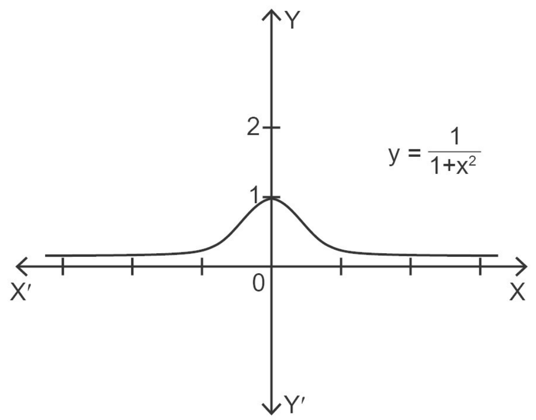


Fig. 14: Graph of $\mathbf{f(x) = \frac{1}{1+x^2}}$

What happens to $f(x)$ as x takes smaller and smaller negative values? Let us make a table (Table 11) to get some idea.

Table 11

x	-10	-100	-1000
$f(x) = \frac{1}{1+x^2}$	$\overline{101}$	$\overline{10001}$	$\overline{1000001}$

We see that as x takes smaller and smaller negative values, $f(x)$ comes closer and closer to zero. In fact $1/(1+x^2) < \varepsilon$ whenever $1+x^2 > 1/\varepsilon$, that is, whenever $x^2 > (1/\varepsilon) - 1$, that is, whenever either $x < -\left|\frac{1}{\varepsilon} - 1\right|^{1/2}$ or $x > \left|\frac{1}{\varepsilon} - 1\right|^{1/2}$. Thus, we find that if we take $K = \left|\frac{1}{\varepsilon} - 1\right|^{1/2}$, then $x < -K \Rightarrow |f(x)| < \varepsilon$. Consequently, $\lim_{x \to -\infty} f(x) = 0$.

In this example, we also find that $\lim_{x \to \infty} f(x) = 0$.

Let us see how $\lim_{x \to \infty} f(x)$ can be interpreted geometrically.

In the above example, we have the function $f(x) = 1/(1+x^2)$, and as $x \to \infty$, or $x \to -\infty$, $f(x) \to 0$. From Fig. 14, you can see that, as $x \to \infty$ or $x \to -\infty$, the curve $y = f(x)$ comes nearer and nearer the straight line $y = 0$, which is the x-axis.

Similarly, if we say that $\lim_{x \to \infty} g(x) = L$, then it means that, as $x \to \infty$ the curve $y = g(x)$ comes closer and closer to the straight line $y = L$.

***

Example 15: Show that $\lim_{x \to \infty} \frac{x^2}{(1+x^2)} = 1$.

Solution: Now, $\left|\frac{x^2}{1+x^2} - 1\right| = \left|\frac{1}{1+x^2}\right| = \frac{1}{1+x^2}$. In the previous example, we have shown that $|1/(1+x^2)| < \varepsilon$ for $x > K$, where $K = |1/\varepsilon - 1|^{1/2}$. Thus, given $\varepsilon > 0$, we choose $K = |1/\varepsilon - 1|^{1/2}$, so that

$$x > K \Rightarrow \left| \frac{x^2}{1+x^2} - 1 \right| < \varepsilon. \text{ This means that } \lim_{x \to \infty} \frac{x^2}{1+x^2} = 1.$$

We show this geometrically in Fig. 15.

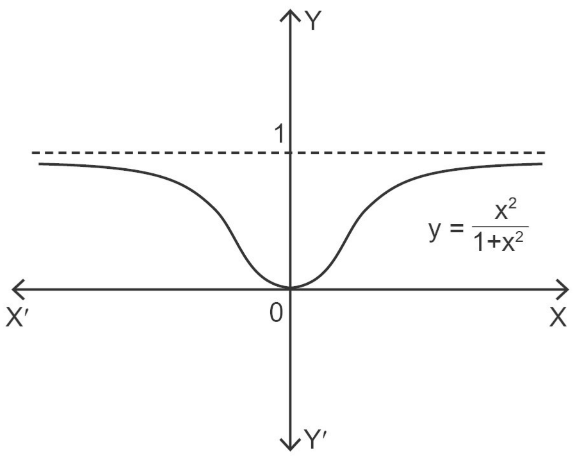


Fig. 15: Graph of $\frac{x^2}{1+x^2}$

***

You can try these exercises.

E12) Show that

i) $\lim_{x \to \infty} 1/x = 0$.

ii) $\lim_{x \to \infty} 1/x^2 = 0$.

E13) i) If for some $\varepsilon > 0$, and for every $K, \exists x > K$ s.t. $|f(x) - L| > \varepsilon$ what will you infer?

ii) If $\lim_{x \to p} f(x) \neq L$, how can you express it in the $\varepsilon - \delta$ form?

We end this section with the following important remark.

Remark 3: In case we have to show that a function f does not tend to a limit L as x approaches a, we shall have to negate the definition of limit. Let us see what this means. Suppose we want to prove that $\lim_{x \to a} f(x) \neq L$. Then, we should find some $\varepsilon > 0$ such that for every $\delta > 0$, there is some $x \in]a - \delta, a + \delta[$ for which $|f(x) - L| > \varepsilon$.

Through our next example we shall illustrate the negation of the definition of the limit of $f(x)$ as $x \to \infty$.

Example 16: Show that $\lim_{x \to \infty} 1/x \neq 1$.

Solution: We have to find some $\varepsilon > 0$ such that for any K (howsoever large) we can always find an $x > K$ such that $|1/x - 1| > \varepsilon$. Take $\varepsilon = 1/4$. Now, for any $K > 0$, if we take $x = \max\{2, K+1\}$, we find that $x > K$ and $|1/x - 1| > 1/4$. This clearly shows that $\lim_{x\to\infty} 1/x \neq 1$.

***

In the following section, we shall discuss theorems about limits.

7.5 THEOREMS ON LIMITS

Before we go further, let us ask, 'Can a function $f(x)$ tend to two different limits as x tends to a', or 'Is the sum of two limits the limit of sum of two functions at a'? These questions will be answered in the following theorems.

Theorem 1 (Uniqueness of Limit): If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, then $L = M$.

Proof: Suppose $L \neq M$, then $|L - M| > 0$. Take $\varepsilon = \frac{|L-M|}{2}$, then $\varepsilon > 0$.

Since $\lim_{x\to a} f(x) = L$, $\exists \delta_1 > 0$ such that

$$|x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$$

Similarly, since $\lim_{x\to a} f(x) = M$, $\exists \delta_2 > 0$ such that

$$|x - a| < \delta_2 \Rightarrow |f(x) - M| < \varepsilon$$

If we choose $\delta = \min\{\delta_1, \delta_2\}$, then $\delta \le \delta_1$ and $\delta \le \delta_2$. So $|x - a| < \delta$ will mean that $|x - a| < \delta_1$ and $|x - a| < \delta_2$. In this case, we will have both $|f(x) - L| < \varepsilon$, as well as, $|f(x) - M| < \varepsilon$.

So that
$$\begin{aligned} |L - M| &= |L - f(x) + f(x) - M| \le |f(x) - L| + |f(x) - M| \\ &< \varepsilon + \varepsilon \quad (\text{using } |a+b| \le |a| + |b|) \\ &= 2\varepsilon = |L - M| \end{aligned}$$

That is, we get $|L - M| < |L - M|$, which is a contradiction. Therefore, our supposition is wrong. Hence $L = M$. ■

Let us state some basic properties of limits in the following theorem without proof.

Theorem 2: Let f and g be two functions such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

i) $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ (Sum rule)

ii) $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ (Difference rule)

iii) $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$ (Product rule)

iv) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$ (Reciprocal rule)

v) $\lim_{x\to a} k = k$, where k is a constant (Constant function rule)

vi) $\lim_{x\to a} x = a$ (Identity function rule)

vii) $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, where n is a positive integer. (Power rule)

In particular, if $f(x) = x$, $\lim_{x\to a} x^n = a^n$, where n is a positive integer.

viii) $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, where n is a positive integer, and if n is even, we assume that $\lim_{x\to a} f(x) > 0$.

The properties given in Theorem 2 are also applicable for one-sided limits. Using the properties that we have just stated, we will calculate the limit in the following example.

Example 17: Find the limits of polynomial function f at $x = a$.

Solution: We know that $\lim_{x\to a} x = a$. Hence

$\lim_{x\to a} x^2 = \lim_{x\to a}(x.x) = \lim_{x\to a} x.\lim_{x\to a} x = a.a = a^2$ and so on.....Thus, we get

$\lim_{x\to a} x^n = a^n$

Now, let a polynomial function f be given by $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$.

Thinking of each of $a_0, a_1x, a_2x^2, ..., a_nx^n$ as a function, we have

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}[a_0 + a_1x + a_2x^2 + ... + a_nx^n] \\
&= \lim_{x\to a} a_0 + \lim_{x\to a} a_1x + \lim_{x\to a} a_2x^2 + ... + \lim_{x\to a} a_nx^n \quad \text{[using Theorem 2 (i),]} \\
&= a_0 + a_1\lim_{x\to a} x + a_2\lim_{x\to a} x^2 + ... + a_n\lim_{x\to a} x^n \quad \text{[using Theorem 2 (iii), and 2 (v)]} \\
&= a_0 + a_1a + a_2a^2 + ... + a_na^n \\
&= f(a)
\end{aligned}$$

(Make sure that you understand the justification for each step in the above!)

***

Example 18: Find the limit of the rational function h defined by $h(x) = \dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials such that $g(x) \neq 0$.

Solution: Here, $\lim_{x\to a} h(x) = \lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{f(a)}{g(a)}$

However, if $g(a) = 0$, there are two cases – (i) when $f(a) \neq 0$ and (ii) when $f(a) = 0$. In the former case, the limit does not exist because non-zero divided by zero is not a real number. In the later case, we can write $f(x) = (x-a)^m f_1(x)$, where m is the maximum of powers of $(x-a)$ in $f(x)$. Similarly, $g(x) = (x-a)^n g_1(x)$ as $g(a) = 0$. Now, if $m > n$, we have

$$\lim_{x\to a} h(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{\lim_{x\to a}(x-a)^m f_1(x)}{\lim_{x\to a}(x-a)^n g_1(x)}$$

$$= \frac{\lim_{x\to a}(x-a)^{(m-1)} f_1(x)}{\lim_{x\to a} g_1(x)} = \frac{0.f_1(a)}{g_1(a)} = 0$$

If $m < n$, the limit is not defined.

***

A general rule that needs to be kept in mind while evaluating limits is the following.

Say, given that the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists and we want to evaluate this. First, we check the value of $f(a)$ and $g(a)$. If both are 0, then we see if we can get the factor which is causing the terms to vanish, i.e., see if we can write $f(x)=f_1(x)f_2(x)$ so that $f_1(a)=0$ and $f_2(a)\neq 0$. Similarly, we write $g(x)=g_1(x)g_2(x)$, where $g_1(a)=0$ and $g_2(a)\neq 0$. Cancel out the common factors from $f(x)$ and $g(x)$ (if possible) and write

$$\frac{f(x)}{g(x)}=\frac{p(x)}{q(x)},\text{ where } q(x)\neq 0.$$

Then, $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{p(a)}{q(a)}$.

Now, let us understood few examples.

Example 19: Find the limits

i) $\lim_{x\to 2}\frac{3x^2+4x}{2x+1}$

ii) $\lim_{x\to 2}\frac{x^2-1}{x^3-2x^2+x}$

Solution: The required limit is of a rational function. Hence, we first evaluate these functions at the prescribed points. If this is of the form $\frac{0}{0}$, we try to rewrite the function cancelling the factors which are causing the limit to be of the form $\frac{0}{0}$.

i) We have, $\frac{\lim_{x\to 2}(3x^2+4x)}{\lim_{x\to 2}(2x+1)}=\frac{\lim_{x\to 2}3x^2+\lim_{x\to 2}4x}{\lim_{x\to 2}2x+\lim_{x\to 2}1}$

$$=\frac{\lim_{x\to 2}3\lim_{x\to 2}x\lim_{x\to 2}x+\lim_{x\to 2}4\lim_{x\to 2}x}{\lim_{x\to 2}2\lim_{x\to 2}x+\lim_{x\to 2}1}$$

$$=\frac{3\times 2\times 2+4\times 2}{2\times 2+1}=\frac{20}{5}=4$$

ii) $\lim_{x\to 1}\frac{x^2-1}{x^3-2x^2+x}=\lim_{x\to 1}\frac{(x+1)(x-1)}{x(x-1)^2}$

$$=\lim_{x\to 1}\frac{(x+1)}{x(x-1)}=\frac{2}{1(1-1)}=\frac{2}{0}\text{ which is not defined.}$$

***

Example 20: Find $\lim_{x\to\infty}\frac{3x+1}{2x+5}$.

Solution: We cannot apply Theorem 2 directly since the limits of the numerator and the denominator, as $x\to\infty$, cannot be found.
Instead, we rewrite the quotient by multiplying the numerator and denominator by $1/x$, for $x\neq 0$.

Then, $\frac{3x+1}{2x+5} = \frac{3+(1/x)}{2+(5/x)}$, for $x \neq 0$. Now, we use that $\lim_{x\to\infty} 1/x = 0$, which you must have proved in E12 (i), to get

$$\lim_{x\to\infty} \frac{3x+1}{2x+5} = \lim_{x\to\infty} \frac{3+(1/x)}{2+(5/x)}$$

$$= \frac{\lim_{x\to\infty}(3+1/x)}{\lim_{x\to\infty}(2+5/x)} = \frac{3+0}{2+0} = \frac{3}{2}$$

***

Next, we will discuss an important theorem.

Theorem 3 (Sandwich theorem or Squeeze theorem): Let f, g and h be real functions defined on an interval I containing a, except possibly at a. Suppose

i) $f(x) \leq g(x) \leq h(x) \ \forall x \in I \setminus \{a\}$

ii) $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$

Then, $\lim_{x\to a} g(x)$ exists and is equal to L. This is illustrated in Fig. 16

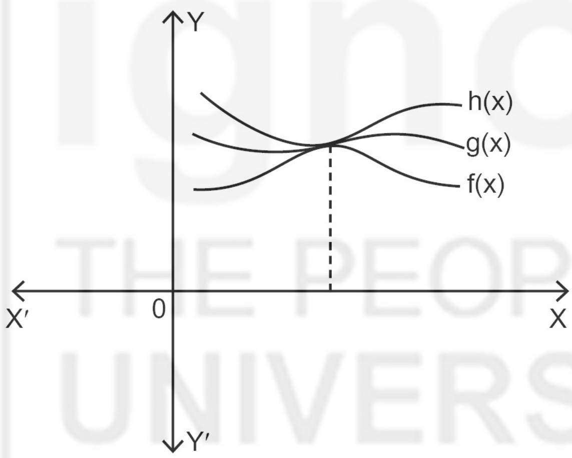


Fig. 16: Graph of three functions

Proof: By the definition of limit, given $\varepsilon > 0$, $\exists \delta_1 > 0$ and $\delta_2 > 0$ such that $|f(x) - L| < \varepsilon$ for $0 < |x-a| < \delta_1$ and $|h(x) - L| < \varepsilon$ for $0 < |x-a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon \text{ and } |h(x) - L| < \varepsilon$$

$$\Rightarrow L - \varepsilon \leq f(x) \leq L + \varepsilon, \text{ and } L - \varepsilon \leq h(x) \leq L + \varepsilon$$

We also have $f(x) \leq g(x) \leq h(x) \ \forall x \in I \setminus \{a\}$.

Thus, we get $0 < |x-a| < \delta \Rightarrow L - \varepsilon \leq f(x) \leq g(x) \leq h(x) \leq L + \varepsilon$.

In other words, $0 < |x-a| < \delta \Rightarrow |g(x) - L| < \varepsilon$.

Therefore, $\lim_{x\to a} g(x) = L$. ■

Remark 4: This is called sandwich theorem as g is being sandwiched between f and h.

Let us see how this theorem can be used to find the limits of trigonometric functions in the following examples.

Example 21: Prove that $\cos x < \frac{\sin x}{x} < 1$ for $0 < |x| < \frac{\pi}{2}$.

Solution: We know that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. Hence, it is sufficient to prove the inequality for $0 < x < \frac{\pi}{2}$.

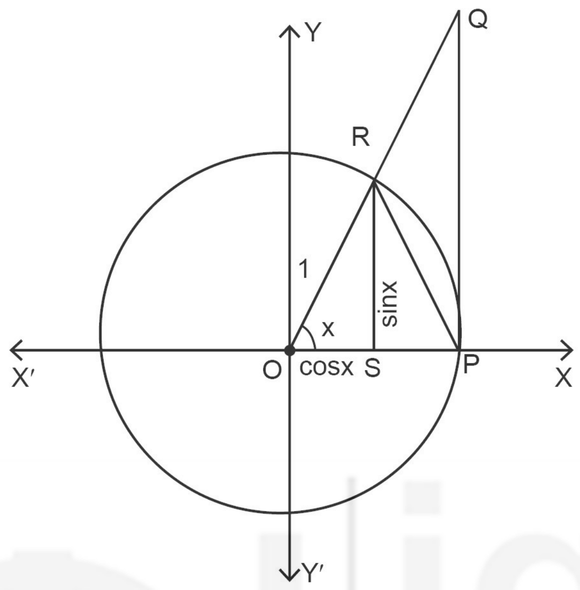


Fig. 17

In the Fig. 17, O is the centre of the unit circle such that the angle POR is x radians and $0 < x < \frac{\pi}{2}$. Line segments QP and RS are perpendicular to OP. Further, join PR. Then Area of ΔOPR < Area of sector OPR < Area of ΔOPQ.

i.e., $\frac{1}{2} OP . RS < \frac{x}{2\pi} . \pi . (OP)^2 < \frac{1}{2} OP . PQ.$

i.e., $RS < x . OP < PQ$ ... (1)

From ΔORS, $\sin x = \frac{RS}{OR} = \frac{RS}{OP}$ and hence $RS = OP \sin x$.

Also from ΔOQP, $\tan x = \frac{PQ}{OP}$ and hence $PQ = OP. \tan x$.

Now substituting the values of RS and PQ in the inequality (1).

Thus, $OP \sin x < OP . x < OP. \tan x$.

Since length OP is positive, we have

$\sin x < x < \tan x$ ... (2)

Since $0 < x < \frac{\pi}{2}$, $\sin x$ is positive and thus, dividing the inequality (2) throughout by $\sin x$, we get $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$. Taking reciprocals throughout, we have

$$\cos x < \frac{\sin x}{x} < 1.$$

Hence proved.

***

Example 22: Find the two important limits.

i) $\lim_{x \to 0} \frac{\sin x}{x} = 1.$ ii) $\lim_{x \to 0} \frac{1-\cos x}{x} = 0.$

Solution: i) The inequality in Example 21 says that the function $\frac{\sin x}{x}$ is sandwiched between the function $\cos x$ and the constant function which takes value 1. Further, since $\lim_{x \to 0} \cos x = 1,$ we see that the proof of (i) is complete by sandwich theorem.

ii) You may recall the trigonometric identity $1-\cos x = 2\sin^2\left(\frac{x}{2}\right).$

Then, $$\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{2\sin^2\left(\frac{x}{2}\right)}{x} = \lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} . \sin\left(\frac{x}{2}\right)$$

$$= \lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} . \lim_{x \to 0} \sin\left(\frac{x}{2}\right) = 1.0 = 0$$

Observe that we have implicitly used the fact that $x \to 0$ is equivalent to $\frac{x}{2} \to 0.$ This may be justified by putting $y = \frac{x}{2}$.

***

Example 23: Evaluate

i) $\lim_{x \to 0} \frac{\sin 3x}{\sin x}$ ii) $\lim_{x \to 0} \frac{\tan x}{x}$

Solution: i) $$\lim_{x \to 0} \frac{\sin 3x}{\sin x} = \lim_{x \to 0} \left[\frac{\sin 3x}{3x} . \frac{3x}{\sin x}\right]$$

$$= 3.\lim_{x \to 0} \left[\left(\frac{\sin 3x}{3x}\right) \div \left(\frac{\sin x}{x}\right)\right]$$

$$= 3.\lim_{3x \to 0} \left[\frac{\sin 3x}{3x}\right] \div \lim_{x \to 0} \left[\frac{\sin x}{x}\right]$$

$$= 3.1.1 = 3 \, (\text{as } x \to 0, 3x \to 0)$$

ii) We have, $$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x} = \lim_{x \to 0} \frac{\sin x}{x} . \lim_{x \to 0} \frac{1}{\cos x} = 1.1 = 1$$

***

You can similarly calculate the limits in the following exercises.

E14) Prove v) and vi) of theorem 2.

E15) Show that $\lim_{x \to 1} \frac{3}{x} = 3$.

E16) Calculate $\lim_{x \to 1} \left[2x + 5\left(\frac{x^2}{1+x^2}\right)\right]$.

Let us see how the concepts of one-sided limit and limit are connected in the following theorem.

Theorem 4: The following statements are equivalent.

i) $\lim_{x \to p} f(x)$ exists.

ii) $\lim_{x \to p^+} f(x)$ and $\lim_{x \to p^-} f(x)$ exist and are equal.

Proof: To show that i) and ii) are equivalent, we have to show that i) $\Rightarrow$ ii) and ii) $\Rightarrow$ i). We first prove that i) $\Rightarrow$ ii). For this, we assume that $\lim_{x \to p} f(x) = L$.

Then given $\varepsilon > 0, \exists \delta > 0$ such that $|f(x) - L| < \varepsilon$ for $0 < |x - p| < \delta$.
Now, $0 < |x - p| < \delta \Rightarrow p < x < p + \delta$ and $p - \delta < x < p$. Thus, we have $|f(x) - L| < \varepsilon$ for $p < x < p + \delta$ and for $p - \delta < x < p$. This means that $\lim_{x \to p^-} f(x) = L = \lim_{x \to p^+} f(x)$.
We now prove the converse, that is, ii) $\Rightarrow$ i). For this, we assume that $\lim_{x \to p^-} f(x) = \lim_{x \to p^+} f(x) = L$. Then, given $\varepsilon > 0, \exists \delta_1, \delta_2 > 0$ such that

$$|f(x) - L| < \varepsilon \text{ for } p - \delta_1 < x < p$$

$$|f(x) - L| < \varepsilon \text{ for } p < x < p + \delta_2$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for both $p - \delta < x < p$ and $p < x < p + \delta$, we have $|f(x) - L| < \varepsilon$. This means that $|f(x) - L| < \varepsilon$, whenever, $0 < |x - p| < \delta$. Hence, $\lim_{x \to p} f(x) = L$.
Thus, we have shown that i) $\Rightarrow$ ii) and ii) $\Rightarrow$ i), proving that they are equivalent. ■

From Theorem 4, we can conclude that if $\lim_{x \to p} f(x)$ exists, then $\lim_{x \to p^+} f(x)$ and $\lim_{x \to p^-} f(x)$ also exist and further

$$\lim_{x \to p} f(x) = \lim_{x \to p^+} f(x) = \lim_{x \to p^-} f(x).$$

Remark 5: If you apply Theorem 4 to the function $f(x) = x - [x]$, you will see that $\lim_{x \to p} \{x - [x]\}$ does not exist as $\lim_{x \to p^+} \{x - [x]\} \neq \lim_{x \to p^-} \{x - [x]\}$.

