Indira Gandhi National Open University
School of Sciences

BMTC - 131
CALCULUS

Block
3
DIFFERENTIATION

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BLOCK 3 DIFFERENTIATION

This is the third of the five blocks which you will be studying for the course "Calculus". We shall begin this block by defining derivatives of various functions which we discussed in Block 2.

In Unit 9 we shall find derivatives of some standard functions using the definitions of derivatives. We shall also discuss algebra of derivatives. In this unit we shall find that continuity is necessary for a function to be differentiable.

In Unit 10 we shall continue our discussion of derivatives to find derivative of logarithmic, exponential and hyperbolic functions. We shall also discuss other differentiation techniques such as method of logarithmic differentiation and implicit differentiation.

In Unit 11 we shall find derivative of a derivative of a function and will extend our discussion to higher order derivatives. We shall apply higher order derivatives to find polynomial approximation.

In Unit 9 to 11, we have included a number of examples. Please go through them carefully they will help you in a better understanding of the concepts discussed and will also serve as a guideline in solving the exercises.

At the end of the block, you will find miscellaneous examples and exercises covering the concepts you have studied across the units. Please solve the exercises on your own. At the end of each unit, and after the miscellaneous exercises, we do provide some solutions/answers to the exercises concerned. These are only as a support for you to be able to check whether you have been able to solve the problem correctly or not. Please do not look at these solutions till you have spent enough time on studying the unit and trying all the exercises.

After the miscellaneous examples and exercises, you will find two appendices.

Appendix 1: Parametric Representation of curves

Appendix 2: Partial Fractions

A word about some signs used in the unit! Throughout each unit, you will find theorems, examples and exercises. To signify the end of the proof of a theorem, we use the sign ■. To show the end of an example, we use ***. Further, equations that need to be referred to are numbered sequentially within a unit, as are exercises and figures. E1, E2 etc. Denote the exercises and Fig. 1, Fig. 2, etc. denote the figures.

NOTATIONS AND SYMBOLS (used in Block 3)

w.r.t.	with respect to
$\frac{dy}{dx}, y^{(1)}, y', D(y)$	the first derivative of y w.r.t. x.
$\frac{d}{dx}(f(x), f'(x)$	the first derivative of $f(x)$ w.r.t x.
$\frac{d^2y}{dx^2}, y^{(2)}, f''(x)$	the second derivative of y or $f(x)$ w.r.t. x.
$\frac{d^ny}{dx^n}, y^{(n)}, f^{(n)}(x)$	the nth derivative of y or $f(x)$ w.r.t. x.
$\approx$	is approximately equal to

Also, see the list of notations and symbols in Block 1 and Block 2.

UNIT 9

AN INTRODUCTION TO DIFFERENTIATION

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9.1 INTRODUCTION

In the previous block, we have introduced the concepts of 'limit' and 'continuity'. We have also talked about the algebra of limits and operations on continuous functions. This unit will build on these concepts and take you a step further in your study of calculus.

In this unit on differentiation, we introduce the concept of a derivative which is a basic tool of calculus. Leibnitz was motivated directly by the problem of finding tangent to a given curve at a given point, a problem which was of great significance for scientific applications. He recognised the derivative as the slope of the tangent to the curve at the given point. Earlier, Newton, on the other hand, arrived at it by considering some physical problems such as determining velocity or acceleration of a particle at a particular instant. He recognised the derivative as rate of change of physical quantities. We shall now show you how both these considerations are built on to the concept of derivative as the limit of a ratio. To understand what a derivative is, you will have to go through Sec. 9.2 thoroughly. In Sec 9.3, you will see that all the

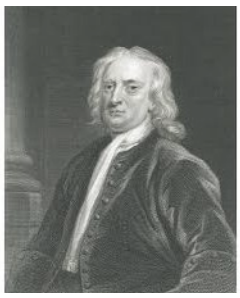

Fig. 1: Newton (1642-1727)

functions which are continuous may not be differentiable.

We will differentiate some standard functions using the definition of the derivative in Sec. 9.4. The algebra of derivatives can be effectively used to write down the derivatives of several functions which are algebraic combinations of these functions which have been discussed in Sec. 9.5. We shall also discuss the chain rule of differentiation which offers an unbelievable simplification in the process of finding derivatives in Sec. 9.6. In Sec.9.7, we will discuss the derivatives of trigonometric functions. In Sec. 9.8, we will go on to study the inverse function theorem, and apply it to find derivatives of inverses of some standard functions.

Fig. 2: Leibnitz (1646-1716)

And now we will list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

Objectives

After studying this unit, you should be able to:

- find the slope of a curve at a given point;
- determine the rate of change of a given quantity with respect to another;
- obtain the derivatives of some simple functions from the first principles;
- find the derivatives of the sum, difference, product and quotient of functions whose derivatives you already know;
- derive, and apply, the chain rule of differentiation for finding the derivatives of the composition of two or more functions;
- find the relationship between continuity and derivability of a function;
- find the derivatives of trigonometric functions;
- state and apply, the inverse function theorem; and
- find the derivatives of inverse trigonometric functions.

9.2 THE DERIVATIVES

There are many real life situations where it is desirable to know the rate of change of a particular parameter with respect to some other parameter. For example, it is important to know the depth of water at several instances of time to predict the overflow in a dam, the change in distance travelled at various times is needed to compute the precise velocity, etc. In this section, we shall find the rate of change of a variable with respect to another variable using the tangent to a curve.

Fig. 3: Euclid

Let us consider the problem of finding a tangent to a given curve at a given point. What do we mean by the tangent to a curve? Euclid (300 B.C.) thought of a tangent to a curve as a line touching the curve exactly at one point. The word 'tangent' derives from the Latin word 'tangentem', meaning 'touch'. A tangent line touches a curve at a single point only, in the same way as the tangent line touches the circle in Fig. 4(a). A line that cuts the curve more than once is called **secant line** as shown in Fig. 4(a). In Fig. 4(b), the line L touches the curve exactly once in the small interval containing P. The point P is called the **point of tangency**. We are not concerned with the behaviour of the line far from the point of tangency. We see that L does pass through the curve elsewhere, but it is still considered a tangent line to the curve at the point P.

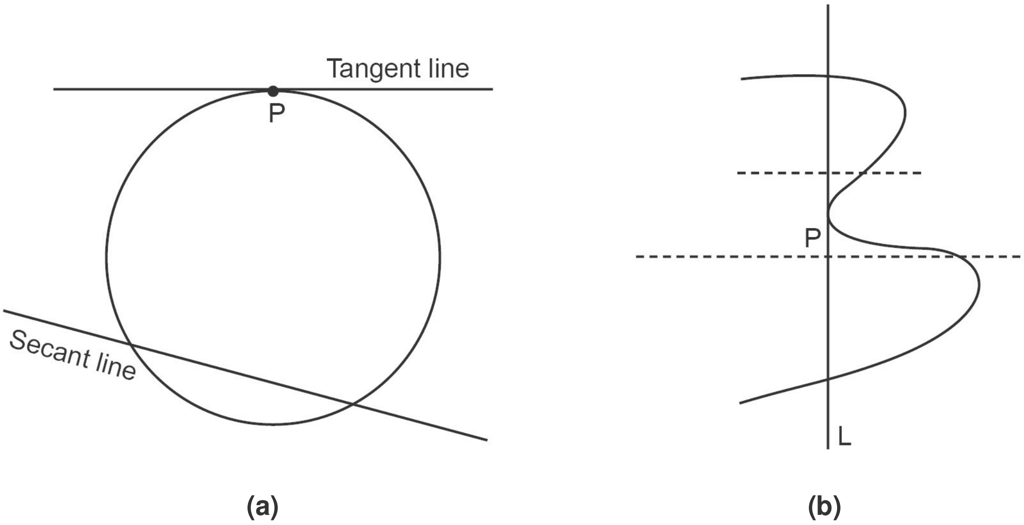

Fig. 4: Some curves and tangents to them

We now define a tangent to a curve at P to be a line which best approximates the curve near P. To do this, we use the notion of limit which you have studied in Unit 7.

Let P be a fixed point on the curve in Fig. 5 (a). To obtain the tangent line to the curve at point P, consider a secant line PQ. As the point Q approaches P through say Q_1, Q_2 and so on, the secant lines, PQ_1, PQ_2 and so on, approach the line L as shown Fig. 5 (a). Each secant line has a slope. The slopes m_1, m_2, m_3 and so on, of the secant lines approach the slope m of line L. Now, we define line L as the tangent line, the line that contains point P and has slope m, where m is the limit of the slopes of the secant lines as the point Q approaches P as shown in Fig. 5(b). Slope m is the instantaneous rate of change of P.

Fig. 5: Tangent line at P

You may think of the sequence of secant lines as an animation as the point Q moves closer to the fixed point P, the resulting secant lines '**lie down**' on the tangent line.

We have said earlier that the tangent at point P is the limiting position of the secant PQ. With reference to a system of coordinate axes OX and OY (Fig. 6), we can also say that the tangent at P is a line through P whose slope is the limiting value of the slope of the secant through P and Q as Q approaches P along the curve. Therefore, the problem of determining the tangent is, then, the problem of finding the slope of the tangent line.

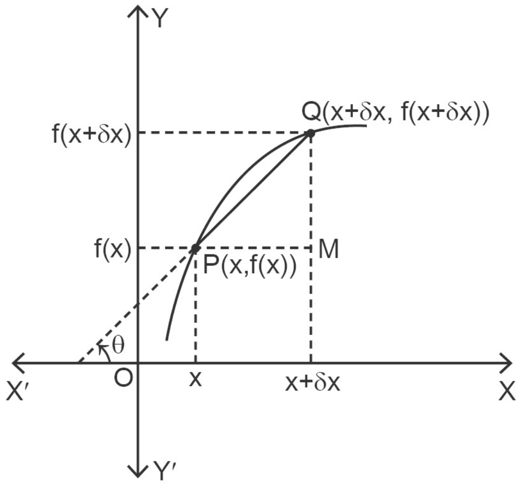

Fig. 6

Caution: δx is one inseparable quantity. It is not $\delta \times x$.

Suppose the curve $y = f(x)$ is given in Fig. 6. Let $(x, f(x))$ be the point P and let $Q(x+\delta x, f(x+\delta x))$ be any other point on the curve near P. (The prefix δ before a variable quantity means a small change in the quantity. Thus, δx means a small change in the variable x.) The coordinates of Q, $(x+\delta x, f(x+\delta x))$ indicate that Q is very very near P along the curve f. If θ is the angle which PQ makes with the x-axis, then the slope of the secant line $PQ = \tan\theta = QM/PM = \dfrac{f(x+\delta x) - f(x)}{\delta x}$.

The limiting value of the slope of PQ, as Q tends to P, (and hence $\delta x \to 0$), then gives us the slope of the tangent at P. Thus, we have the following:

The slope of the tangent line at $(x, f(x)) = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x}$ ... (1)

This limit is also the instantaneous rate of change of $f(x)$ with respect to x.

This indicates that the tangent line will exist only if the limit of $\dfrac{f(x+\delta x) - f(x)}{\delta x}$ exists as $\delta x \to 0$.

Remark 1: In Fig. 6 we have taken δx to be positive. But our discussion is valid even for negative values of δx.

We shall find the slope of a tangent line to the graph of a function f in the following example.

Example 1: Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = 3x - x^2$. Find the instantaneous rate of change of f at $x = 2$.

Solution: Here, $f(x) = 3x - x^2$. When $x = 2$, $f(2) = 2$.

Thus, $\dfrac{f(2+\delta x) - f(2)}{\delta x} = \dfrac{\left[3(2+\delta x) - (2+\delta x)^2\right] - 2}{\delta x}$

$$= \frac{-\delta x^2 - \delta x}{\delta x} = -\delta x - 1$$

Therefore, $\lim\limits_{\delta x \to 0} \dfrac{f(2+\delta x) - f(2)}{\delta x} = -1$.

Hence, we can say that

- The tangent line to the curve at the point $(2, 2)$ has slope -1 as shown in Fig. 7.
- The instantaneous rate of change at $x = 2$ is -1.

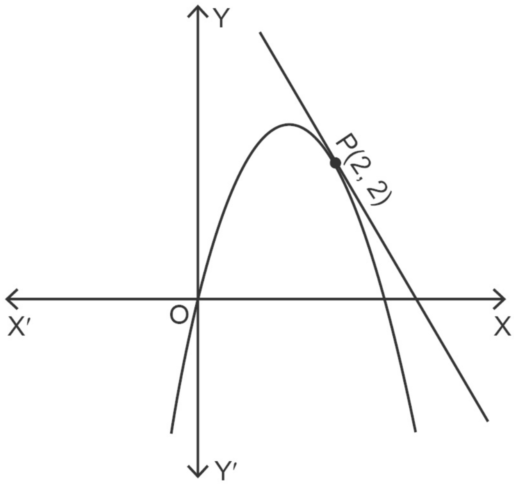

Fig. 7

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Example 2: Find the slope of the tangent line to the curve $f(x) = \frac{1}{x}$ at $x = 2$.

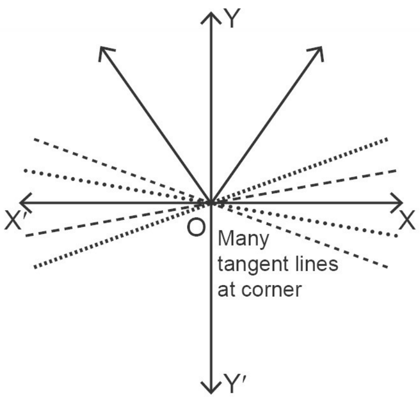
Solution: We have $\frac{f(2+\delta x) - f(2)}{\delta x} = \frac{[1/2+\delta x]-[1/2]}{\delta x} = \frac{-1}{2(2+\delta x)}$

We want to find $\lim_{\delta x \to 0} \frac{f(2+\delta x) - f(2)}{\delta x} = \lim_{\delta x \to 0} \frac{-1}{2(2+\delta x)} = -\frac{1}{4}$

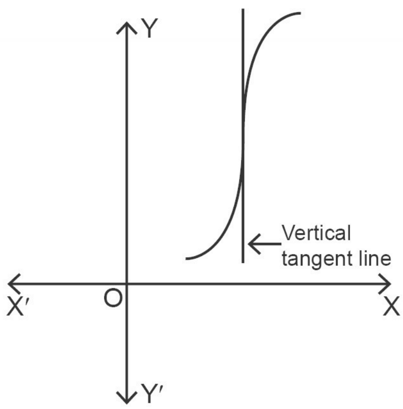
This is the slope of the tangent line at $x = 2$.

***

It may not always be possible to have tangents at some points. In fact, there are curves which do not have a tangent at any point. For example, $f(x) = |x|$ has a corner (not smooth) at $x = 0$, and would seem to have many tangent lines at $(0, 0)$, and thus many slopes as shown in Fig. 8(a). Also, the limit of the slope of the vertical tangent at any point as shown in Fig.8 (b) is undefined.



(a)



(b)

Fig. 8

Now, try the following exercises.

E1) Find the slope of the tangent to the following curves at the given points

i) $y = 1/x$ at $(2, 1/2)$.

ii) $y = x^3$ at $(1, 1)$.

E2) Does the slope of the tangent stay the same at different points? Justify your answer.

Now let us consider another problem, that is, the rate of change from the Newtonian view. Let us begin with an example. Consider that a car travels $150\,\text{km}$ in $3\,\text{h}$. Its average rate of change (speed) is $150\,\text{km}/3\text{h}$. Suppose that the car is on a free way and the driver begins accelerating. Looking at the speedometer, it is seen that at that instant the instantaneous rate of change is $50\,\text{km}/\text{h}$.

Now, suppose a function f is defined by $f(x)$, and consider a small change in the value of x that is δx. The change in x is $(x + \delta x) - x$, and the change in f due to the change in x is $f(x + \delta x) - f(x)$. The average rate of change of f with respect to x as x changes from x to $x + \delta x$, is the ratio of the change in f to the change in x. Thus, we have the following:

The average rate of change of f due of change δx in x is $\dfrac{f(x+\delta x) - f(x)}{(x + \delta x) - x}$, where $\delta x \neq 0$.

The average rate of f with respect to x is also called the **difference-quotient**.

But this does not give the instantaneous rate of change. How do we calculate this?

If δx is very small, then $x + \delta x$ is very near x and so the average rate of change would be the instantaneous rate of change at

$$x = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

Thus, the instantaneous rate of change of f with respect to x.

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \qquad \ldots(2)$$

If we look at the graph of the function in Fig. 9, we see that the average rate of change is the slope of the secant line from P to Q. Thus, the slope of the secant line is interpreted as the average rate of change of f from x to $x + \delta x$. Also, the instantaneous rate of change is the slope of the tangent line.

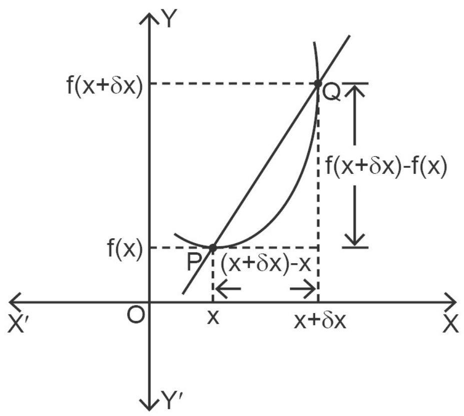


Fig. 9

We find average rate of change in the following examples.

Example 3: Find the average rate of change of the function f defined by $f(x) = x + 2 \ \forall x \in \mathbb{R}$, at $x = 0$.

Solution: We shall first calculate the average rate of change of f in an interval $[0, \delta x]$, i.e. $\delta x > 0$.

This average rate of change of f in $[0, \delta x]$ is

$$\frac{f(0+\delta x) - f(0)}{(0+\delta x) - 0} = \frac{f(\delta x) - f(0)}{\delta x} = \frac{\delta x + 2 - 2}{\delta x} = \frac{\delta x}{\delta x} = 1.$$

Hence, the rate of change of f at $x = 0$, which is the limiting value of this average rate as $\delta x \to 0$, is

$$= \lim_{\delta x \to 0} \frac{f(0+\delta x) - f(0)}{\delta x} = \lim_{\delta x \to 0} 1 = 1.$$

You can check that a similar argument provides the rate of change of f at $x = 0$ as 1 using $\delta x < 0$.

***

Example 4: Suppose a particle is moving along a straight line and the distance s (in metres) covered in time t (in seconds) is given by the equation $s = (1/2)t^2$. Find the velocity of the particle after 2 seconds.

Solution: We know that $s(t) = \frac{1}{2}t^2$. The velocity $v(t)$, after t seconds, is given by $v(t) = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}$ m/s

$$= \lim_{\delta t \to 0} \frac{\frac{1}{2}(t+\delta t)^2 - \frac{1}{2}t^2}{\delta t} \text{ m/s}$$

$$= \lim_{\delta t \to 0} \left(\frac{\delta t}{2} + t \right) \text{m/s}$$

$= t$ m/s.

$\therefore$ The velocity after 2 sec is $v(2) = 2\,\mathrm{m/s}$.

***

Remark 2: i). If the path of a particle moving according to $s = f(t)$, is shown in the ts-plane and if P and Q are points on the path which correspond to $t = t_1$ and $t = t_2$, then the average velocity of the particle in time $(t_2 - t_1)$ is given by the slope of PQ and the velocity at time t_1 is given by the slope of the tangent at P.

ii) Distance is always measured in units of length (metres, centimetres) and so velocity v really means v units of distance per unit of time. The slope of the tangent is a dimensionless number, while the velocity has the dimension of length/time.

Now you can try some exercises.

E3) Consider the curve $s = \left(1/2\right)t^2$ in its ts-plane. Find the slope of the tangent of the curve at $t = 2$. Also, compare your result with the velocity found in Example 4.

E4) A particle is thrown vertically upwards in the air. The distance it covers in time t is given by $s(t) = ut - (1/2)\, gt^2$ where u is the initial velocity and g denotes the acceleration due to gravity. Find the velocity of the particle at any time t.

E5) The area of a circle is a function of its radius. Find the rate of change of the area of a circle with respect to its radius when the radius is 2 cm.

E6) Find the average rate of change of the function f, defined by $f(x) = 2x^2 + 1 \;\forall\, x \in \mathbb{R}$ in the interval $[1, 1+h]$ and, hence, evaluate the rate of change of f at $x = 1$.

We have seen that the slope of a tangent and the instantaneous rate of change have a common concept behind them. Won't it be better, then, to give a separate name to this concept, and study it independently of its diverse applications? The name of it is "derivative", and we define it now.

The notation for dy/dx is due to Leibnitz and $f'(x)$ is due to Lagrange. $f'(x)$ is read as "f dash x".

Definition: Let f be a real-valued function whose domain is a subset D of $\mathbb{R}$ and let $x \in D$. Then $\lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x}$, **if this limit exists**, is called the **derivative of f at x**. This is usually denoted by $f'(x)$ or Df. Sometimes we denote $f(x)$ by y. Then, we denote $f'(x)$ by $\dfrac{dy}{dx}$ or $\dfrac{df}{dx}$.

Now, if we write $f(x + \delta x) = y + \delta y$, then derivative of $f = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$, where δy denotes the change in y caused by a change δx in x.

Note that f' is also a function and the value of $f'(x)$ at a point x_0 is denoted by $f'(x_0)$. Thus,

$$\mathbf{f'(x_0) = \lim_{\delta x \to 0} \frac{f(x_0 + \delta x) - f(x_0)}{\delta x}}, \text{ if limit exists.}$$

Here $f'(x_0)$ quantifies the change in $f(x)$ at x_0 with respect to x. We write $x = x_0 + \delta x$, then $\delta x = x - x_0$ and $\delta x \to 0 \Leftrightarrow x \to x_0$. Therefore, an equivalent way of stating the definition of the derivative is

$$\mathbf{f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}.$$

If $y = f(x)$, then, we also use the notations $\left.\dfrac{dy}{dx}\right|_{x=x_0}$ or $\left.\dfrac{dy}{dx}\right]_{x=x_0}$ for $f'(x_0)$, that is, the derivative of the function f at $x = x_0$.