In Unit 6, we introduced polynomial functions and rational functions and trigonometric functions. In the following section, we will complete our list of the elementary functions typically used in calculus by using limits to introduce exponential functions and their inverses, the logarithmic functions.

7.6 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

So far, we have learnt some aspects of different types of functions like modulus function, greatest integer function, polynomial function, rational function and trigonometric functions. In this section, we shall discuss about a new type of function called exponential function and its inverse function called logarithmic function.

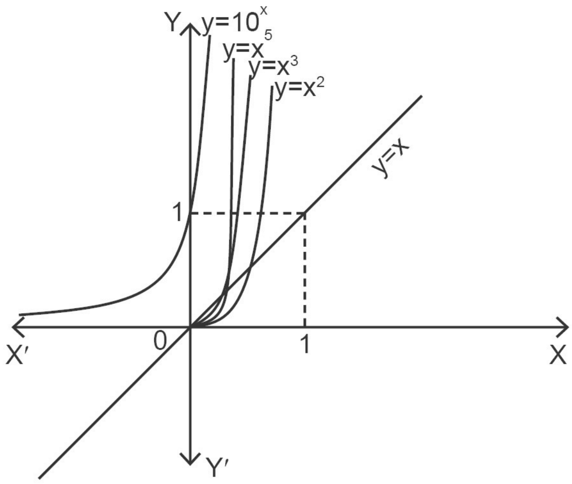


Fig. 18

Fig. 18 shows the graph of $y = f_1(x) = x, y = f_2(x) = x^2, y = f_3(x) = x^3$ and $y = f_4(x) = x^4$. Observe that the curves gets steeper as the power of x increases. Steeper the curve, faster is the rate of growth. What does this mean? It means that for a fixed increment in the value of $x(>1)$, the increment in the value of $y = f_n(x)$ increases as n increases for $n = 1, 2, 3, 4$. It is conceivable that such a statement is true for all positive values of n, where $f_n(x) = x^n$. Essentially, this means that the graph of $y = f_n(x)$ leans more towards the $y-$axis as n increases. For example, consider $f_5(x) = x^5$ and $f_{10}(x) = x^{10}$. If x increases from 1 to 4, f_5 increases from 1 to 4^5 whereas f_{10} increases from 1 to 4^{10}. Thus, for the same increment in x, f_{10} grows faster than f_5.

From the above discussion, it is clear that the growth of polynomial functions is dependent on the degree of the polynomial function that is higher the degree, greater is the growth. Is there a function which grows faster than any polynomial function? The answer is yes and an example of such a function is $y = f(x) = 10^x$.

Our claim is that this function f grows faster than $f_n(x) = x^n$ for any positive integer n. For example, we can prove that 10^x grows faster than $f_{100}(x) = x^{100}$. For large values of x like $x = 10^3$, note that $f_{100}(x) = (10^3)^{100} = 10^{300}$ whereas $f(10^3) = 10^{10^3} = 10^{1000}$. Clearly $f(x)$ is much greater than $f_{100}(x)$. It is not difficult to prove that for all $x > 10^3, f(x) > f_{100}(x)$. But, we will not attempt to give a proof of this here. Similarly, by choosing large values of x, one can verify that $f(x)$ grows faster than $f_n(x)$ for any positive integer n. This leads to the following definition.

Definition: The exponential function with positive base a is the function f defined by $y = f(x) = a^x$.

Let us explain it.

If x is a positive integer, then $a^n = \underbrace{a.a....a}_{\text{n times}}$.

If x is 0, then $a^o = 1$.

If x is a negative integer, then $a^{-n} = \frac{1}{a^n}$.

If x is a rational number that is $x = \frac{p}{q}$, where p and q are integers and $q \neq 0$, then $a^x = a^{p/q} = (a^p)^{1/q} = (a^{1/q})^p = (\sqrt[q]{a})^p$.

Now, the question is how would we define a^x, if x is an irrational number? For example, $2^\pi, 5^{\sqrt{3}}, 10^{\sqrt{2}}$, etc.

To answer this, look at the graph of $y = 10^x$ in Fig. 18, where x is rational. If we expand the domain of $y = 10^x$ from rational to both rational as well as irrational, then, there will be holes in the graph corresponding to irrational values of x. How do we fill these holes? For this, consider
$10^{1.4} < 10^{\sqrt{2}} < 10^{1.5}$ [Since $1.4 < \sqrt{2} < 1.5$]

We further approximate $\sqrt{2}$, then, from left side, we get $1.4, 1.41, 1.414, 1.4142, 1.41423, \ldots$ and from the right side we get 1.5, 1.42, 1.418, 1.4143,...
Using the approximation process, we can write $10^{\sqrt{2}} \approx 10^{1.4142}$.
Also, the holes can be filled by these approximations. Thus, we can define $a^x = \lim_{r \to x} a^r$, where r is rational and for any irrational x. For example, 2^π is the limit of the sequence of numbers $2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, \ldots$

Now, let us draw the graph of a^x for different positive values of x. Fig. 19 shows these graphs.

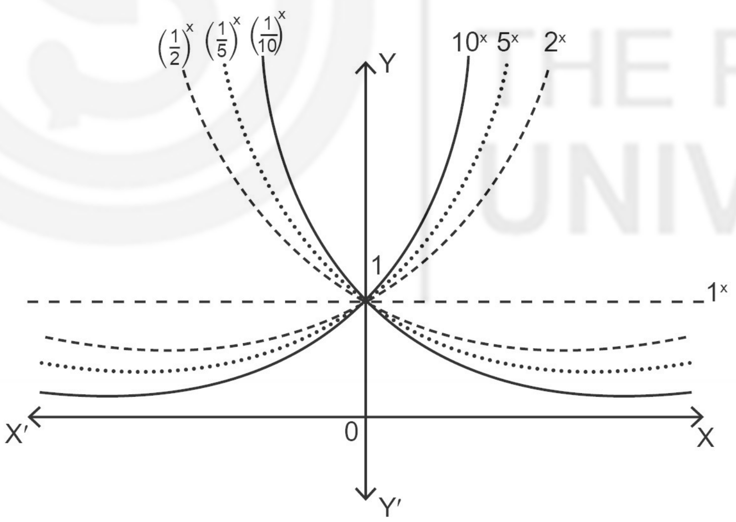


Fig. 19: Graph of a^x for $a = 1, 2, 5, 10, 1/2, 1/5, 1/10$

Fig. 19 shows three types of exponential functions $y = a^x$. These are

i) When $0 < a < 1$, the exponential function decreases.

ii) When $a = 1$, the exponential function is constant.

iii) When $a > 1$, the exponential function increases.

You may notice that the graph of $(1/a)^x$ is just the reflection of the graph of a^x about $y-$axis. This is because $(1/a)^x = \frac{1}{a^x} = a^{-x}$. You may also notice that all the graphs pass through the same point $(0,1)$ because $a^o = 1$ for $a \neq 0$.

We are giving some of the properties of the exponential functions.

i) Domain of the exponential function is $\mathbb{R}$,

ii) Range of the exponential function is $]0, \infty[$.

iii) $a^x > 0$ for all $x \in \mathbb{R}, a > 0$.

iv) The point $(0,1)$ is always on the graph of the exponential function, since $a^o = 1$ for every real $a > 1$.

v) Exponential function is increasing i.e., as we move from left to right, the graph rises above for $a > 1$. The function is decreasing for $0 < a < 1$. Since, the exponential function is monotonic, therefore, it is one-one and onto.

vi) $a^0 = 1,\ a^{-m} = \frac{1}{a^m},\ a^{1/m} = \sqrt[m]{a},\ a^{m/n} = \left(a^{1/n}\right)^m,$ where m and n are reals.

vii) If $a \neq 1$, then $a^m = a^n$ iff $m = n$.

viii) $a^m a^n = a^{m+n}$

ix) $\frac{a^m}{a^n} = a^{m-n}$

x) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

xi) If $m > n$ and $a > 1$, then $a^m > a^n$. Also, If $m > n$ and $0 < a < 1$, then $a^m < a^n$.

The exponential function whose base is 10, is called the **common exponential function.**

Let us discuss the following example to understood more.

Example 24: Solve each of the following equations.

i) $3^{x^2+3} = 81$ ii) $2^{x+1}\ 3^x = 432$ iii) $\left(\sqrt{2}\right)^x = \frac{4^x}{2}$

Solution: i) Writing 81 as 3 raised to the power 4, so that powers can be equated on same base, we get.

$$3^{x^2+3} = 3^4$$
$$x^2 + 3 = 4$$

or $x^2 = 1$, hence, $x = \pm 1$

ii) $2^{x+1}\ 3^x = 432$

$2^x\ .\ 2\ .\ 3^x = 432$

$2^x\ .\ 3^x = 216$

$6^x = 216$

$6^x = 6^3$

or $x = 3$

iii) $\left(\sqrt{2}\right)^x = \frac{4^x}{2}$

$$(2)^{\frac{x}{2}} = \frac{4^x}{2}$$

$$(2)^{\frac{x}{2}} = \frac{2^{2x}}{2}$$

$$(2)^{\frac{x}{2}} = 2^{2x-1}$$

$$\frac{x}{2} = 2x - 1$$

or $x = \frac{2}{3}$.

***

We can read the following limits from the graphs given in Fig. 19.

i) $\lim\limits_{x\to\infty} a^x = \infty$, when $a > 1$.

ii) $\lim\limits_{x\to-\infty} a^x = 0$, when $a > 1$.

iii) $\lim\limits_{x\to\infty} a^x = 0$, when $0 < a < 1$.

iv) $\lim\limits_{x\to-\infty} a^x = \infty$, when $0 < a < 1$.

You may find these limits using the definition of a limit at infinity.

Example 25: Find $\lim\limits_{x\to\infty}(5^{-x} - 1)$. Also, draw the graph of $5^{-x} - 1$.

Solution: Let $y = 5^{-x} - 1$.

$$\lim_{x\to\infty}(5^{-x} - 1) = \lim_{x\to\infty}(5^{-x}) - \lim_{x\to\infty}(1) = \lim_{x\to\infty}\left(\frac{1}{5}\right)^x - 1 = 0 - 1 = -1.$$

Fig. 20 shows the graph of $5^{-x} - 1$.

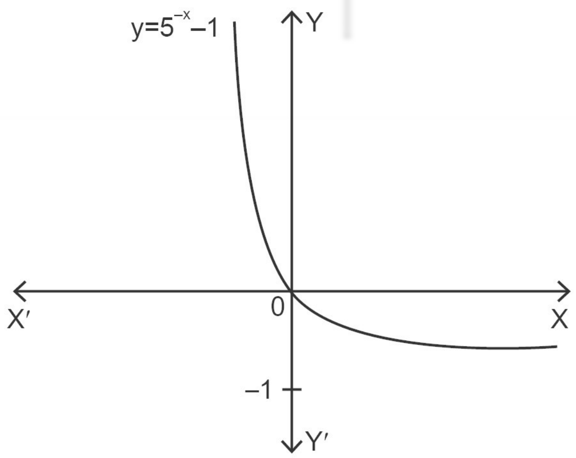


Fig. 20: Graph of $5^{-x} - 1$

***

One of the most convenient base chosen for calculus is a number e, which is defined as $e = \lim\limits_{x\to 0}(1+x)^{1/x}$. Accordingly $a^x = e^x$, when $a = e$. Let us find the value of e intuitively. Table 12 shows the value of $(1+x)^{1/x}$ corresponding to the values of x close to 0.

Table 12

x	1	0.1	0.01	0.0001	0.000001	0.0000001
$(1+x)^{1/x}$	2	2.5937	2.7048	2.7181	2.71828	2.71828

Since $e = \lim\limits_{x\to 0}(1+x)^{1/x} \approx 2.71828$

You may notice that the decimal places of e are non repeating because e is an irrational number. This leads to the following definition.

Definition: The function f defined as $f(x) = e^x$ is called the **natural exponential function** with domain $\mathbb{R}$ and range $]0,\infty[$. Thus, $e^x > 0$ for all x. Fig. 21 shows the graph of e^x. You may observe that the graph of e^x lies between the graphs of 2^x and 3^x as e lies between 2 and 3.

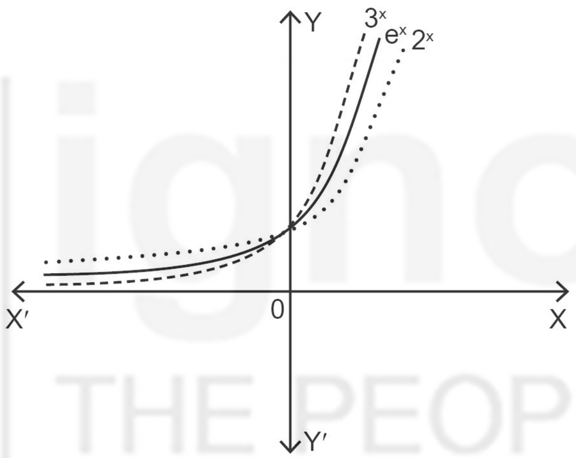


Fig. 21: Graph of e^x

From the graph of e^x, given in Fig. 21, it is clear that

i) The graph passes through the point $(0,1)$.

ii) $\lim\limits_{x\to -\infty} e^x = 0$

iii) $\lim\limits_{x\to \infty} e^x = \infty$.

The natural exponential function is commonly used in calculus and its applications. Now, let us find the limit of exponential function in the following example.

Example 26: Find $\lim\limits_{x\to 2^+} e^{3/(2-x)}$.

Solution: Let $t = \dfrac{3}{2-x}$.

As $x \to 2^+ \Rightarrow 2-x \to 0^- \Rightarrow \dfrac{3}{2-x} \to -\infty$.

Therefore, $\lim\limits_{x\to 2^+} e^{\frac{3}{(2-x)}} = \lim\limits_{t\to -\infty} e^t = 0$.

***

Try the following exercises.

E17) Solve $3^{x^2-x}=9$.

E18) Find $\lim\limits_{x\to\infty}\dfrac{e^{2x}}{e^{2x}+1}$.

E19) Compare the graphs of $f(x)=x^2$ and $g(x)=2^x$.

It would be interesting to know if the inverse of the exponential function exists and has nice interpretation. If $a>0$ and $a\neq 1$, the exponential function $f(x)=a^x$ is either increasing or decreasing and so it is one-one and onto. Therefore, it has an inverse function, which is called the **logarithmic function** with base a and is denoted by $\log_a$. This gives the following definition.

The letter e was chosen to represent this number in honour of the great Swiss mathematician Leonhard Euler, who discovered many of its special properties and investigated applications in which e plays a vital role.

Definition: Let $a>0$ and $a\neq 1$, then $\log_a x$ is called the **logarithmic function**. It is read as logarithm of x to base a.

Thus, $\log_a x=y$ if $a^y=x$. That is $f^{-1}(x)=y \Leftrightarrow f(y)=x$. Thus, if $x>0$, then $\log_a x$ is the exponent to which the base a must be raised to give x. For example, $\log_{10}100=2$ because $10^2=100$. Let us work with a few explicit examples to get a feel for this. We know that $3^2=9$. In terms of logarithms, we can rewrite this as $\log_3 9=2$. Similarly, $10^3=1000$ is equivalent to saying $\log_{10}1000=3$. Also, $256=2^8=4^4$ is equivalent to saying $\log_2 256=8$ or $\log_4 256=4$ or if we fix a base $a>1$, we may look at logarithm as a function from positive real numbers to all real numbers. This function is called **logarithmic function**, and is $f:]0,\infty[\to\mathbb{R}$ defined by $f(x)=\log_a x=y$ if $a^y=x$.

If the base $a=10$, we say it is **common logarithmic function** and if $a=e$, then we say it is **natural logarithmic function**. Often natural logarithm is denoted by $\ln$. Fig. 22 gives the plot of logarithm function to base 2, $e,5$ and 10.

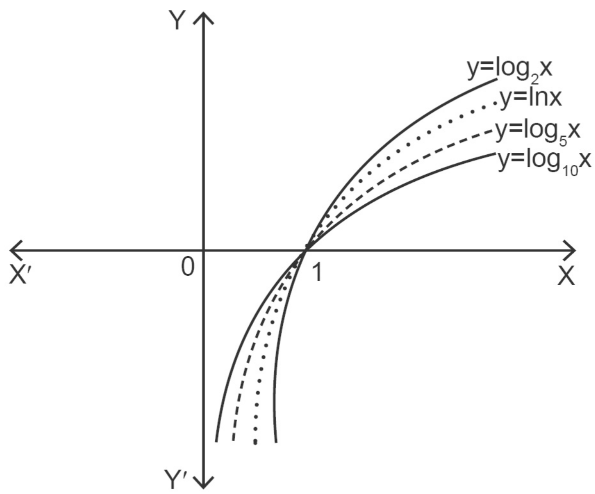


Fig. 22

Here, we are listing the properties of the logarithmic function to any base $a > 1$.

i) The domain of log function is $]0, \infty[$, since, there cannot be a meaningful definition of logarithm of non-positive reals.

ii) The range of logarithmic function is the set of all real numbers.

iii) The graph of logarithmic function $\log_a x$ is the reflection of the graph of $y = a^x$ about the line $y = x$. The point $(1,0)$ is always on the graph of the logarithmic function.

iv) The logarithmic function is always increasing, i.e., as we move from left to right the graph rises.

v) For x very near to zero, the value of $\log x$ can be made lesser than any given real number. In other words, in the fourth quadrant the graph approaches $y-$axis (but never meets it).

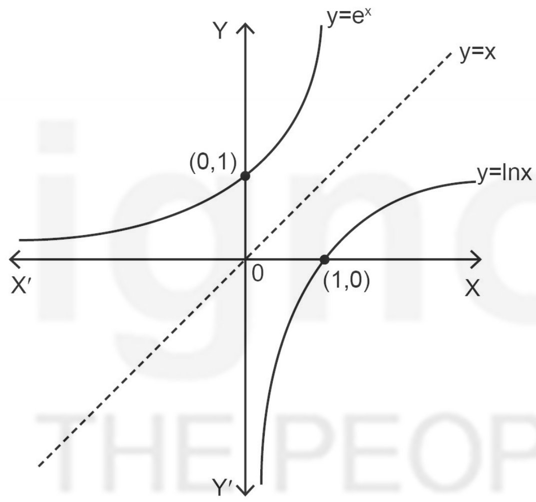


Fig. 23: Graph of e^x and $\ln x$

vi) Fig. 23 shows the plot of $y = e^x$ and $y = \ln x$. It is of interest to observe that the two curves are the mirror images of each other reflected along the line $y = x$.

vii) $\log_a x = \dfrac{\log_b x}{\log_b a}$ or $\log_a x . \log_b a = \log_b x$.[Change of base property]

viii) $\log_a (xy) = \log_a x + \log_a y$

ix) $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

x) $\log_a (x^r) = r \log_a x,$ where r is a real number

xi) $\ln(e^x) = x$, for all $x \in \mathbb{R}$

xii) $e^{\ln x} = x$ for $x > 0$.

xiii) $\ln e = 1$.

Let us attempt few examples based on this.

Example 27: Solve the equation $\log_5(2x+1)-2\log_5(x-3)=1$

Solution: Given that $\log_5(2x+1)-\log_5(x-3)^2=1$

$\log_5\frac{(2x+1)}{(x-3)^2}=1$ [using (ix)]

$5^1=\frac{2x+1}{(x-3)^2}$ [using definition]

$5(x^2-6x+9)=2x+1$

$5x^2-32x+44=0$

$x=2,\frac{22}{5}$

Note that for $x=2, x-3<0$. Thus, the given logarithmic equation has only $x=\frac{22}{5}$ as a solution.

***

Example 28: Is it true that $x=e^{\log x}$ for all real x? Justify.

Solution: First, observe that the domain of log function is set of all positive real numbers. So, the above equation is not true for non-positive real numbers. Now, let $y=e^{\log x}$. If $y>0$, we may take logarithm which gives us $\log y=\log(e^{\log x})=\log x.\log e=\log x$. Thus, $y=x$. Hence, $x=e^{\log x}$ is true only for positive values of x.

***

Example 29: Find x, if $\ln x=8$.

Solution: Here, $\ln x=8$ means $e^8=x$.
Therefore, $x=e^8$

***

Example 30: Solve $e^{2x+3}=1$.

Solution: $e^{2x+3}=1$.
Taking natural logarithm both the sides, we get

$$\ln e^{2x+3}=\ln 1$$
$$2x+3=0$$
$$x=-\frac{3}{2}$$

***

Now, try the following exercises.

E20) Solve the equation $\ln\left(\frac{x^2}{1-x}\right)=\ln x+\ln\left(\frac{2x}{1+x}\right)$.

E21) Find $\lim_{x\to 0}\log_{10}(\tan^2 x)$.

In the next section, we shall discuss hyperbolic functions and their inverses.

7.7 HYPERBOLIC FUNCTIONS AND THEIR INVERSE FUNCTIONS

In applications of mathematics to other sciences, we, very often, come across certain combinations of e^x and e^{-x}. Because of their importance, these combinations are given special names, like the hyperbolic sine, the hyperbolic cosine etc. These names suggest that they have some similarity with the trigonometric functions. Let's look at their precise definitions and try to understand the points of similarity and dissimilarity between the hyperbolic and the trigonometric functions.

Definition: The hyperbolic sine function is defined by $\sinh x = \frac{e^x - e^{-x}}{2}$ for all $x \in \mathbb{R}$. The range of this function is $\mathbb{R}$. You will notice that

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = -\frac{e^x - e^{-x}}{2} = -\sinh(x).$$

Definition: The hyperbolic cosine function is defined by $\cosh x = \frac{e^x + e^{-x}}{2}$ for all $x \in \mathbb{R}$. The range of this function is $[1, \infty]$. You will notice that

$$\cosh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = -\frac{e^x - e^{-x}}{2} = \cosh x.$$

It is clear that, the hyperbolic sine is an odd function, while the hyperbolic cosine is an even function. Fig. 24(a) and (b) show the graphs of these two functions respectively.

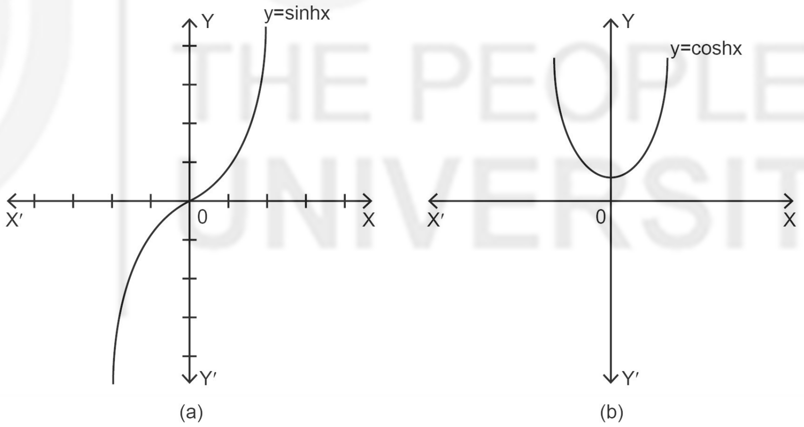


Fig. 24: Graph of (a) $\sinh x$ (b) $\cosh x$

Example 31: Prove that $\cosh^2 x - \sinh^2 x = 1$.

Solution: Here $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4} = 1$$

This identify refers to the reason for the name **hyperbolic** function. We represent a point $(\cos t, \sin t)$ on a unit circle $x^2 + y^2 = 1$. Likewise, if t is any real number, then, the point $P(\cosh t, \sinh t)$ lies on the right branch of the hyperbola $x^2 - y^2 = 1,$ because $\cosh^2 t - \sinh^2 t = 1$ and $\cosh t \geq 1$. Fig. 25 shows the point P.

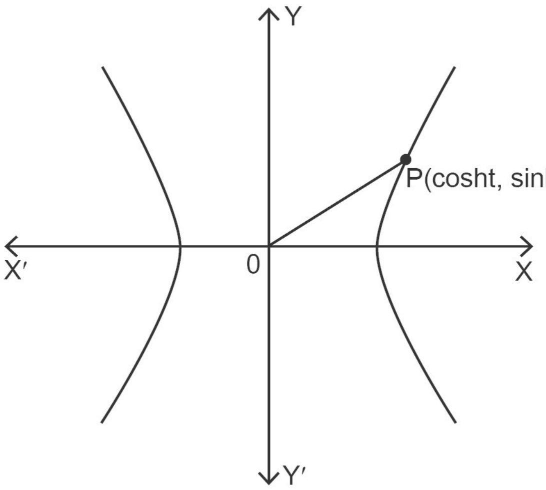


Fig. 25: A point on hyperbola

We also define four other hyperbolic functions as given below.

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},\ \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}},\ \operatorname{cosech} x = \frac{2}{e^x - e^{-x}}.$$

Let us discuss the inverse hyperbolic sine function.

From Fig. 24(a) you can see that the hyperbolic sine is a strictly increasing function. This means that its inverse exists, and

$$y = \sinh^{-1} x \Leftrightarrow x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$\Leftrightarrow 2x = e^y - e^{-y}$$

$$\Leftrightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Leftrightarrow (e^y)^2 - 2xe^y - 1 = 0$$

$$\Leftrightarrow e^y = x + \left(\sqrt{1+x^2}\right)$$

$$\Leftrightarrow y = \ln\left(x + \sqrt{1+x^2}\right)$$

[We have used the formula for finding the roots of a quadratic equation here. Note that if $e^y = x - \sqrt{1+x^2}$, then $e^y < 0$, which is impossible. Therefore, we ignore this root.]

Thus, $\sinh^{-1} x = \ln\left(x + \sqrt{1+x^2}\right), x \in]-\infty, \infty[$. In Fig. 26, we have drawn the graph of $\sinh^{-1} x$.

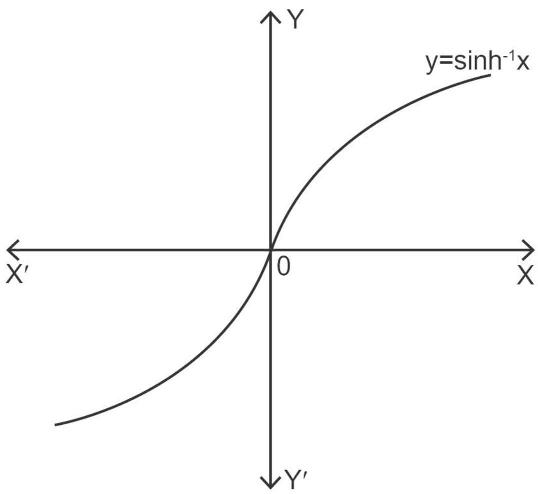


Fig. 26: Graph of $\sinh^{-1}x$

In the case of the hyperbolic cosine function, we see from Fig. 24(b), that its inverse will exist if we restrict its domain to $[0, \infty[$. The domain of this inverse function will be $[1, \infty[$, and its range will be $[0, \infty[$.

Now $y = \cosh^{-1} x \Leftrightarrow x = \cosh y = \dfrac{e^y - e^{-y}}{2}$

$$\Leftrightarrow e^{2y} - 2xe^y + 1 = 0$$

$$\Leftrightarrow e^y = x + \sqrt{x^2 - 1}$$

$$\Leftrightarrow y = \ln(x + \sqrt{x^2 - 1})$$

[Again we ignore the root $e^y = x - \sqrt{x^2 - 1}$, because then $e^y < 1$, which is impossible since $y > 0$.]

Thus, $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), x \geq 1.$

Fig. 27 shows the graph of $\cosh^{-1} x$.

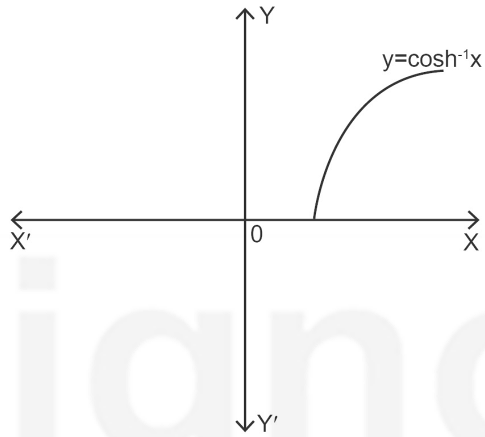
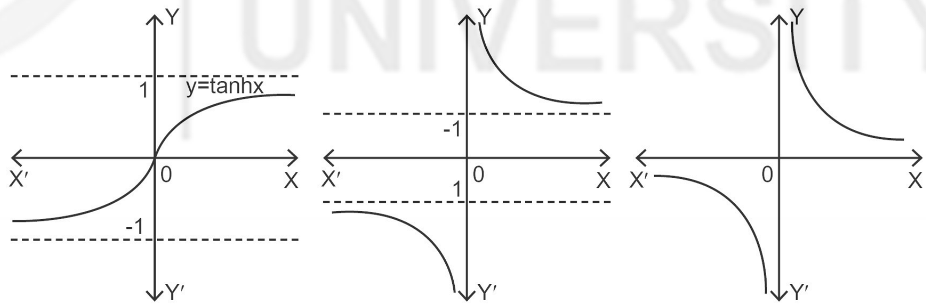


Fig. 27: Graph of $\cosh^{-1} x$

Fig. 28 (a), (b) and (c) show the graphs of $\tanh x, \coth x$ and $\text{cosech}\, x$. You can see that each of these functions is one-one and strictly monotonic. Thus, we can talk about the inverse in each case.



(a) (b) (c)

Fig. 28: Graph of (a) $\tanh x$, (b) $\coth x$, (c) $\text{cosech}\, x$

Arguing as for $\sinh^{-1} x$ and $\cosh^{-1} x$, we get

$$y = \tanh^{-1} x \Leftrightarrow x = \tanh y \Leftrightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), |x| < 1$$

$$y = \coth^{-1} x \Leftrightarrow x = \coth y \Leftrightarrow y = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), |x| > 1$$

$$y = \text{cosech}^{-1} x \Leftrightarrow x = \text{cosech}\, y \Leftrightarrow y = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{1}\right), x \neq 0$$

Since, $\operatorname{sech} x = \frac{1}{\cosh x}$, we shall have to restrict the domain of each $\operatorname{sech} x$ to $[0, \infty[$ before talking about its inverse, as we did for $\cosh x$, $\operatorname{sech}^{-1} x$ is defined all $x \in]0,1]$, and we can write $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).0 < x \le 1$.

Now, try the following exercises.

E22) Verify that

i) $\tanh x = \frac{\sinh x}{\cosh x}$

ii) $1 - \tanh^2 x = \operatorname{sech}^2 x.$

E23) Derive an identify connecting $\coth x$ and $\operatorname{cosech} x$.

Now, let us summarise the unit.

7.7 SUMMARY

We end this unit by summarising what we have covered in it.

1. **Informal view of limit:** If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a. (but not equal to a), then $\lim_{x\to a} f(x) = L.$

2. One-sided limits.

3. The limit of a function f at a point p of its domain is L if given $\varepsilon > 0, \exists\, \delta > 0$, such that $|f(x) - L| < \varepsilon$ whenever $|x - p| < \delta$.

4. $\lim_{x\to p} f(x)$ exists if and only if $\lim_{x\to p^+} f(x)$ and $\lim_{x\to p^-} f(x)$ both exist and both are equal.

5. Uniqueness of limit i.e.
If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, then $L = M$.

6. Algebra of limits:
Let f and g be two functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.
Then

i) $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ (Sum rule)

ii) $\lim_{x\to a}[f(x) - g(x)]^a = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ (Difference rule)

iii) $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$ (Product rule)

iv) $\lim_{x\to a} \frac{f}{g(x)} = \frac{f}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \ne 0$ (Reciprocal rule)

v) $\lim_{x\to a} k = k$ (Constant function rule)

vi) $\lim_{x\to a} x = a$ (Identity function rule)

vii) $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, where n is a positive integer. (Power rule)

In particular, if $f(x) = x$, $\lim_{x\to a} x^n = a^n$, where n is a positive integer.

viii) $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, where n is a positive integer, and if n is even, we assume that $\lim_{x\to a} f(x) > 0$.

7. **Sandwich theorem or Squeeze theorem:**
Let f, g and h be functions defined on an interval I containing a, except possibly at a. Suppose

i) $f(x) \le g(x) \le h(x)\ \forall x \in I\setminus\{a\}$

ii) $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$

Then $\lim_{x\to a} g(x)$ exists and is equal to L.

8. The exponential function f defined by $f(x) = a^x, a > 1$, and the natural exponential function f is defined by $f(x) = e^x$.

9. The logarithmic function f is defined by $f(x) = \log_a x,\ a > 1$, and the natural logarithmic function f is defined by $f(x) = \ln x$.

10. The hyperbolic functions are

i) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

ii) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

iii) $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

iv) $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

v) $\operatorname{sech} x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

vi) $\operatorname{cosech} x = \dfrac{2}{e^x - e^{-x}}$.

11. Inverse hyperbolic functions are

i) $\sinh^{-1} x = \ln\left(x + \sqrt{1 + x^2}\right), x \in\ ]-\infty, \infty[$.

ii) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), x \ge 1$

iii) $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), |x| < 1$

iv) $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), |x| > 1$

v) $\operatorname{cosech}^{-1}x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{1}\right), x \neq 0$

vi) $\operatorname{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right). 0 < x \leq 1.$

7.8 SOLUTIONS/ANSWERS

E1) i) $f(x) = \frac{x^2 - 9}{x-3}$

$= \frac{(x+3)(x-3)}{(x-3)}$

$= (x+3)$ [because $x \neq 3, \text{so } x - 3 \neq 0$]

Table 13

$x < 3$	2.5	2.9	2.95	2.99	$x > 3$	3.01	3.05	3.1	3.5
$f(x)$	5.5	5.9	5.95	5.99	$f(x)$	6.01	6.05	6.1	6.5

x ⟶ ⟵ x

In the above table, it is clear that as x gets closer to 3, the corresponding value of $f(x)$ also gets closer to 6.
However, in this case $f(x)$ is not defined at $x = 3$. The idea can be expressed by saying that the limiting value of $f(x)$ is 6 when x approaches to 3.

ii) $f(x) = \frac{x^2 - 2x}{x^2 - 4} = \frac{x(x-2)}{(x-2)(x+2)} = \frac{x}{x+2}$ [as $x - 2 \neq 0$]

Now, we substitute values of x close to 2 but not equal to 2.

Table 14

$x > 2$	3.0	2.5	2.1	2.01	2.001	$x < 2$	1.999	1.99	1.9	1.5	1.0
$f(x)$	0.60	0.56	0.51	0.50	0.500	$f(x)$	0.500	0.501	0.52	0.55	0.33

x ⟶ ⟵ x

We see in the above that the value of $f(x)$ gets closer to 0.5 as x gets values closer to 2.

iii) For finding the limit, we assign values to x from the left and also from the right of 0.

Table 15

$x < 0$	- 0.5	- 0.1	- 0.01	- 0.001	- 0.001
$f(x) = 4x - 1$	- 3	- 1.4	- 1.04	-1.004	-1.004

Table 16

x	0.5	0.1	- 0.01	- 0.001	- 0.0001
$4x - 1$	1	- 0.6	- 0.96	-0.996	-0.9996

It is clear from the above table that the limit of $4x-1$ as x approaches 0 is -1.

i.e. $\lim_{x\to 0}(4x-1)=-1$

E2) i) This function being the constant function takes the same value (3, in this case) everywhere, i.e., its value at points close to 2 is 3. Hence

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=\lim_{x\to 2} f(x)=3$$

Graph of $f(x)=3$ is anyway the line parallel to $x-$axis passing through (0, 3) and is shown in Fig. 29. From this it is also clear that the required limit is 3. In fact, it is easily observed that $\lim_{x\to a} f(x)=3$ for any real number a.

ii) We tabulate the values of $f(x)$ near $x=1$ in Table

Table 17

x	0.9	0.99	0.999	1.01	1.1	1.2
$f(x)$	1.71	1.9701	1.997001	2.0301	2.31	2.64

From this it is reasonable to deduce that $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x)=2.$

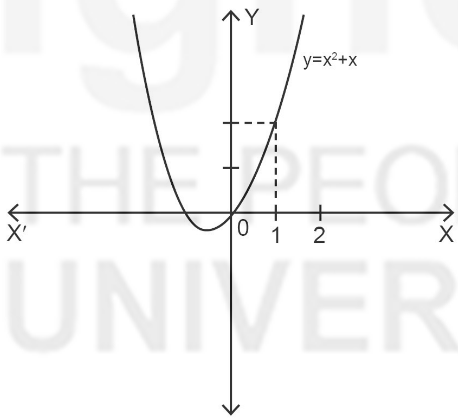


Fig. 29

From the graph of $f(x)=x^2+x$ shown in the Fig. 29, it is clear that as x approaches 1, the graph approaches (1, 2).

Here, again we observe that the $\lim_{x\to 1} f(x)=f(1)$.

iii) Here we tabulate the (approximate) value of $f(x)$ near 0 (Table).

Table 18

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.9850	0.98995	0.9989995	1.0009995	1.00995	1.0950

From the Table , we may deduce that

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=\lim_{x\to 0} f(x)=1$$

In this case too, we observe that $\lim_{x\to 0} f(x)=f(0)=1.$

E3)

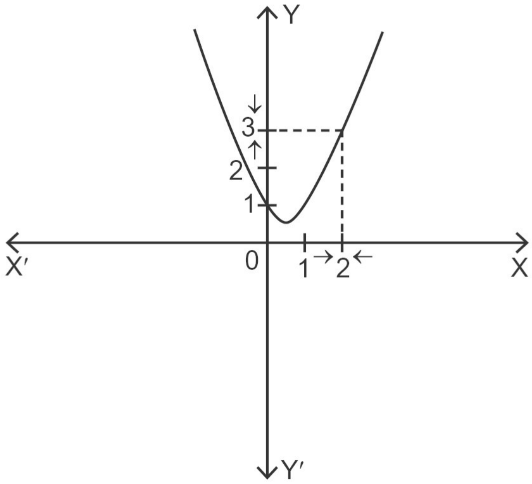


Fig. 30: Graph of $x^2 - x + 1$.

Table 19

x	1.0	1.5	1.9	1.95	1.99	1.995	1.999
$f(x)$	1.000	1.750	2.710	2.853	2.970	2.985	2.997

⟶ Left sided limits

Table 20

x	3.0	2.5	2.1	2.05	2.01	2.005	2.001
$f(x)$	7.000	4.750	3.310	3.153	3.030	3.115	3.003

⟶ Right sided limits

It is clear that $\lim_{x \to 2} (x^2 - x + 1) = 3$

E4)

Table 21 ($x \to 0^-$)

x	-1	-0.5	-0.1	-0.01	-0.001	0
$f(x) = \frac{1}{x^2}$	1	4	100	10000	1×10^6	Undefined

Table 22 ($x \to 0^+$)

x	1	0.5	0.1	0.01	0.001	0
$f(x) = \frac{1}{x^2}$	1	4	100	10000	1×10^6	Undefined

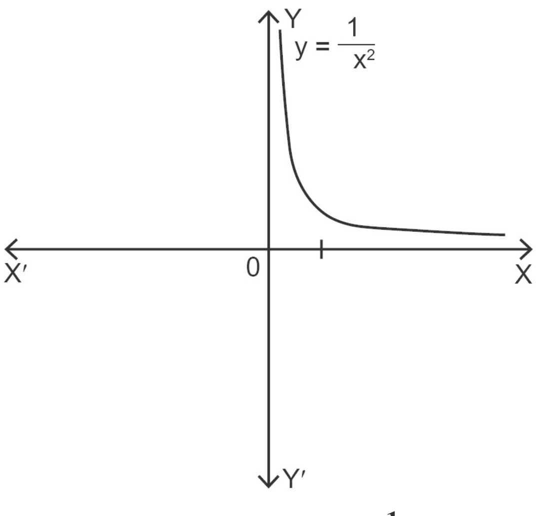


Fig. 31: Graph of $\frac{1}{x^2}$

It is clear from the graph that $y = f(x)$ rises without bound $x \to 0$.

Therefore, $\lim_{x\to 0} \frac{1}{x^2}$ does not exist.

E5) As usual we make a table of x near 0 with $f(x)$. Observe that for negative of x we need to evaluate $x - 2$ and for positive values, we need to evaluate $x + 2$.

Table 23

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-2.1	-2.01	-2.001	2.001	2.01	2.1

From the first three entries of Table 23, we deduce that the value of the function is decreasing to -2 and hence

$$\lim_{x\to 0^-} f(x) = -2$$

From the last three entries of the table, we deduce that the value of the function is increasing from 2 and hence

$$\lim_{x\to 0^+} f(x) = 2$$

Since, the left and right hand limit at 0 do not coincide, we say that the limit of the function at 0 does not exist.

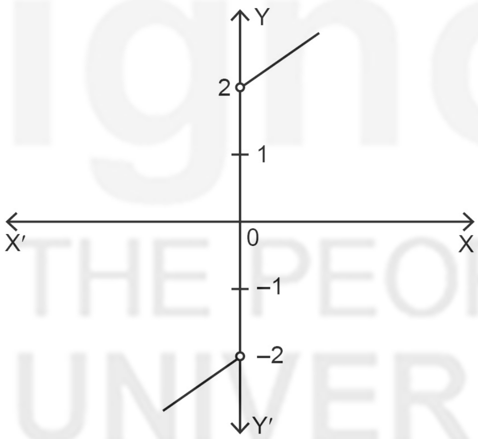


Fig. 32

Graph of this function is given in Fig. 32. Here, we remark that the value of the function at $x = 0$ is not even defined.

E6) $\text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{h\to 0^+} f(1-h)$

$$= \lim_{h\to 0} \frac{|1-h-1|}{1-h-1}$$

$$= \lim_{h\to 0} \frac{|0-h|}{0-h} = -1$$

$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{h\to 0^+} f(1+h)$

$$= \lim_{h\to 0} \frac{|1+h-1|}{1+h-1}$$

$$= \lim_{h\to 0} \frac{|h|}{h} = 1.$$

$\lim_{x\to 1} \frac{|x-1|}{x-1}$ does not exist.

E7) You may like to try it yourself.

E8) i) To prove, if $\varepsilon > 0, \exists \delta > 0,$ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < \in$ whenever $0 < |x-2| < \delta.$

Let $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$

$$\Rightarrow \frac{1}{2} - \varepsilon < \frac{1}{x} < \frac{1}{2} + \varepsilon$$

$$\Rightarrow \frac{1-2\varepsilon}{2} < \frac{1}{x} < \frac{1+2\varepsilon}{2}$$

$$\Rightarrow \frac{2}{1+2\varepsilon} < x < \frac{2}{1-2\varepsilon}$$

$$\Rightarrow \frac{2}{1+2\varepsilon} - 2 < x - 2 < \frac{2}{1-2\varepsilon} - 2$$

$$\Rightarrow \frac{-4\varepsilon}{1+2\varepsilon} < x - 2 < \frac{4\varepsilon}{1-2\varepsilon}$$

Now let $\delta = \min\left\{\frac{4\varepsilon}{1+2\varepsilon}, \frac{4\varepsilon}{1-2\varepsilon}\right\} = \frac{4\varepsilon}{1+2\varepsilon}$

Then, $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$, whenever $|x-2| < \delta = \frac{4\varepsilon}{1+2\varepsilon}.$

Thus, $\lim\limits_{x\to 2} \frac{1}{x} = \frac{1}{2}.$

ii) $\frac{x^3-1}{x-1} - 3 = \frac{x^3 = 3x+2}{x-1} = (x-1)\,(x+2)$, if $x \neq 1$.

Given $\varepsilon > 0$, if we choose $\delta = \min\{(2/7)\,\varepsilon, 1/2\}$, then $|\,x-1\,| < 1/2 \Rightarrow x < 3/2 \Rightarrow x+2 < 7/2$ and

$$\left|\frac{x^3-1}{x-1} - 3\right| = |\,(x-1)(x+2)\,| < (7/2)|\,x-1\,| < (7/2).2/7.\varepsilon = \varepsilon.$$

That is, $|\,x-1\,| < \delta \Rightarrow \left|\frac{x^3-1}{x-1} - 3\right| < \varepsilon$

Hence, $\lim\limits_{x\to 1} \frac{x^3-1}{x-1} = 3$.

E9) $|f(x) - L| = \left|\sin\frac{1}{x} - 0\right|$ (if $x \neq 0$)

$$= \left|\sin\frac{1}{x}\right| \qquad ....(3)$$

Now, clearly $0 < |x-0| < \delta$ or $0 < |x| < \delta$ ....(4)

Take $\varepsilon = 1/2$, if δ be any positive number, then, it is possible to find a positive integer n, such that

$$\delta > \frac{2}{(4x+1)\pi} \text{ or } \frac{1}{\delta} < \left(2n\pi + \frac{\pi}{2}\right)$$

Now, from (3) and (4), we get $|f(x)-L| = \left|\sin\left(2n\pi + \frac{\pi}{2}\right)\right| = 1 > \varepsilon$

Hence, $f(x)$ does not tend to 0 as $x \to 0$.