Caution: (i) 'dy' and 'dx' in the expression $\dfrac{dy}{dx}$ are **not separate entities**.

That is dy is meaningless without dx. In fact, it is $\dfrac{d}{dx}(y)$ and means the derivative of y with respect to x.

(ii) You cannot cancel 'd' from dy/dx to get y/x. The notation only suggests the fact that the derivative is obtained as a ratio.

Let us state another definition.

Definition: Let $f : D \to \mathbb{R}$ be a function. The function f is **derivable** (**or differentiable**) at x if $f'(x)$ exists. When f is differentiable at each point of its domain D, then f is said to be a **differentiable function.** The process of obtaining the derivative is called **differentiation**. The process of finding the derivative of a function by actually calculating the limit of the ratio $\dfrac{f(x+\delta x)-f(x)}{\delta x}$ is called **differentiating from first principles,** or differentiating ***ab initio***.

'Ab initio' is Latin term used for 'from the beginning'.

Let us find more derivatives in the following examples.

Example 5: Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ at $x = 1$.

Solution: We have $f'(1) = \lim\limits_{\delta x \to 0} \dfrac{f(1+\delta x)-f(1)}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{2(1+\delta x)-2}{\delta x} = 2.$

Therefore, the derivative of f at $x = 1$ is 2.

***

Example 6: Find $f'(0)$ and $f'(-1)$ for the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + x - 5$.

Solution: We have $f'(0) = \lim\limits_{\delta x \to 0} \dfrac{f(0+\delta x)-f(0)}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{[(\delta x)^2 + \delta x - 5]-[-5]}{\delta x}$

$$= \lim_{\delta x \to 0} (1+\delta x) = 1$$

and $f'(-1) = \lim\limits_{\delta x \to 0} \dfrac{f(-1+\delta x)-f(-1)}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{[(-1+\delta x)^2 + (-1+\delta x) - 5] - [(-1)^2 + (-1) - 5]}{\delta x}$$

$$= \lim_{\delta x \to 0} (\delta x - 1) = -1$$

***

Now, try the following exercise.

E7) Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 8x + 9$ at the point c.

The function f' which associates to each point x of D, the derivative $f'(x)$ at x, is called the **derived function** of f, because it has been derived from f by limiting operation. Thus, f' is a function from D' to $\mathbb{R}$, where $D' = \{x \in D : f'(x)$ exists$\}$. The domain of f' may be smaller than the domain of f.

We are talking about the function f', now it is important to see the relationship between the graph of a function f and its derivative f'. From the definition of f', it is clear that at point x, when $f'(x) > 0$, the tangent line must be tilted upward and the graph is rising as tangent line has positive slope. Similarly, when $f'(x) < 0$, that is the slope of the tangent line is negative, the tangent line is tilted downward, and the graph is falling. But when $f'(x) = 0$, the tangent line is horizontal at x, so the graph flattens. Fig. 10 illustrates the same.

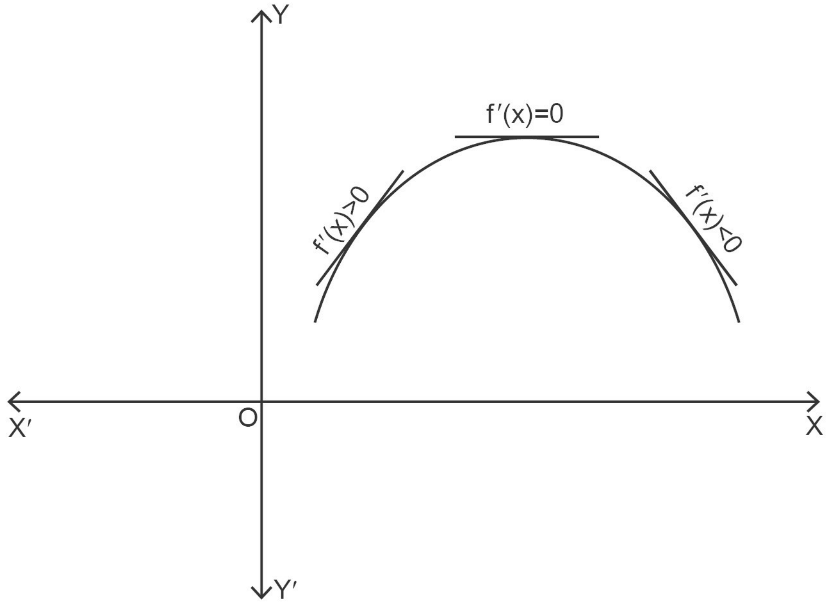


Fig. 10: Sign of f'

We will see the graphs of f and f' in the following examples.

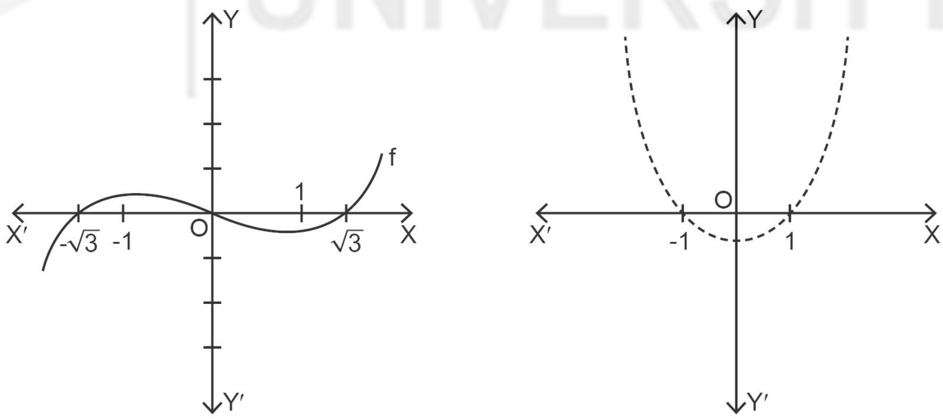
Example 7: If $f(x) = x^3 - 3x$, find a formula for $f'(x)$. Also, compare the graphs of f and f'.

Solution: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h \to 0} \frac{[(x+h)^3 - 3(x+h)] - [x^3 - 3x]}{h}$$

$$= 3x^2 - 3 = 3(x^2 - 1)$$

Fig. 11(a) shows the graph of the function f.



(a) Graph of f **(b) Graph of f'**

Fig. 11

Notice that the graph of f is rising for $x < -1$ and $x > 1$ and it is falling for $-1 < x < 1$ and it has horizontal tangent lines at $x = -1$ and 1. Thus, the graph of f' is above the $x-$axis that is $f'(x) > 0$ for $x < -1$ and $x > 1$, it is below the $x-$axis for $-1 < x < 1$ that is $f'(x) < 0$ and cuts the $x-$axis at $x = 1$ and at $x = -1$. One possible graph of these features is shown in Fig. 11(b).

***

Example 8: Check whether or not the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = |x|$, is differentiable. If f' exists, compare the graphs of f and f'.

Solution: i) If $x > 0$, then $|x| = x$ and we can choose δx small enough such that $x + \delta x > 0$ and hence $|x + \delta x| = x + \delta x$. Therefore, for $x > 0$, we have $f'(x) = \lim_{\delta x \to 0} \frac{|x+\delta x| - |x|}{\delta x} = \lim_{\delta x \to 0} \frac{(x+\delta x) - x}{\delta x} = \lim_{\delta x \to 0} \frac{\delta x}{\delta x} = \lim_{\delta x \to 0} 1 = 1$

and so f is differentiable for any $x > 0$.

ii) Similarly, for $x < 0$ we have $|x| = -x$ and δx can be chosen small enough that $x + \delta x < 0$ and so $|x + \delta x| = -(x + \delta x)$. Therefore, for $x < 0$, the derivative is

$$f'(x) = \lim_{\delta x \to 0} \frac{|x+\delta x| - |x|}{\delta x} = \lim_{\delta x \to 0} \frac{-(x+\delta x) - (-x)}{\delta x} = \lim_{\delta x \to 0} \frac{-\delta x}{\delta x} = -1$$

and so f is differentiable for any $x < 0$.

iii) When $x = 0$, we have to find $f'(0) = \lim_{\delta x \to 0} \frac{f(0+\delta x) - f(0)}{\delta x}$, if it exists.

Here, $\lim_{\delta x \to 0^+} f'(0) = 1$ and $\lim_{\delta x \to 0^-} f'(0) = -1$. Since these limits are different, $f'(0)$ does not exist. Thus, f is differentiable at all x except 0. From above, the function f' is defined as

$$f' = \begin{cases} 1, \text{ if } x > 0 \\ -1, \text{ if } x < 0 \end{cases}.$$

The graphs of f and f' are given in Fig. 12 (a) and (b) respectively.

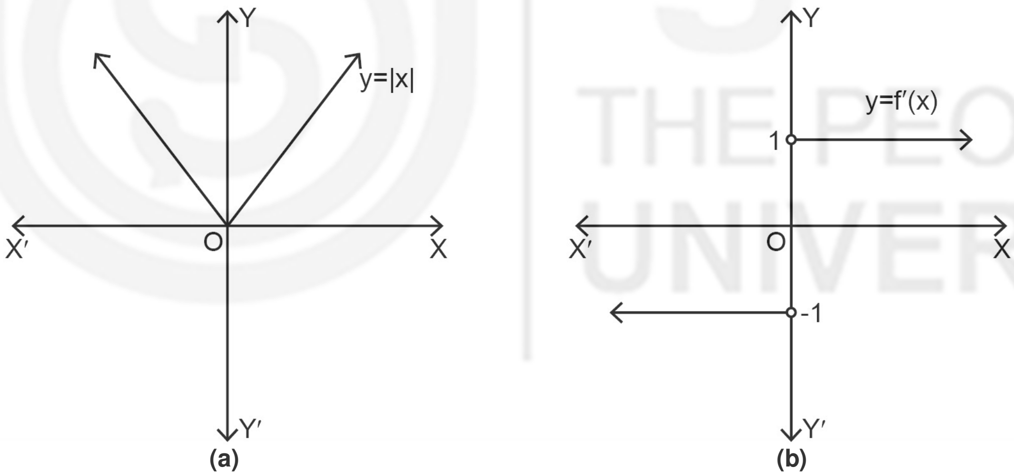


Fig. 12: Graphs of (a) f and (b) f'

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You can now try the following exercises.

E8) Use the definition of derivative to find $f'(x)$ for the function f given by $f(x) = 3x^2 + 2x + 1$. Also, draw the possible graph of f'.

E9) Find dy/dx, wherever it exists, using first principles, for each of the following:

i) $y = x^3$

ii) $y = |x+1|$

iii) $y = \sqrt{2x+1},\ x \ge -\dfrac{1}{2}$

E10) Show that each of the following functions is derivable at $x = 2$. Find $f'(2)$ in each case.

i) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$

ii) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$, where a and b are fixed real numbers.

So far, we have discussed the derivatives. Now we will establish an important relation between continuity and differentiability in the following section.

9.3 CONTINUITY VERSUS DERIVABILITY

Let us begin with an example that is $f(x) = |x|$. In Unit 8, we have proved that the function $y = |x|$ is continuous $\forall x \in \mathbb{R}$, and in Example 8, we saw that this function is derivable at every point except at $x = 0$. This means that the function $y = |x|$ is continuous at $x = 0$, but is not derivable at this point. This shows that a function can be continuous at a point without being derivable at that point. However, we will now prove that if a function is derivable at a point, then it must be continuous at that point or derivability $\Rightarrow$ continuity. In other words, we expect the following theorem to be true.

Theorem 1: Let f be a function defined on an interval I. If f is derivable at a point $x_0 \in I$, then it is continuous at x_0.

Proof: To prove that f is continuous at x_0, we have to show that $\lim\limits_{x \to x_0} f(x) = f(x_0)$. We do this by showing that the difference $f(x) - f(x_0)$ approaches 0. If $x \ne x_0$ but $x \to x_0$, then, we divide and multiply $f(x) - f(x_0)$ by $(x - x_0)$, and we may write

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{(x - x_0)}(x - x_0).$$

Since, f is derivable at x_0, $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists and equals $f'(x_0)$.

Thus, taking limit as $x \to x_0$, we have,

$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim_{x \to x_0} \left\{ \frac{f(x) - f(x_0)}{x - x_0} . (x - x_0) \right\}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0)$$

$$= f'(x_0) \times 0 = 0$$

Therefore, $\lim\limits_{x \to x_0} f(x) - \lim\limits_{x \to x_0} f(x_0) = 0$.

That is, $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} f(x_0) = f(x_0)$

Consequently, f is continuous at x_0. ■

You may note that the converse of Theorem 1 is not true. There are functions that are continuous but not differentiable. For example, the function $y = |x|$ is continuous at 0 but not derivable at only one point, $x = 0$. But, there are some continuous functions which are not derivable at infinitely many points. For instance, look at Fig. 13.

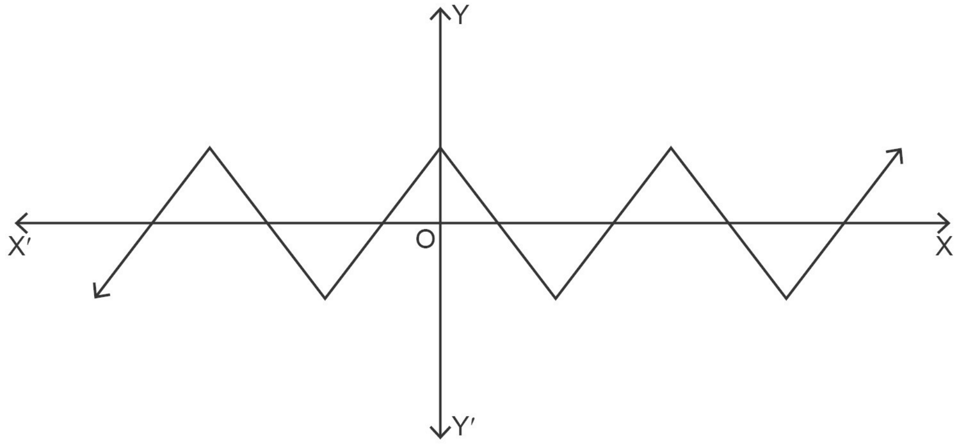

Fig. 13

Fig. 13 shows the graph of a continuous function which is not derivable at infinitely many points. Can you mark those infinitely many points at which this function is not derivable? You can take your hint from the graph of the function $y = |x|$.

Note that a function is differentiable only if the limit in the definition of derivative exists. At points where a function f is not differentiable, we say that the limit of the derivative of f does not exist at those points. Three common reasons for a derivative to not exist at a point x_0 in the domain of f are as given below:

i) If the graph of a function f has a 'corner' in it, then the graph of f has no tangent at this point and f is not differentiable there. Fig. 14 (a) shows a corner in a graph of a function.

ii) If the curve has a vertical tangent line when $x = x_0$, that is f is continuous at x_0 and $\lim\limits_{x \to x_0} f'(x) = \infty$ or $\lim\limits_{x \to x_0} f'(x) = -\infty$. This means that the tangent lines become steeper and steeper as $x \to x_0$. Fig. 14 (b) shows this.

iii) Theorem 1 gives another reason for a function not to have a derivative. It says that if f is not continuous at x_0, then f is not derivable at x_0. So at any discontinuity, f will not be differentiable. Fig. 14 (c) strengthens this.

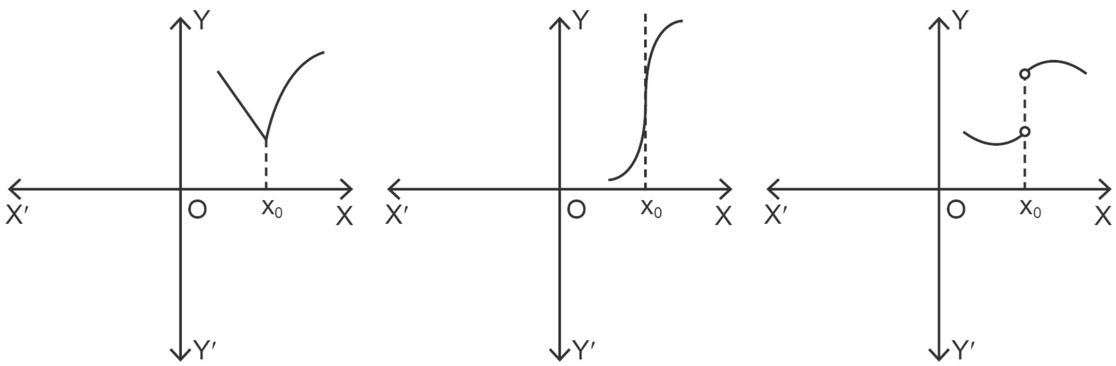

Fig. 14: Graph of functions not differentiable at x_0 ((a) A corner, (b) A vertical tangent, and (c) A discontinuity)

We now give another example of a function, which is continuous but not differentiable.

Example 9: Let $f(x) = \begin{cases} -2x & , \text{ if } x < 1 \\ \sqrt{x} - 3, & \text{if } x \geq 1 \end{cases}$

i) Draw the graph of f ;

ii) Show that f is continuous, but not differentiable, at $x = 1$.

Solution: i) The graph of f is shown in Fig. 15

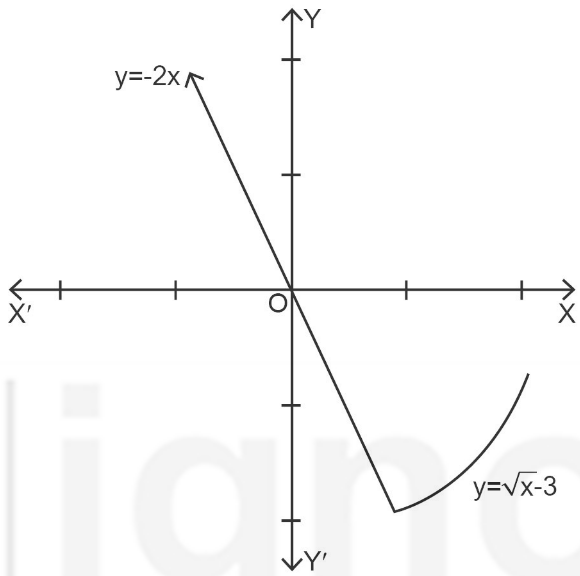

Fig. 15: Graph of f

ii) The function f is defined at 1 and $f(1) = -2$, then, $\lim_{x \to 1^-} f(x) = -2$ and $\lim_{x \to 1^+} f(x) = -2$. Therefore, $\lim_{x \to 1} f(x) = f(1)$ and the function f is continuous at $x = 1$. Since, the graph of the function f has a corner at $x = 1$, therefore, the graph cannot have any tangent at $x = 1$. Thus, f is not differentiable at $x = 1$. So, the function f is continuous at $x = 1$ but not differentiable at $x = 1$.

***

There are functions which are **continuous everywhere** but **differentiable nowhere**. The discovery came as a surprise to the nineteenth century mathematicians who believed, till then, that if a function is such that it is not derivable at any point, then it cannot be continuous at every point. The first such function was put forth by the mathematician Weierstrass (although he is said to have attributed the discovery to Riemann) in 1872. He showed that the function f given by $f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$, where a is an odd integer and b is a positive constant between 0 and 1 such that $ab > 1 + 3\pi/2$, is a function which is continuous everywhere, but derivable nowhere. We will not prove this assertion in this course.

Try the exercises now.

E11) Is the function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \frac{(2x+3)^{50}}{9x+2}$ continuous at $x = 0.1$?

E12) Give an example of a function that is continuous on $]-\infty,\infty[$ but is not differentiable at $x=5$.

In the following section, you will study some basic differentiation rules for constant function and power function.

9.4 SOME SIMPLE DIFFERENTIATION

As we will see later, it is not always necessary to find a derivative from the first principles. We shall develop certain rules which can be used to write down the derivatives of some functions. Some such rules are obtained as given below.

9.4.1 Derivative of a Constant Function

Let $f:\mathbb{R}\to\mathbb{R}$ be a constant function f defined as $f(x)=c$ for all $x\in\mathbb{R}$, where c is a real number. Let us see, if f' exists.

$$f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}=\lim_{\delta x\to 0}\frac{c-c}{\delta x}=\lim_{\delta x\to 0}0=0.$$

Hence, a **constant function is differentiable** and its derivative is equal to **zero** at each point of its domain.

Thus, $\frac{d}{dx}(c)=0\ \forall\, x\in\mathbb{R}$. You may recall the constant function and its geometric representation from Unit 2. In Fig. 16, if we join any two points, P and Q, the line PQ is the line $y=c$, and is parallel to the x-axis. Hence, the angle made by PQ with the x-axis is zero. This means that the slope of PQ is $\tan 0=0$. Since, $f'(x)$ is the limit of this slope as $Q\to P$, we get $f'(x)=0$. This is true for any x in the domain of f.

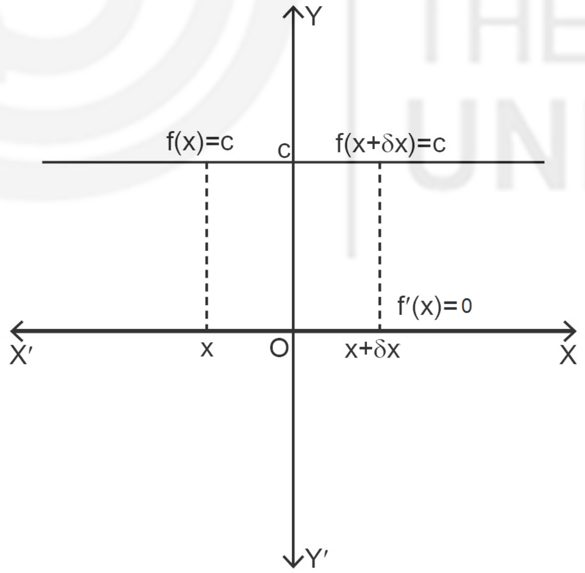

Fig. 16: Graph of f and f'

We will find derivative of few constant functions in the following example.