E10) $|f(x)-L| = \left|x\sin\frac{1}{x} - 0\right| < \in$

$\Rightarrow \left|x\sin\frac{1}{x}\right| < \in$

$\Rightarrow |x|\left|\sin\frac{1}{x}\right| < \in$

$\Rightarrow 0 < |x| < \in$ [$\because \left|\sin\frac{1}{x}\right| \le 1$ except for $x = 0$. At, $x = 0$, it is indefined but we are not concerned with]

$\Rightarrow 0 < |x| < \delta$

Where $\delta = \in$

Thus, $\lim_{x\to 2} x\sin\frac{1}{x} = 0.$

E11) i) Since $x-[x] = x-2, 2 \le x < 3$,

$\lim_{x\to 3^-} x-[x] = \lim_{x\to 3^-} x-2 = 1$

ii) $\lim_{x\to 0^+} |x|/x = \lim_{x\to 0^+} x/x = 1, |x| = x$ for $x > 0$.

iii) $\lim_{x\to 0^-} \frac{(x^2+2)|x|}{x} = \lim_{x\to 0^+} \frac{(x^2+2)(-x)}{x}$ since $|x| = -x$ for $x < 0$

$= \lim_{x\to 0^-} -(x^2+2) = -2$

E12) i) Given $\varepsilon > 0$ if we choose $K = 1/\varepsilon$, then

$x > K \Rightarrow |1/x - 0| = |1/x| < 1/K = \varepsilon$.

Thus, $\lim_{x\to\infty} 1/x = 0$.

ii) Given $\varepsilon > 0$, if we choose $K = 1/\sqrt{\varepsilon}$, then

$x > K \Rightarrow |1/x^2 - 0| = |1/x^2| < 1/K^2 = \varepsilon$

Hence, $\lim_{x\to\infty} 1/x^2 = 0$

E13) i) $\lim_{x\to 1} f(x) \ne L$

ii) $\exists\, \varepsilon > 0$, s.t. $\forall\, \delta > 0\, x$ s.t. $|x-p| < \delta$ and $|f(x)-L| > \varepsilon$.

E14) Theorem 2 v) Here, $|f(x)-L| = |k-k| = 0 < \varepsilon$, whatever be the value of δ.

Theorem 2 vi) $|f(x)-L| = |x-p| < \varepsilon$ whenever $|x-p| < \delta$, if we choose $\delta = \varepsilon$.

E15) $\lim_{x\to 1} 3/x = \frac{\lim_{x\to 1} 3}{\lim_{x\to 1} x} = 3/1 = 3$

E16) $\lim\limits_{x\to 1} 2x + 5\left(\frac{x^2}{1+x^2}\right) = 2\lim\limits_{x\to 1} x + \frac{5\lim\limits_{x\to 1} x^2}{1+\lim\limits_{x\to 1} x^2}$

$$= 2 + \frac{5\times 1}{1+1} = 2 + 5/2 = 9/2$$

E17) $3^{x^2-x} = 3^2$

$x^2 - x = 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$.

E18) We divide the numerator and denominator by e^{2x}

$$\lim_{x\to\infty} \frac{e^{2x}}{e^{2x}+1} = \lim_{x\to\infty} \frac{1}{1+e^{-2x}}$$

$$= \frac{1}{1+\lim\limits_{x\to\infty} e^{-2x}} = \frac{1}{1+0} = 1$$

We have used the fact that $t = -2x \to -\infty$ as $x \to \infty$ and so $\lim\limits_{x\to\infty} e^{-2x} = \lim\limits_{t\to -\infty} e^{t} = 0$.

E19)

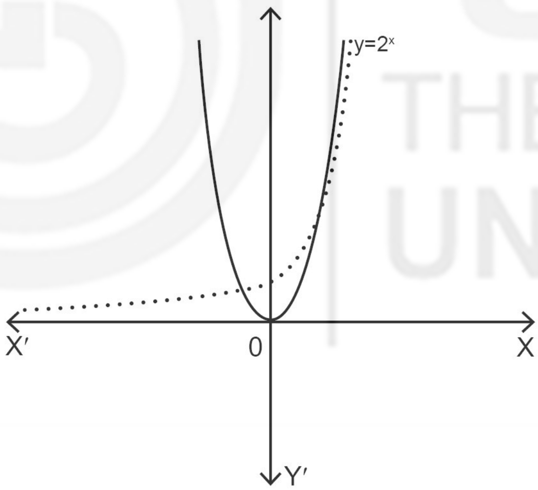


Fig. 33

Fig. 33 shows how the exponential function $y = 2^x$ compares with the power function $y = x^2$. These graphs intersect three times, but ultimately the exponential curve $y = 2^x$ grows far more rapidly than the parabole $y = x^2$.

E20) $\ln \frac{x^2}{1-x^2} = \ln x . \frac{2x}{1+x}$

$$\frac{x^2}{1-x^2} = \frac{2x^2}{1+x}$$

$$x^2(1+x)=2x^2(1-x^2)$$

$$x^2(1+x)(1-2+2x)=0$$

$$x=0,\ -1,\ \frac{1}{2}$$

x cannot be $0,-1$ as $\ln$ will be undefined for these. Thus, $x=\frac{1}{2}$ is the only solution of the given equation.

E21) As $x\to 0, \tan^2 x \to \tan^2 0 = 0$

Now, $\lim\limits_{x\to 0}\log_{10}(\tan^2 x)=\lim\limits_{t\to 0^+}\log_{10} t\,[t=\tan^2 x]=-\infty$.

E22) i) $$\frac{\sinh x}{\cosh x}=\frac{\frac{e^x+e^{-x}}{2}}{\frac{e^x-e^{-x}}{2}}=\frac{e^x+e^{-x}}{e^x-e^{-x}}=\tanh x$$

ii) $$1-\frac{\sinh^2 x}{\cosh^2 x}=\frac{\cosh^2 x-\sinh^2 x}{\cosh^2 x}$$

$$=\frac{1}{\cosh^2 x}=\operatorname{sec h}^2 x$$

E23) $$\coth^2 x-1=\frac{e^{2x}+e^{-2x}+2}{e^{2x}+e^{-2x}-2}-1$$

$$=\frac{4}{e^{2x}+e^{-2x}-2}=\operatorname{cosech}^2 x.$$

UNIT 8

CONTINUITY

Structure — Page No.

8.1 INTRODUCTION

In this unit, we will use the concept of limit to study "Continuity". The word 'continuous' means unbroken, that is without gaps, breaks, or holes. If we use continuous for the graphs of the real-valued functions, we mean the graphs without gaps, breaks. In Sec. 8.2, we shall learn about continuity and develop some fundamental properties of continuous functions in Sec. 8.4.

In Sec. 8.3, we shall discuss the type of discontinuity. In Sec. 8.5, we shall use continuity to state 'Intermediate Value Theorem' for continuous function with its applications.

And, now, we shall list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

Objectives

After reading this unit, you should be able to:

- define continuity of a function at given point;
- check whether the function is continuous or not in the given domain;
- identify the type of discontinuity;
- apply various properties of continuity; and
- state and apply intermediate value theorem.

Let us begin with continuity.

8.2 CONTINUITY

Continuous functions play a very important role in calculus. As you proceed, you will be able to see that many theorems which we shall state in this course are true only for continuous functions. You will also see that continuity is a necessary condition for the differentiability of a function.

We begin the section with two examples to get a feel of continuity. Consider, the function f defined by

$$f(x) = \begin{cases} 2, \text{ if } x \le 0 \\ 4, \text{ if } x > 0 \end{cases}$$

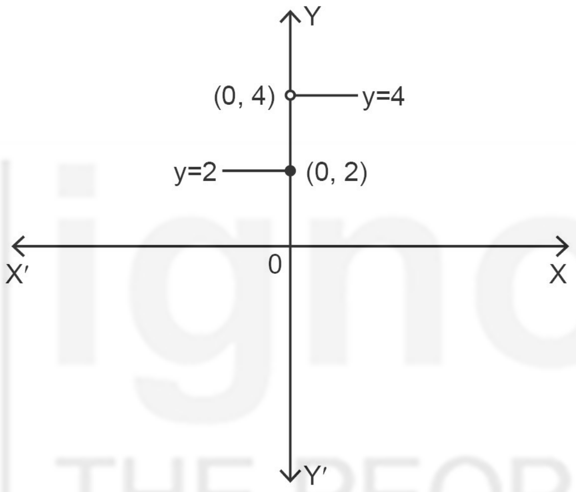


Fig. 1: Graph of f

You may observe that this function is defined at every real number. Fig. 1 shows the graph of this function. From the graph, it is clear that the value of the function at nearby points on $x-$axis remain close to each other except at $x=0$. At the points near and to the left of 0, i.e., at points like $-0.1, -0.01, -0.001,$ the value of the function is 2. At the points near and to the right of 0, i.e., at the points like $0.1,\ 0.01,\ 0.001,$ the value of the function is 4. Using the concept of left and right hand limits, we say that the left hand limit of f at 0 is 2 and the right hand limit of f at 0 is 4. It is clear that the left and right hand limits do not coincide. We also observe that the value of the function at $x=0$ coincides with the left hand limit.

You will notice that when we try to draw the graph, we cannot draw it in one stroke, i.e., without lifting the pen from the plane of the paper. In fact, we need to lift the pen when we come to 0 from right and 0 from left. This is one instance of function being not continuous at $x=0$.

Now, consider the function defined as

$$f(x) = \begin{cases} 2, \text{ if } x \ne 0 \\ 4, \text{ if } x = 0 \end{cases}$$

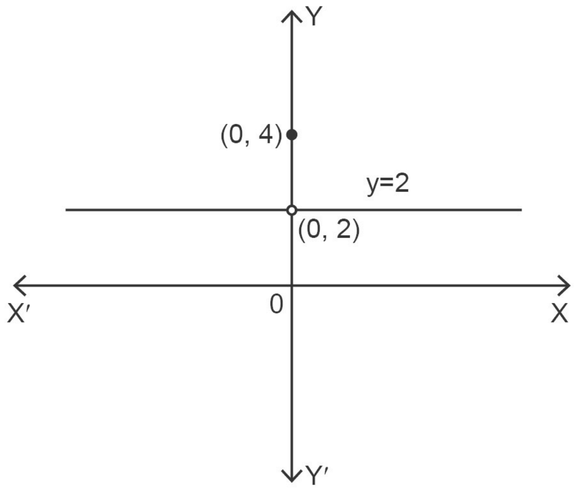


Fig. 2: Graph of f

It is clear that this function is defined at every point. Left hand limit and the right hand limit of the function f at $x = 0$ are equal to 2. But, the value of the function at $x = 0$ is equal to 4 which does not coincide with the common value of the left and right hand limits. Again, we note that we cannot draw the graph of the function without lifting the pen. This is another instance of a function being not continuous at $x = 0$.

Here, we may say that a function is continuous at a fixed point if we can draw the graph of the function around that point in one stroke that is without lifting the pen from the plane of the paper.

From the above illustrations, we can say that a continuous process is one that goes on smoothly without any abrupt change. Continuity of a function can also be interpreted in a similar way. Look at Fig. 3. The graph of the function f in Fig. 3(a) has an abrupt cut at the point $x = a$, whereas the graph of the function g in Fig. 3(b) proceeds smoothly. We say that the function g is continuous, while f is not.

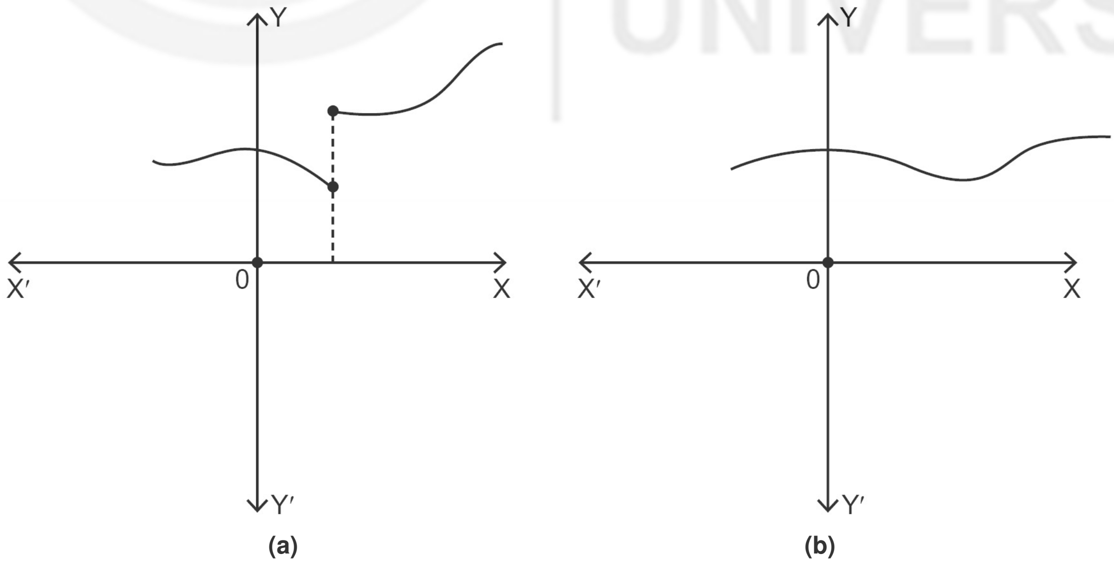


Fig. 3: (a) Graph of f (b) Graph of g

We have seen in Unit 7, that the limit of a function as x tends to a can often be found simply by calculating the value of the function at the values close to number a. If the value of a function at a is equal to the value of the limit of the function as x tends to a, then the function it is called **continuous** at a. This can be written as in the following definition:

Definition: Suppose f is a real function on a subset of the real numbers and let a be a point in the domain of f. Then, f is continuous at a if $\lim_{x \to a} f(x) = f(a).$

This definition implicitly includes the following three criteria.

i) The function f defined by $f(x)$ is defined at $x = a$.

ii) The limit $\lim_{x \to a} f(x)$ exists.

iii) The value of the limit of the function f at $x = a$ is equal to the value of the function at a, that is $\lim_{x \to a} f(x) = f(a)$.

Thus, unlike limit, for continuity it is essential for the function to be defined at that particular point. If f is not continuous at a, we say f is **discontinuous** at a (or f has a discontinuity at a or f is not continuous at a), and a is called a **point of discontinuity** of f.

Example 1: Check the continuity of the following functions at the specified point.

i) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3,$ at $x = 1.$

ii) The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \begin{cases} \frac{x}{2} + 3, \text{ if } x < -2 \\ x - 1, \text{ if } x \geq -2 \end{cases}$, at $x = -2$

iii) The function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2 - 5,$ at $x = 3.$

Solution: i) Let us check all the conditions of continuity as given in its definition.

i) The function f is given by $f(x)$, which is defined at the given point $x = 1$.

ii) The limit of the function at $x = 1$ is $\lim_{x \to 1} f(x) = \lim_{x \to 1} (2x + 3) = 2(1) + 3 = 5$.

Therefore, $\lim_{x \to 1} f(x)$ exists.

iii) The value of the function at $x = 1$ is 5. Thus, $\lim_{x \to 1} f(x) = 5 = f(1)$.

Hence, the function f is continuous at $x = 4$. The graph of f is shown in Fig. 4. From Fig. 4, it is clear that the function f is continuous at $x = 1$.

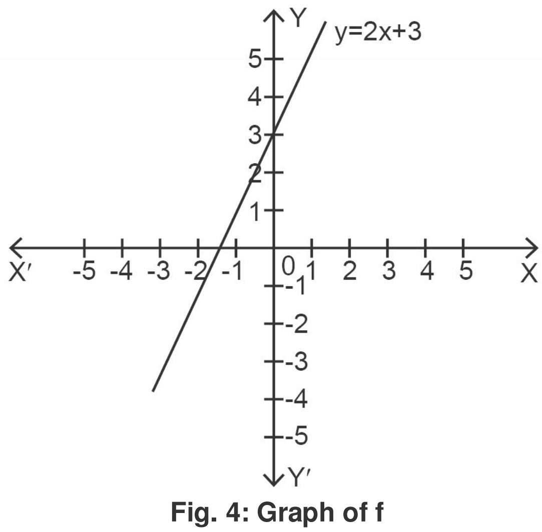


Fig. 4: Graph of f

ii) First note that the function g is defined at the given point $x = -2$ and $g(-2) = -3$. Then, $\lim_{x \to -2^+} g(x) = -3$ and $\lim_{x \to -2^-} g(x) = 2$.

Since $\lim_{x\to -2^+} g(x) \neq \lim_{x\to -2^-} g(x)$, therefore, $\lim_{x\to -2} g(x)$ does not exist. Thus, g is not continuous at $x = -2$. The graph of g can be visualized in Fig. 5. It shows that there is a jump at $x = -2$, which shows that g is not continuous at $x = -2$ or g has a discontinuity at $x = -2$.

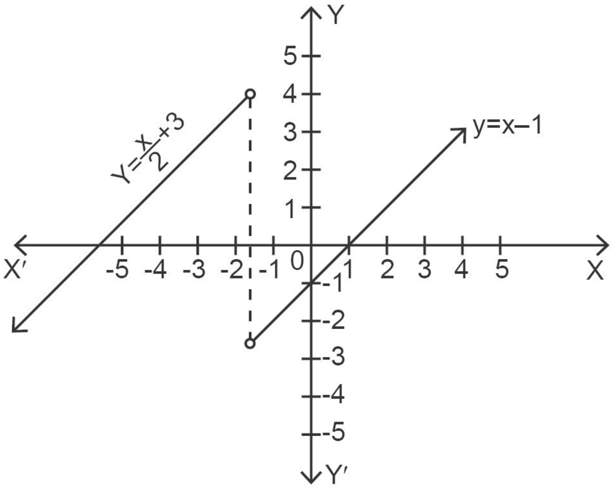


Fig. 5: Graph of g

iii) Clearly, the function h is defined at $x = 3$, and $\lim_{x\to 3} h(x)$ exists, and $\lim_{x\to 3} h(x) = 4 = h(3)$. Therefore, h is continuous at $x = 3$. The graph of this function is shown in Fig. 6.

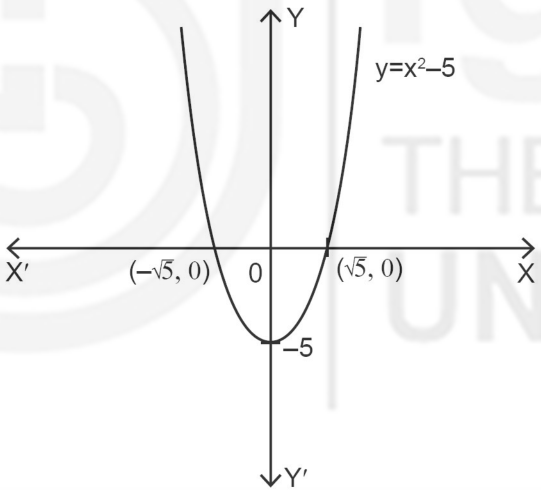


Fig. 6: Graph of h

***

Example 2: Check whether the following functions are continuous at the specified point.

i) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{x^2 - 2x - 3}{x - 3}$ at $x = 3$

ii) The function f on $\mathbb{R}$ defined by $f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x - 3}, & x \neq 3 \\ 1, & x = 3 \end{cases}$ at $x = 3$

Solution: i) The function f given by $f(x) = \dfrac{x^2 - 2x - 3}{x - 3}$ is not defined at $x = 3$, therefore, f is not continuous at $x = 3$.

ii) Clearly the function f is defined at $x = 3$ and $f(3) = 1$. Now,

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{(x-3)(x+1)}{(x-3)} = \lim_{x \to 3} (x+1) = 4.$$

Since the limit of f at $x = 3$ does not concide with $f(3)$, the function f is not continuous at $x = 3$.

***

Remark 1: You may note that in both (i) and (ii) of Example 2, the function is not continuous at $x = 3$, but the reason is different.

Example 3: Check whether function f defined by $f(x) = \frac{1}{x}$ is continuous at $x = 0$ or not.

Solution: To check the continuity of the function f given by $f(x) = \frac{1}{x}$, we try to find its right hand limit as well as left hand limit close to 0. We tabulate these values in the Table 1.

Table 1

x	$f(x)$	x	$f(x)$
0.5	2	−0.5	−2
0.3	3.33	−0.3	−3.33
0.1	10	−0.1	−10
0.01	100	−0.01	−100
0.001	1000	−0.001	−1000
0.00001	100000	−0.00001	−100000

From Table 1, we observe that as x gets closer to 0 from the right, the value of $f(x)$ shoots up higher. Hence, the value of $f(x)$ may be made larger than any given real number by choosing a positive real number very close to 0.
We can write as $\lim_{x \to 0^+} f(x) = +\infty$. You may recall that $+\infty$ is **NOT** a real number, thus the right hand limit of f at 0 does not exist.
Similarly, from Table 1, it is clear that $\lim_{x \to 0^-} f(x) = -\infty$. We say that the left hand limit of f at 0 does not exist as $-\infty$ is **NOT** a real number.
Since, the limit of f does not exist at 0, therefore, f is not continuous at 0.

***

Example 4: A shopkeeper sells an item by kilograms (kg), charging ₹15 per kg for quantities upto and including 10kg. Above 10 kg, the shopkeeper charges ₹12 per kg and a surcharge of c. If x represents the number of kg, the price function (p) is given by:

$$p(x) = \begin{cases} 15x & ; \text{if } x \le 10 \\ 12x + c & ; \text{if } x > 10 \end{cases}$$

i) Find c such that the price p is continuous at $x = 10$.

ii) Why it is preferable to have continuity at $x = 10$.

Solution: i) If p is continuous at $x = 10$, its limit must exist there as well and must be equal to $p(10)$. Therefore, we must have

$$\lim_{x \to 10^-} p(x) = \lim_{x \to 10^+} p(x) = p(10)$$

$$\lim_{x\to 10^-} p(x) = \lim_{h\to 0} p(10-h) = \lim_{h\to 0} 15(10-h) = 150 = p(10)$$

$$\lim_{x\to 10^+} p(x) = \lim_{h\to 0} p(10+h) = \lim_{h\to 0} 12(10+h) + c = 120 + c$$

This simplifies to $c = 30$. Thus if $c = 30$, the price function p will be continuous at $x = 10$.
The graph of p is given in Fig. 7.

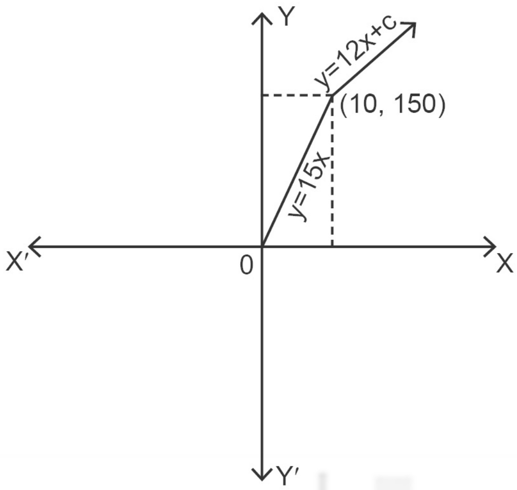


Fig. 7: Graph of p(x)

ii) The graph given in Fig. 8 shows that the customer still get a cheaper rate per kg once x is above 10 kg. Also, the shopkeeper does not lose revenue. This is fair compromise, therefore continuity is preferred.

***

Now try the following exercises.

E1) Check for the continuity of the function f given by $f(x) = |x|$ at $x = 0$.

E2) Give an example of any function which is not continuous at 5.

E3) Check whether the following functions are continuous at the given number. Also, verify graphically.

i) The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) = \begin{cases} x & ; x < 0 \\ x^2; & x \geq 0 \end{cases}$ at $x = 0$.

ii) The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) = \begin{cases} \dfrac{1}{x-2} & ; x \neq 2 \\ 1 & ; x = 2 \end{cases}$ at $x = 2$.

iii) The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) = \dfrac{2x-3x^2}{1+x^3}$, at $x = 1$.

Now, you know how to test the continuity of a function at a point, we make a natural extension of this definition to discuss the continuity of a function in an interval.

Definition: A real function f defined on an interval, is said to be **continuous** on the interval, if it is continuous at every point in the interval.

This definition requires a bit of elaboration. Suppose f is a function defined on a closed interval $[a, b]$, then for the function f to be continuous, it needs to be continuous at every point in $[a, b]$ including the end points a and b. Continuity at all end-points means one-sided continuity, which is defined as follows:

Continuity of f at a means $\lim_{x\to a^+} f(x) = f(a)$ and f is said to be continuous from the right at $x = a$. Similarly, continuity of f at b means $\lim_{x\to b^-} f(x) = f(b)$ and f is also said to be continuous from the left at $x = b$. You may observe that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to b^+} f(x)$ do not make sense. From this, we can say that if the function f is defined only at one point, it is continuous there that is if the domain of the function f is a singleton, f is continuous function.

You may note that if the function f is defined only on one endpoint of the interval that in on an half open or half close interval, then it is continuous from the left or from the right according to the endpoint at which f is defined.

Let us see some more examples.

Example 5: Is the identity function defined on the set of real numbers by $f(x) = x$, continuous at each real number? Justify.

Solution: Let a be any arbitrary real number. The function f is clearly defined at every point and $f(a) = a$ for each real number a.
Also, $\lim_{x\to a} f(x) = \lim_{x\to a} x = a$
Therefore, $\lim_{x\to a} f(x) = a = f(a)$ and the function f is continuous at each real number. Hence, f is continuous at $\mathbb{R}$.

***

Example 6: Check whether the constant function f defined by $f(x) = c$ is continuous.

Solution: Clearly, the function f defined by $f(x) = c$ is defined for every real number. Let a be an arbitrary real. Then $\lim_{x\to a} f(x) = \lim_{x\to a} c = c$.
Since, $\lim_{x\to a} f(x) = c = f(a)$ for every real number a, therefore, the function f is continuous at every real number.

***

Example 7: Check for the continuity of the function f on $\mathbb{R}$ defined by $f(x) = \frac{1}{x}, x \neq 0$.

Solution: The given function f defined by $f(x) = \frac{1}{x}, x \neq 0$ is defined for every real number except 0. Let a be any non-zero real number. Then,

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{1}{x} = \frac{1}{a}. \text{ Since, } a \neq 0,\ f(a) = \frac{1}{a} = \lim_{x\to a} f(x).$$

Hence, f is continuous at every non-zero real number. Thus, f is a continuous function. Fig. 8 illustrates this.

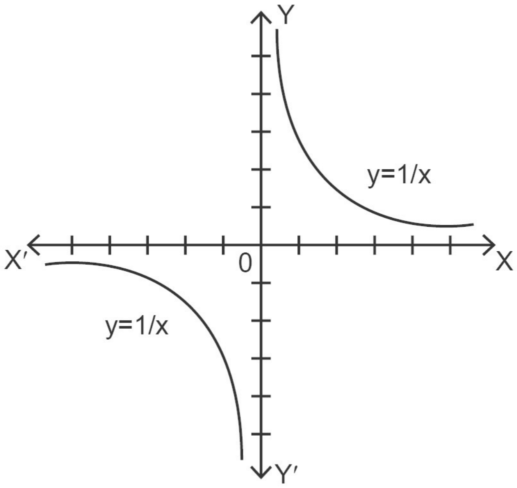


Fig. 8: Graph of $\frac{1}{x}$

***

Example 8: Check for the continuity of the function f defined by $f(x) = x^n$ for all $x \in \mathbb{R}$ and any $n \in \mathbb{Z}^+$.

Solution: The function f is defined on $\mathbb{R}$ and let a be any real number. Then, $\lim_{x\to a} x^n = a^n$ for any $a \in \mathbb{R}$. Also, $f(a) = a^n$

Since, $\lim_{x\to a} f(x) = \lim_{x\to a}(x^n) = a^n = f(a)$. Hence, f is continuous at every number in $\mathbb{R}$. Therefore, we can say that f is continuous on $\mathbb{R}$.

***

Example 9: The greatest integer function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is not continuous. Verify.

Solution: The function f is defined for all real numbers. Let us consider two cases:

i) When a is an integer, $\lim_{x\to a^+} f(x) = \lim_{x\to a^+}[x] = a$ and $\lim_{x\to a^-} f(x) = \lim_{x\to a^-}[x] = a-1$.
Since, the left hand limit and right hand limit do not coincide, therefore, f is not continuous at any integer.

ii) When a is real but not equal to an integer. The function f is defined and $f(a) = [a]$. Also, $\lim_{x\to a} f(x) = \lim_{x\to a}[x] = [a]$. Since, $\lim_{x\to a} f(x) = [a] = f(a)$, therefore, f is continuous at all real numbers not equal to integers. Fig. 9 shows that f is not continuous at every integral point.

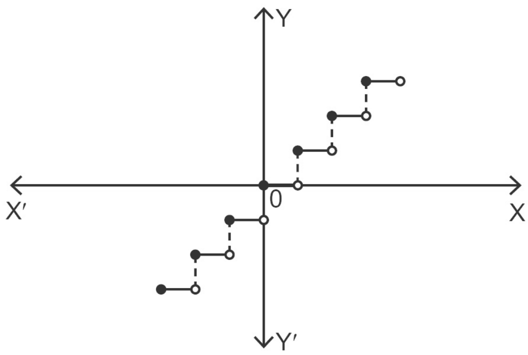


Fig. 9: Graph of $[x]$

***

Example 10: Check for the continuity of the function f defined by $f(x) = |x|$ on $\mathbb{R}$.

Solution: The function f can be rewritten as $f(x) = x$, if $x \geq 0$, and $f(x) = -x$ if $x < 0$.

We consider three cases:

i) When $a > 0$, the function f is defined and $f(a) = |a| = a$.

Also, $\lim_{x \to a} f(x) = \lim_{x \to a} x = a$

Since, $\lim_{x \to a} f(x) = a = f(a)$, therefore, f is continuous at every real a.

ii) When $a = 0$, the function f is defined and $f(0) = 0$.

$\lim_{x \to 0} f(x) = \lim_{x \to 0} |0| = 0$.

Since, $\lim_{x \to 0} f(x) = 0 = f(0)$, therefore, f is continuous at $x = 0$.

iii) When $a < 0$, the function f is defined for every a and $f(a) = -a$.

Now, $\lim_{x \to a} f(x) = \lim_{x \to a} f(-x) = -a$

Since, $\lim_{x \to a} f(x) = -a = f(-x)$, therefore f is continuous at every $x = a$.

Hence, from (i), (ii) and (iii), we can say that the function f is continuous on $\mathbb{R}$.

***

Example 11: Check for the continuity of the function f is defined by $f(x) = 1 - \sqrt{1 - x^2}$ in the interval $[-1, 1]$.

Solution: Let a be any arbitrary real number and $a \in]-1, 1[$. The function f is defined on $]-1, 1[$ and $f(a) = 1 - \sqrt{1 - a^2}$.

Then, we have
$$\lim_{x \to a} f(x) = \lim_{x \to a} (1 - \sqrt{1 - x^2})$$
$$= 1 - \lim_{x \to a} \sqrt{1 - x^2}$$
$$= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} \text{ [Using Theorem 2 (viii), Unit 7]}$$
$$= 1 - \sqrt{1 - a^2}.$$

Since, $\lim_{x \to a} f(x) = 1 - \sqrt{1 - a^2} = f(a)$, therefore, f is continuous at every a in the interval $]-1, 1[$.

Now, let us check the continuity at the endpoints of the interval. For this, we find the right hand limit at -1 and the left hand limit at $+1$, which are as follows:

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (1 - \sqrt{1 - x^2})$$
$$= \lim_{h \to 0} (1 - \sqrt{1 - (-1 + h)^2})$$
$$= \lim_{h \to 0} (1 - \sqrt{1 - 1}) = 1 = f(-1)$$

Therefore, f is right continuous at one end of the interval that is at -1.

Now, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1 - \sqrt{1 - x^2})$
$$= \lim_{h \to 0} (1 - \sqrt{1 - (1 - h)^2}) = 1 = f(1)$$

From the discussion above, we can say that f is continuous on $]-1,1[$, and f is also continuous from the right at -1, and from the left at 1. Therefore, f is continuous in the interval $[-1,1]$.

***

Example 12: Check the continuity of the function f defined by

$$f(x)=\begin{cases} x+1, \text{ if } x<0 \\ -x+1, \text{ if } x>0 \end{cases}.$$

Solution: Clearly the function f is defined at all real numbers except 0.

i) When $a<0$, the function is defined and $f(a)=a+1$.
Since, $\lim_{x\to a} f(x)=a+1=f(a)$, therefore, f is continuous for every negative real number.

ii) When $a>0$, the function is defined and $f(a)=-a+1$, then,
$\lim_{x\to a} f(x)=\lim_{x\to a}(-x+1)=-a+1$.
Since, $\lim_{x\to a} f(x)=-a+1=f(a)$, therefore, f is continuous for every positive real number.

From (i) and (ii), we can say that f is continuous at all points in the domain of f. Hence, f is continuous.

Fig. 10 shows the graph of f. You may note that we need to lift the pen while drawing the graph of f, but still the function is continuous. This is because we need to do that only for those points which are in the domain of the function f.

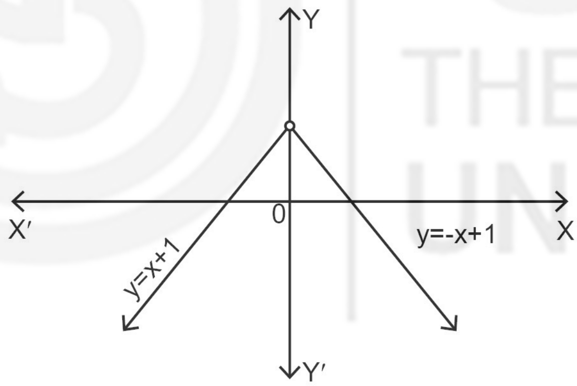


Fig. 10: Graph of f

***

Now, try the following exercises.

E4) Prove that the polynomial function p defined by $p(x)=a_o+a_1x+\cdots+a_nx^n$, where $a_o, a_1\ldots a_n\in\mathbb{R}$, is continuous on $\mathbb{R}$.

E5) Show that the function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=1/(x^2-9)$ is continuous at all point of $\mathbb{R}$ except at $x=3$ and $x=-3$.

E6) Check for the continuity of the function f given by

$$f(x)=\begin{cases} x, \text{ if } x\geq 0 \\ x^2, \text{if } x<0 \end{cases}.$$

So far, we have been seeing the continuity of rational functions. Now, let us check continuity of other functions.

You may recall the graphs of $\sin x$ and $\cos x$ given in Unit 6, they are continuous curves as again given in Fig. 11 (a) and Fig. 11 (b) respectively.

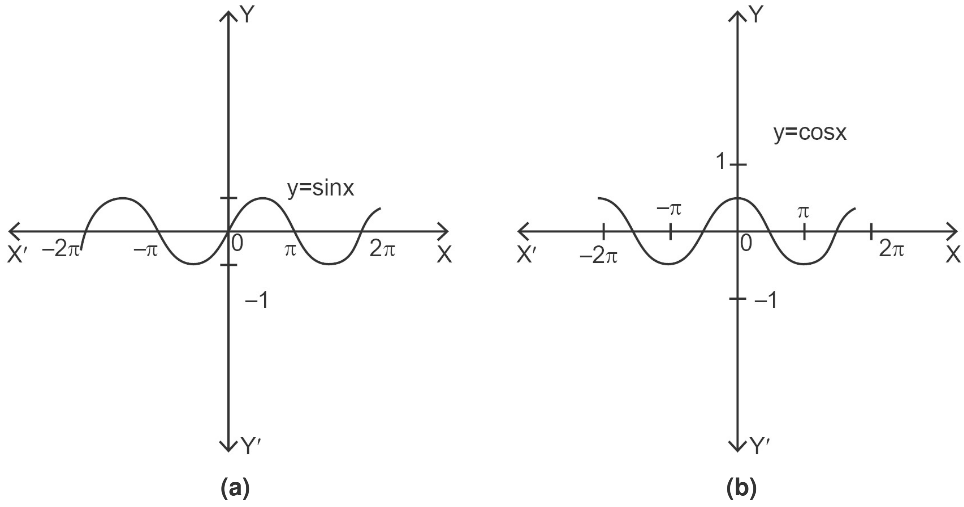


Fig. 11: Graph of (a) $\sin x$ and (b) $\cos x$

To prove that these functions are continuous everywhere, we must show that the following conditions hold for every real number a.

$$\lim_{x\to a} \sin x = \sin a \text{ and } \lim_{x\to a} \cos x = \cos a.$$

We are proving these results by considering the behaviour of the point $P(\cos x, \sin x)$ as it moves around a unit circle with centre O, where x is the angle made by OP from $x-$axis, and is measured in radians. Let $Q(\cos a, \sin a)$ be the corresponding point on the unit circle, where a is a fixed angle in radian measure. As the angle x tends to angle a that is $x \to a$, the point P moves along the circle towards Q and this implies that the coordinates of P approach to the corresponding coordinates of Q. We can rewrite as $\cos x \to \cos a$ and $\sin x \to \sin a$ as shown in Fig. 12.

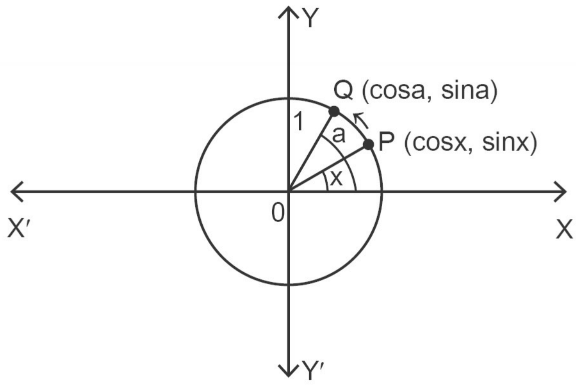


Fig. 12: Points P and Q on unit circle

Here, we can say that sine and cosine both functions are defined at each a in $\mathbb{R}$. Also, $\lim_{x\to a} \sin x = \sin a =$ value of the sine function at a and

$\lim_{x\to a} \cos x = \cos a =$ value of the cosine function at a.

Thus, $\sin x$ and $\cos x$ are continuous in $\mathbb{R}$.

Example 13: Check the continuity of (i) $\sin\left(\frac{1}{x}\right)$ at $x = 0$ and (ii) $x\sin\left(\frac{1}{x}\right)$ at $x = 0$.

Solution: (i) As $x \to 0^+, \frac{1}{x} \to +\infty$, we can see that $\sin\left(\frac{1}{x}\right)$ as the sine of an angle that increases indefinitely as $x \to 0^+$. As this angle increases, the function defined by $\sin\left(\frac{1}{x}\right)$ keeps oscillating between -1 and 1 without approaching a limit.

Similarly, there exists no limit of $\sin\left(\frac{1}{x}\right)$ from the left since $\frac{1}{x} \to -\infty$ as $x \to 0^-$ as again the function keeps oscillating between -1 and 1.

Thus, the function defined by $\sin\left(\frac{1}{x}\right)$ is not continuous at $x = 0$. We can verify these results by the graph given in Fig. 13 (a). You may observe that the oscillations become more and more rapid as x approaches 0 because $\frac{1}{x}$ increases (or decreases) more and more rapidly as x approaches 0.

ii) when $x > 0$, we know $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ and $-x \le x\sin\left(\frac{1}{x}\right) \le x$. Similarly, when $x < 0, x \le x\sin\left(\frac{1}{x}\right) \le -x$.

Thus, for $x \ne 0, |-x| \le x\sin\left(\frac{1}{x}\right) \le |x|$.

Since both $\lim\limits_{x\to 0}|x| = 0$ and $\lim\limits_{x\to 0} -|x| = 0$, therefore by Squeeze Theorem, we can conclude that $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

Fig. 13 (b) shows the graph in support of this. It is clear that $f(0) = 0$ because $\sin\left(\frac{1}{x}\right)$ oscillates between -1 and 1 and $x\sin\left(\frac{1}{x}\right)$ will be 0 at $x = 0$. Thus $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0 = f(0)$ and the given function is continuous at $x = 0$.

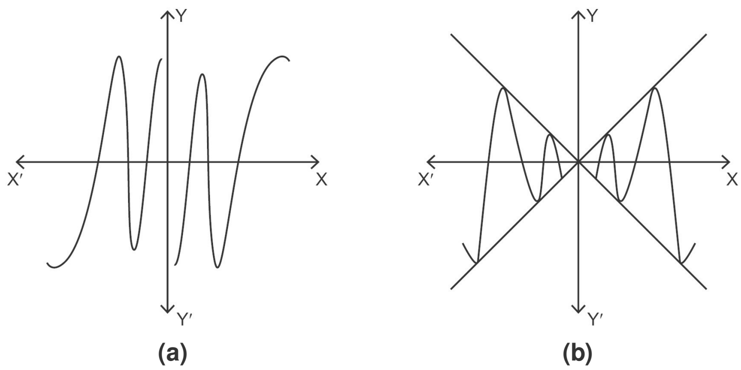


Fig. 13: Graph of (a) $\sin\left(\frac{1}{x}\right)$ and (b) $x \to x\sin\left(\frac{1}{x}\right)$

***

Now, try the following exercises.

E7) Check the continuity of the function f defined by

$$f(x)=\begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0 &, \text{if } x=0 \end{cases} \quad \text{at } x=0.$$

E8) Check the continuity of the following functions:

i) Exponential function

ii) Natural exponential function

iii) Logarithmic function

iv) Natural Logarithmic function.

Let us go back to the definition of continuity, where we have stated three cirteria for a function to be continuous at a number a. If any one or more condition is violated, the function is said to be discontinuous. You will see in the following section that the different types of discontinuity occur on the failure of any of these conditions.

8.3 TYPES OF DISCONTINUITY

Let us observe the graph of a function f given in Fig. 14. We develop an intuitive, geometric feel for where a function is continuous and where it is discontinuous.

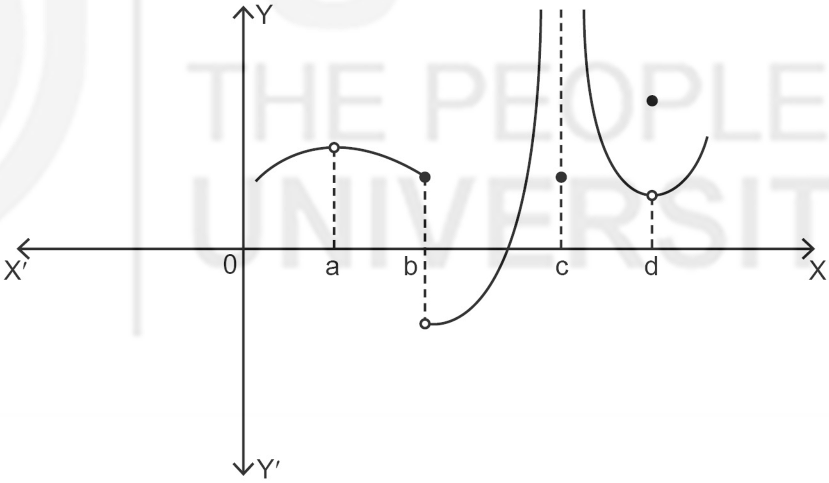


Fig. 14: Graph of f

Here, we see that the function f has breaks at four numbers a, b, c, d either due to a hole, or a jump or due to oscillation. But all these breaks are different. It looks as if there is a discontinuity when $x=a$. The reason is that the function $f(a)$ is not defined, thus, fails to satisfy the first condition of continuity. The graph also breaks at $x=b$, but the reason for discontinuity is different. Here, $\lim_{x\to b} f(x)$ does not exist (because the left and right limits are different). So, f is discontinuous at b, and failing the second condition of the continuity. Now let us see discontinuity at $x=c$. Again, $\lim_{x\to c} f(x)$ does not exist, because left and right limits are different. At $x=d$, the function is defined at d and $\lim_{x\to d} f(x)$ exists (because left and right hand limits are equal), but

$\lim_{x \to d} f(x) \neq f(d)$, failing the third condition of the continuity and f is discontinuous at $x = d$. You notice that the function f is discontinuous at different points a, b, c, and d but the reason of discontinuity is different at each point. This leads to the following types of discontinuity:

1. **Removable (point) Discontinuity:** If $\lim_{x \to a} f(x)$ exists but f is not continuous at $x = a$, either because $f(a)$ is not defined at a or $f(a) \neq \lim_{x \to a} f(x)$, then it is **removable discontinuity or point discontinuity**.

   The terminology removable discontinuity is appropriate because a removable discontinuity of a function at $x = a$ can be removed by redefining the function or defining the function in case it is not defined at the point of discontinuity, that is $f(x)$ at $x = a$ is $\lim_{x \to a} f(x)$. Fig. 15 shows removable discontinuity.

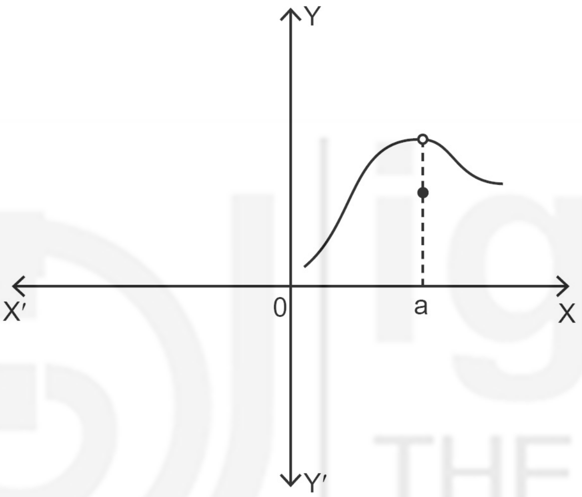
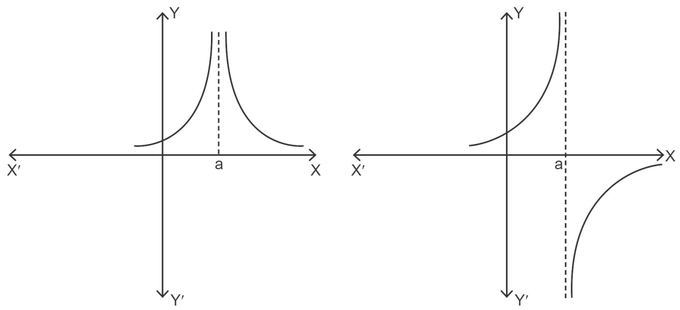


Fig. 15: Removable Discontinuity

2. **Jump Discontinuity:** In jump discontinuity, the left hand limit and the right hand limit of a function f at $x = a$, that is $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ are not equal. The function is approaching the different values depending on the direction a is coming from. For example, in the greatest integer function the function has jump discontinuity at each integer. This is because the left hand limit is always 1 less than the right hand limit at each integer.