Example 10: Find the derivatives of the following with respect to x:

(i) $f(x)=\pi\ \forall\, x\in\mathbb{R}$

(ii) $f(x)=10\ \forall\, x\in\mathbb{R}$

Also, define the domain and range f' in each case and draw the graph of f' in each case.

Solution: (i) $\frac{d}{dx}(f(x)) = \frac{d}{dx}(\pi) = 0$ (since π is a constant)

Domain of f' is $\mathbb{R}$ and range of f' is 0.

(ii) $\frac{d}{dx}(f(x)) = \frac{d}{dx}(10) = 0$

Domain of f' is $\mathbb{R}$, and range of f' is 0.

***

Now try the following exercise.

E13) Find the derivatives of the following with respect to x.

i) $y = \sqrt{2}$

ii) $y = e$.

Now in the following subsection we will differentiate the power function.

Binomial Theorem:
$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}\, b^2 + \ldots + {}^nC_n\, b^n$.
Where n is a non-negative integer and a, b are arbitrary.

9.4.2 Derivative of the Power Function (x^n)

If n is a positive integer, let us find the derivative of x^n. According to the definition of the derivatives, $\frac{d}{dx}(x^n) = \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{\delta x}$.

Instead of δx, we can also use the letter h to denote the small change in the variable x. We are, in fact, free to use any notation, but δx or h are the most commonly used ones. Then, we rewrite the derivative of x^n by replacing δx with h.

$$\frac{d}{dx}(x^n) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{(x^n + {}^nC_1 hx^{n-1} + {}^nC_2\, h^2 x^{n-2} + \cdots + h^n) - x^n}{h} \text{ (using the binominal theorem)}$$

$$= \lim_{h\to 0} \left\{nx^{n-1} + {}^nC_2\, hx^{n-2} + \cdots + h^{n-1}\right\}$$

$$= nx^{n-1} \text{ [since } \lim_{h\to 0}\left({}^nC_2\, h\, x^{n-2} + \ldots + h^{n-1}\right) = 0]$$

Therefore, $\frac{d}{dx}(x^n) = D(x^n) = nx^{n-1}$.

Let us find the derivative of x^n in the following example.

Example 11: Find the derivatives of x^6 and x^{11} with respect to x.

Solution: $\frac{d}{dx}\underbrace{(x^6)}_{x^n} = \underbrace{6}_{n}\underbrace{x^{6-1}}_{x^{n-1}} = 6x^5$

$\frac{d}{dx}\underbrace{(x^{11})}_{x^n} = \underbrace{11}_{n}\,\underbrace{x^{11-1}}_{x^{n-1}} = 11\,x^{10}$

***

We shall show later, that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all $x > 0$ if n is any real number.

If $n = 0$, then $x^n = 1$ for all x and hence, $\frac{d}{dx}(x^n) = 0$ for all $x \in \mathbb{R}$. This means that the result is trivially true for $n = 0$. Now, we are in a position to prove this result for $n = 1/2$. That is, $\frac{d}{dx}\left(\sqrt{x}\right) = \left(\frac{1}{2}\right)x^{-1/2}$, and this we do in the following example.

$\sqrt{x}$ is not defined for $x < 0$.

Example 12: Show that the function f defined by $f(x) = \sqrt{x},\ x \geq 0,$ is differentiable.

Solution: We have, $\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

[multiplying numerator and denominator by $(\sqrt{x+h} + \sqrt{x})$].

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2},\ x > 0.$$

Thus, $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}},\ x > 0.$ You may note that $f(x) = \sqrt{x}$ is defined for all $x \geq 0,$ whereas its derivative $f'(x) = \frac{1}{2\sqrt{x}}$ is defined for $x > 0.$ This shows that a function need not be differentiable throughout its entire domain. What happens at $x = 0,$ the function f has vertical tangent at $x = 0,$ and the slope of a vertical tangent is undefined.

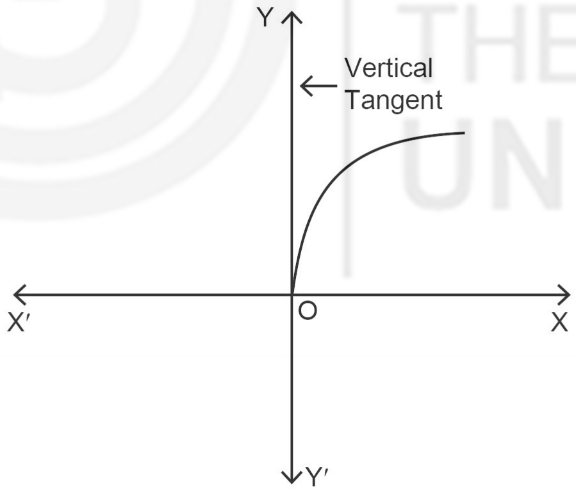

Fig. 17: Graph of $f(x) = \sqrt{x}$

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Now, try the following exercise.

E14) Find the derivative of the following with respect to x.

i) $y = x^8$

ii) $y = x^{1000}$

iii) $y = 10$.

A function is said to be differentiable in an interval $[a, b]$ if it is differentiable at every point of $[a, b]$. As in the case of continuity, at the end points a and b, we take the right hand limit and left hand limit, which are nothing but the left hand derivative and the right hand derivative of the function at b and a respectively. Similarly, a function is said to be differentiable in an interval $]a, b[$ if it is differentiable at every point of $]a, b[$.

That is $f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$ is a two-sided limit and only a one-sided limit makes sense at an endpoint. To deal with this situation, we define derivatives from the left and right. These are denoted by f'_- and f'_+, respectively, and are defined in the following definition.

Definition: (i) $f'_+(a) = Rf'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$, if it exists, is called the **right hand derivative** of $f(x)$ at $x = a$ and is written as $Rf'(a)$ or $f'_+(a)$. At points where $Rf'(a)$ exists we say that the function f is differentiable from the right.

(ii) $f'_-(a) = Lf'(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$ if it exists, is called the **left hand derivative** of $f(x)$ at $x = a$ and is written as $Lf'(a)$ or $f'_-(a)$.

At points where $Lf'(a)$ exists, we say that the function f is differentiable from the left. If we say derivative of function at any point exists, that means left hand derivative and right hand derivative of the function f at that point must be equal to the value of the derivative of the function at that point. That is if $f'(a)$ exists, we must have $Rf'(a) = Lf'(a) = f'(a)$. Geometrically, $Rf'(a)$ is the limit of the slopes of the secant lines approaching a from the right, and $Lf'(a)$ is the limit of the slopes of the secant lines approaching a from the left. Fig. 18 shows the geometrical representations of $Rf'(a)$ and $Lf'(a)$.

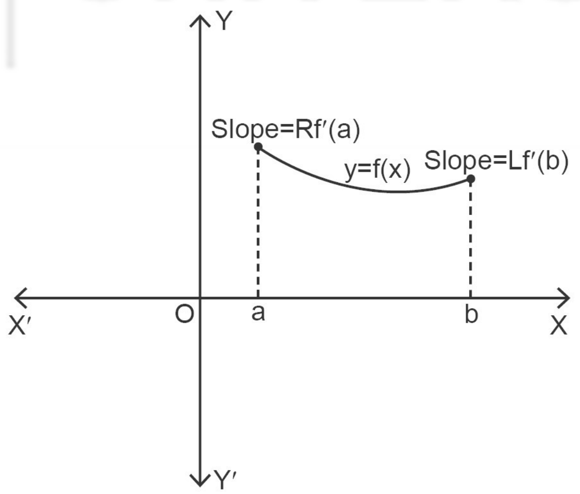

Fig. 18: Right and Left Derivatives

Example 13: Sketch the graph of the function f defined by $f(x) = x^{2/3}$. Also check whether f is differentiable at $x = 0$ or not.

Solution: Fig. 19 shows the graph of f.

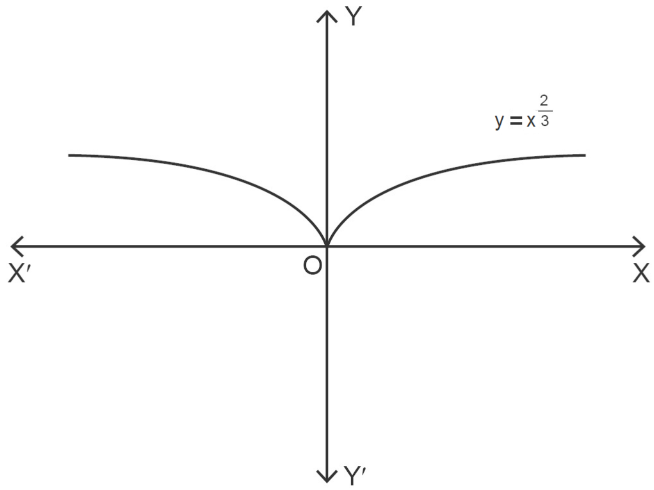

Fig. 19: Graph of $f(x) = x^{2/3}$

Now, let us find its left and right hand derivatives.

$$Rf'(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{(h)^{2/3} - 0^{2/3}}{h} = \lim_{h\to 0^+} \frac{1}{h^{1/3}} = +\infty$$

$$Lf'(0) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{h^{2/3} - 0^{2/3}}{h} = \lim_{h\to 0^-} \frac{1}{h^{1/3}} = -\infty$$

Thus, f is not differentiable at $x = 0$. Fig. 19 shows that graph of $f(x) = x^{2/3}$, which has a corner at $x = 0$ and a vertical tangent as well. Therefore, the function f is not differentiable at $x = 0$.

***

Now you can try following exercise.

E15) Check whether the function f given by $f(x) = x^{1/3}$ is differentiable at $x = 0$ or not.

So far, we have obtained derivatives of certain functions by differentiating from the first principles. When new functions are formed from old functions by adding, subtracting, multiplying by a constant, their derivatives can be calculated in terms of derivatives of the old functions. We will find the derivatives of these new functions in the following section.

9.5 ALGEBRA OF DERIVATIVES

Consider the function f defined by $f(x) = \dfrac{2x^3 + 3x^2}{x^4 - 1}$. If we try to find the derivative of this function from the first principles, we will have to do lengthy, complicated calculations. However, a close look at this function reveals that it is composed of several functions: constant function like 2, 3 and -1, and power functions like x^3, x^2 and x^4. We already know the derivatives of these functions. Can we use this knowledge to find the derivative of $f(x)$? In this section, we shall state and prove some theorems related to this.

Theorem 2 (The Constant Multiple Rule): If f is a differentiable function and $c \in \mathbb{R}$ then cf is differentiable, and $\dfrac{d}{dx}[cf(x)] = c\dfrac{d}{dx}[f(x)]$.

Proof: Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function and let $c\in\mathbb{R}$. Then, consider the function $y=cf$. We know that f is derivable. So, $f'(x)$ exists. Now the derivative of y with respect to x is

$$\frac{dy}{dx}=\frac{d\,[cf(x)]}{dx}=\lim_{h\to0}\frac{(cf)(x+h)-cf(x)}{h}=\lim_{h\to0}\frac{cf(x+h)-cf(x)}{h}$$
$$=\lim_{h\to0}c\left[\frac{f(x+h)-f(x)}{h}\right]$$
$$=c\left[\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\right] \qquad \text{[as c is constant]}$$
$$=cf'(x)=c\frac{d}{dx}[f(x)]$$

Thus, cf is differentiable and $(cf)'=cf'$. ■

Example 14: Find $\frac{dy}{dx}$, if

i) $y=-x^2$

ii) $y=\frac{3}{4}x^{10}$

iii) $y=\frac{1}{2x^2}$

Solution:

i) $\frac{dy}{dx}=\frac{d}{dx}\underbrace{(-x^2)}_{c.f(x)}=\underbrace{(-1)}_{c}\frac{d}{dx}\underbrace{(x^2)}_{f(x)}=-2x$

ii) $\frac{dy}{dx}=\frac{d}{dx}\underbrace{\left(\frac{3}{4}x^{10}\right)}_{c.f(x)}=\underbrace{\frac{3}{4}}_{c}.\frac{d}{dx}\underbrace{(x^{10})}_{f(x)}=\frac{3}{4}.10x^9=\frac{15}{2}x^9$

iii) $\frac{d}{dx}\left(\frac{1}{2x^2}\right)=\frac{d}{dx}\underbrace{\left(\frac{1}{2}x^{-2}\right)}_{c.f(x)}=\underbrace{\frac{1}{2}}_{c}\frac{d}{dx}\underbrace{(x^{-2})}_{f(x)}=\frac{1}{2}.(-2)x^{-2-1}=-x^{-3}$

A common mistake is to write an expression such as $\frac{1}{2x^2}$ as $(2x)^{-2}$, which is incorrect. The exponent 2 applies only to x and the 2 is the part of the coefficient $\frac{1}{2}$.

***

Example 15: The volume V of a spherical tumor can be approximated by $V(r)=\frac{4}{3}\pi r^3$, where r is the radius of the tumor, in cm.

i) Find the rate of change of the volume with respect to the radius.

ii) Find the rate of change of volume at $r=1.5\,cm$.

Solution: i) $\frac{dV}{dr}=\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)=\frac{4}{3}\cdot\pi\cdot3r^2=4\pi r^2$

ii) $V'(1.5)=4\pi(1.5)^2\approx28\,cm^2$

Thus, when the radius is $1.5\,\text{cm}$, the volume is changing at the rate of $28\,\text{cm}^3$ for every change of $1\,\text{cm}$ in the radius.

***

Try the following exercise now.

E16) Differentiate the following with respect to x, using Theorem 2.

i) $y = (5/3)x^3$ for all $x \in \mathbb{R}$.

ii) $y = 8\sqrt{x}$ for x.

iii) $y = \sqrt{2}|x|$ for all $x \in \mathbb{R}$.

E17) Differentiate $y = 7|x|$ with respect to x.

Now, let us discuss the derivative of sum of two or more functions.

Theorem 3 (The Sum Rule): The sum of two differentiable functions f and g is a differentiable function and $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$.

Proof: Let f and g be differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Let us examine whether $f+g$, the sum of the functions f and g, is differentiable. Now,

$$\frac{d}{dx}[f(x)+g(x)] = \lim_{h\to 0}\frac{\{f(x+h)+g(x+h)\}-\{f(x)+g(x)\}}{h}$$

$$= \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right\}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \frac{df(x)}{dx}+\frac{dg(x)}{dx}$$

$$= f'(x)+g'(x)$$

Thus, we have proved the theorem. ■

The above result can be easily extended to the sum of any number of functions, that is,

$$\frac{d}{dx}(f_1+f_2+\cdots+f_n) = \frac{df_1}{dx}+\frac{df_2}{dx}+\cdots+\frac{df_n}{dx},$$

where $f_1, \ldots, f_n$ are differentiable functions.

Remark 3: From Theorem 2 and Theorem 3, it follows that if f and g are differentiable functions, then $f-g$ is also a differentiable function (since $-g = (-1)g$ and $f-g = f+(-g)$), and $(f-g)'(x) = f'(x)-g'(x)$.

Let us see how Theorem 2 and Theorem 3 are is useful in the following example.

Example 16: Differentiate the following w.r.t. x.

i) $x^6+6x^5-4x^3+2x^2+10$

ii) $12x-\sqrt{x}+\frac{5}{x}$

Solution: i) $\frac{d}{dx}(x^6+6x^5-4x^3+2x^2+10)$

$=\frac{d}{dx}(x^6)+\frac{d}{dx}(6x^5)+\frac{d}{dx}(-4x^3)+\frac{d}{dx}(2x^2)+10$ [applying Theorem 3]

$=6x^5+6\frac{d}{dx}(x^5)+(-4)\frac{d}{dx}(x^3)+2\frac{d}{dx}(x^2)+\frac{d}{dx}(10)$ [applying Theorem 2]

$=6x^5+6(5x^4)-4(3x^2)+2(2x)+0$

$=6x^5+30x^4-12x^2+4x$

Thus, $\frac{d}{dx}(x^6+6x^5-4x^3+2x^2+10)=6x^5+30x^4-12x^2+4x$.

ii) $\frac{d}{dx}\left(12x-\sqrt{x}+\frac{5}{x}\right)=12.\frac{d}{dx}(x)-\frac{d}{dx}(x^{1/2})+5\frac{d}{dx}\left(\frac{1}{x}\right)$

$=12.1-\frac{1}{2}x^{\frac{1}{2}-1}+5(-1)x^{-1-1}$

$=12-\frac{1}{2\sqrt{x}}-\frac{5}{x^2}$

Thus, $\frac{d}{dx}\left(12x-\sqrt{x}+\frac{5}{x}\right)=12-\frac{1}{2\sqrt{x}}-\frac{5}{x^2}$.

***

You are now in a position to solve these exercises.

E18) Differentiate the following with respect to x.

i) $5x^3-2x+6$.

ii) $a_0+a_1x+a_2x^2+\cdots+a_nx^n$, where $a_1\in\mathbb{R}$ for $i=1, 2, \ldots, n$.

E19) Find the points on the curve $y=x^4-8x^2+4$ where the tangent line is horizontal.

Now derivative of product of two functions is in the following theorem.

Theorem 4 (The Product Rule): The product of two differentiable functions is again a differentiable function and its derivative at any point x is given by the formula

$$\frac{d}{dx}[f(x)g(x)]=f'(x)g(x)+f(x)g'(x)$$

Proof: Let f and g be two differentiable functions on $\mathbb{R}$. We want to find out whether their product fg is also differentiable.

$$\frac{d}{dx}[f(x).g(x)]=\lim_{h\to0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to0}\frac{\{f(x+h)-f(x)\}g(x+h)+\{g(x+h)-g(x)\}f(x)}{h}$$

[We have added and subtracted $f(x)g(x+h)$]

$$=\lim_{h\to0}\left\{\left(\frac{f(x+h)-f(x)}{h}\right)g(x+h)\right\}+\lim_{h\to0}\left\{\left(\frac{g(x+h)-g(x)}{h}\right)f(x)\right\}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)\lim_{h\to 0} g(x+h) + \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)\lim_{h\to 0} f(x)$$

$$= \frac{df(x)}{dx} g(x) + \frac{d\,g(x)}{dx} f(x)$$

$$= f'(x)g(x) + g'(x)f(x)$$

Therefore, $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$. ■

We can extend this result to the product of three differentiable functions. This gives us $(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

You see, you have to differentiate only one function at a time. This result can also be extended to the product of any finite number of differentiable functions. Thus, if $f_1, \ldots, f_n$ are differentiable functions, then

$$(f_1 f_2 \ldots f_n)'(x) = f_1'(x) f_2(x) \ldots f_n(x) + f_1(x) f_2'(x) f_3(x) \ldots f_n(x) + \cdots + f_1(x) f_2(x) \ldots f_n'(x)$$

Caution: Unlike the sum rule of differentiation, in the product rule $[f(x)g(x)]' \neq f'(x)g'(x)$. It is different.

Theorem 4 is very useful in simplifying calculations, as you can see in the following example.

Example 17: Differentiate the following with respect to x.

i) $x^2(x+4)$

ii) $(x^2+2x-9)(3x^3-\sqrt{x})$

Solution: i) We take $f(x) = x^2$ and $g(x) = x+4$. We differentiate each of these, obtaining $f'(x) = \frac{d}{dx}(x^2) = 2x$ and

$g'(x) = \frac{d}{dx}(x+4) = \frac{d(x)}{dx} + \frac{d(4)}{dx} = 1+0 = 1$.

By the product rule, the derivative of the given function is the

$$\frac{d}{dx}[f(x).g(x)] = f(x).g'(x) + g'(x).f'(x)$$

$$\therefore \frac{d}{dx}(x^2(x+4)) = 2x(x+4) + 1\times x^2$$

$$= 2x^2 + 8x + x^2$$

$$= 3x^2 + 8x.$$

ii) $$\frac{d}{dx}[(x^2+2x-7)(3x^3-\sqrt{x})] = (x^2+2x-7).\frac{d}{dx}(3x^3-\sqrt{x}) + (3x^3-\sqrt{x}).\frac{d}{dx}(x^2+2x-7)$$

$$= (x^2+2x-7).\left(9x^2 - \frac{1}{2\sqrt{x}}\right) + (3x^3-\sqrt{x}).(2x+2)$$

***

Remark 4: You could also have differentiated $x^2(x+4)$ without using Theorem 4, as follows:

$$x^2(x+4) = x^3 + 4x^2$$

Therefore, $\frac{d}{dx}(x^2(x+4)) = \frac{d}{dx}(x^3 + 4x^2)$

$$= \frac{d}{dx}(x^3) + \frac{d}{dx}(4x^2) \text{ [By Theorem 3]}$$

$$= 3x^2 + 4(2x) = 3x^2 + 8x$$

This shows that the same function can be differentiated by using different methods. You may use any method that you find convenient. This observation should also help you to check the correctness of your result. (We assume that you would not make the same mistake while using two different methods!)

***

Example 18: Differentiate the function f defined by $f(t) = \sqrt{t}(2+3t), t > 0$ w.r.t. t.

Solution: $\frac{d}{dt} f(t) = \frac{d}{dt}\left[\sqrt{t}\,(2+3t)\right]$

$$= \frac{d}{dt}(2\sqrt{t} + 3t^{3/2})$$

$$= \frac{2}{2\sqrt{t}} + \frac{3\sqrt{t}}{3/2}$$

$$= \frac{1}{\sqrt{t}} + 2\sqrt{t}.$$

***

You may try following exercises.

E20) Using Theorem 4, differentiate the following functions. Also, differentiate these functions without using Theorem 4, and compare the results.

i) $x\sqrt{x}$.

ii) $(x^5 + 2x^3 + 5)^2$

iii) $(x+1)(x+2)(x+3)$.

E21) If $f(x) = x\,g(x)$ and it is known that $g(3) = 2$ and $g'(3) = 5$, find $f'(3)$.

We shall discuss derivative of the quotient of two functions in the following theorem.

Theorem 5 (The Quotient of Rule): The quotient f/g of two differentiable functions f and g such that $g(x) \neq 0$, for any x in its domain, is again a differentiable function and its derivative at any point x is given by $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

This can also be written as

$$\frac{d}{dx}\left(\frac{\text{Numerator}}{\text{Denominator}}\right)=$$

$$\frac{(\text{Denominator})\ (\text{derivative of Numerator})-(\text{Numerator})\ (\text{derivative of Denominator})}{(\text{Denominator})^2}$$

Proof: Let $\phi=f/g$, where f and g are differentiable functions on $\mathbb{R}$, and $g(x)\neq 0$. Then,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{d\phi(x)}{dx}=\lim_{h\to 0}\frac{\phi(x+h)-\phi(h)}{h}=\lim_{h\to 0}\frac{(f/g)(x+h)-(f/g)(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}\right]$$

$$=\lim_{h\to 0}\frac{g(x)f(x+h)-f(x)g(x+h)}{h\big(g(x)g(x+h)\big)}$$

$$=\lim_{h\to 0}\frac{\left[g(x)\left\{\frac{f(x+h)-f(x)}{h}\right\}-f(x)\left\{\frac{g(x+h)-g(x)}{h}\right\}\right]}{g(x+h)\ g(x)}$$ [by adding and subtracting $f(x)g(x)$ in the numerator]

$$=\frac{\lim\limits_{h\to 0}\left[\left\{g(x)\frac{f(x+h)-f(x)}{h}\right\}-f(x)\left\{\frac{g(x+h)-g(x)}{h}\right\}\right]}{\lim\limits_{h\to 0}g(x+h)\ g(x)}$$

$$=\frac{\lim\limits_{h\to 0}\left[g(x)\left\{\frac{f(x+h)-f(x)}{h}\right\}\right]-\lim\limits_{h\to 0}\left[f(x)\left\{\frac{g(x+h)-g(x)}{h}\right\}\right]}{\lim\limits_{h\to 0}g(x+h)\lim\limits_{h\to 0}g(x)}$$

$$=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

Thus, $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{\big(g(x)\big)^2}$, if $g(x)\neq 0$. ■

We will obtain an important corollary to Theorem 5 now.

Corollary 1: If g is a function such that $g(x)\neq 0$ for any x in its domain, then

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right)'=\frac{-g'(x)}{(g(x))^2}.$$

Proof: In the result of Theorem 5, take f to be the constant function 1. Then $f'(x)=0$ for all x.

Therefore,

$$\left(\frac{1}{g(x)}\right)'=\frac{g(x)\times 0-1\times g'(x)}{(g(x))^2}\ \text{as}\left(\left(\frac{f(x)}{g(x)}\right)'=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}\ \text{where}\ f(x)=1\right)$$

$$=\frac{-g'(x)}{(g(x))^2}.$$ ■

As we said earlier that the derivative of x^n can be found for any real n. In the next example, we will find the derivative of x^n for any negative integer n using the quotient rule.

Example 19: Show that $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is a negative integer and $x \neq 0$.

Solution: Consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = x^{-m}$, where $m \in \mathbb{N}$. Then $f(x) = 1/x^m \forall x \in \mathbb{R}$. Thus, $f = 1/g$, where $g(x) = x^m$ for all $x \in \mathbb{R}, x \neq 0$. g is a differentiable function and $g(x) \neq 0$ if $x \neq 0$. So, except at $x = 0$, we find that

$$f'(x) = \frac{-g'(x)}{\{g(x)\}^2} \text{ (from Corollary 1)}$$

$$= \frac{-mx^{m-1}}{(x^m)^2} [g'(x) = mx^{m-1}]$$

$$= \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1}$$

Denoting $-m$ by n, we get $f(x) = x^n$, and $f'(x) = nx^{n-1}$.

***

In the next example, we will apply the quotient rule to find the derivative of a rational function.

Example 20: Differentiate the function f given by $f(x) = (x^3 + 2)/(x^2 + 2x)$ w.r.t. x.

Solution: We can write f as the quotient g/h where $g(x) = (x^3 + 2)$ and $h(x) = x^2 + 2x$. Here $h(x) = 0,$ when $x = 0, -2$. Thus, the function f is differentiable except $x = 0, -2$.

Now, $g'(x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(2) = 3x^2 + 0 = 3x^2$.

Also, $h'(x) = 2x + 2$.

Therefore, $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$

$$= \frac{(x^2 + 2x)(3x^2) - (x^3 + 2)(2x + 2)}{(x^2 + 2x)^2}$$

$$= \frac{3x^4 + 6x^3 - 2x^4 - 2x^3 - 4x - 4}{(x^2 + 2x)^2}$$

$$= \frac{x^4 + 4x^3 - 4x - 4}{(x^2 + 2x)^2}, \text{ if } x \neq 0, -2.$$

***

Caution: You should not use the quotient rule every time, wherever you see a quotient. Sometimes it is easier to rewrite a quotient first to put it in a form that is simpler for the purpose of differentiation. For example, $f(x) = \frac{2x^3 + 3x}{\sqrt{x}}$ can be rewritten as $f(x) = 2x^{5/2} + 3x^{1/2}$, and is easier to differentiate the later form of $f(x)$.

Example 21: The population P, in thousand, of a small city is given by

$P(t)=\frac{500t}{2t^2+9}$, where t is the time, in years.

i) Find the growth rate.

ii) Find the population after 12 years.

iii) Find the growth rate at $t=12$ yrs.

Solution: i) The growth rate $P'(t)=\frac{4500-1000t^2}{(2t^2+9)^2}$.

ii) $P(12)=\frac{6000}{288+9}=\frac{6000}{297}\approx 22$

iii) $P'(12)=\frac{4500-14000}{(297)^2}=-\frac{139500}{88209}\approx -1.6$

The population function is given in Fig. 20.

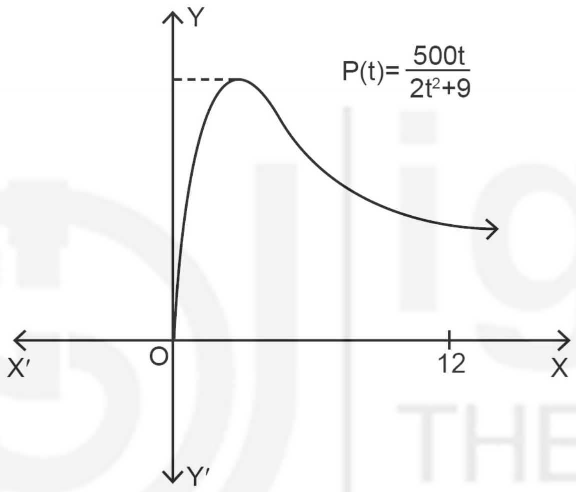

Fig. 20

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Now, try the following exercises.

E22) Differentiate following w.r.t. x.

i) $\frac{2x+1}{x+5}$, if $x\neq 5$

ii) $\frac{1}{a+bx+cx^2+dx^3}$ where a, b, c, d are fixed real numbers, if $a+bx+cx^2+dx^3\neq 0$.

iii) $\frac{2x^3+3x^2}{x^4-1}$, if $x^4-1\neq 0$.

E23) Obtain the derivative of $1/f(x)$ by differentiating from first principles, assuming that $f(x)\neq 0$, for any x.

E24) Differentiate $f(x)=\frac{2+5x+7x^{-1}}{x^5}$ by three different methods.

Sometimes we are asked to differentiate the functions for which we cannot apply any rule of differentiation directly. For example, if $f(x)=\sqrt{x+1}$, we cannot find the derivative of $f(x)$ directly using any rule. But if you observe $f(x)$, you will find that it is a composite function [Recall Unit 2 for composite functions]. That is $g(t)=\sqrt{t}$ and $h(x)=x+1$. We can also write $f=g \circ h$. If the functions g and h are differentiable and we can find their derivatives, then using the chain rule, we can find the derivative of f. Therefore, the chain rule of differentiation is a rule for differentiating composite functions; it is a remarkable rule which helps us to differentiate complicated functions in an easy and elegant way. In the following section, we will discuss the chain rule.

9.6 THE CHAIN RULE

We establish the rule in the following theorem.

Theorem 6 (The Chain Rule): Let $y=g(u)$ and $u=f(x)$. If both dy/du and du/dx exist, then $\frac{dy}{dx}$ exists and is given by $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$.

Proof: We first note that $y=g(u)=g(f(x))=(gof)\ (x)$, so that y is the composite function gof. We are given that y, regarded as a function of u and is differentiable w.r.t. u. We want to prove that y, regarded as a function of x, is also differentiable. To do this, we must show that $\lim\limits_{\delta x\to 0} \delta y/\delta x$ exists, where δy is the change in the variable y corresponding to a change δx in the variable x.

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$
$$=\lim_{\delta x\to 0}\frac{\delta y}{\delta u}\cdot\frac{\delta u}{\delta x}$$
$$=\lim_{\delta x\to 0}\frac{\delta y}{\delta u}\cdot\lim_{\delta x\to 0}\frac{\delta u}{\delta x}$$
$$=\lim_{\delta u\to 0}\frac{\delta y}{\delta u}\cdot\lim_{\delta x\to 0}\frac{\delta u}{\delta x} \qquad \left[\because \lim_{\delta x\to 0}\delta u=\lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x}\delta x\right)=\lim_{\delta x\to 0}\frac{\delta u}{\delta x}\times\lim_{\delta x\to 0}\delta x\right.$$
$$\left.=\frac{du}{dx}\times 0=0\text{. This means that }\delta u\to 0\text{ as }\delta x\to 0.\right]$$
$$=\frac{dy}{du}\cdot\frac{du}{dx}$$

Therefore,

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

■

Hence dy/dx exists and is equal to $\frac{dy}{du}\times\frac{du}{dx}$, that is the product of derivative of y w.r.t. u and of u w.r.t. x.

You may find it more convenient to remember and use the rule in the following form:

If $h(x)=g(f(x))$ is the composite of two differentiable functions g and f, then, h is differentiable and $h'(x)=g'(f(x))f'(x)$.

To clarify this rule let us try the following example.

Example 21: Differentiate $y=(2x+1)^3$ with respect to x.

Solution: Let $u=2x+1$. Then $y=(2x+1)^3=u^3$.
Now y is a differentiable function of u and u is a differentiable function of x.

Here, $\frac{dy}{du}=3u^2$ and $\frac{du}{dx}=2$. Hence, we can use the chain rule to get

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=3u^2.2=6u^2=6(2x+1)^2.$$

***

You might be thinking that there was really no necessity of using the chain rule here. We could simply expand $(2x+1)^3$ and then write the derivative. But the situation is not always as simple as in this example. You would appreciate the power of the chain rule after using it in the next example.

Example 22: Differentiate $(x^3+2x^2-1)^{100}$ with respect to x.

Solution: Let $y=(x^3+2x^2-1)^{100}$ and let $u=(x^3+2x^2-1)$. Then, $y=u^{100}$.
Since $\frac{dy}{du}$ and $\frac{du}{dx}$ both exist, and $\frac{dy}{du}=100u^{99}$ and $\frac{du}{dx}=3x^2+4x$, therefore, by chain rule,

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}.\frac{du}{dx}\\ &=100u^{99}.(3x^2+4x)\\ &=100(x^3+2x^2-1)^{99}(3x^2+4x).\end{aligned}$$

***

Our next example illustrates that this rule can be extended to composition of three functions.

Example 23: Differentiate $\{(5x+2)^{10}+3\}^4$ with respect to x.

Solution: We write $y=\{(5x+2)^{10}+3\}^4$, $u=(5x+2)^{10}+3$ and $v=5x+2$.
Then $y=u^4$ and $u=v^{10}+3$. That is, y is a function of u, u is a function of v, and v is function of x. By extending the chain rule, we get

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

This gives,

$$\begin{aligned}\frac{dy}{dx}&=(4u^3)(10v^9).(5)=200u^3v^9\\ &=200[(5x+2)^{10}+3]^3(5x+2)^9\end{aligned}$$

***

This example illustrates that there may by situations in which we may go on using chain rule for a function of a function of a function…, and so on. This perhaps justifies the name ‘chain’ rule. Thus, if $g_1, \ldots, g_n$ and h are functions such that $h=(g_1og_2o\ldots og_n)(x)$, then

$$h'(x)=g_1'(g_2o\ldots og_n)(x))g_2'(g_3o\ldots og_n)(x)\ldots g_{n-1}'(g_n)(x).g_n'(x)$$

Example 24: A spherical balloon is filled with air. At any time t, the volume of the balloon is $V(t)$ and its radius is $r(t)$.

i) What do the derivatives $\frac{dV}{dr}$ and $\frac{dV}{dt}$ represent?

ii) Express $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$.

Solution: i) $\frac{dV}{dr}$ is the rate of change of volume with respect to radius and $\frac{dV}{dt}$ is the rate of change of volume with respect to time.

ii) We know that the volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.

Differentiating V with respect to t, we get

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$ (using the chain rule)

$$= \frac{d\left(\frac{4}{3}\pi r^3\right)}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

***

Now, try the following exercises.

E25) Find dy/dx for each of the following using the chain rule:

i) $\frac{5}{1+5x+7x^2}$.

ii) $\frac{(2x+3)^2}{1+(2x+3)^3}$

iii) $\{(9x+5)^3 + (9x+5)^{-3}\}^7$.

E26) Find $f'(x)$, if $f(x) = \sqrt{1+x}$.

E27) A particle moves along a straight line with displacement $s(t)$ and velocity $v(t)$. Explain the difference between the meanings of the derivatives $\frac{dv}{dt}$ and $\frac{dv}{ds}$. Also, express $\frac{dv}{dt}$ in terms of $\frac{dv}{ds}$.

Let us find derivatives of the trigonometric function in the following section.

9.7 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

In this section, we shall calculate the derivatives of the six trigonometric functions: $\sin x, \cos x, \tan x, \cot x, \sec x$ and $\operatorname{cosec} x$. You already know that these six functions are related to each other. For example, we have: i) $\sin^2 x + \cos^2 x = 1$ ii) $\tan x = \sin x / \cos x$, and many more identities which express the relationships between these functions. As you will soon see, our job of finding the derivatives of all trigonometric functions becomes a lot easier because of these identities.

We shall now find out the derivative of $\sin x$ from the first principles. If $y = f(x) = \sin x$, then by definition

$$f'(x) = \frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{2\sin(h/2)\cos(x+h/2)}{h} \quad \left[\text{Since } \sin A - \sin B = 2\sin\frac{(A-B)}{2}\cos\frac{(A+B)}{2}\right]$$

$$= \lim_{h\to 0}\frac{\sin(h/2)}{h/2}\lim_{h\to 0}\cos(x+h/2)$$

$$= 1\times\cos x = \cos x \qquad \text{[Recall Example 25 of Unit 7, } \lim_{x\to 0}\frac{\sin x}{x} = 1\text{]}$$

Thus, we get

$$\frac{d}{dx}(\sin x) = \cos x$$

Now, let us consider the cosine function given by $y = f(x) = \cos x$ and find its derivative. In this case,

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin(h/2)\sin(x+h/2)}{h} \quad \left[\text{Since } \cos A - \cos B = 2\sin\frac{B-A}{2}\sin\frac{A+B}{2}\right]$$

$$= -\lim_{h\to 0}\frac{\sin(h/2)}{h/2}\lim_{h\to 0}\sin(x+h/2)$$

$$= -1\cdot\sin x = -\sin x$$

Thus, we get $\frac{d}{dx}(\cos x) = -\sin x$

Actually, having first calculated $\frac{d}{dx}(\sin x)$, we could also have found out the derivative of $\cos x$ by using the formulae $\cos x = \sin(x+\pi/2)$. This gives us,

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}(\sin(x+\pi/2))$$

$$= \frac{d}{dt}(\sin t)\times\frac{dt}{dx} \text{ where } t = x+\pi/2 \quad \text{[using chain rule]}$$

$$= \cos t\times 1 = \cos t = \cos(x+\pi/2) = -\sin x.$$

Let us solve a few examples.

Example 25: Find $\frac{dy}{dx}$ for the following

i) $y = \sin 2x$,

ii) $y = \cos^2 x$,

iii) $y = 5\sin^7 x \sin 3x$,

iv) $y = \sin\left(1/x^2\right)$.

Solution: i) $\frac{dy}{dx} = \frac{d}{dx}(\sin 2x) = \frac{d}{dt}(\sin t)\times\frac{dt}{dx}$ where $t = 2x$

$$= \cos t\times\frac{d}{dx}(2x) = 2\cos 2x.$$

ii) $\frac{dy}{dx} = \frac{d}{dx}(\cos^2 x)$

$$= \frac{dt^2}{dt} \times \frac{dt}{dx} \quad \text{where } t = \cos x$$

$$= 2t \times \frac{d}{dx}(\cos x) = -2\cos x \sin x$$

iii) $\frac{dy}{dx} = \frac{d}{dx}(5\sin^7 x \ \sin 3x)$

$$= 5\left[\frac{d}{dx}(\sin^7 x) \times \sin 3x + \sin^7 x \times \frac{d}{dx}(\sin 3x)\right]$$

$$= 5\left[\frac{dt^7}{dt} \times \frac{dt}{dx} \times \sin 3x + \sin^7 x \times \frac{d}{du}(\sin u) \times \frac{du}{dx}\right]$$

where $t = \sin x$ and $u = 3x$

$$= 5[7\sin^6 x \times \cos x \times \sin 3x + \sin^7 x \times \cos 3x \times 3]$$

$$= 35\sin^6 x \cos x \sin 3x + 15\sin^7 x \cos 3x.$$

iv) $\frac{dy}{dx} = \frac{d(\sin t)}{dt} \cdot \frac{dt}{dx}$ where $t = \frac{1}{x^2}$

$$= \cos t \cdot \frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$= \frac{-2\cos(1/x^2)}{x^3}.$$

***

Before, we find the derivatives of the other four trigonometric functions by using a similar formula, it is time to do some exercises.

E28) Find dy/dx of the following:

i) $y = x^3 \cos 9x$ ii) $y = \cos(\sin x)$

iii) $y = \sin(x^2 + 1)$ iv) $y = \sin\sqrt{1 + \cos x}$

Let us now find the derivatives of other four trigonometric functions

i) Suppose $f(x) = \tan x$, where $x \in]-\pi/2, \pi/2[$. We know that

$\tan x = \frac{\sin x}{\cos x}$

Then $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$ and $\cos x \neq 0$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$ [quotient rule of derivatives]

$$= \frac{\cos x.\cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Hence, $\frac{d}{dx}(\tan x) = \sec^2 x$.

ii) Now, suppose $y = f(x) = \cot x$. Since $\cot x = 1/\tan x$, we get

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = \frac{\tan x . \frac{d}{dx}(1) - 1 . \frac{d}{dx}(\tan x)}{\tan^2 x}$$

$$= \frac{\tan x . 0 - 1 . \sec^2 x}{\tan^2 x} = \frac{-\sec^2 x}{\tan^2 x} = -\operatorname{cosec}^2 x$$

Thus, $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

iii) Now, let $y = f(x) = \sec x$. Since, we know that $\sec x = 1/\cos x$, proceeding as in ii), we get

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cos x . \frac{d}{dx}(1) - 1 . \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x . 0 - 1 . (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

Thus, $\frac{d}{dx}(\sec x) = \sec x \tan x$.

iv) Now, let $y = f(x) = \operatorname{cosec} x$. Since $\operatorname{cosec} x = 1/\sin x$, we get

$$\frac{d(\operatorname{cosec} x)}{dx} = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{\sin x . \frac{d}{dx}(1) - 1 . \frac{d}{dx}(\sin x)}{\sin^2 x} = \frac{\sin x . 0 - 1 . \cos x}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin^2 x} = -\operatorname{cosec} x \cot x$$

Thus, $\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x$

When you memorise the derivatives of the trigonometric functions, you may notice that the minus signs go with the derivatives of the 'co-functions' that is cosine, cosecant and cotangent.

Let us summarise our results in Table 1.

Table 1

Function	Their Derivatives
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Remark 5: Here again we note that the angle is measured in radians. Thus,

$$\frac{d}{dx}(\sin x^{\circ}) = \frac{d}{dx}\left(\sin \frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos x^{\circ}$$

We shall now see how we can use these results to find the derivatives of some more complicated functions. The chain rule and the algebra of derivatives with which you must have become quite familiar by now, will come in handy again.

Example 26: Differentiate following with respect to x :

i) $\sec^3 x$ ii) $\sec x \tan x + \cot x$, iii) $\sqrt{x^3 + \operatorname{cosec} x}$

Solution: i) Let $y = \sec^3 x$. If we write $u = \sec x$, we get $y = u^3$. Thus,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
$$= 3u^2 \sec x \tan x$$
$$= 3\sec^3 x \tan x$$

ii) If $y = \sec x \tan x + \cot x$, then

$$\frac{dy}{dx} = \frac{d}{dx}(\sec x \tan x) + \frac{d}{dx}(\cot x)$$
$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) - \operatorname{cosec}^2 x$$
$$= \sec x(\sec^2 x + \tan^2 x) - \operatorname{cosec}^2 x .$$

iii) $$\frac{d\sqrt{x^3 + \operatorname{cosec} x}}{dx}$$

$$= \frac{d\sqrt{u}}{du} \cdot \frac{du}{dx} \text{ where } u = x^3 + \operatorname{cosec} x$$
$$= \frac{1}{2\sqrt{u}} \cdot \frac{d(x^3 + \operatorname{cosec} x)}{dx}$$
$$= \frac{1}{2\sqrt{x^3 + \operatorname{cosec} x}} \cdot (3x^2 - \operatorname{cosec} x \cot x)$$
$$= \frac{3x^2 - \operatorname{cosec} x \cot x}{2\sqrt{x^3 + \operatorname{cosec} x}}$$

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Example 27: A mass on a spring vibrates horizontally on a smooth level surface as shown in Fig. 21. Its equation of motion is $x(t) = 8\sin t$ where t is in seconds and x in centimetres.