3. **Infinite Discontinuity:** In this type of discontinuity, the function rises infinitely large at $x = a$ either from left or from right or from both left and right side. Fig. 16 (a) and Fig. 16 (b) illustrate infinite discontinuity.



(a) **(b)**

Fig. 16: Infinite Discontinuity

Now, let us understand these through the following examples.

Example 14: Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^3 + 2, & \text{if } x < 2 \\ 5, & \text{if } x = 2 \\ x^2 + 6, & \text{if } x > 2 \end{cases}$$

Check its continuity on $\mathbb{R}$, and comment on the type of discontinuity, if any.

Solution: When $x < 2$, the function f is a polynomial function, therefore, f is continuous. When $x > 2$, the function f is a polynomial function and hence it is continuous. Now let us check its continuity at $x = 2$. Clearly, the function f is defined at $x = 2$ and $f(2) = 5$. Then,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 + 2) = 10$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 6) = 10$$

Since, $\lim_{x \to 2} f(x) = 10 \neq f(2)$, therefore, f the function is not continuous at $x = 2$, and the discontinuity is removable discontinuity.

The removable discontinuity can be removed by redefining the function f defined by as

$$f(x) = \begin{cases} x^3 + 2, & \text{if } x < 2 \\ 10, & \text{if } x = 2 \\ x^2 + 6, & \text{if } x > 2 \end{cases}$$

***

Example 15: Examine the type of discontinuity at $x = 1$ for the function f defined by $f(x) = \dfrac{x}{x-1}$.

Solution: The function f given by $f(x) = \dfrac{x}{x-1}$ is continuous at every real number except at 1. This is because the function f is a rational function and the denominator is non-zero except at $x = 1$. Now at $x = 1$, we have

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x}{x-1} = +\infty$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x}{x-1} = -\infty$$

Thus, the function f is discontinuous at $x = 1$ and it has infinite discontinuity at $x = 1$.

***

You may now try the following exercises:

E9) Find k which makes the function f defined by

$$f(x) = \begin{cases} x^2 - 2, & \text{if } x < 1 \\ kx - 4, & \text{if } x \geq 1 \end{cases}$$ continuous at $x = 1$.

E10) Check the continuity of the function f defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

Also, comment of the type of discontinuity.

E11) Check the continuity of the following functions at $x = 0$;

i) The function f defined by $f(f(x))$, where $f(x) = \frac{1}{x}$.

ii) The function g defined by $g(x) = \sin\left(\frac{1}{x}\right)$

iii) The function h defined by $h(x) = \frac{x}{|x|}$

Also, identify the type of discontinuity, if they are not continuous.

E12) Check the continuity of the function $f(x) = \sqrt{16 - x^2}$, and comment on the type of discontinuity, if any.

Now we know how to check whether a function is continuous or not, and if the function is not continuous, we can identify its type of discontinuity. Let us go further, and talk about the continuity of some combinations of functions in the following section.

8.4 ALGEBRA OF CONTINUOUS FUNCTIONS

In unit 7, we learnt some algebra of limits. Now we will study some algebra of continuous functions. Let f and g be functions defined and continuous on a common domain $D \subseteq \mathbb{R}$, and let k be any real number. In Unit 2, we defined the functions $f + g$, fg, f/g (provided $g(x) \neq 0$ everywhere in D), kf and $|f|$. The following theorem tells us about the continuity of these functions.

Theorem 1: Let f and g be the functions defined and continuous on a common domain D, and let k be any real number. The functions $f + g$, kf, $|f|$ and fg are all continuous on D. If $g(x) \neq 0$ everywhere in D, then, the function f/g is also continuous on D.

We shall not prove this theorem.

In Theorem 2, we will talk about the continuity of the composition of two continuous functions. Here again, we shall state the theorem without proof.

Theorem 2: Let $f : D_1 \to D_2$ and $g : D_2 \to D_3$ be continuous on their domains. Then, gof is continuous on D_1, $(D_1, D_2, D_3 \subseteq \mathbb{R})$.

Example 16: Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x^2 + 1)^3$ is continuous at $x = 0$.

Solution: We consider the functions $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$ and $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2 + 1$.
You can check that $f(x) = goh(x)$.
The functions g and h are continuous because they are polynomials. Further, by Theorem 2, $goh = f$ is continuous on $\mathbb{R}$.

***

Let us see if the converse of the above theorems is true. For example, if f and g are defined on an interval $[a, b]$ and if $f+g$ is continuous on $[a, b]$, does that mean that f and g are continuous on $[a, b]$?

No. Consider the function f and g over the interval $[0, 1]$ given by

$$f(x)=\begin{cases}0, & 0\le x\le 1/2\\ 1, & 1/2<x\le 1\end{cases} \text{ and } g(x)=\begin{cases}1, & 0\le x\le 1/2\\ 0, & 1/2<x\le 1\end{cases}$$

Then, neither f nor g is continuous at $x=1/2$. (Why?) But $(f+g)(x)=1\,\forall x\in[0, 1]$. Therefore, $f+g$ is continuous on $[0, 1]$.

Now if $|f|$ is continuous at a point a, must f also be continuous at a? Again, the answer is **No.** Take, for example, the function $f:\mathbb{R}\to\mathbb{R}$ given by

$$f(x)=\begin{cases}-1 & \text{for } x\le 0\\ 1 & \text{for } x>0\end{cases}$$

Then, $|f(x)|=1$ in $\mathbb{R}$ and hence, $|f|$ is continuous.

But f is not continuous at $x=0$ (Why?)

Example 17: Prove that every rational function is continuous.

Solution: The rational function f is given by $f(x)=\dfrac{p(x)}{q(x)}, q(x)\ne 0$, where p and q are polynomial functions. The domain of f is all real numbers except point at which q is zero. Since, polynomial functions are continuous, f is continuous by Theorem 1.

***

Example 18: Find the interval in which the function defined by $f(x)=\dfrac{\cos x+x}{x^2-1}$ is continuous.

Solution: We know that x and $\cos x$ are continuous for all $x\in\mathbb{R}$.

The denominator of the function f defined by $f(x)=x^2-1$, is a polynomial, so it is continuous everywhere.

Now, using Theorem 1 (for f/g, $g(x)\ne 0$), the function f is continuous in $\mathbb{R}$ except where the denominator is 0 that is $x^2-1=0$.

Hence, f is continuous in $\mathbb{R}-\{-1,1\}$.

***

Example 19: Prove that the function f defined by $f(x)=\tan x$ is a continuous function.

Solution: The function $f(x)=\tan x=\dfrac{\sin x}{\cos x}$. This is defined for all real numbers such that $\cos x\ne 0$, i.e., $x\ne(2n+1)\dfrac{\pi}{2}$. We have just proved that both sine and cosine functions are continuous. Thus, $\tan x$ being a quotient of two continuous functions is continuous wherever it is defined.

***

Example 20: Check the continuity of function f defined by $f(x)=\sin(x^2)$.

Solution: Observe that the function is defined for every real number. The function f may be thought of as a composition goh of the two functions g and h, where $g(x)=\sin x$ and $h(x)=x^2$. Since, both g and h are continuous functions, therefore, by Theorem 2, it can be deduced that f is a continuous function.

***

Try the following exercises.

E13) Check for the continuity of sine function using algebra of continuity.

E14) Comment on the continuity of the function f defined by $f(x) = \cos(x^2)$.

E15) Check the continuity of the function f defined by $f(x) = \left|1 - x + |x|\right|$, where x is any real number.

Now, we shall state an important theorem concerning functions. Once again, we won't prove this theorem here. But try to understand its statement because we shall use it in subsequent units.

8.5 INTERMEDIATE VALUE THEOREM FOR CONTNUOUS FUNCTIONS

Consider a function f defined by $f(x)$, which is continuous on the closed interval $[a, b]$ as given in Fig. 17. We draw any horizontal line, say $y = c$, such that $f(a)$ and $f(b)$ lie on two opposite sides of line $y = c$. Then, this line must intersect the graph of $f(x)$ at least once in $[a, b]$. In other words, we can say that if f is continuous on $[a, b]$, then, the function f must take on every value c between $f(a)$ and $f(b)$ at least once as x takes values from a to b.

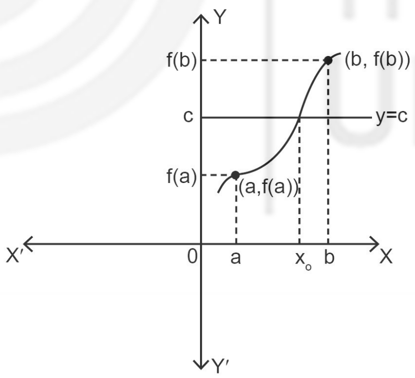


Fig. 17: Graph of a continuous function

For example, consider the polynomial function f defined by $f(x) = 2x^2 + x - 6$ has the value -3 at $x = 1$ and the value 4, at $x = 2$. Thus, it follows from the equation $2x^2 + x - 6 = k$ has at least one solution in the interval $[1,2]$ for every value of k.

This idea leads to the following theorem.

Theorem 3 (Intermediate Value Theorem): If f is continuous on a closed interval $[a, b]$, and c is a real number lying between $f(a)$ and $f(b)$, both inclusive, (that is $f(a) \le c \le f(b)$ or $f(b) \le c \le f(a)$), then, there exists at least one number x_o in the interval $[a, b]$, such that $f(x_o) = c$.

This theorem can easily be interpreted geometrically as shown in Fig. 17. We know that the graph of a continuous function is smooth. It does not have any breaks or jumps. This theorem says that, if the point $(a, f(a))$ and $(b, f(b))$ lie on the opposite sides of a line $y = c$, then the graph must cross the line $y = c$.

You may note that this theorem guarantees only the existence of the number x_o. It does not tell us how to find it. Another thing is that this x_o need not be unique.

This theorem can be used to show that there is a root of the given equation in the given interval. For this, we check for the change in sign in the values of the function and say the function must be zero. Fig. 18 illustrates that since $f(a)$ and $f(b)$ have opposite signs, therefore, 0 is between $f(a)$ and $f(b)$. Thus, by Intermediate-value theorem there is at least one number x_0 in $[a, b]$ such that $f(x_0) = 0$.

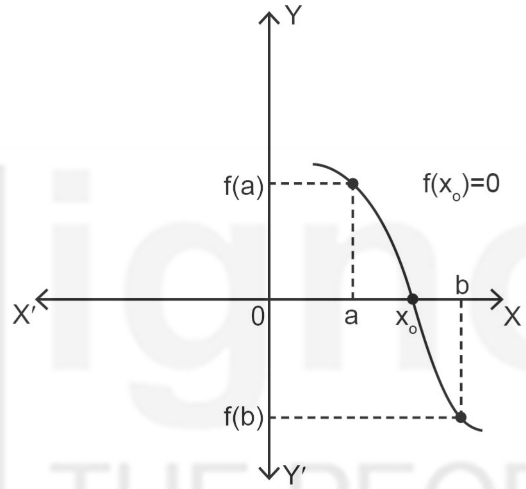


Fig. 18: Showing $x_0 \in [a, b]$

Let us apply this theorem in the following examples.

Example 21: Comment on the roots of the equation $x^5 - 2x^4 - x - 3 = 0$ between 2 and 3.

Solution: Let $f(x) = x^5 - 2x^4 - x - 3$. $f(2) = -5$ and $f(3) = 75$,which shows that a root x_0 lies in the interval $[2, 3]$. To find the approximate value of the root, we divide the interval $[2, 3]$ in 5 equal parts and evaluate at each point of the subdivision. These values are given in Table 2.

Table 2

x	2	2.2	2.4	2.6	2.8	3
$f(x)$	-5	-0.51	7.6	21.81	43.37	76

In Table 2, the values of $f(2.2)$ and $f(2.4)$ have opposite signs, so using intermediate value theorem, we know that the root lies in the interval $[2.2, 2.4]$. The length of this interval is still large, so we can continue the process of dividing the interval $[2.2, 2.4]$ into 10 subdivisions and can approximate the root of the given equation.

***

Example 22: If $f(x) = x^3 - x^2 + x$, show that there is $x_0 \in \mathbb{R}$ such that $f(x_0) = 10$.

Solution: We want the point x_0 such that the value of the function at x_0 is 10. For, using intermediate-value theorem, we can find the two values of $f(x)$, out of which one value is less than 10 and which other value is greater than 10. Let us begin with 0 and continue for other integers. $f(0) = 0,\ f(1) = 1,\ f(2) = 6,$ $f(3) = 21$. So, it is clear that $f(2) < 10 < f(3)$. Since, the function f being polynomial, is continuous everywhere, therefore, there must lie $x_0 \in \mathbb{R}$ such that $f(x_0) = 10$.

***

Now, try the following exercises

E16) Show that there is a positive number c such that $c^2 = 2$.

E17) Find an interval in which the smallest positive root of the equation $x^2 = \sqrt{x+1}$ lie.

E18) Let f be a continuous function on $[0,1]$. Show that if $-1 \le f(x) \le 1$ for all $x \in [0,1]$ then there is $c \in [0,1]$ such that $[f(c)]^2 = c$.

E19) Use the intermediate value theorem to show that there is a root of the given equation in the specified interval.

i) $\cos x = x, \quad [0,1]$

ii) $x^4 + x - 3 = 0,\ [1,2]$

iii) $x = (1-x)^3, \quad [0,1]$.

This brings us to the end of the unit.

8.6 SUMMARY

We end this unit by summarising what we have covered in it.

1. A function f is continuous at a point $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

2. The function f defined on domain D is said to be continuous on D, if it is continuous at every point of D.

3. The following three types of discontinuity:
   i) Removable Discontinuity.
   ii) Jump Discontinuity
   iii) Infinite Discontinuity.

4. If the function f and g are continuous on D, then so are the function $f + g, fg, |f|, kf$ (where $k \in \mathbb{R}$) f/g where $g(x) \neq 0$ in D).

5. The Intermediate Value Theorem: If f is continuous on $[a, b]$ and if $f(a) < c < f(b)$ (or $f(a) > c > f(b)$), then $\exists x_0 \in]a, b[$ such that $f(x_0) = c$.

8.7 SOLUTIONS / ANSWERS

E1) By definition $f(x) = \begin{cases} -x, \text{ if } x < 0 \\ x, \text{ if } x \geq 0 \end{cases}$ clearly the function is defined at 0 and $f(0) = 0$. Left hand limit of f at 0 is $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = 0$. Right hand limit f at 0 is $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x) = 0$.
Thus, the left hand limit, right hand limit and the value of the function concide at $x = 0$. Hence, f is continuous at $x = 0$.

E2) For example $f(x) = \dfrac{x^2 + x + 1}{x - 5}$.
Here $f(x)$ is discontinuous at $x = 5$ as it is not defined at $x = 5$.

E3) i) For f to be continuous, let us check each of the conditions of continuity.

a) $f(x)$ is defined at $x = 0$

b) $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0 - h) = \lim_{h\to 0}(0 - h) = 0$

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0 + h) = \lim_{h\to 0}(0 + h)^2 = 0$

We see that $\lim_{x\to 0} f(x)$ exists $x \to 0$.

c) Also, $\lim_{x\to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$. You can visualize its graph in Fig. 19.

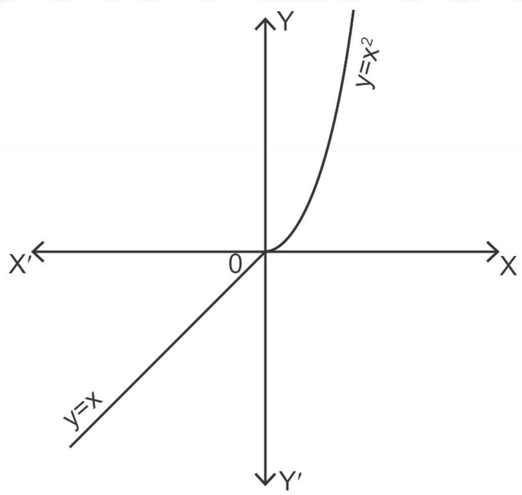


Fig. 19

ii) a) $f(x)$ is defined at $x = 2$

b) $\lim_{x\to 2} f(x)$ does not exist

Therefore, f is not continuous at $x = 2$. Fig. 20 shows the corresponding graph, and it is clear that $\lim_{x\to 2} f(x)$ does not exist.

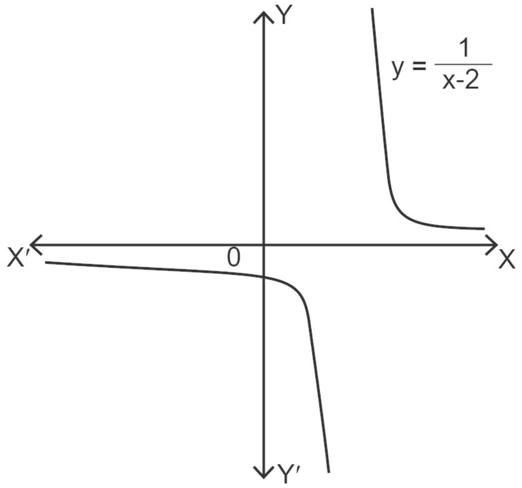


Fig. 20

iii), iv) You may like to try these out yourself.

E4) Let the function f defined by $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$.
Let $a \in \mathbb{R}$, clearly f is defined for every real number a and
$f(a) = a_0 + a_1a + a_2a^2 + ... + a_na^n$.

Then, $\lim_{x \to a} f(x) = \lim_{x \to a}[a_0 + a_1x + ... + a_nx^n]$

$= a_0 + a_1a + ... + a_na^n$

$= f(a)$

Thus, a polynomial function is continuous.

E5) The function f is defined for each real number except 3 and -3.
Let $a \in \mathbb{R} - \{3, -3\}$, then $f(a)$ is defined and given is $f(x) = \frac{1}{x^2 - 9}$.

$$\lim_{x \to a} = \lim_{x \to a} \frac{1}{x^2 - 9}$$

$$= \frac{1}{a^2 - 9} = f(a)$$

E6) Clearly the function is defined at every real number. Graph of the function is given in Fig. 21.

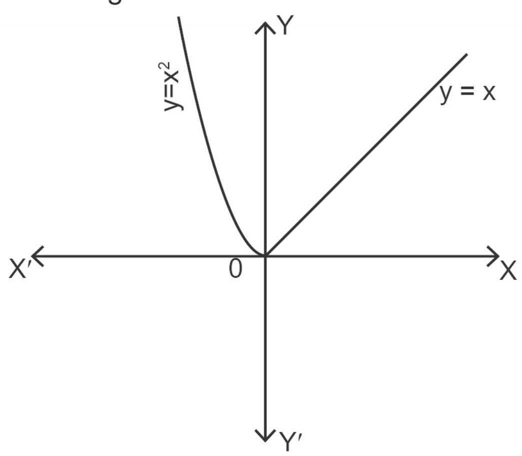


Fig. 21

i) When $a < 0$, the function is defined and $f(a) = a^2$.

$\lim_{x \to a} f(x) = \lim_{x \to a}(a^2) = a^2$.

ii) When $a > 0$, we have $f(x) = x$ and it is easy to see that it is continuous.

iii) When $a = 0$, the value of the function at 0 is $f(0) = 0$.

The left hand limit of f at 0 is

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0^2 = 0$$

The right hand limit of f at 0 is

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$$

Thus, $\lim_{x\to 0} f(x) = 0 = f(0)$ and hence f is continuous at 0. This means that f is continuous at every point in its domain and hence, f is a continuous function.

E7) Try yourself using Squeezing theorem.

E8) i) Let $f(x) = a^x, a > 0, x \in \mathbb{R}$.

Let k be any arbitrary real. $f(x)$ exists for all k and $f(k) = k^x$.

$$\lim_{x\to k} f(x) = \lim_{x\to k} a^x = a^k.$$

Thus, $\lim_{x\to k} f(x) = f(k)$. Therefore, exponential function is continuous. You may verify this with the graph of exponential function also. As, the graph of $a^x, a > 0$ has no break, it is a continuous function.

ii) You may try this yourself.

iii) From the graph of $\log_a x, a > 0$, it is clear that there is no break.

Now let k be any arbitrary real and $f(x) = \log_a x, a > 0$. Here, $f(k)$ exists and $f(k) = \log_a k, a > 0$. Now,

$$\lim_{x\to k} f(x) = \lim_{x\to k} \log_a x = \log_a k.$$

Therefore, $\lim_{x\to k} f(x) = f(k)$ and logarithmic function is a continuous function.

iv) You may like to solve it yourself.

E9) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 - 2) = -1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (kx - 4) = k - 4$

Since $f(x)$ is continuous, therefore $\lim_{x\to 1} f(x) = f(1)$, which gives

$k - 4 = -1 \Rightarrow k = 3.$

E10) This function is not continuous because close to every real number there exist infinite rational and irrational numbers, and the left and right hand limits of this function do not exist. It has jump discontinuity.

E11) i) $f(x) = \dfrac{1}{x}$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{1}{h} = \infty$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} -\frac{1}{h} = -\infty$$

This limit does not exist, therefore $f(x)=\frac{1}{x}$ is discontinuous at $x=0$, It is infinite discontinuity. Now for composite function

$$f(f(x))=f\left(\frac{1}{x}\right)=\frac{1}{\frac{1}{x}}=x$$ which is continuous at $x=0$.

ii) $g(x)=\sin\left(\frac{1}{x}\right)$

$$\lim_{x\to 0} g(x)=\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$$

The limit of $\sin\left(\frac{1}{x}\right)$ as $x\to 0$ does not attain a unique value, because as $x\to 0, \frac{1}{x}\to\infty$ and the value of the function $\sin\left(\frac{1}{x}\right)$ will fluctuates in the interval $[-1,1]$. Therefore, $\lim_{x\to 0} g(x)$ does not exist, and the function is not continuous at $x=0$. Thus, it has jump discontinuity at $x=0$.

Therefore, $g(x)$ is discontinuous at $x=0$.

iii) $h(x)=\frac{x}{|x|}$

$$h(x)=\begin{cases} 1\ ;\ \text{if } x>0 \\ -1\ ;\ \text{if } x<0. \end{cases}$$

$h(x)$ is not defined at $x=0$. Therefore, it is discontinuous at $x=0$. This is removable discontinuity.