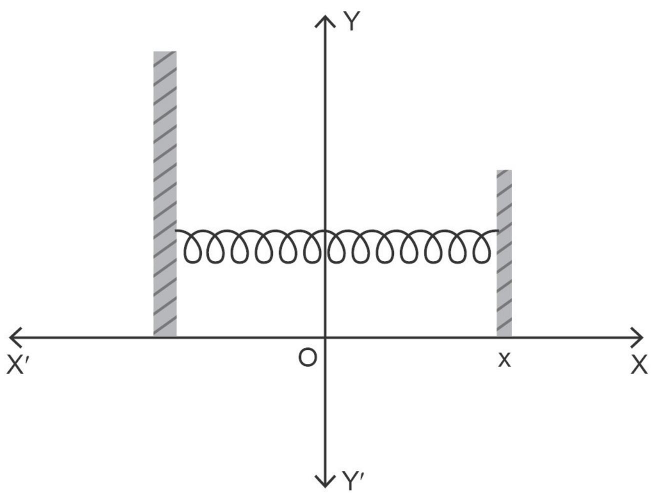

Fig. 21

i) Find the velocity at time t.

ii) Find the position and velocity of the mass at time $t = \frac{2\pi}{3}$. In which direction is it moving at that time?

Solution: i) Given is $x(t) = 8\sin t$

The velocity at time $t = \frac{dx(t)}{dt}$

$$= \frac{d(8\sin t)}{dt} = 8\cos t$$

ii) The position at time $\frac{2\pi}{3} = x\left(\frac{2\pi}{3}\right)$

$$= 8\sin\left(\frac{2\pi}{3}\right)$$

$$= 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3} \text{ cm.}$$

The velocity at time $\frac{2\pi}{3} = 8\cos\left(\frac{2\pi}{3}\right)$

$$= 8\left(\frac{-1}{2}\right) = -4 \text{ cm/s.}$$

It is moving in the left direction as $x'(t) < 0$.

***

Remark 6: The functions $\sin x, \cos x, \sec x, \operatorname{cosec} x$ are periodic functions with period 2π. Their derivatives are also periodic with period 2π. $\tan x$ and $\cot x$ are periodic with period π. Their derivatives are also periodic with period π.

We have been considering variables which are dimensionless. Actually, in practice, we may have to consider variables having dimensions of mass, length, time etc., and we have to be careful in interpreting their derivatives. Thus, we may be given that the distance x travelled by a particle in time t is $x = a\cos bt$. Here, since bt is dimensionless (being an angle), b must have the dimension $\frac{1}{T}$. Similarly, $x/a = \cos bt$ has to be dimensionless. This means that a must have the same dimension as x. That is dimension of a is L.

Now $\frac{dx}{dt} = -ab\sin bt$ has the dimension of $ab = L \times 1/T = L/T$, which is not unexpected, since $\frac{dx}{dt}$ is nothing but the velocity of that particle.

See if you can do these exercises now.

E29) Find the derivatives of the following with respect to x.

i) $\operatorname{cosec} 2x$ ii) $\cot x + \sqrt{\operatorname{cosec} x}$

iii) $5\cot 9x$ iv) $(1 + x^5 \cot x)^{-8}$

E30) An object with mass m is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is $F = \frac{\mu mg}{\mu \sin\theta + \cos\theta}$, where μ is a constant called coefficient of friction.

i) Find the rate of change of F w.r.t. θ.

ii) When is the rate of change equal to 0?

E31) Use the chain rule to show that if θ is measured in degrees, then $\frac{d}{d\theta}(\sin\theta) = \frac{\pi}{180^o}\cos\theta$.

Now, in the following section, we shall find the derivatives of inverse trigonometric functions.

9.8 DERIVATIVES OF INVERSE FUNCTIONS

Recall Unit 6, wherein we discussed that the graphs of a function and its inverse are very closely related to each other. If we are given the graph of a function, we have only to take its reflection in the line $y = x$, to obtain the graph of its inverse. In this section, we shall establish a relation between the derivatives of a function and its inverse.

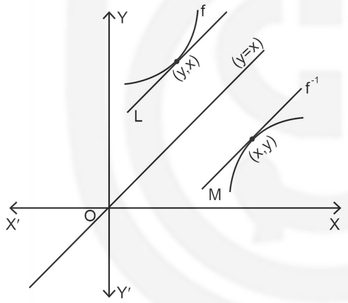

Fig. 22

Consider a function f defined by $f(x) = 5x + 3$. You may check that this function is one-one and onto and hence its inverse exists and is $f^{-1}(x) = \frac{x-3}{5}$. Both f and f^{-1} are linear functions. You may note that the slope of the graph of f is 5 and the slope of the graph of f^{-1} is $1/5$. This is because, when we reflect across $y = x$, we take the reciprocal of the slope. This geometrical observation gives the differentiation formula for inverse functions. From Fig. 22, we have

$$\frac{d}{dx}(f^{-1}(x)) = \text{Slope of } M$$

$$= \frac{1}{\text{Slope of } L}$$

$$= \frac{1}{f'(y)}$$

$$= \frac{1}{f'(f^{-1}(x))}$$

So, we have been able to find some relation between the derivatives of these inverse functions. Let us state our results more precisely.

Theorem 7 (The Inverse Function Theorem) : Let f be differentiable and strictly monotonic on an interval I. If $f'(x) \neq 0$ at a certain x in I, then f^{-1} is differentiable at $y = f(x)$ and

$$(f^{-1})'(y) = 1/f'(x) \text{ or } \frac{d}{dy}[f^{-1}(y)] = \frac{1}{f'[f^{-1}(x)]}$$

Thus, the inverse function rule gives

$$(f^{-1})'(y) = \frac{1}{f'(x)}, \text{ or } \frac{d}{dy}(f^{-1})(y) = \frac{1}{f'(x)} \text{ or } \frac{dx}{dy} = \frac{1}{dy/dx} \quad [\text{as } f^{-1}(y) = x]$$

The derivative of the inverse function is the reciprocal of the derivative of the given function.

Soon we shall see that this rule is very useful if we want to find the derivative of a function when the derivative of its inverse function is already known. This will become clear when we consider the derivatives of the inverses of some standard functions. But first, let us use this rule to find the derivative of $f(x) = x^r$, where r is a rational number.

Example 28: If $y = f(x) = x^r$, where r is a rational number for which x^r and x^{r-1} are both defined, then show that $\frac{d}{dx}(x^r) = rx^{r-1}$.

x^r may not be always defined. For example, if $x = -1$ and $r = 1/2$, $x^r = \sqrt{-1}$ is not defined in $\mathbb{R}$.

Solution: Let us first consider the case when $r = 1/q$, where q is any non-zero integer. In this case, $y = f(x) = x^{1/q}$. Its inverse function g will be given by $x = g(y) = y^q$. This means

$$\frac{dx}{dy} = \frac{d}{dy}g(y) = g'(y) = qy^{q-1}$$

Thus, by the inverse function theorem, we get

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{qy^{q-1}}$$

$$= \frac{1}{q(x^{1/q})^{q-1}} \text{ [Since } y = x^{1/2}]$$

$$= \frac{1}{qx^{(q-1)/q}} = \frac{1}{q}x^{-(q-1)/q}$$

$$= \frac{1}{q}x^{(1/q)-1} = rx^{r-1}.$$

So far, we have seen that the theorem is true when r is of the form $1/q$, where q is a non-zero integer. Now, having proved this, let us take the general case when $r = p/q,\ p, q \in \mathbb{Z}(q$ is, of course non-zero). Here

$$y = f(x) = x^r = x^{p/q}$$

So, $\frac{dy}{dx} = \frac{d}{dx}(x^{p/q}) = \frac{d}{dx}(x^{1/q})^p$

Now, $\frac{d}{dx}(x^{1/q})^p = p(x^{1/q})^{p-1}\frac{d}{dx}(x^{1/q})$, [using chain rule]

$$= p(x^{1/q})^{p-1}(1/q)x^{(1/q)-1}$$

$$= (p/q)x^{(p/q)-1}$$

Thus, $\frac{dy}{dx} = \frac{d}{dx}(x^r) = (p/q)x^{(p/q)-1} = rx^{r-1}$

***

Example 29: Differentiate $y = (x^{5/6} + \sqrt{x})^{1/11}$ with respect to x.

Solution: We write $u = x^{5/6} + \sqrt{x}$. This gives us, $y = u^{1/11}$

By chain rule, we get

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{d}{du}(u^{1/11}).\frac{d}{dx}(x^{5/6}+x^{1/2})=\left(\frac{1}{11}u^{\frac{1}{11}-1}\right).\left(\frac{5}{6}x^{\frac{5}{6}-1}+\frac{1}{2}x^{\frac{-1}{2}}\right)$$

$$=\frac{1}{11}(x^{5/6}+\sqrt{x})^{(1/11)-1}\left(\frac{5}{6}x^{(5/6)-1}+\frac{1}{2}x^{-1/2}\right)$$

Thus,

$$\frac{dy}{dx}=\frac{1}{66}(x^{5/6}+\sqrt{x})^{-10/11}(5x^{-1/6}+3x^{-1/2}) .$$

***

Why don't you try these exercises now?

E32) Differentiate following with respect to x.

i) $5(x^3+x^{1/3})$ ii) $\left(\sqrt[5]{x}-\sqrt[9]{x}\right)x^2$

We have seen how the inverse function theorem helps us in finding the derivative of x^n, where n is a rational number. We shall now use this theorem to find the derivatives of inverse trigonometric functions.

We have noted in Unit 6, that sometimes when a given function is not one-one, we can still talk about its inverse, provide we restrict its domain suitably. Now, $\sin x$ is neither a one-one, nor an onto function from $\mathbb{R}$ to $\mathbb{R}$. But if we restrict its domain to $[-\pi/2, \pi/2]$, and co-domain to $[-1, 1]$, then it becomes a one-one and onto function, and hence the existence of its inverse is assured. In a similar manner, we can talk about the inverses of the remaining trigonometric functions if we place suitable restrictions on their domains and co-domains.

Now that we are sure of the existence of inverse trigonometric functions, let us go ahead and find their derivatives.

Let us consider the function $y=f(x)=\sin x$ in the domain $[-\pi/2, \pi/2]$. Its inverse is given by $g(y)=\sin^{-1}(y)=x$. We can see clearly that $\sin x$ is strictly increasing on $[-\pi/2, \pi/2]$ (Refer Unit 6).

We also know that the derivative $\frac{d}{dx}(\sin x)=\cos x$ exists and is non-zero for all $x\in]-\pi/2, \pi/2[$.

This means that $\sin x$ satisfies the conditions of the inverse function theorem. We can, therefore, conclude that $\sin^{-1} y$ is differentiable on $]-1, 1[$, and

$$\frac{d}{dy}(\sin^{-1}(y))=\frac{1}{f'(x)}=\frac{1}{\cos x}$$

$$=\frac{1}{\sqrt{1-y^2}} \quad \text{[Since } \cos x = y,\ \sin x=\sqrt{1-y^2} \text{ for } 0<x<\pi.]$$

Thus, we have the result $\frac{d}{dt}(\sin^{-1} t)=\frac{1}{\sqrt{1-t^2}}$.

Remember, $\sin^{-1} x$ is not the same as $(\sin x)^{-1}=1/\sin x$ or $\sin x^{-1}=\sin 1/x$.

We shall follow exactly the same steps to find out the derivative of the inverse cosine function.

Let's start with the function $y = f(x) = \cos x$, and restrict its domain to $[0, \pi]$ and its co-domain to $[-1, 1]$. Its inverse function $g(y) = \cos^{-1} y$ exists and the graphs of $\cos x$ and $\cos^{-1} x$ are continuous (Refer Unit 7).

As in the earlier case, we can now check that the conditions of the inverse function theorem are satisfied and conclude that $\cos^{-1} y$ is differentiable in $]-1, 1[$. Further

$$\frac{d}{dy}(g(y)) = \frac{d}{dy}(\cos^{-1} y) = \frac{1}{f'(x)} = \frac{1}{-\sin x}$$

$$= \frac{-1}{\sqrt{1-y^2}} \qquad [\text{Since } \cos x = y,\ \sin x = \sqrt{1-y^2} \text{ for } 0 < x < \pi.]$$

This gives us the result $\frac{d}{dt}(\cos^{-1} t) = \frac{-1}{\sqrt{1-t^2}}$.

Example 30: Find $\frac{dy}{dx}$, if

i) $y = \cos^{-1}(3x)$

ii) $y = \sin^{-1}(\sqrt{x^3})$

Solution: i) $\frac{dy}{dx} = \frac{d\cos^{-1}(u)}{du} \cdot \frac{du}{dx}$ where $u = 3x$

$$= \frac{-1}{\sqrt{1-4^2}} \times 3 = \frac{-1}{\sqrt{1-9x^2}} \times 3 = \frac{-3}{\sqrt{1-9x^2}}$$

ii) $\frac{dy}{dx} = \frac{d}{dt}(\sin^{-1}(t)) \cdot \frac{dt}{dx}$ where $t = \sqrt{x^3}$

$$= \frac{1}{\sqrt{1-t^2}} \cdot \frac{d}{dx}(x^{3/2})$$

$$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{3}{2}(x)^{1/2}$$

$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}.$$

***

You can apply these two results to get the derivatives in the following exercises.

E33) Differentiate following with respect to x.

i) $\sin^{-1}(5x)$ ii) $\cos^{-1}\sqrt{x}$

iii) $\sin x \cos^{-1}(x^3 + 2)$.

After finding the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$, we shall find the derivatives of $\tan^{-1} x$ and $\cot^{-1} x$.

Derivatives of $\tan^{-1}x$ and $\cot^{-1}x$

Consider the graph of $\tan x$ given in Fig. 23, let us indicate the interval to which the domain of $\tan x$ should be restricted so that the existence of its inverse is guaranteed.

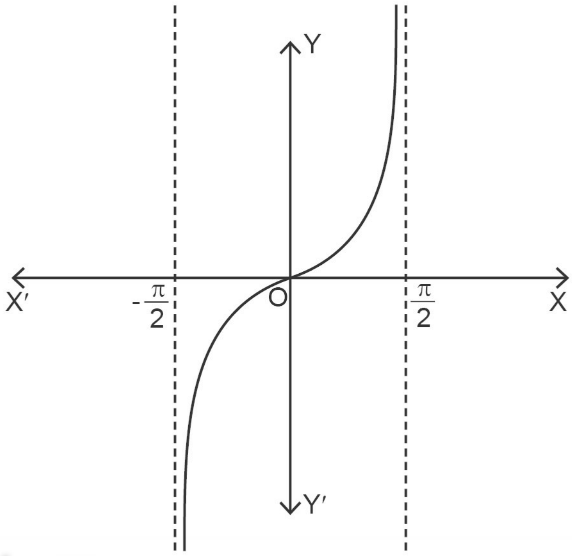

Fig. 23: Graph of $\tan x$

Fig. 23 shows that $\tan x$ restricted to $]-\pi/2, \pi/2[$ is strictly increasing one-one function of x. Thus, its inverse exists when restricted to $]-\pi/2, \pi/2[$.

The domain of $\tan^{-1} x$ is $]-\infty, \infty[$.

If $y = f(x) = \tan x$, then $\frac{d}{dy}(\tan^{-1} y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{1+y^2}$ $[\because \sec^2 x = 1 + \tan^2 x]$

Hence $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

You may check for the conditions of inverse function theorem and find the interval in which $\cot x$ has inverse.

$y = f(x) = \cot x$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{1}{f'(x)} = \frac{-1}{\operatorname{cosec}^2 x} = \frac{-1}{1+y^2}$$

Now, we shall find derivatives of two remaining inverse trigonometric functions.

Derivatives of $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$

Let's find the inverse of the remaining two trigonometric functions now.

If $y = \sec^{-1} x$, then $\sec y = x$ or $1/\cos y = x$, which means that $1/x = \cos y$.

This gives us $y = \cos^{-1}(1/x)$, where $|x| \geq 1$.

We have seen that $\cos^{-1} t$ is defined in the interval $[-1, 1]$.

Thus, $y = \sec^{-1} x = \cos^{-1}(1/x), |x| \geq 1$

From this we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1}(1/x))$$

$$= \frac{-1}{\sqrt{1-1/x^2}} \frac{d}{dx}(1/x) \quad \text{[using chain rule]}$$

$$= \frac{-|x|}{\sqrt{x^2-1}}(-1/x^2)$$

$$= \frac{1}{|x|\sqrt{x^2-1}}, |x|>1$$

Thus, we have

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x|>1$$

Note that although $\sec^{-1} x$ is defined for $|x| \geq 1$, the derivative of $\sec^{-1} x$ does not exist when $x = 1$.

Following exactly similar steps, we can find the derivative of $\operatorname{cosec}^{-1}x$.

$y = \operatorname{cosec}^{-1}x \Rightarrow \operatorname{cosec} y = x \Rightarrow \sin y = 1/x \Rightarrow y = \sin^{-1}(1/x)$ where $|x| \geq 1$.

Thus, $\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}(1/x))$

$$= \frac{1}{\sqrt{1-1/x^2}}(-1/x^2) = \frac{|x|}{\sqrt{x^2-1}}(-1/x^2)$$

$$= \frac{-1}{|x|\sqrt{x^2-1}}, |x|>1$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}, |x|>1.$$

Let us now apply these in the following example.

Example 31: Find the derivative of $y = \sec^{-1} 2\sqrt{x}$ with respect to x.

Solution: $\frac{dy}{dx} = \frac{d}{dx}(\sec^{-1} 2\sqrt{x})$

$$= \frac{d}{dt}(\sec^{-1} t) \times \frac{dt}{dx} \quad \text{where } t = 2\sqrt{x} \text{ [using chain rule]}$$

$$= \frac{1}{|t|\sqrt{t^2-1}} \times \frac{dt}{dx} = \frac{1}{2\sqrt{x}\sqrt{4x-1}} \frac{d}{dx}\left(2\sqrt{x}\right)$$

$$= \frac{1}{2\sqrt{x}\ \sqrt{4x-1}} \times \frac{1}{\sqrt{x}} = \frac{1}{2x\sqrt{4x-1}}$$

***

Now, you will be able to solve these exercises using the results about the derivatives of inverse trigonometric functions.

E34) Differentiate following with respect to x.

i) $\cot^{-1}(x/2)$ ii) $\frac{\cot^{-1}(x+1)}{\tan^{-1}(x+1)}$

iii) $\cos^{-1}(5x+4)$ iv) $\sec^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right)$, θ is a constant.

v) $\operatorname{cosec}^{-1}(x+1) + \sec^{-1}(x-1)$

We conclude this unit by summarising what we have covered in it. We have discussed following points.

9.9 SUMMARY

We have discussed followings points in this unit.

1. For any function $y = f(x)$, $f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$ (if it exists) is called the **derivative** of f at x, denoted by $f'(x)$. The function f' is the derived function. The derivative $f'(x)$ is the slope of the tangent to the curve $y = f(x)$ at the point (x, y). The derivative also gives the average rate of change of the function with respect to the independent variable.

2. There are three steps in calculating a derivative using first principle.
   - Write difference quotient $\frac{f(x+\delta x) - f(x)}{\delta x}$
   - Simplify difference quotient.
   - Find the limit as δx approaches 0.

3. Every derivable function is continuous. The converse is not true, that is, there exist functions which are continuous but not differentiable.

4. The derivative of a constant function is 0, that is, $\frac{d}{dx}(c) = 0,$ where $c \in \mathbb{R}$.

5. $\frac{d}{dx}(x^n) = nx^{n-1}$,
   Where n is any integer (and $x \neq 0$ if $n < 0$).

6. $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$, where $x > 0$

7. The function $y = |x|$ is derivable at every point except at $x = 0$.

8. The derivatives of the following:

   i) $\frac{d}{dx}[c.f(x)] = c.\frac{d}{dx}f(x)$

   ii) $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

   iii) $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

   iv) $\frac{d}{dx}[f(x).g(x)] = f(x).\frac{d}{dx}g(x) + g(x).\frac{d}{dx}f(x)$

   v) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2}$, provided $g(x) \neq 0$

   vi) $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-f'(x)}{[f(x)]^2}$, provided $f(x) \neq 0$.

9. The chain rule: If $y = f(x)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

10. The derivatives of trigonometric functions:

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

11. The inverse function theorem states the rule $\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(y)}$, where f is differentiable and strictly monotonic and $f'(y) \neq 0$.

12. $\frac{d}{dx}(x^r) = rx^{r-1}$, where r is a rational number and x is non-zero.

13. The derivatives of inverse trigonometric functions using the inverse function theorem are as follows:

Function	Derivative
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$
$\tan^{-1} x$	$\frac{1}{1+x^2}, x \in \mathbb{R}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}, x \in \mathbb{R}$
$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}, \lvert x\rvert > 1$
$\operatorname{cosec}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}, \lvert x\rvert > 1$

9.10 SOLUTIONS/ANSWERS

E1) i) Let $y = f(x) = \frac{1}{x}$, then slope at $\left(2, \frac{1}{2}\right)$ is

$$= \lim_{\delta x \to 0} \frac{f(2+\delta x) - f(2)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\frac{1}{2+\delta x} - \frac{1}{2}}{\delta x} = \lim_{\delta x \to 0} \frac{-1}{2+\delta x} = -\frac{1}{2}$$

ii) Let $y = f(x) = x^3$, then slope at $(1,1)$

$$= \lim_{\delta x \to 0} \frac{f(1+\delta x) - f(1)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(1+\delta x)^3 - 1}{\delta x} = 3$$

E2) No, give justification.

E3) To calculate the slope of the tangent at the point $P(2, s(2))$, we choose a point $R\left(2+\delta t, \frac{1}{2}(2+\delta t)^2\right)$ on the curve, near P. Then, the required rate of change is $\lim_{\delta t \to 0}$ (slope of PR)

$$= \lim_{\delta t \to 0} \frac{\frac{1}{2}(2+\delta t)^2 - 2}{(2+\delta t) - 2} = \lim_{\delta t \to 0} \frac{\delta t(4+\delta t)}{2\delta t} = 2.$$

Fig. 24 shows the curve represented by $s = \frac{1}{2}t^2$ taking time along $x-$axis and distance along $y-$axis.