E12) The domain of this function is $[-4,4]$, therefore we will need to check the continuity of f on the open interval $]-4,4[$ and at the two endpoints.

$$\lim_{x\to a} f(x)=\lim_{x\to a}\sqrt{16-x^2}=\sqrt{16-a^2}=f(a)$$

which proves that f is continuous at each point in the interval $]-4,4[$.

Now, at end points,

$$\lim_{x\to 4^-} f(x)=\lim_{x\to 4^-}\sqrt{16-x^2}=0=f(4)$$

$$\lim_{x\to -4^+} f(x)=\lim_{x\to 4^+}\sqrt{16-x^2}=0=f(-4)$$

Thus, f is continuous on the interval $[-4,4]$.

E13) To see this we use the following facts.

$$\lim_{x\to 0}\sin x=0$$

We have not proved it, but it is intuitively clear from the graph of $\sin x$ near 0. Now, observe that $f(x)=\sin x$ is defined for every real number. Let c be a real number. Put $x=c+h$. If $x\to c$ we know that $h\to 0$. Therefore,

$$\lim_{x\to c} f(x)=\lim_{x\to c}\sin x$$

$$=\lim_{h\to 0}\sin(c+h)$$

$$= \lim_{h\to 0}[\sin c \cos h + \cos c \sin h]$$
$$= \lim_{h\to 0}[\sin c \cos h] + \lim_{h\to 0}[\cos c \sin h]$$
$$= \sin c + 0 = \sin c = f(c)$$

Thus, $\lim_{x\to c} f(x) = f(c)$ and hence f is a continuous function.

E14) i) Let $g(x) = x^2$ and $h(x) = \cos x$

Now define $f = hog$ we know that $g(x)$ and $h(x)$ are continuous in $\mathbb{R}$. Using theorem 2, we say that f is continuous in $\mathbb{R}$.

ii) $h(x) = \tan x = \dfrac{\sin x}{\cos x}, \cos x \neq 0, \text{if } x = \dfrac{\pi}{2}, -\dfrac{\pi}{2}$, therefore using theorem 1, $\tan x$ is continuous in $\left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$.

E15) Define g by $g(x) = 1 - x + |x|$ and h by $h(x) = |x|$ for all real x. Then

$$(h\,o\,g)(x) = h(g(x))$$
$$= h(1 - x + |x|)$$
$$= |1 - x + |x|| = f(x)$$

It is clear that h is a continuous function. Hence, g being a sum of a polynomial function and the modulus function is continuous. But then f being a composite of two continuous functions is continuous.

E16) Let $f(x) = x^2$, then, f is continuous and $f(0) = 0 < 2 < 4 = f(2)$. By the intermediate value theorem, there is $c \in [0,2]$ such that $c^2 = f(c) = 2$.

E17) Let $f(x) = x^2 - \sqrt{x+1}$. Then, f is continuous for all $x > -1$, $f(1) = 1 - \sqrt{2}$ and $f(2) = 4 - \sqrt{3}$. Therefore, $f(1) < 0 < f(2)$ and by the intermediate value theorem there is $x \in [1,2]$ such that $f(x) = 0$. But then $x^2 - \sqrt{x+1} = 0$ or $x^2 - \sqrt{x+1}$.

E18) If $f(x)$ is continuous on $[0,1]$ then so is $[f(x)]^2$. Set $g(x) = [f(x)]^2 - x$. Now $g(0) = [f(0)]^2 - 0 = [f(0)]^2 \geq 0$ and $g(1) = [f(1)]^2 - 1 \leq 0$. By intermediate value theorem, there is $c \in [0,1]$ such that $g(c) = 0$. Then, $[f(c)]^2 - c = 0$ or $[f(c)]^2 = c$.

E19) i) Let $f(x) = \cos x - x$

We know $f(x)$ is continuous in $[0,1]$.

$f(0) = 1$

$f(1) = \cos(1) - 1 < 0$

Using intermediate value theorem, we can say that there exists a number x_0 between 0 and 1, such that $f(x_0) = 0$,

ii) & iii) you may like to work out on your own.

MISCELLANEOUS EXAMPLES AND EXERCISES

The examples and exercises given below cover the concepts and processes you have studied in this block. Doing them will give you a better understanding of the concepts concerned, as well as practice in solving such problems.

Example 1: Write the interval notation for each of the following set:

i) $\{x \mid -4 < x < 5\}$ ii) $\{x \mid x \geq -2\}$

Solution:

i) $\{x \mid -4 < x < 5\} =]-4, 5[$ ii) $\{x \mid x \geq -2\} = [-2, \infty[$

***

Example 2: Write interval notation for each of the graph of Fig. 1.

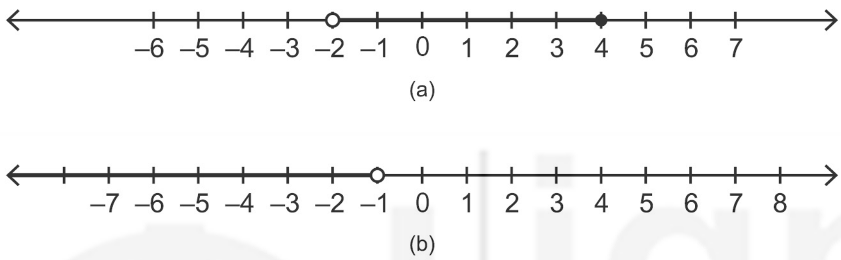


Fig. 1

Solution: i) $]-2, 4]$

ii) $]-\infty, -1[$.

***

Example 3: Find the domain of the function f defined by $f(x) = |x|$.

Solution: Is there any number x for which we cannot calculate $|x|$? The answer is no. Thus, the domain of f is the set of all real numbers.

***

Example 4: Suppose that ` 500 is invested at the rate of interest 6%, compounded quarterly in a bank for t years. The amount in the account after t years is given by $A(t) = 500\left(1 + \frac{0.06}{4}\right)^{4t} = 500(1.015)^{4t}$.

The amount A is a function of the number of years for which the money is invested. Determine the domain.

Solution: We can substitute any real number for t into the formula, but a negative number of years is not meaningful. The context of the application excludes negative numbers. Thus, the domain is the set of all non-negative numbers, $[0, \infty[$.

***

Example 5: Using the same set of axes, draw the graphs of the functions defined by $f(x) = x^2$ and $g(x) = x^3$.

Solution: First, we set up table of values, plot the points, and then draw the graphs.

Table 1

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x^2	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4
x^3	-8	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	8

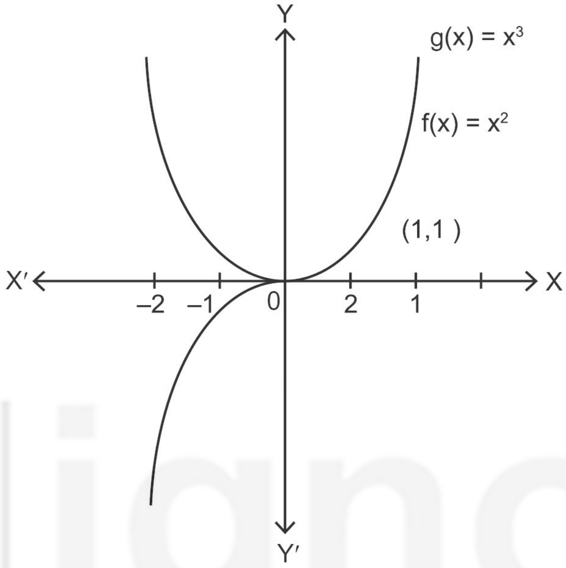


Fig. 2: Graph of f and g

***

Example 6: Draw the graph of the function f defined by $f(x) = 1/x$.

Solution: We make a table of values, plot the points, and then draw the graph.

Table 2

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
f(x)	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-4	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$

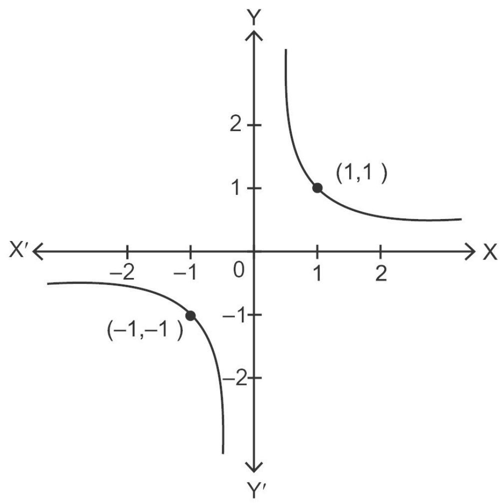


Fig. 3: Graph of f

***

Example 7: Draw the graph of the function f defined by $f(x) = -\sqrt{x}$.

Solution: The domain of this function is the set of all non-negative real numbers, that is the interval $[0, \infty[$. we can find approximate values of square roots to draw the graph. We set up a table values, plot the points, and then draw the graph.

Table 3

$\mathbf{x}$	0	1	2	3	4	5
$\mathbf{f(x) = -\sqrt{x}}$	0	-1	-1.4	-1.7	-2	-2.2

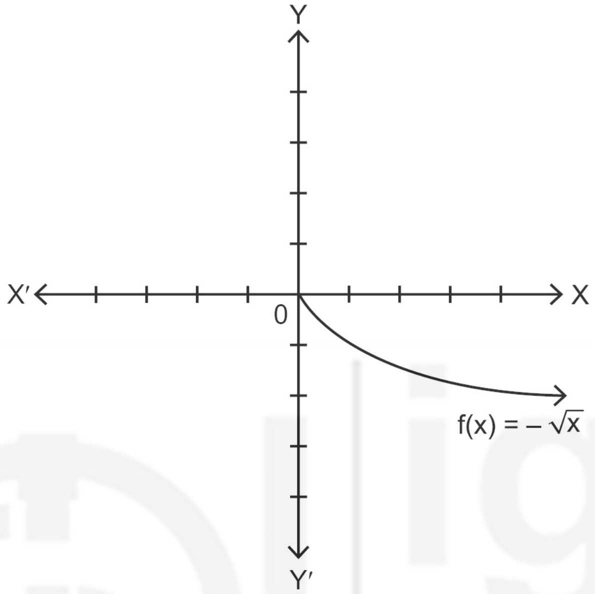


Fig. 4: Graph of f

***

Example 8: Find the equilibrium point for the demand and supply functions for the Ultra-Fine coffee maker. Here q represents the number of coffee makers produced, in hundreds, and x is the price, in rupees.

Demand: $q = 50 - \frac{1}{4}x$

Supply: $q = x - 25$

Solution: To find the equilibrium point, the quantity demanded must match the quantity supplied. Therefore, we get

$$50 - \frac{1}{4}x = x - 25 \Rightarrow 60 = x$$

Thus, at $x = 60$, to Find q, we substitute x into either function. We select the supply function, we get $q = x - 25 = 60 - 25 = 35$.

Thus, the equilibrium quantity is 3500 units, and the equilibrium point is (60, 3500).

***

Example 9: Consider the function f given by $f(x) = \frac{1}{x-2} + 3$.

Draw the graph of the function, and find the following limits, if they exist. Also, find the following limits intuitively.

i) $\lim_{x \to 3} f(x)$ ii) $\lim_{x \to 2} f(x)$ iii) $\lim_{x \to \infty} f(x)$

Solution: The graph of $f(x)$ is shown in Fig. 5. You may note that it is the same as the graph of $f(x) = \frac{1}{x}$ given in Fig. 3 but shifted 2 units to the right and 3 units up.

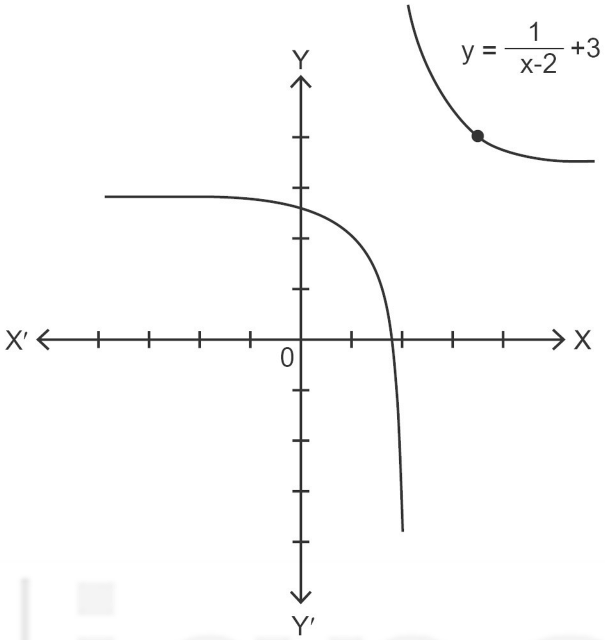


Fig. 5: Graph of f

i) As x approaches 3 from the left, $f(x)$ approaches 4, so the limit from the left is 4. As x approaches 3 from the right, $f(x)$ also approaches 4. Since the limit from the left, 4, is the same as the limit from the right, we have $\lim_{x \to 3} f(x) = 4$.

Table 4

$x \to 3^- \ (x < 3)$	$f(x)$	$x \to 3^+ \ (x > 3)$	$f(x)$
2.1	13	3.5	3.667
2.5	5	3.2	3.8333
2.9	4.1111	3.1	3.9090
2.99	4.0101	3.01	3.9901

Fig. 5 also strengthens that the limit of $f(x)$ from the left as well as from the right of 3 approaches 4.

ii) As x approaches 2 from the left, $f(x)$ becomes more and more negative, without bound. These numbers do not approach any real number, although it might be said that the limit from the left is negative infinity, $-\infty$. That is, $\lim_{x \to 2^-} f(x) = -\infty$

As x approaches 2 from the right, $f(x)$ becomes larger and larger, without bound. These numbers do not approach any real number, although it might be said that the limit from the right is infinity, ∞. That is, $\lim_{x \to 2^+} f(x) = \infty$

Because the left-sided limit differs from the right-sided limit, $\lim_{x \to 2} f(x)$ does not exist. Table 5 shows that $\lim_{x \to 2} f(x)$ does not exist.

Table 5

$x \to 2^-$ $(x < 2)$	$f(x)$	$x \to 2^+$ $(x > 2)$	$f(x)$
1.5	1	2.5	5
1.9	−7	2.1	13
1.99	−97	2.01	103
1.999	−997	2.001	1003

Fig. 6 also strengthens that the limit of f at $x = 2$ does not exist.

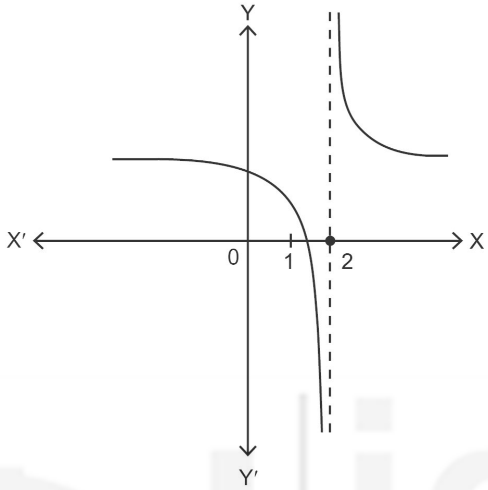


Fig. 6

iii) As x gets larger and larger, $f(x)$ gets closer and closer to 3. We have $\lim_{x\to\infty} f(x) = 3$.

Table 6

$x \to \infty$	$f(x)$
5	3.3333
10	3.125
100	3.0102
1000	3.0010

Fig. 7 also shows that $\lim_{x\to\infty} f(x) = 3$.

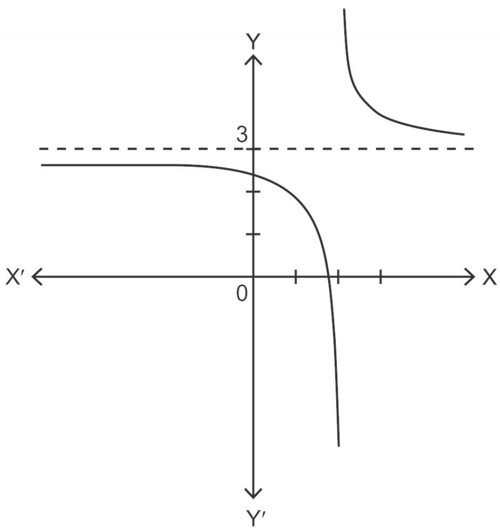


Fig. 7

***

Example 10: Find these limits and write limit property you use at each step.

i) $\lim_{x\to 1}(2x^3+3x^2-6)$

ii) $\lim_{x\to 4}\left(\frac{2x^2+5x-1}{3x-2}\right)$

iii) $\lim_{x\to -2}\sqrt{1+3x^2}$

Solution: i) $\lim_{x\to 1}(2x^3-3x^2-6)=\lim_{x\to 1}(2x^3)+\lim_{x\to 1}(-3x^2)+\lim_{x\to 1}(-6)[\lim_{x\to a}[f(x)+g(x)]$

$=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)]$

$$=2\lim_{x\to 1}(x^3)-3\lim_{x\to 1}(x^2)-\lim_{x\to 1}(6) \quad [\lim_{x\to a}Cf(x)=\lim_{x\to a}Cf(x)]$$
$$=2-3-6$$
$$=-7$$

ii) $\lim_{x\to 4}\frac{2x^2+5x-1}{3x-2}=\frac{\lim_{x\to 4}(2x^2+5x-1)}{\lim_{x\to 4}(3x-2)}$ [Applying limit of quotient]

$$=\frac{2\lim_{x\to 4}(x^2)+5\lim_{x\to 4}(x)+\lim_{x\to 4}(-1)}{3\lim_{x\to 4}(x)+\lim_{x\to 4}(-2)}$$
$$=\frac{2\times 16+5\times 4-1}{3\times 4-2}=\frac{51}{10}$$

iii) $\lim_{x\to -2}\sqrt{1+3x^2}$

$$=\sqrt{\lim_{x\to -2}(1+3x^2)} \quad [\because \lim_{x\to a}\sqrt{f(x)}=\sqrt{\lim_{x\to a}f(x)}]$$
$$=\sqrt{\lim_{x\to -2}1+3\lim_{x\to -2}(x^2)}$$
$$=\sqrt{1+3(4)}$$
$$=\sqrt{13}$$

***

Example 11: Is the function g, given by

$$g(x)=\begin{cases}\frac{1}{2}x+3, & \text{for } x<-2,\\ x-1, & \text{for } x\ge -2,\end{cases}$$

continuous at $x=-2$? Why or why not?

Solution: To find out if g is continuous at -2, we must determine whether $\lim_{x\to -2}g(x)=g(-2)$. Thus, we first note that $g(-2)=-2-1=-3$. To find $\lim_{x\to -2}g(x)$, we find left-and right-hand limits $\lim_{x\to -2^-}g(x)=\frac{1}{2}(-2)+3=-1+3=2$ and $\lim_{x\to -2^+}g(x)=-2-1=-3$.

Since $\lim_{x\to -2^-}g(x)\ne\lim_{x\to -2^+}g(x)$, we see that $\lim_{x\to -2}g(x)$ does not exist. Thus, g is not continuous at -2. It is continuous at all other $x-$values. Fig. 8 also strengthens this.

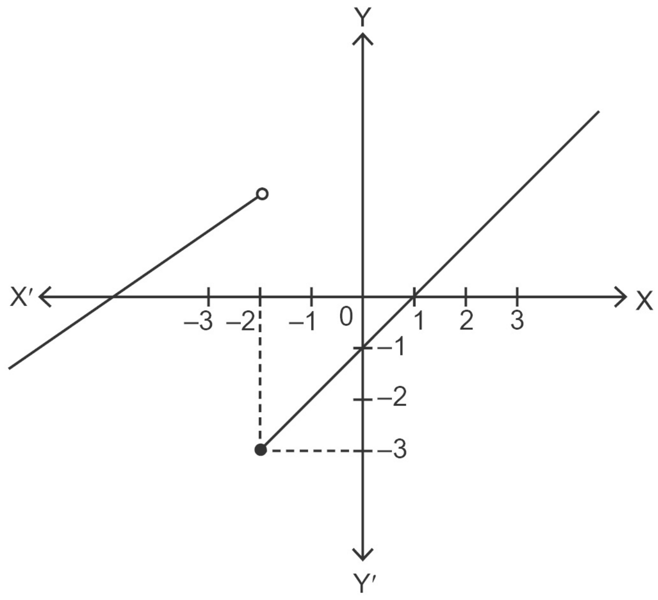


Fig. 8

***

Example 12: A shopkeeper sells an item in bulk quantities. For quantities up to and including 500 kg, the shopkeeper charges ` 2.50 per kg. For quantities above 500 kg, he charges ` 2 per kg. The price function is stated as a piecewise function defined by $p(x) = \begin{cases} 2.50x, & \text{for } 0 < x \le 500, \\ 2x, & \text{for } x > 500. \end{cases}$

where p is the price in rupees and x is the quantity in kg. Is the price function $p(x)$ continuous at $x = 500$? Why or why not?

Solution: The graph of $p(x)$ follows.

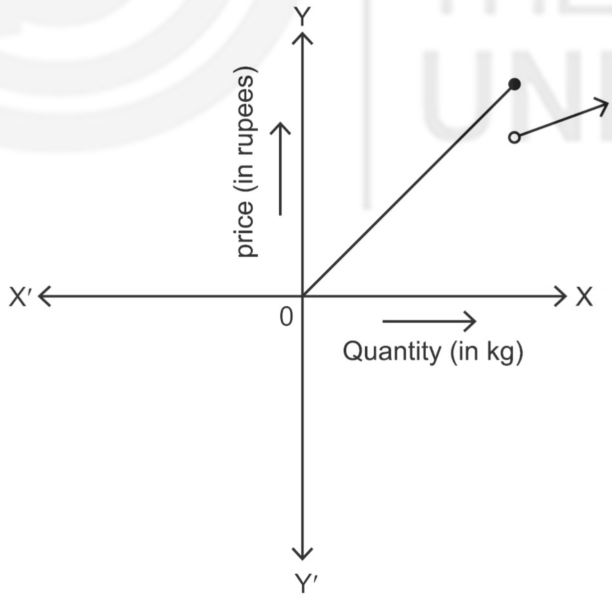


Fig. 9

As x approaches 500 from the left, we have $\lim_{x \to 500^-} p(x) = 1250$, and when x approaches 500 from the right, we have $\lim_{x \to 500^+} p(x) = 1000$. Since the left-hand and right-hand limits are not equal, the limit $\lim_{x \to 500} p(x)$ does not exist. Thus, the function is not continuous at $x = 500$. Fig. 9 shows a price "break."

The value of the function value at $x = 500$ is $p(500) = 1250$, but this fact plays no role with regard to whether or not the limit exists.

***

Example 13: A reservoir is empty at time $t = 0$ minutes. It fills at a rate of 3 gallons of water per minute for 30 minutes. At 30 minutes, the reservoir is no longer being filled and a value is opened, allowing water to escape at a rate of 4 gallons per minute. The volume v after t minutes is given by the function

$$v(t) = \begin{cases} 3t, & \text{for } 0 \le t \le 30, \\ k - 4t, & \text{for } t > 30. \end{cases}$$

Determine k such that the volume function v is continuous at $t = 30$. Explain why this must be true.