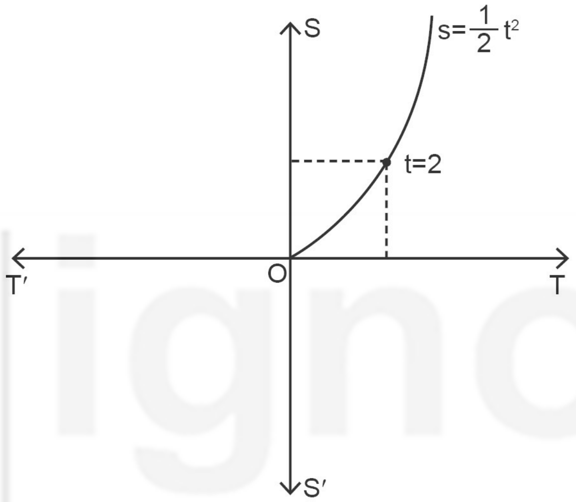

Fig. 24

Hence, the velocity at $t = 2$ is the same as the slope of the tangent at $P(2, s(2))$.

E4) $$v = \lim_{\delta t \to 0} \frac{s(t+st) - s(t)}{\delta t} = \lim_{\delta t \to 0} \frac{u(t-\delta t) - \frac{1}{2}g(t+\delta t)^2 - ut + \frac{1}{2}gt^2}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta t(u - gt - \frac{1}{2}g\delta t)}{\delta t} = u - gt$$

E5) $A(r) = \pi r^2$

Rate of change of area of a circle with respect to its radius when radius is 2 cm $= \lim_{\delta r \to 0} \frac{\pi(2+\delta r)^2 - \pi.2^2}{\delta r} = 4cm^2$

E6) Average rate of change of f in $[1, 1+h] = \frac{f(1+h) - f(1)}{h}$

$$= \frac{2(1+h)^2 + 1 - (2 \times 1^2 + 1)}{h}$$

$$= 4 + 2h$$

Rate of change of f at $(x = 1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ [where h may be positive or negative]

$$= \lim_{h \to 0}(4 + 2h) = 4$$

E7) $f'(c) = \lim_{\delta x \to 0} \frac{f(c+\delta x) - f(c)}{\delta x}$

$= \lim_{\delta x \to 0} \frac{(c+\delta x)^2 - 8(c+\delta x) + 9 - c^2 + 8c - 9}{\delta x}$

$= \lim_{\delta x \to 0} \frac{2c\delta x + \delta x^2 - 8\delta x}{\delta x}$

$= \lim_{\delta x \to 0} (2c + \delta x - 8)$

$= 2c - 8$.

E8) $f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

$= \lim_{\delta x \to 0} \frac{[3(x+\delta x)^2 + 2(x+\delta x) + 1] - (3x^2 + 2x + 1)}{\delta x}$

$= 6x + 2$

$f'(x) > 0$, when $x > -\frac{1}{3}$

$f'(x) < 0$, when $x < -\frac{1}{3}$

$f'(x) = 0$, when $x = -\frac{1}{3}$

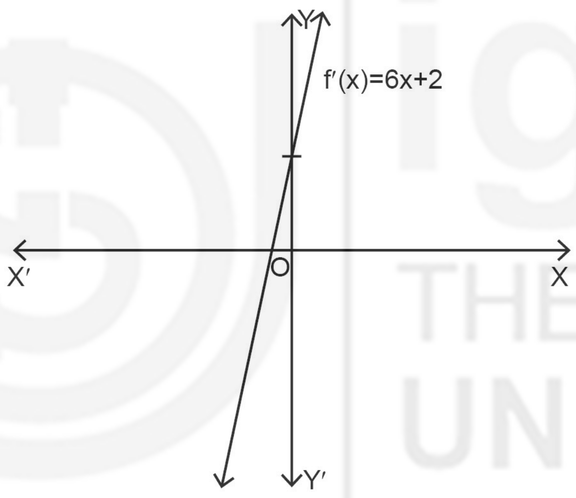

Fig. 25: Graph of f'

E9) i) $\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$

ii) If $x > -1$, $x + 1 > 0$, choose $h > 0$ s.t. $h < |x+1|$

then, $x + h + 1 > 0$ and $\lim_{h \to 0} \frac{|x+h+1| - |x+1|}{h}$

$= \lim_{h \to 0} \frac{(x+h+1) - (x+1)}{h} = 1$

If $x < -1$, $\lim_{h \to 0} \frac{|x+h+1| - |x+1|}{h} = -1$

Thus, dy/dx exists, when $x > -1$ or when $x < -1$. It does not exist, when $x = -1$, since, $Rf'(-1) = 1$ and $Lf'(-1) = -1$

iii) $$\frac{dy}{dx}=\lim_{h\to 0}\frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h}\times\frac{\sqrt{2(x+h)+1}+\sqrt{2x+1}}{\sqrt{2(x+h)+1}+\sqrt{2x+1}}$$

$$=\lim_{h\to 0}\frac{2}{\sqrt{2(x+h)+1}+\sqrt{2x+1}}=\frac{1}{\sqrt{2x+1}}$$

E10) i) $$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{2+h-2}{h}=1$$

ii) $$f'(2)=\lim_{h\to 0}\frac{a(2+h)+b-(a\times 2+b)}{h}$$

$$=\lim_{h\to 0}\frac{ah}{h}=a$$

E11) $$f'(x)=\frac{50(2x+3)^{49}\,2(9x+2)-9(2x+3)^{50}}{(9x+2)^2}$$

exists at $x=0.1$. Hence, the function is continuous at $x=0.1$.

E12) $$\frac{d}{dx}\left(\frac{1}{f(x)}\right)=\lim_{h\to 0}\frac{\frac{1}{f(x+h)}-\frac{1}{f(x)}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\,\frac{f(x)-f(x+h)}{f(x)f(x+h)}$$

$$=\frac{\lim\limits_{h\to 0}\frac{f(x)-f(x+h)}{h}}{f(x)\lim\limits_{h\to 0}f(x+h)}$$

$$=\frac{-f'(x)}{f(x)^2}$$

E13) i) 0 ii) 0

E14) i) $$\frac{d}{dx}(x^8)=8x^{8-1}=8x^7$$

ii) $$\frac{d}{dx}(x^{1000})=1000\,x^{1000-1}=1000\,x^{999}$$

iii) $$\frac{d}{dx}(10)=0$$

E15) $$Lf'(0)=\lim_{h\to 0^-}\frac{(0+h)^{1/3}-0^{1/3}}{h}$$

$$=\lim_{h\to 0^-}\frac{1}{h^{2/3}}=-\infty$$

$$Rf'(0)=\lim_{h\to 0^+}\frac{(0+h)^{1/3}-0^{1/3}}{h}$$

$$=\lim_{h\to 0^+}\frac{1}{h^{2/3}}=+\infty$$

Therefore, f is not differentiable at $x=0$.

E16) i) $\frac{d}{dx}\left(\frac{5}{3}x^3\right)=\frac{5}{3}\frac{d}{dx}(x^3)=\frac{5}{3}\times 3x^2=5x^2$

ii) $\frac{d}{dx}\left(8\sqrt{x}\right)=8\frac{d}{dx}\left(\sqrt{x}\right)=\frac{4}{\sqrt{x}}$

iii) $\frac{d}{dx}\sqrt{2}|x|=\begin{cases}\sqrt{2}\ ; & \text{when } x>0\\ -\sqrt{2}\ ; & \text{when } x<0\\ \text{does not exist}\ ; & \text{when } x=0\end{cases}$

E17) We apply the scalar multiple rule obtained in Theorem 2 at all points where | x | is differentiable and get

$\frac{d}{dx}(7|x|)=7\frac{d}{dx}(|x|)$

But, in view of Example 9, when $x=0, \frac{d}{dx}(|x|)$ does not exist.

When $x>0, \frac{d}{dx}(|x|)=\frac{dx}{dx}=1$

and, when $x<0, \frac{d}{dx}(|x|)=\frac{d(-x)}{dx}=-\frac{d(x)}{dx}=-1$.

Therefore, $\frac{d}{dx}(7|x|)=7\frac{d}{dx}(|x|) \quad =\begin{cases}7; & \text{when } x>0\\ -7; & \text{when } x<0\\ \text{does not exist;} & \text{when } x=0\end{cases}$

E18) i) $15x^2-2$

ii) $a_1+2a_2x+\cdots+na_nx^{n-1}$

E19) Horizontal tangent occurs when the derivatives is 0. Thus,

$\frac{dy}{dx}=\frac{d}{dx}(x^4)-8\frac{d}{dx}(x^2)+\frac{d}{dx}(4)$

$=4x^3-16x$.

Since, $\frac{dy}{dx}=0,$ therefore, $4x^3-16x=0,$ that is $x=0,2,-2.$ So, the given curve has horizontal tangents when $x=0,2$ and -2. The corresponding points are $(0,4),(2,-12)$ and $(-2,-12)$.

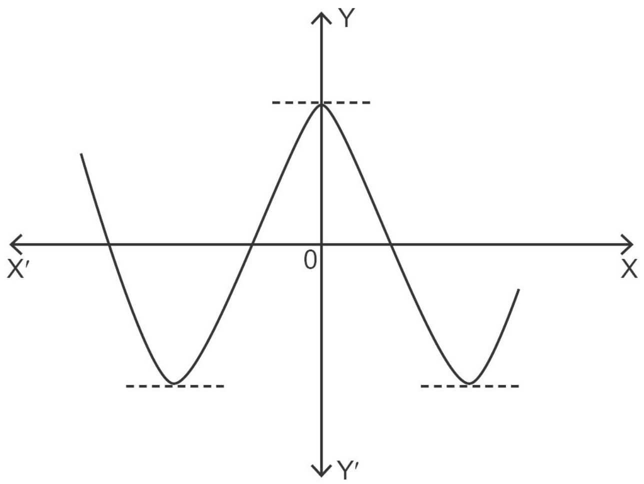

Fig. 26

E20) i) Using Theorem 4, $\frac{d}{dx}\left(x\sqrt{x}\right)=\frac{d}{dx}(x).\sqrt{x}+x.\frac{d}{dx}\sqrt{x}$

$$=1.\sqrt{x}+x.\frac{1}{2}x^{-1/2}$$

$$=\sqrt{x}+\frac{1}{2}\sqrt{x}=\frac{3}{2}\sqrt{x}$$

Without using Theorem 4, $\frac{d}{dx}(x\sqrt{x})=\frac{d}{dx}(x)^{3/2}=\frac{3}{2}x^{\frac{1}{2}}$

ii) $\frac{d}{dx}\{(x^5+2x^3+5)(x^5+2x^3+5)\}$

$$=(x^5+2x^3+5)(5x^4+6x^2)+(5x^4+6x^2)\,(x^5+2x^3+5)$$

$$=2(x^5+2x^3+5)\,(5x^4+6x^2)$$

iii) $dy/dx=(x+2)\,(x+3)+(x+1)(x+3)+(x+1)\,(x+2)$

E21) $f'(x)=\frac{d}{dx}[xg(x)]=\frac{d(x)}{dx}g(x)+x\frac{d\,g(x)}{dx}$

$$=(1)\,g(x)+x\,g'(x)$$

Put $x=3$

$f'(3)=g(3)+3g'(3)$

$$=2+3(5)=17.$$

E22) i) $\frac{d}{dx}\left(\frac{2x+1}{x+5}\right)=\frac{2(x+5)-(2x+1)}{(x+5)^2}=\frac{9}{(x+5)^2}$, where $x\neq 5$

ii) $\frac{d}{dx}\left(\frac{1}{a+bx+cx^2+dx^3}\right)=\frac{-(b+2cx+3dx^2)}{(a+bx+cx^2+dx^3)^2}$

iii) $\frac{d}{dx}\left(\frac{2x^3+3x^2}{x^4-1}\right)$

$$=\frac{(x^4-1)\,(6x^2+6x)-(2x^3+3x^2)\,(4x^3)}{(x^4-1)^2}$$, where $x^4-1\neq 0$

$$=\frac{6x(x+1)(x^4-1)-4x^5(2x+3)}{(x^4-1)^2}$$

E23) $\frac{d}{dx}\left(\frac{1}{f(x)}\right)=\frac{\frac{d}{dx}(1).f(x)-(1).\frac{d}{dx}f(x)}{(f(x))^2}$

$$=\frac{0-f'(x)}{(f(x))^2}$$

$$=-\frac{f'(x)}{(f(x))^2},f(x)\neq 0$$

E24) i) As a sum of three terms, we get $f(x)=2x^{-5}+5x^{-4}+7x^{-6}$

$$f'(x)=-10x^{-6}-20x^{-5}-42x^{-7}$$

ii) Applying quotient rule; we get

$$f'(x)=\frac{x^5(5-7x^{-2})-5x^4(2+5x+7x^{-1})}{x^{10}}$$

$$=x^{-10}(-20x^5-42x^3-10x^4)$$

$$=-20x^{-5}-42x^{-7}-10x^{-6}$$

iii) As a product of two functions, $f(x) = x^{-5}(2+5x+7x^{-1})$

$$f(x) = x^{-5}(5-7x^{-2}) - 5x^{-6}(2+5x+7x^{-1})$$
$$= -20x^{-5} - 42x^{-7} - 10x^{-6}$$

E25) i) $u = 1+5x+7x^2,\ y = 5/u$

$$dy/dx = (-5/u^2)\,(5+14x) = \frac{-25-70x}{(1+5x+7x^2)}$$

ii) $u = 2x+3,\ y = u^2/(1+u^3)$

$$\frac{dy}{dx} = \frac{2u(1+u^3)-3u^4}{(1+u^3)^2}\times 2$$
$$= \frac{4(2x+3)\,[1+(2x+3)^3]-6(2x+3)^4}{[1+(2x+3)^3]^2}$$

iii) $u = 9x+5,\ v = u^3+u^{-3},\ y = v^7$

$$dy/dx = 7v^6(3u^2-3u^{-4})\times 9$$
$$= 63[(9x+5)^3+(9x+5)^{-3}]^6[3(9x+5)^2-3(9x+5)^{-4}]$$

E26) Let $f(x) = goh(x)$, where $g(t) = \sqrt{t}$ and $h(x) = x+1$

$g'(t) = \dfrac{1}{2\sqrt{t}}$ and $h'(x) = 1$

$$f'(x) = g'(h(x)).\ h'(x)$$
$$= \frac{1}{2\sqrt{x+1}}\cdot 1 = \frac{1}{2\sqrt{x+1}}, \text{if } x > -1.$$

E27) $\dfrac{dv}{dt}$ represents the rate of change in velocity with respect to time and $\dfrac{dv}{ds}$ represents the rate of change in velocity with respect to displacement.

Also, $\dfrac{dv}{dt} = \dfrac{dv}{ds}\cdot\dfrac{ds}{dt}$

E28) i) $$\frac{dy}{dx} = \frac{d}{dx}(x^3\cos 9x)$$
$$= \frac{d}{dx}(x^3)\times\cos 9x + x^3\times\frac{d}{dx}(\cos 9x)$$
$$= 3x^2\times\cos 9x + x^3(-9\sin 9x)$$
$$= 3x^2\cos 9x - 9x^3\sin 9x.$$

ii) $$\frac{dy}{dx} = \frac{d}{dx}(\cos(\sin x))$$
$$= \frac{d}{dt}(\cos t)\times\frac{dt}{dx} \quad \text{where } t = \sin x$$
$$= -\sin t\times\frac{d}{dx}(\sin x) = -\sin(\sin x).\cos x$$

iii) $$\frac{dy}{dx} = \frac{d}{dx}(\sin(x^2+1))$$

$$= \frac{d}{dt}(\sin t).\frac{dt}{dx} \text{ where } t = x^2 + 1$$

$$= \cos t . \frac{d(x^2+1)}{dx}$$

$$= \cos(x^2+1).(2x)$$

$$= 2x\cos(x^2+1)$$

iv) $$\frac{dy}{dx} = \frac{d(\sin t)}{dt} \cdot \frac{dt}{dx} \text{ where } t = \sqrt{1+\cos x}$$

$$= \cos t . \frac{d\sqrt{u}}{du} . \frac{du}{dx} \text{ where } u = 1 + \cos x$$

$$= -\frac{\sin x \cos\sqrt{1+\cos x}}{2\sqrt{1+\cos x}}$$

E29) i) $$\frac{d}{dx}(\operatorname{cosec} 2x) = \frac{d}{d(2x)}(\operatorname{cosec} 2x) \times \frac{d}{dx}(2x)$$

$$= -2\operatorname{cosec} 2x \cot 2x.$$

ii) $$\frac{d}{dx}(\cot x + \sqrt{\operatorname{cosec} x}) = \frac{d}{dx}(\cot x) + \frac{d}{dx}(\sqrt{\operatorname{cosec} x})$$

$$= -\operatorname{cosec}^2 x + \frac{d}{dt}(\sqrt{t}) \times \frac{dt}{dx} \text{ where } t = \operatorname{cosec} x$$

$$= -\operatorname{cosec}^2 x + \frac{1}{2\sqrt{t}} \times \frac{d}{dx}(\operatorname{cosec} x)$$

$$= -\operatorname{cosec}^2 x + \frac{1}{2\sqrt{\operatorname{cosec} x}} \times -\operatorname{cosec} x.\cot x$$

$$= -\operatorname{cosec}^2 x - \frac{\operatorname{cosec} x \cot x}{2\sqrt{\operatorname{cosec} x}}$$

iii) $$\frac{d}{dx}[5(\cot 9x)] = 5\frac{d}{dx}(\cot 9x)$$

$$= 5(-\operatorname{cosec}^2 9x) \times \frac{d}{dx}(9x)$$

$$= -5 \operatorname{cosec}^2 9x \times 9$$

$$= -45 \times \operatorname{cosec}^2 9x .$$

iv) $$\frac{d}{dx}\left[(1+x^5 \cot x)^{-8}\right]$$

$$= \frac{du^{-8}}{du} . \frac{du}{dx} \text{ where } u = 1 + x^5 \cot x$$

$$= -8u^{-9} . \frac{d(1+x^5 \cot x)}{dx}$$

$$= -8(1+x^5 \cot x)^{-9}.[x^5(-\operatorname{cosec}^2 x) + 5x^4(\cot x)]$$

$$= (1+x^5 \cot x)^{-9}.(8x^5 \operatorname{cosec}^2 x - 40x^4 \cot x)$$

E30) i) The rate of change of F w.r.t. θ is $\frac{dF}{d\theta} = \frac{-\mu mg(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2}$

ii) $\frac{dF}{d\theta} = 0 \Rightarrow \mu\cos\theta - \sin\theta = 0 \Rightarrow \tan\theta = \mu \Rightarrow \theta = \tan^{-1}(\mu)$

E31) $\frac{d}{d\theta}(\sin\theta)$, where θ is in degree

$$= \frac{d}{d\theta}\left(\sin\theta . \frac{\pi}{180^o}\right)$$

$$= \frac{\pi}{180^o}\cos\theta$$

E32) i)

$$\frac{d}{dx}5(x^3 + x^{\frac{1}{3}}) = 5\frac{d}{dx}(x^3 + x^{\frac{1}{3}})$$

$$= 5\left[\frac{d}{dx}x^3 + \frac{d}{dx}(x^{\frac{1}{3}})\right]$$

$$= 5\left[3x^2 + \frac{1}{3}x^{\frac{1}{3}-1}\right]$$

$$= 15x^2 + \frac{5}{3}x^{-\frac{2}{3}}.$$

ii)

$$\frac{d}{dx}[\sqrt[5]{x} - \sqrt[9]{x}]x^2 = \frac{d}{dx}[x^{1/5} - x^{1/9}]x^2 = \frac{d}{dx}[x^{11/5} - x^{19/9}]$$

$$= \frac{d}{dx}(x^{11/5}) - \frac{d}{dx}(x^{19/9})$$

$$= \frac{11}{5}x^{\frac{11}{5}-1} - \frac{19}{9}x^{\frac{19}{9}-1}$$

$$= \frac{11}{5}x^{6/5} - \frac{19}{9}x^{10/9}.$$

E33) i)

$$\frac{d}{dx}[\sin^{-1}(5x)] = \frac{d}{dt}(\sin^{-1}t) \times \frac{dt}{dx} \quad \text{where } t = 5x$$

$$= \frac{1}{\sqrt{1-t^2}} \times \frac{d}{dx}(5x)$$

$$= \frac{1}{\sqrt{1-25x^2}} \times 5$$

$$= \frac{5}{\sqrt{1-25x^2}}$$

ii)

$$\frac{d}{dx}(\cos^{-1}\sqrt{x}) = \frac{d}{dt}(\cos^{-1}t) \times \frac{dt}{dx} \quad \text{where } t = \sqrt{x}$$

$$= \frac{-1}{\sqrt{1-t^2}} \times \frac{d\sqrt{x}}{dx}$$

$$= \frac{-1}{\sqrt{1-x}} \times \frac{1}{2}x^{-1/2}$$

$$= \frac{-1}{2\sqrt{x(1-x)}}.$$

iii) $$\frac{d}{dx}(\sin x\ \cos^{-1}(x^3+2))$$

$$=\sin x.\frac{d}{dx}(\cos^{-1}(x^3+2))+\frac{d}{dx}(\sin x).\cos^{-1}(x^3+2)$$

$$=\sin x.\frac{d}{dt}(\cos^{-1}t)\times\frac{dt}{dx}+\cos x.\cos^{-1}(x^3+2) \quad \text{where } t=x^3+2$$

$$=\sin x.\frac{-1}{\sqrt{1-t^2}}\times 3x^2+\cos x.\ \cos^{-1}(x^3+2)$$

$$=-\frac{3x^2\sin x}{\sqrt{1-(x^3+2)^2}}+\cos x\cos^{-1}(x^3+2)$$

E34) i) $$\frac{d}{dx}(\cot^{-1}(x/2))=\frac{d}{dt}\cot^{-1}t\times\frac{dt}{dx} \quad \text{where } t=x/2$$

$$=\frac{-1}{1+t^2}.\frac{1}{2}$$

$$=\frac{-1}{2(1+x^2/4)}$$

ii) $$\frac{d}{dx}\left(\frac{\cot^{-1}(x+1)}{\tan^{-1}(x+1)}\right)$$

$$=\frac{\tan^{-1}(x+1)\frac{d}{dx}(\cot^{-1}(x+1))-\frac{d}{dx}(\tan^{-1}(x+1)).\cot^{-1}(x+1)}{(\tan^{-1}(x+1))^2}$$

$$=\frac{-\tan^{-1}(x+1)\left(\frac{1}{1+(x+1)^2}\right)-\cot^{-1}(x+1)\left(\frac{1}{1+(x+1)^2}\right)}{(\tan^{-1}(x+1))^2}$$

iii) $$\frac{d}{dx}\cos^{-1}(5x+4)=\frac{d}{dt}(\cos^{-1}t)\times\frac{dt}{dx} \quad \text{where } t=5x+4$$

$$=\frac{-5}{\sqrt{1-(5x+4)^2}}$$

iv) $$\frac{d}{dx}\left(\sec^{-1}\frac{x\sin\theta}{1-x\cos\theta}\right)=\frac{d}{dt}\sec^{-1}t\times\frac{dt}{dx} \quad \text{where } t=\frac{x\sin\theta}{1-x\cos\theta}$$

$$=\frac{1}{\left|\frac{x\sin\theta}{1-x\cos\theta}\right|\sqrt{\frac{x^2\sin^2\theta}{(1-x\cos\theta)}}}\left[\frac{\sin\theta(1-x\cos\theta)+x\sin\theta\cos\theta}{(1-x\cos\theta)^2}\right]$$ [apply quotient rule to find out $\frac{dt}{dx}$]

v) $$\frac{d}{dx}(\operatorname{cosec}^{-1}(x+1)+\sec^{-1}(x-1))$$

$$=\frac{d}{dx}(\operatorname{cosec}^{-1}(x+1))+\frac{d}{dx}(\sec^{-1}(x-1))$$

$$=\frac{-1}{|x+1|\sqrt{(x+1)^2-1}}+\frac{1}{|x-1|\sqrt{(x-1)^2-1}}$$

UNIT 10

SOME MORE DERIVATIVES

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10.1 INTRODUCTION

In Unit 7, we discussed exponential functions, which occupy an important place in pure and applied science. Laws of growth and decay are very often expressed in terms of these functions. In Sec. 10.2, we will study the derivatives logarithmic of exponential functions. The inverse function theorem which we stated in Unit 9 will then help us to differentiate their inverse, the exponential functions in Sec. 10.3. In particular, you will find that the natural exponential function is its own derivative.

Further, we will differentiate hyperbolic functions and their inverse functions in Sec. 10.4. In Sec. 10.5, we will study logarithmic method of differentiation. We will extend our process of differentiation to differentiate implicit functions in Sec. 10.6, and at the end, we will study other methods of differentiation in Sec. 10.7.

With this unit we come to the end of our quest for the derivatives of some standard, frequently used functions. And now we shall list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

Objectives

After reading this unit you should be able to:

- find the derivatives of exponential and logarithmic functions;

- use the method of logarithmic differentiation;
- differentiate hyperbolic functions and inverse hyperbolic functions;
- differentiate implicit functions; and
- compute the derivatives of those functions which are defined with the help of a parameter.

10.2 DERIVATIVE OF LOGARITHMIC FUNCTION

In this section, we will find the derivative of a logarithmic function. For this, you first recall the definition of a logarithmic function, discussed in Unit 7. To find the derivative of the logarithmic function, we will use the definition of the derivative and the fact that the logarithmic function is a continuous function. We will also use the limit $\lim\limits_{n\to 0}(1+n)^{1/n} = e$.

Let $f(x) = \log_a x$, where $x > 0$, then

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h}$$

$$= \lim_{h\to 0}\frac{\log_a\left(\frac{x+h}{x}\right)}{h} \quad \text{[Since, } \log_a m - \log_a n = \log_a \frac{m}{n}\text{]}$$

$$= \lim_{h\to 0}\frac{\log_a\left(1+\frac{h}{x}\right)}{h}$$

$$= \frac{1}{x}\lim_{h\to 0}\frac{x}{h}\log_a\left(1+\frac{h}{x}\right) \quad \text{[Multiplying and divide by } x\text{]}$$

$$= \frac{1}{x}\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{x/h} \quad \text{[Since, } m\log_a n = \log_a n^m\text{]}$$

$$= \frac{1}{x}\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{\frac{1}{h/x}}$$

$$= \frac{1}{x}\log_a\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h/x}}\right] \quad \text{[Since, } \log_a m \text{ is continuous]}$$

$$= \frac{1}{x}\log_a e \quad \text{[Since } \lim_{n\to 0}(1+n)^{1/n} = e \text{ and } n = h/x, \text{ as } h\to 0, n\to 0\text{]}$$

Therefore, $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e,\ x > 0$.

We can also write $\log_a e$ in terms of the natural logarithmic function, which gives

$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e = \frac{1}{x}.\frac{\ln e}{\ln a} \text{ [changing base]}$$

$$= \frac{1}{x\ln a},\ x > 0. \text{ [Since } \ln e = 1\text{]}$$

In particular, if $a = e$, then we get the derivative of the natural logarithmic function, which is

$$\frac{d}{dx}(\ln x) = \frac{1}{x},\ x > 0$$

Thus, among all possible bases, the base $a = e$ produces the simplest formula for the derivative of $\log_a x$. This is one of the reasons why the natural logarithm function is preferred over other logarithms in calculus. Fig.1 shows the graphs of $y = \ln x, x > 0$ and its derivative.

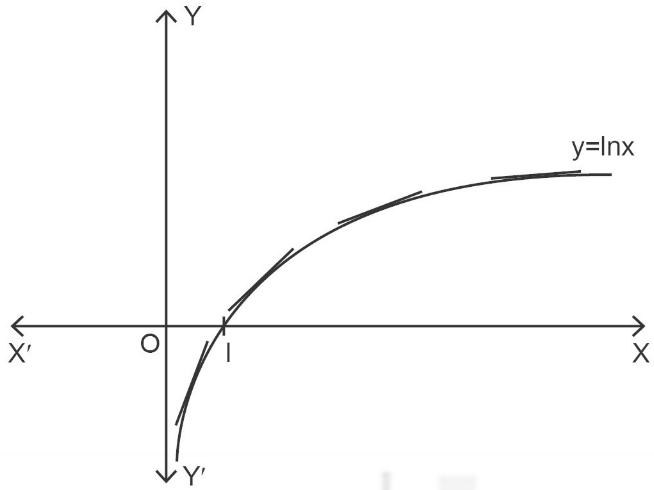

Fig. 1: Graph of $\ln x$ with tangent lines

The slopes of the tangent lines at the points $x = \frac{1}{3}, \frac{1}{2}, 1, 3$ and 5 are $1/x = 3, 2, 1, \frac{1}{3}$ and $\frac{1}{5}$, which is consistent with Fig.1 also. From the graph it does not appear that there are any horizontal tangent lines. This is confirmed by the fact that $\frac{dy}{dx} = \frac{1}{x}$ is not equal to 0 for any real value of x.

We give a few examples to find the derivatives of the logarithmic functions.

Example 1: Differentiate $y = \ln(x^2 - 2x + 2)$ with respect to x.

Solution: Since, $x^2 - 2x + 2 = (x-1)^2 + 1$ and hence, is positive for all x, therefore, $\ln(x^2 - 2x + 2)$ is well defined.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\ln(x^2 - 2x + 2)) \\ &= \frac{d}{dt}(\ln t).\frac{dt}{dx} \text{ where } t = x^2 - 2x + 2 \\ &= \frac{1}{t}\frac{d}{dx}(x^2 - 2x + 2) \\ &= \frac{2x-2}{x^2 - 2x + 2}\end{aligned}$$

***

Example 2: Find the derivative of $\ln|x|$ w.r.t. x.

Solution: The function $\ln|x|$ is defined for all x, except $x = 0$. Therefore, we will consider two cases $x > 0$ and $x < 0$ separately.

i) When $x > 0$, $y = \ln|x| = \ln x$

Here, $\frac{dy}{dx} = \frac{d}{dx}(\ln x)$

$= \frac{1}{x}$, if $x > 0$.

ii) When $x < 0,\ y = \ln|x| = \ln(-x)$.

Here $\frac{dy}{dx} = \frac{d}{dx}[\ln(-x)] = \frac{d}{dx}\ln(t).\frac{dt}{dx}$, where $t = -x$

$$= \frac{1}{t}.(-1) = \frac{-1}{-x} = \frac{1}{x}.$$

Therefore, $\frac{d}{dx}\left(\ln|x|\right) = \begin{cases} \frac{1}{x}, \text{ when } x > 0 \\ \frac{1}{x}, \text{ when } x < 0 \end{cases}$.

Thus, $\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$ for all $x \neq 0$.

***

Example 3: Differentiate $y = \ln\left|\frac{1+x^2}{1-x^2}\right|, |x| \neq 1$ with respect to x.

Solution: If we want to differentiate $y = \ln\left|\frac{1+x^2}{1-x^2}\right|$, $|x| \neq 1$, we will consider two cases: i) $|x| > 1$ and ii) $|x| < 1$.

i) If $|x| > 1$, we get $\left|\frac{1+x^2}{1-x^2}\right| = \frac{1+x^2}{-(1-x^2)}$ [Since, $|x| > 1$, therefore $(1-x^2)$ is negative]

$$= \frac{x^2+1}{x^2-1}.$$

So, in this case, $\frac{dy}{dx} = \frac{d}{dx}\left(\ln\left(\frac{x^2+1}{x^2-1}\right)\right)$

$$= \frac{d}{dt}(\ln t).\frac{dt}{dx}, \text{ where } t = \frac{x^2+1}{x^2-1}$$

$$= \frac{x^2-1}{x^2+1}\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

After simplification, we get $\frac{dy}{dx} = \frac{4x}{1-x^4}$

ii) When $|x| < 1$, $\left|\frac{1+x^2}{1-x^2}\right| = \frac{1+x^2}{1-x^2}$ and so,

$$\frac{dy}{dx} = \frac{1-x^2}{1+x^2}\frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right)$$

$$= \frac{4x}{1-x^4}$$

So, we see that $\frac{dy}{dx} = \frac{4x}{1-x^4}$ for all x such that $|x| \neq 1$.

***

Example 4: Differentiate $y = \log_7(\tan^3 x)$ w.r.t. x.

Solution: $\frac{dy}{dx} = \frac{d}{dx}(\log_7(\tan^3 x))$

$$= \frac{d}{dt}(\log_7 t).\frac{dt}{dx}, \text{ where } t = \tan^3 x$$

$$= \frac{1}{t}\log_7 e.\frac{d}{dx}(\tan^3 x)$$

$$= \log_7 e\frac{1}{\tan^3 x}.3\tan^2 x.\frac{d}{dx}(\tan x) \text{ [By chain rule]}$$

$$= \log_7 e\frac{1}{\tan^3 x}\, 3\tan^2 x \sec^2 x$$

$$= 3\log_7 e\frac{\sec^2 x}{\tan x}$$

***

Example 5: Differentiate $f(x) = \sqrt{\ln x}$ with respect to x when $x \geq 1$.

Solution: $\frac{d}{dx}\left(\sqrt{\ln x}\right) = \frac{d}{dt}\left(\sqrt{t}\right).\frac{dt}{dx}$, where $t = \ln x$

$$= \frac{1}{2\sqrt{t}}.\frac{1}{x}$$

$$= \frac{1}{2x\sqrt{\ln x}}$$

***

Example 6: A particle is moving along the curve $y = x\ln x$. Find all the values of x at which the rate of change of y with respect to time is three times that of x.

Solution: $\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$

$$= \left(x.\frac{1}{x} + 1.\ln x\right)\frac{dx}{dt}$$

$$= (1 + \ln x)\frac{dx}{dt} \qquad \text{... (1)}$$

Given is $\frac{dy}{dt} = 3\frac{dx}{dt}$ ... (2)

Comparing (1) and (2), we get

$(1 + \ln x) = 3$ (assuming that $\frac{dx}{dt} \neq 0$)

$\Rightarrow \ln x = 2$

$\Rightarrow x = e^2$

***

Example 7: An aeroplane takes off from an airport at sea level and its altitude (in meters) at time t (in minutes) is given by $h = 500\ln(t+1)$. Find the rate of climb at time $t = 3$ min. Also, compare the graphs of h and h'.

Solution: Given is $h = 500\ln(t+1)$. To find, the rate of climb, we need to find the first derivative.

$$\frac{dh}{dt} = \frac{d}{dt}(500\ln(t+1))$$
$$= \frac{500}{1+t}$$

At $t = 3,$ we have $\frac{dh}{dt} = 125\,m/min$

So, the required rate of climb is 125 m/min. Fig. 2 shows the graph of h (solid line) and h' (dotted line).

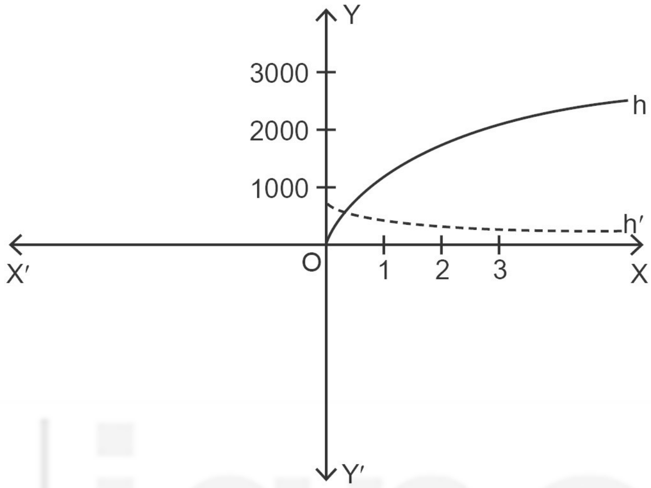

Fig. 2: Graph of h and h'

The graph shows that at low altitudes the rate of climb is good, but as you go higher, the rate decreases.

***

Example 8: Find the equation of the tangent line to the graph of $y = (x^2 - x)\ln(6x)$ at $x = 2$.

Solution: To find the equation of the tangent at any point on the graph of function, we need slope of the tangent at that point. We have $x = 2$, which gives $y = 2\ln(12)$. Thus, the point is $(2, 2\ln(12))$.

$$\text{Slope} = \left[\frac{dy}{dx}\right]_{\text{at }(2,2\ln(12))}$$
$$= \left[(2x-1)\ln(6x) + (x^2 - x).\frac{1}{6x}.6\right]_{\text{at }(2,2\ln(12))}$$
$$= 3\ln(12) + 1$$

Thus, the equation of the tangent is

$y - 2\ln(12) = [3\ln(12) + 1](x - 2)$ [Recall Unit 3 for equation of a line]

***

Now, try the following exercises.

E1) Find the derivative of the following w.r.t. x.

i) $\log_2 2x$ ii) $7\log_{11}(5x^2 + 2)$

iii) $x^2 \ln x$ iv) $\ln\left(\frac{1+x}{1-x}\right), |x| < 1$

v) $\ln(\sin^4 x)$ vi) $\log_{10}(2 + \sin x)$

E2) Let the sound pressure P for a given sound is given by

$P = 10\log\frac{w}{w_o}$, where w is the sound power and w_o is a constant used for lowest threshold. Find the rate of change of the sound pressure P with respect to time t, if $w = 7.2, w_o = 10^{-12}$ watt / m^2 and $\frac{dw}{dt} = 0.5$ at some given time t.

In the following section, we will use the inverse function theorem to find the derivatives of the exponential function and natural exponential function.

10.3 DERIVATIVE OF EXPONENTIAL FUNCTION

You may recall the definitions of exponential and logarithmic functions. These functions are inverses of each other. To find the derivative of the exponential functions a^x, where $a > 0$, we will use the inverse function theorem. Consider $y = a^x$. We know that the logarithmic function is differentiable, and its derivative is non-zero. We know that the inverse of $\log_a x$ is differentiable that is a^x is differentiable and using the inverse function theorem, the derivative is

$$\frac{d}{dx}(a^x) = \frac{1}{\frac{d}{dy}(\log_a y)} \quad \text{[Since, } y = a^x\text{, therefore, } x = \log_a y\text{]}$$

$$= \frac{1}{\frac{1}{y}\log_a e} = y\ln a = a^x \ln a.$$

Therefore, $\frac{d}{dx}(a^x) = a^x \ln a$.

In a particular, if $a = e$, then $\frac{d}{dx}(e^x) = e^x \ln e = e^x$.

You may observe that natural exponential function is its own derivative. This is the simplest differentiation formula and is used often in calculus. The natural exponential function gives its value 1, at $x = 0$, that is,

$f(x) = e^x$, $f'(x) = e^x$ and $f'(0) = e^0 = 1$.

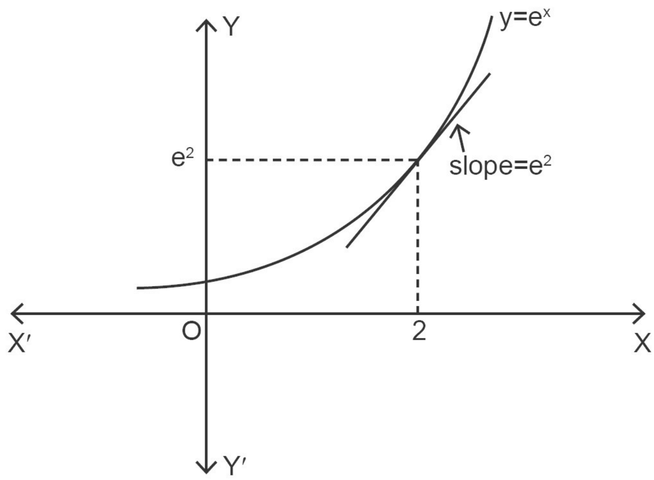

Fig. 3: Graph of e^x slope of tangent at $x = 2$

We can say that the slope of the tangent, where it cuts the $y-$axis, is 1. Also, the slope of a tangent to the curve $y = e^x$ at any point x is equal to the $y-$coordinate of the point. Let us take the example when $x = 2$. At this point, the $y-$value is $e^2 \approx 7.39$. Since the derivative of e^x is e^x, then the slope of the tangent at $x = 2$ is also $e^2 \approx 7.39$. We can see that this is true in Fig.3.

Let us now see if it is true at some other values of x. For example at $x = 4, e^4 \approx 54.6$ and at $x = 5, e^5 \approx 148.4$. We can also verify it by finding the slope of the tangent of the function $y = e^x$ at $x = 4$ and at $x = 5$.

Slope of tangent at $x = 4$ is $\lim\limits_{\delta x \to 0} \dfrac{e^{4+\delta x} - e^4}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{e^4(e^{\delta x} - 1)}{\delta x} = e^4 \approx 54.6$

Similarly, slope of tangent at $x = 5$ is $\lim\limits_{\delta x \to 0} \dfrac{e^5(e^{\delta x} - 1)}{\delta x} = e^5 \approx 148.4$

Thus, we can say that the value of the derivative and the slope of the tangent are the same. Fig. 4 shows these values.

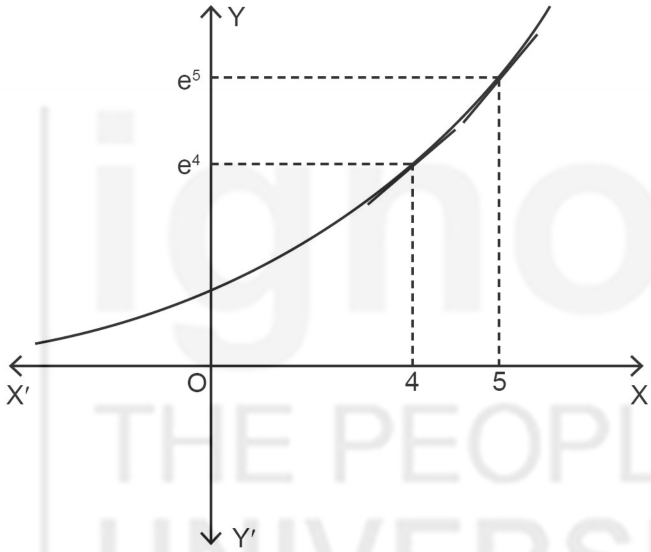

Fig. 4: Graph of e^x and the tangents at $x = 4$ and $x = 5$

Remark 1: It is important to distinguish between differentiation of the exponential function a^x (variable exponent and constant base) and the power function x^a (variable base and constant exponent).
For example,

$\dfrac{d}{dx}(x^{10}) = 10x^9$ and

$\dfrac{d}{dx}(10^x) = 10^x \ln 10$.

Now you can find derivatives of exponential functions in the following examples.

Example 9: Find the derivatives of the following w.r.t. x

i) $e^{(x^2+2x)}$ ii) $\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ iii) $a^{\sin^{-1} x}$

Solution: i) Let $y = e^{(x^2+2x)}$. Then, by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}, \text{ where } t = x^2 + 2x$$
$$= e^{(x^2+2x)}(2x + 2)$$

Hence, $\frac{d}{dx}[e^{(x^2+2x)}] = 2(x+1)e^{(x^2+2x)}$.

ii) $\frac{d}{dx}\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right) = \frac{(e^x-e^{-x})\frac{d}{dx}(e^x+e^{-x})-(e^x+e^{-x})\frac{d}{dx}(e^x-e^{-x})}{(e^x-e^{-x})^2}$ [using quotient rule of derivatives]

$$= \frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})\,(e^x+e^{-x})}{(e^x-e^{-x})^2} \quad [\because \frac{d}{dx}e^{-x} = -e^{-x}]$$

$$= \frac{(e^x-e^{-x})^2-(e^x+e^{-x})^2}{(e^x-e^{-x})^2}$$

$$= \frac{-4}{(e^x-e^{-x})^2}$$

iii) We apply the chain rule again to differentiate $a^{\sin^{-1}x}$

$$\frac{d}{dx}(a^{\sin^{-1}x}) = \frac{d}{dt}(a^t).\frac{dt}{dx} \text{ where } t = \sin^{-1}x$$

$$= a^{\sin^{-1}x}\ln a\frac{d}{dx}(\sin^{-1}x)$$

$$= \frac{1}{\sqrt{1-x^2}}a^{\sin^{-1}x}\ln a$$

***

Example 10: Find y', if $y = e^{-2x}\sin 3x$.