Solution: $v(t)$ is defined at $x = 30$ and $v(30) = 90$ gallons. Now, $\lim_{t \to 30^-} v(t) = 90$ and $\lim_{t \to 30^+} v(t) = k - 120$.

Since $v(t)$ is continuous at $t = 30$, therefore $\lim_{t \to 30} v(t) = 30 \Rightarrow k - 120 = 90 \Rightarrow$ $k = 210$.

The amount of water (in gallons) is continuous as a function of time (t).

***

Example 14: Consider the function $C(x) = \begin{cases} -1, & \text{for } x < 2, \\ 1, & \text{for } x \ge 2. \end{cases}$

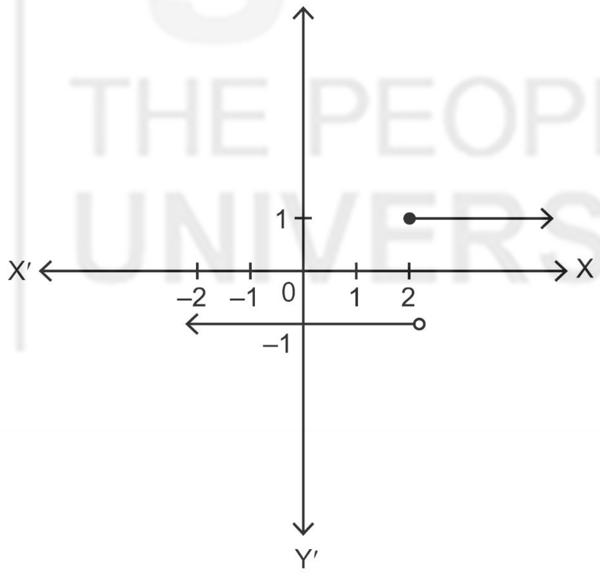


Fig. 10

i) Find $\lim_{x \to 2^+} C(x)$.

ii) Find $\lim_{x \to 2^-} C(x)$.

iii) Find $\lim_{x \to 2} C(x)$.

iv) Find $C(2)$.

v) Is C continuous at $x = 2$? Why or why not?

vi) Is C continuous at $x = 1.95$? Why or why not?

Solution: i) $\lim_{x \to 2^+} C(x) = 1$

ii) $\lim_{x \to 2^-} C(x) = -1$

iii) $\lim_{x \to 2} C(x)$ does not exist.

iv) $C(2) = 1$

v) Since $\lim_{x \to 2} C(x)$ does not exist. C is not continuous at $x = 2$.

vi) Since C is constant function when x is less than 2, therefore C is continuous at $x = 1.95$.

***

Example 15: Show that $\lim_{x \to 2}(4x - 3) = 5$ using $\varepsilon - \delta$ definition.

Solution: Here $f(x) = 4x - 3$, and $L = 5$.

$$|f(x) - L| = |4x - 3 - 5|$$
$$= |4x - 8|$$

$= 4|x - 2|$ whenever $|x - 2| < \delta$ for a given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{4}$, then

$$|f(x) - L| = 4|x - 2| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

***

Example 16: If $y = |x| + |x - 1|$, show that

$$y = \begin{cases} 1 - 2x, & \text{if } x \le 0 \\ 1, & \text{if } 0 < x < 1 \\ 2x - 1, & \text{if } x \ge 1 \end{cases}$$

Solution: i) When, $x \le 0 \Rightarrow |x| = -x$ and $x - 1 \le 0 \Rightarrow |x - 1| = -(x - 1)$, therefore $y = (-x) + (-x(x - 1) = -x - x + 1 = 1 - 2x$.

ii) When, $0 < x < 1 \Rightarrow |x| = x$ and $x - 1 < 0 \Rightarrow |x - 1| = -(x - 1)$, therefore $y = x + (-x(x - 1)) = x - x + 1 = 1$.

iii) When, $x \ge 1 \Rightarrow |x| = x$ and $x - 1 \ge 0 \Rightarrow |x - 1| = x - 1$, therefore, $y = x + (x - 1) = 2x - 1$.

Hence, $y = \begin{cases} 1 - 2x, & \text{if } x \le 0 \\ 1, & \text{if } 0 < x < 1 \\ 2x - 1, & \text{if } x \ge 1 \end{cases}$

***

Example 17: Find $\lim_{x \to 0}(x \sin x)$.

Solution: i) When $x > 0, 0 < x < \pi/2 \Rightarrow \sin x > 0$ for $0 < x < \pi/2$, we have $\sin x < x \Rightarrow x \sin x < x^2$.

Let $\varepsilon > 0$, for values of x which are positive and less than $\sqrt{\varepsilon}$, we have $x^2 < \varepsilon$. Thus $0 < x \sin x < \varepsilon$ when $0 < x < \sqrt{\varepsilon}$. It follows that $\lim_{x \to 0^+} x \sin x = 0$.

ii) When $x < 0, -\pi/2 < x < 0 \Rightarrow \sin x < 0$.

The values of the function for two values of x whis are equal in magnitude but opposite in signs and equal. Hence, as in (i), we see that for any value of x in the interval $]-\sqrt{\varepsilon},0[$, the numerical value of the difference between $x\sin x$ and 0 is less than ε.

Thus $\lim\limits_{x\to 0^-} x\sin x = 0$

Combining the conclusion arrived at in (i) and (ii), we see that the corresponding to any positive ε, there exists an interval $]-\sqrt{\varepsilon},\sqrt{\varepsilon}[$ around 0, such that for any x belonging to this interval, the numerical value of the difference between $x\sin x$ and 0 is $<\varepsilon$, i.e. $|x\sin x - 0| < \varepsilon$. Thus $\lim\limits_{x\to 0} x\sin x = 0$.

***

Example 18: Check the continuity of the function f defined by

$$f(x) = \begin{cases} (1+3x)^{1/x}, & \text{if } x \neq 0 \\ e^3, & \text{if } x = 0 \end{cases} \text{ at } x = 0.$$

Solution: $f(x)$ is defined at $x = 0$ and $f(0) = e^3$. Now

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[(1+3x)^{1/3x}\right]^3$$

$$= \lim_{x\to 0}\left[(1+3x)^{1/3x}\right]^3$$

$$= e^3 \quad [\because x \to 0 \Rightarrow 3x \Rightarrow 0]$$

$$= f(0)$$

Hence f is continuous at $x = 0$

***

Now you may try the following exercises.

E1) Sketch the graph of $y = 4 - |x-3|$.

E2) Determine whether f is even, odd or neither even nor odd. Verify your answer with the graph of f for the following functions.

i) $f(x) = x|x|$

ii) $f(x) = x^2 + 2x + 1$

E3) A rectangular box with volume $2\,m^3$ has a square base. Express the surface area of the box as a function of the length of a side of the base.

E4) Prove that $\lim\limits_{x\to 0^+} \sqrt{x} = 0$.

E5) Guess the value of the limit, if it exists by calculating the function f given by $f(x) = \dfrac{x(x-2)}{x^2-x-2}$ at $x = 2.5, 2.2, 2.1, 2.01, 2.005, 2.001, 2.0001, 1.5, 1.8, 1.9, 1.95, 1.99, 1.995, 1.999$.

E6) Use the graph of the function f given by $f(x) = \sqrt{x}$ to find a number δ such that if $|x-9| < \delta$, then $|\sqrt{x} - 3| < 0.1$.

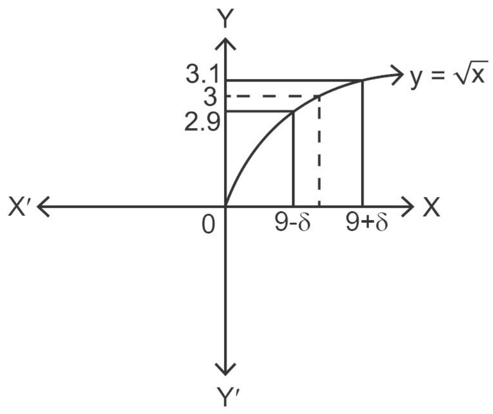


Fig. 11

E7) Let a function f defined by $f(x) = \frac{x^2 + x - 6}{|x-2|}$, find

i) LHL and RHL of f at $x = 2$,

ii) Does $\lim_{x\to 2} f(x)$ exists? Give reasons for your answer.

iii) Sketch the graph of f .

E8) Is the function f defined by $f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$ continuous at $x = 0$? At $x = 1$? At $x = 2$? Give reasons.

E9) Check whether the function f defined by $f(x) = x^2 - \sin x + 5$ is continuous at $x = \pi$.

E10) For what value of k is the function f defined by
$f(x) = \begin{cases} k(x^2 - 1), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$ continuous at $x = 0$? What about the continuity at $x = 1$?

E11) Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x\to 2}[3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

E12) The gravitational force exerted by the Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \frac{GMr}{R^3}, & \text{if } r < R \\ \frac{GM}{r^2}, & \text{if } r \ge R \end{cases}$$

where M is the mass of the Earth, R is its radius, and G is the gravitational constant. Check whether F is continuous function at $r = R$ or not.

E13) Check which of the following function is continuous or discontinuous. Give reasons.

i) The temperature of a specific place as a function of time.

ii) The fare paid for a taxi as a function of distance travelled.

iii) The current in the circuit for the lights in a room as a function of time.

E14) Find $\lim\limits_{x\to\infty}(2^{-x}-1)$.

E15) Find the domain of the functions given below:

i) $f(x)=\sqrt{1-2^x}$ ii) $f(x)=\sin(e^{-x})$

iii) $\dfrac{1+x}{e^{\cos x}}$ iv) $f(x)=e^{\sqrt{5x-3-2x^2}}$

E16) Find $\lim\limits_{x\to0}\log_{10}(\tan^2 x)$.

E17) Check whether the function f given by $f(x)=\cos(\log(x+\sqrt{x^2+1}))$ is even or not.

E18) Solve the inequality $|x-1|+|x+1|<4$.

E19) Find the period of the following functions:

i) $f(x)=5\sin(7x-3)$

ii) $f(x)=\sin x^2$

iii) $f(x)=\sqrt{\tan x}$

iv) $f(x)=|\sin x|+|\cos x|$.

E20) Check whether the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=|\sin x|$ is continuous or not.

SOLUTIONS/ANSWERS

E1) The graph can be obtained by two transaltions that is first translate the graph of $y=|x|$ right 3 units to obtain the graph of $|x-3|$, then translate this graph up 4 units to obtain the graph of $y=|x-3|+4$ as shown in Fig. 12.

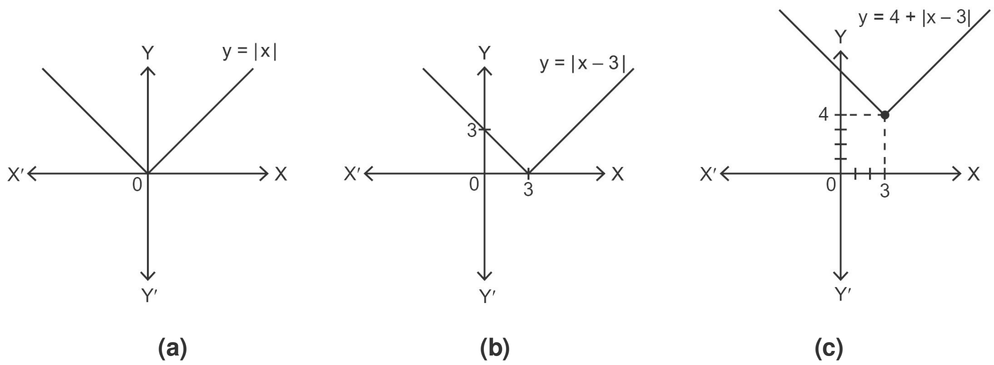


Fig. 12: (a) $y=|x|$, (b) $y=|x-3|$, (c) $y=|x-3|+4$

E2) i) Given is $f(x)=x|x|$, and $f(-x)=-x|-x|=-x|x|=-f(x)$

Since $f(x) = -f(-x)$, therefore, f is odd function. We can rewrite the function f as $f(x) = \begin{cases} x^2, x \geq 0 \\ -x^2, x < 0 \end{cases}$.

Fig. 13 shows the graph of the function f defined by $f(x) = x|x|$. The graph also strengthens that f is odd.

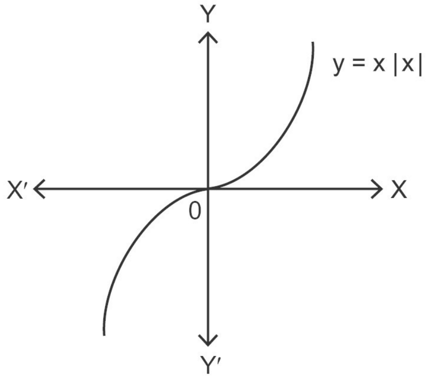


Fig. 13: Graph of $x|x|$.

ii) $f(-x) = x^2 - 2x + 1$ The function f is neither even nor odd. The graph of f is given in Fig. 14, from which it is verified that f is neither even nor odd.

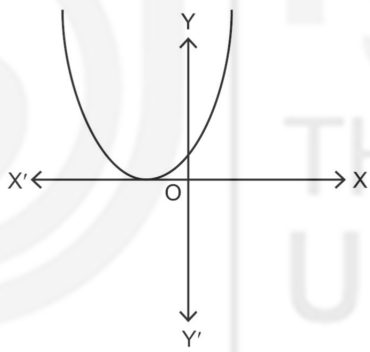


Fig. 14: Graph of $x^2 + 2x + 1$

E3) Let the side of the base area be a and height be h. Then the volume $V = a^2h$. Since volume $= 2m^3$, therefore, $a^2h = 2$, which gives $h = 2/a^2$. Thus, Surface area $S = 2a^2 + 4ah = 2a^2 + 4a.\dfrac{2}{a^2} = 2a^2 + \dfrac{8}{a}$.

Therefore, the surface area of the box is $2a^2 + \dfrac{8}{a}$, where a is the side of the square base of the box.

E4) Let ε be given as a positive number, and we want to find δ such that if $0 < x < \delta$, then $|\sqrt{x} - 0| < \varepsilon$ that is $\sqrt{x} < \varepsilon$. But $\sqrt{x} < \varepsilon$ or $x < \varepsilon^2$.

Hence $\lim_{x \to 0^+} \sqrt{x} = 0$. Fig 15 shows this limit.

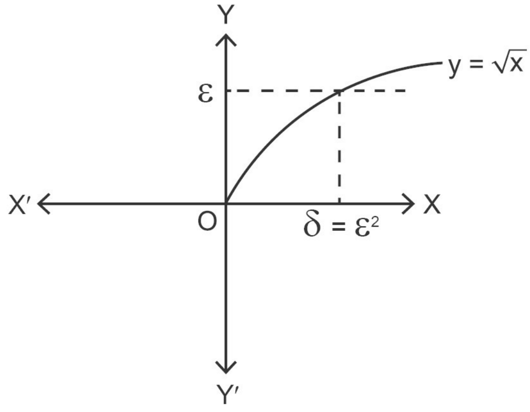


Fig. 15

E5) **Table 7**

x	2.5	2.2	2.1	2.01	2.005	2.001	2.0001
f(x)	0.714	0.688	0.677	0.668	0.667	0.667	0.667
x	1.5	1.8	1.9	1.95	1.99	1.995	1.999
f(x)	0.6	0.643	0.655	0.661	0.666	0.666	0.667

Therefore, $\lim_{x\to 2} f(x) = 0.667$.

E6) Given is, $\left|\sqrt{x} - 3\right| < 0.1$

$$-0.1 < \sqrt{x} - 3 < 0.1$$
$$2.9 < \sqrt{x} < 3.1$$
$$(2.9)^2 < x < (3.1)^2$$
$$8.41 < x < 9.61$$
$$-0.59 < x - 9 < 0.59 < 0.61$$
$$|x - 9| < 0.59$$

Therefore, $\delta = 0.59$ or smaller positive number.

E7) i) $\text{LHL} = \lim_{x\to 2^-} f(2) = \lim_{h\to 0} f(2-h)$

$$= \lim_{h\to 0} \frac{(2-h)^2 + (2-h) - 6}{|2-h-2|}$$

$$= \lim_{h\to 0} \frac{4 + h^2 - 4h + 2 - h - 6}{|-h|}$$

$$= \lim_{h\to 0} \frac{h^2 - 5h}{h} = \lim_{h\to 0} h - 5 = -5$$

$$\text{RHL} = \lim_{x+2^+} f(2) = \lim_{h\to 0} f(2+h)$$

$$= \lim_{h\to 0} \frac{(2+h)^2 + (2+h) - 6}{|2+h-2|}$$

$$= \lim_{h\to 0} \frac{4 + h^2 + 4h + 2 + h - 6}{|h|}$$

$$= \lim_{h\to 0} \frac{h^2 + 5h}{h} = 5$$

ii) Since $\text{LHL} \neq \text{RHL}$, therefore, $\lim_{x\to 2} f(x)$ does not exist.

iii) We can rewrite the function f as $f(x)=\begin{cases} x+3\ ;\ \text{if}\ \ x>2 \\ -(x+3);\ \text{if}\ \ x<2 \end{cases}$. Fig. 16 shows the graph of f.

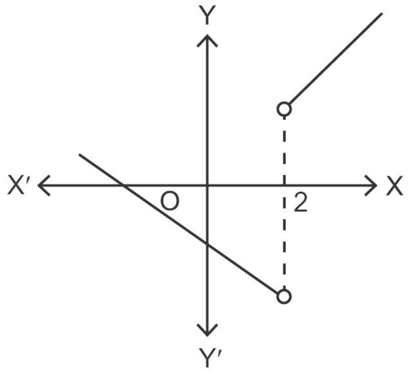


Fig. 16

E8) At $x=0$, the function f is defined and $f(0)=0$. $\text{LHL}=0,\ \text{RHL}=0,$ so continuous at $x=0$.
At $x=1$, the function f is defined and $f(1)=1$. $\text{LHL}=1,\ \text{RHL}=5,$ not continuous.
At $x=2$, the function f is defined and $f(2)=5$. $\text{LHL}=5,\ \text{RHL}=5,$ therefore, $\lim\limits_{x\to 2} f(x)=f(2)$ and f is continuous at $x=2$.

E9) The function f is defined at $x=\pi$ and $f(\pi)=\pi^2-0+5=\pi^2+5$.
Also, $\lim\limits_{x\to\pi} f(x)=\pi^2+5=f(\pi)$. Therefore, f is continuous at $x=\pi$.

E10) At $x=0, f(0)=-k$. $\lim\limits_{x\to 0^-} f(x)=-k,\ \lim\limits_{x\to 0^+} f(x)=1$. Since f is continuous at $x=0,$ therefore $k=-1$.
At $x=1,\ f(1)=5$. $\lim\limits_{x\to 1^-} f(x)=5$ and $\lim\limits_{x\to 1^+} f(x)=5$. The function f is continuous at $x=1$.

E11) $\lim\limits_{x\to 2}\left[3f(x)+f(x)g(x)\right]=36$

$\lim\limits_{x\to 2} 3f(x)+\lim\limits_{x\to 2} f(x)g(x)=36$

$3f(2)+f(2)g(2)=36$ [Since $f(x)$ and $g(x)$ are continuous, therefore $\lim\limits_{x\to 2} f(x)=f(2)$ and $\lim\limits_{x\to 2} g(x)=g(2)$].

$3f(2)+f(2).6=36$

$f(2)=4$.

E12) $f(R)=\dfrac{GM}{R^2}$

$$\lim_{r\to R^-} f(r)=\lim_{r\to R^-}\frac{GMr}{R^3}=\frac{GM}{R^2}$$

$$\lim_{r\to R^+} f(r)=\lim_{r\to R^+}\frac{GM}{r^2}=\frac{GM}{R^2}.$$

Since $\lim\limits_{x\to R} f(r)=\dfrac{GM}{R^2}=f(R),$ therefore f is continuous at $r=R$.

E13) i) continuous.

ii) continuous.

iii) discontinuous.

E14) $\lim_{x\to\infty}(2^{-x}-1)=\lim_{x\to\infty}(2^{-x})-\lim_{x\to\infty}(1)$

$=\lim_{t\to-\infty}(2^{t})-1=0-1=-1.$

E15) i) Domain $=]-\infty,0]$.

ii) Domain $=\mathbb{R}$

iii) Domain $=\mathbb{R}$

iv) Domain $=\left[1,\frac{3}{2}\right]$.

E16) $\lim_{x\to 0}\log_{10}(\tan^2 x)=\log_{10}\lim_{x\to 0}(\tan^2 x)$

$=\log_{10}(0)$

$=-\infty$.

E17) $f(-x)=\cos\{\log\{(-x)+\sqrt{(-x)^2+1}\}$

$=\cos[\log\{-x\sqrt{x^2+1}\}]$

$=\cos[-\log\{x+\sqrt{x^2+1}\}]$ [After raitonalization]

$=\cos[\log(x+\sqrt{x^2+1})]=f(x)$

Since $f(-x)=f(x)$, therefore, f is even.

E18) i) When $x<-1, -(x-1)-(x+1)<4$

$\Rightarrow -2x<4\Rightarrow x>-2$

Therefore, $-2<x<-1$.

ii) When $-1\le x<1$, we have

$(x-1)+(x+1)<4$

$\Rightarrow 2x<4\Rightarrow x<2$

Therefore, $1\le x<2$.

Thus, the inequality holds when $x\in]-2,2[$.

E19) i) Since, $\sin x$ has a period 2π, so, $5\sin(7x-3)$ will have its period $\frac{2\pi}{7}$.

ii) Let the period of $f(x)$ be T, then $f(x+T)=f(x)$

$\Rightarrow \sin(x+T)^2=\sin x^2$

$\Rightarrow (x+T)^2=2n\pi\pm x^2$

$\Rightarrow (x+T)^2=n\pi+(-1)^n\sqrt{x^2}$ .....(1)

The only non-negative values of T that satisfies the equation (1) and is independent of x is 0. Therefore $f(x)=\sin x^2$ is not a periodic function.

iii) Let the period of $\sqrt{\tan x}$ be T, then $\sqrt{\tan(T+x)}=\sqrt{\tan x}$

$\Rightarrow \tan(T+x)=\tan x\Rightarrow T+x=n\pi+x, n\in\mathbb{Z}$.

Therefore the lease positive value of T is π.

So, the period of $\sqrt{\tan x}$ is π.

iv) The period of the function $|\sin x| + |\cos x|$ is $\frac{\pi}{2}$.

E20) $|\sin x|$ is defined for all real x and at each point.

Let $a \in \mathbb{R}$, $f(a) = |\sin a|$

$$\lim_{x\to a} f(x) = \lim_{x\to a} |\sin x| = \left|\lim_{x\to a} \sin x\right|$$

$$= |\sin a|.$$

Therefore, it is continuous everywhere.