Solution: $y' = \frac{dy}{dx} = \frac{d}{dx}(e^{-2x}\sin 3x)$

$$= e^{-2x}.\frac{d}{dx}(\sin 3x) + \frac{d}{dx}(e^{-2x}).\sin 3x$$ [applying product rule of derivatives]

$$= e^{-2x}.\cos 3x.3 + e^{-2x}(-2).\sin 3x$$ [applying chain rule]

$$= e^{-2x}[3\cos 3x - 2\sin 3x].$$

***

Example 11: The growth in length of a particular plant at time t is $L(t) = \frac{1}{1+ae^{-kt}}$, where t is time and a and k are constants. Find the rate of growth with respect to time.

Solution: The rate of growth $= L'(t) = \frac{d}{dt}\left(\frac{1}{1+a\,e^{-kt}}\right)$

$$= -\frac{1}{(1+a\,e^{-kt})^2}.\left(ae^{-kt}(-k)\right)$$

$$= \frac{ake^{-kt}}{\left(1+ae^{-kt}\right)^2}$$

***

Example 12: The charge q of a capacitor in a circuit containing a capacitor of capacitance C, a resistance R and a source of voltage E is given by

$q = CE(1 - e^{-t/RC})$ coulomb.

Show that q satisfies the equation $R\frac{dq}{dt} + \frac{q}{c} = E$.

Solution: Given is $q = CE(1 - e^{-t/RC})$

so $\frac{dq}{dt} = \frac{CE}{RC}(e^{-t/RC})$

$= \frac{E}{R}e^{-t/RC}$

Now $R\frac{dq}{dt} + \frac{q}{c} = Ee^{-t/RC} + E(1 - e^{-t/RC})$

$= E(e^{-t/RC} + 1 - e^{-t/RC})$

$= E$.

***

Example 13: A glass of lemonade with a temperature of 4° C is left to sit in a room whose temperature is a constant 25° C. Using a principle of physics, called Newton's law of cooling, one can show that if the temperature of the lemonade reaches 15° C in 1 hour, then the temperature T of the lemonade as a function of the elapsed time t is modelled by the equation.

$T = 25 - 21e^{-0.5t}$

where T is in $^{\circ}$C and t is in hours. The graph of this equation is shown in Fig.5, which also confirms to our everyday experience that the temperature of the lemonade gradually approaches the temperature of the room.

i) What happens to the rate of temperature rise over time?

ii) Use a derivative to confirm your conclusion in i).

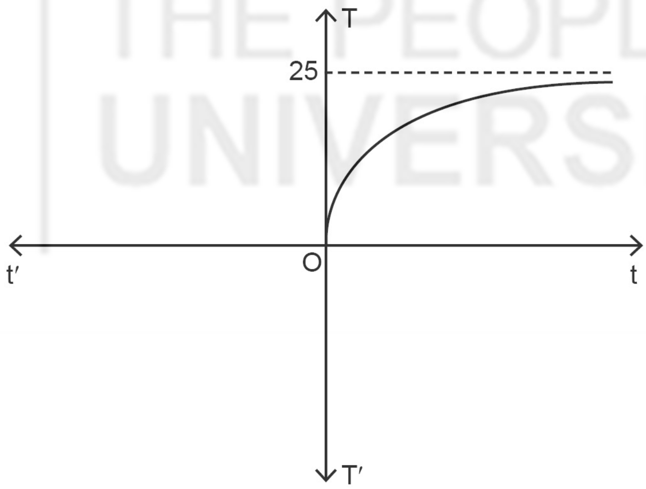

Fig. 5

Solution: i) The rate of change of temperature with respect to time is the slope of the curve $T = 25 - 21e^{-0.5t}$. As t increases, the curve rises such that its slope decreases to 0. Thus, the temperature rises at an over-decreasing rate.

ii) The rate of change of temperature with respect to time is $\frac{dT}{dt} = \frac{d}{dt}(25 - 21e^{-0.5t}) = -21(-0.5)e^{-0.5t} = 10.5e^{-0.5t}$. As t increases, this derivative decreases, which confirms the conclusion in part i).

***

Although we do not prove it here, the exponential function $f(x) = ce^{kx}$ is the only function for which the derivative is a constant times the function itself. That is $\frac{dy}{dx} = ky$ or $f'(x) = k \cdot f(x)$, where $y = c\,e^{kx}$ or $f(x) = c\,e^{kx}$ for some constant c.

For example, if $\frac{dy}{dx} = 5y$, then $y = c\,e^{5x}$, where c is an arbitrary constant. We may check this, as we get $y = c\,e^{5x}$, then $y' = c.5.e^{5x} = 5.y$. We can say that, the solution of $2x + 5 = 11$ is the number 3, and the solution of the equation $\frac{dy}{dx} = ky$ is the function $y(x) = c\,e^{kx}$. Now, let us see its application in the following example.

Example 14: The population of a particular species was approximately 8.04 thousand at the beginning of 2010. From the estimation, it is known that the population is growing exponentially at the rate of 0.02 or 2% per year. Thus, $\frac{dP}{dt} = 0.02P$, where P is the growth and t is the time in years.

i) Find P as a function of t.

ii) What would be the population at the beginning of 2050.

iii) After what period of time will the population be double of what it was in 2010?

Solution: i) $P(t) = c\,e^{kt} = 8.04\,e^{0.02\,t}$

ii) $P(40) = 8.04\,e^{0.02\,(40)} = 8.04\,e^{0.8} \approx 17.89 \approx 18$ thousand

iii) Let the time period, in which population would be double, is t, then $e^{0.02\,t} = 2$

Taking $\ln$ both the sides, we get $0.02t = \ln 2$, and $t = \frac{\ln 2}{0.02} \approx 33$ yrs.

Thus, the population in 2010 would double itself in 2043.

***

See if you can solve this exercise now.

E3) Find the derivative of:

i)	$5e^{(x^2-2)T}$, where T is a constant	ii)	$e^{(x+1)/x}$
iii)	$(x+2)e^{\sqrt{x}}$	iv)	$e^{-m\tan^{-1}x}$, where m is a constant
v)	2^{2x}	vi)	$7^{\cos x}$

E4) How much faster is $f(x) = 2^x$ increasing at $x = 1/2$ than at $x = 0$?

E5) Find the function that satisfies $\frac{d}{dt}[f(t)] = -3f(t)$.

In the following section, we shall find the derivatives of hyperbolic functions.

10.4 DERIVATIVE OF HYPERBOLIC FUNCTIONS AND THEIR INVERSE FUNCTIONS

Now, let us find derivatives of hyperbolic functions, which you have studied in Unit 7. Recall that the hyperbolic functions are defined in terms of the natural exponential function, whose derivative we already know, therefore, it is very easy to calculate their derivatives. For example, $\sinh x = \frac{e^x - e^{-x}}{2}$. This means,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\frac{d}{dx}(e^x - e^{-x}) = \frac{e^x + e^{-x}}{2} = \cosh x$$

Similarly, $\cosh x = \frac{e^x + e^{-x}}{2}$ gives us $\frac{d}{dx}(\cosh x) = \frac{e^x - e^{-x}}{2} = \sinh x$

In the case of $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, we get

$$\frac{d}{dx}(\tanh x) = \frac{(e^x + e^{-x})\,(e^x + e^{-x}) - (e^x - e^{-x})\,(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= 1 - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2} = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2$$

$$= 1 - \tanh^2 x = \text{sech}^2 x$$

We can adopt the same method for finding the derivatives of $\coth x$, $\text{sech} x$ and $\text{cosech}\, x$. You may like to find the derivatives of these. You would observe that the derivatives of functions with co are negative as in case of derivatives of trigonometric functions. In Table 1 we have collected all these results.

Table 1: Derivatives of Hyperbolic Functions

Function	Derivative
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\coth x$	$-\text{cosech}^2 x$
$\text{sech} x$	$-\text{sech}\, x \tanh x$
$\text{cosech}\, x$	$-\text{cosech}\, x \coth x$

Try some examples.

Example 15: Find dy/dx when $y = \tanh(1 - x^2)$.

Solution: $\frac{dy}{dx} = \text{sech}^2(1 - x^2).\frac{d}{dx}(1 - x^2)$

$$= -2x\ \text{sech}^2(1 - x^2)$$

***

Example 16: Differentiate $\cosh\sqrt{x}$ w.r.t. x.

Solution: $\frac{d}{dx}(\cosh\sqrt{x}) = \sinh\sqrt{x}.\frac{d}{dx}\left(\sqrt{x}\right) = \frac{\sinh\sqrt{x}}{2\sqrt{x}}$.

***

Example 17: Find the slope of the tangent of the catenary $y = 10\cosh\left(\frac{x}{10}\right)$ at $x = 5$.

Solution: $\frac{dy}{dx} = \frac{d}{dx}\left(10\cosh\left(\frac{x}{10}\right)\right)$

$$= 10.\sinh\left(\frac{x}{10}\right) \times \frac{1}{10}$$

$$= \sinh\left(\frac{x}{10}\right)$$

$$\left(\frac{dy}{dx}\right)_{\text{at } x=5} = \sinh\left(\frac{1}{2}\right)$$

$\therefore$ The required slope $= \frac{e^{1/2} - e^{-1/2}}{2}$

***

See if you can solve these exercises on your own.

E6) Find $f'(x)$ when $f(x) =$

i) $\tanh\frac{4x+1}{5}$ ii) $\sinh e^{2x}$

iii) $\coth(1/x)$ iv) $\operatorname{sech}(\ln x)$

v) $e^x \cosh x$

E7) At what point of the curve $y = \cosh x$ does the tangent have slope 1?

Now, we shall find derivatives of inverse hyperbolic functions. Recall Unit 7 for inverse hyperbolic functions.

Let us start with the inverse hyperbolic sine function.

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{d}{dx}\left(\ln\left(x + \sqrt{1+x^2}\right)\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}}\ \frac{d}{dx}\left(x + \sqrt{1+x^2}\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{x}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}$$

Thus, $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} = \frac{1}{\sqrt{x^2+1}}$

Further, $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{x+\sqrt{x^2-1}}\ \frac{d}{dx}\left(x + \sqrt{x^2-1}\right)$

$$= \frac{1}{\sqrt{x^2 - 1}}, \ x > 1.$$

Note that the derivative of $\cosh^{-1} x$ does not exist at $x = 1$.

Now, we can find the derivatives of each of these inverse hyperbolic functions. We proceed exactly as we did for the inverse hyperbolic sine and cosine functions and the derivatives are given in Table 2.

Table 2: Derivatives of inverse hyperbolic functions

Function	Derivative
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}, x > 1$
$\tanh^{-1} x$	$\frac{1}{1-x^2}, \lvert x \rvert < 1$
$\coth^{-1} x$	$\frac{1}{1-x^2}, \lvert x \rvert > 1$
$\operatorname{sech}^{-1} x$	$\frac{-1}{x\sqrt{1-x^2}}, 0 < x < 1$
$\operatorname{cosech}^{-1} x$	$\frac{-1}{\lvert x \rvert \sqrt{1+x^2}}, x \neq 0$

Let us use these results to solve some problems now.

Example 18: Find the derivatives of
i) $f(x) = \sinh^{-1}(\tan x)$, and ii) $g(x) = \tanh^{-1}(\cos e^x)$ with respect to x.

Solution: i) Let's start with $f(x) = \sinh^{-1}(\tan x)$.

$$f'(x) = \frac{1}{\sqrt{\tan^2 x + 1}} \frac{d}{dx}(\tan x)$$

$$= \frac{1}{|\sec x|} \sec^2 x = |\sec x|$$

ii) Now if $g(x) = \tanh^{-1}(\cos e^x)$, this means that

$$g'(x) = \frac{1}{1 - \cos^2 e^x} \frac{d}{dx}(\cos e^x)$$

$$= \frac{1}{\sin^2 e^x}(-\sin e^x).e^x$$

$$= \frac{-e^x}{\sin e^x} = -e^x \operatorname{cosec} e^x$$

***

We are now listing some functions for you to differentiate.

E8) Differentiate the following functions on their respective domains.

i) $\operatorname{cosech}^{-1}\left(5\sqrt{x}\right)$ ii) $[\operatorname{sech}^{-1}(\cos^2 x)]^{1/3}$

iii) $\coth^{-1}(e^{(x^2+5x-6)})$ iv) $\tanh^{-1}(\coth x)+\coth^{-1}(2x)$

v) $\sinh^{-1}\sqrt{x}+\cosh^{-1}(2x^2)$

In the following section, we shall study different methods of finding derivatives.

10.5 METHODS OF LOGARITHMIC DIFFERENTIATION

In this section, we shall study different methods of finding derivatives. We shall also see that the problem of differentiating some functions is greatly simplified by using these methods. Some of the results we derived in the earlier sections will be useful to us here.

In Unit 9 we have seen that $\frac{d}{dx}(x^r)=rx^{r-1}$ when r is a rational number. Now, we are in a position to extend this result to the case when r is any real number. So, if $y=x^r$, where $x>0$ and $r\in\mathbb{R}$, we can write this as $y=e^{\ln x^r}=e^{r\ln x}$, since the natural exponential and logarithmic functions are inverses of each other.

Thus, $\frac{dy}{dx}=\frac{d}{dx}(e^{r\ln x})=e^{r\ln x}\frac{d}{dx}(r\ln x)$

$$=re^{r\ln x}\frac{1}{x}=\frac{rx^r}{x}=rx^{r-1}$$

If $x<0$, x^r may not be a real number. For example $-3^{1/2}=\sqrt{-3}\notin\mathbb{R}$.

This proves that

$$\frac{d}{dx}(x^r)=rx^{r-1} \text{ for } x>0,\ r\in\mathbb{R}.$$

We are sure, you will be able to solve this exercise now.

E9) Differentiate

i) $x^{\sqrt{2}}$ ii) x^e

Sometimes we find that the process of taking derivatives becomes simple if we take logarithms before differentiating. Here, we shall illustrate this point through some examples. But to take the logarithm of any quantity we have to be sure that it is non-negative. To overcome this difficulty, let us first try to find the derivatives of $\ln(|x|)$ not by the definition of modulus function as we did in Example 2.

You may recall from Unit 6, that $|x|=\sqrt{x^2}$.

Therefore, $\ln(|x|)=\ln\sqrt{x^2}$, and

$$\frac{d}{dx}\ln|x|=\frac{d}{dx}\ln\sqrt{x^2}=\frac{1}{\sqrt{x^2}}\frac{d}{dx}\left(\sqrt{x^2}\right)$$

$$=\frac{1}{\sqrt{x^2}}\frac{x}{\sqrt{x^2}}=\frac{x}{x^2}=\frac{1}{x}$$

Thus, we get,

$$\frac{d}{dx}\ln(|x|) = \frac{1}{x}.$$

Using chain rule we can now conclude that if u is any function of x, then $\frac{d}{dx}\ln(|u|) = \frac{1}{u}.\frac{du}{dx}$. Here we can say that whenever we calculate derivatives of the functions involving products, quotients or powers, we can use the method of logarithmic differentiation. For this we follow the following steps:

i) Take natural logarithm of both the sides of $y = f(x)$.

ii) Simplify the equations using various properties of logarithms.

iii) Differentiate with respect to x.

iv) Solve the result to obtain y'.

Let us see how this result helps us in simplifying the differentiation of some functions.

Example 19: Differentiate $\frac{(x+1)^4 (x^2-3)^{1/2}}{(x-2)^{3/4}(x^3+x+3)^{-1/5}}$ with respect to x.

Solution: Let $y = \frac{(x+1)^4 (x^2-3)^{1/2}}{(x-2)^{3/4}(x^3+x+3)^{-1/5}}$. In y, all the terms except $(x^3+x+3)^{-1/5}$ are positive. Thus, before taking the logarithm, we take modulus of y.

Thus, $|y| = \frac{|x+1|^4 |x^2-3|^{1/2}}{|x-2|^{3/4}|x^3+x+3|^{-1/5}}$

Then, taking logarithms of both sides, we get

$$\ln|y| = \ln(|x+1|^4|x^2-3|^{1/2}) - \ln(|x-2|^{3/4}|x^3+x+3|^{-1/5}) \quad [\because \ln(a/b) = \ln a - \ln b]$$

$$= \ln(|x+1|^4) + \ln(|x^2-3|^{1/2}) - \ln(|x-2|^{3/4}) - \ln(|x^3+x+3|^{-1/5}) [\because \ln(ab) = \ln a + \ln b]$$

$$= 4\ln|x+1| + \frac{1}{2}\ln|x^2-3| - \frac{3}{4}\ln|x-2| + \frac{1}{5}\ln|x^3+x+3| \quad [\because \ln(a^b) = b\ln a]$$

Differentiating throughout we get,

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{x+1} + \frac{1}{2(x^2-3)}(2x) - \frac{3}{4(x-2)} + \frac{3x^2+1}{5(x^3+x+3)}$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{4}{x+1} + \frac{x}{x^2-3} - \frac{3}{4(x-2)} + \frac{3x^2+1}{5(x^3+x+3)}\right]$$

$$= \frac{(x+1)^4(x^2-3)^{\frac{1}{2}}}{(x-2)^{\frac{3}{4}}(x^3+x+3)^{-\frac{1}{5}}}\left[\frac{4}{x+1} + \frac{x}{x^2-3} - \frac{3}{4(x-2)} + \frac{3x^2+1}{5(x^3+x+3)}\right].$$

***

Example 20: Differentiate $x^{\sin x}, x > 0$, with respect to x.

Solution: Let us write $y = x^{\sin x}$. Since $x > 0$, therefore, $y > 0$ and so we can take logarithms of both sides to the base e and write

$$\ln y = \ln x^{\sin x} = \sin x.\ln x$$

Differentiating throughout, we get,

$$\frac{1}{y}\frac{dy}{dx} = \sin x \frac{1}{x} + \cos x \ln x$$

$$= \frac{\sin x}{x} + \cos x \ln x$$

Therefore, $\frac{dy}{dx} = y\left(\frac{\sin x}{x} + \cos x \ln x\right)$

or, $\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \ln x\right)$

***

Example 21: Differentiate $x^{\cos x} + (\cos x)^x$ with respect to x.

Solution: Let $f(x) = x^{\cos x}$ and $g(x) = \cos x^x$. To ensure that $f(x)$ and $g(x)$ are well defined, let us restrict their domain to $[0, \pi/2]$.

$y = x^{\cos x} + (\cos x)^x = f(x) + g(x) > 0$ for $x \in [0, \pi/2]$

Let us differentiate both $f(x)$ and $g(x)$ by taking logarithms.

We have, $f(x) = x^{\cos x}$

Therefore, $\ln f(x) = \cos x \ln x$.

Thus, $\frac{1}{f(x)} f'(x) = -\sin x \ln x + \cos x \frac{1}{x}$

That is, $f'(x) = f(x)\left(-\sin x \ln x + \frac{\cos x}{x}\right)$

$$= x^{\cos x}\left(\frac{-x \sin x \ln x + \cos x}{x}\right)$$

$$= x^{\cos x - 1}(\cos x - x \sin x \ln x)$$

Similarly, $g(x) = (\cos x)^x$ and so $\ln g(x) = x \ln \cos x$

Then, $\frac{1}{g(x)} g'(x) = \ln \cos x + \frac{x}{\cos x}(-\sin x)$

$\Rightarrow \quad g'(x) = (\cos x)^x \left(\frac{\cos x \ln \cos x - x \sin x}{\cos x}\right)$

$$= (\cos x)^{x-1}(\cos x \ln \cos x - x \sin x)$$

Hence, $\frac{dy}{dx} = f'(x) + g'(x)$

$$= x^{\cos x - 1}(\cos x - x \sin x \ln x) + \cos x^{x-1}(\cos x \ln \cos x - x \sin x)$$

***

If you have followed these examples you should have no difficulty in solving these exercises by the same method.

E10) Differentiate the following with respect to x.

i) $(x^2 - 1)(x^2 + 2)^6(x^3 - 1)^5$ ii) $\frac{1}{(x-1)^5(x-2)^6(x-3)^7}$

iii) $(\sin x)^x + (\cos x)^{\tan x}$ iv) $(x^x)^x + x^{(x^x)}$

v) $(\sin x)^{\ln x} + x^x$

It is not always necessary to express y explicitly in terms of x (as in $y = f(x)$) to find its derivative. We shall now see how to differentiate a function defined implicitly by a relation in x and y (such as, $f(x, y) = 0$) in the following section.

10.6 IMPLICIT DIFFERENTIATION

So far, we saw that most functions were written in the form $y = f(x)$, that is dependent variable is expressed in terms of independent variable. In such functions, y is said to be an explicit function of x. Sometimes, with an equation like $y^4 + x^2y^2 - x^4y = 21$, it may be cumbersome or nearly impossible to express y in terms of x. In such a case, we have an implicit relation between the variables x and y. Then, we can find the derivative of y with respect to x using a process called implicit differentiation. Not only in case of implicit function, sometimes implicit differentiation also allows us to find dy/dx without solving for y.

For example, consider the equation $y^3 = x$. This equation can be solved for y in terms of x, but we can use implicit differentiation to find dy/dx. To do so, we use chain rule.

$$\frac{d}{dx}y^3 = \frac{d}{dx}x$$

$$\Rightarrow 3y^2\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3y^2} = \frac{1}{3}(x^{1/3})^{-2} = \frac{1}{3}x^{-2/3}.$$

Now, consider another equation $x^2 + y^2 = 16$. On solving the equation, we get $y = \pm\sqrt{16 - x^2}$, where $y = \sqrt{16 - x^2}$ represents the top half of the circle and $y = -\sqrt{16 - x^2}$ represents the bottom half. So y is a function of x on the top half and y is a different function of x on the bottom half.

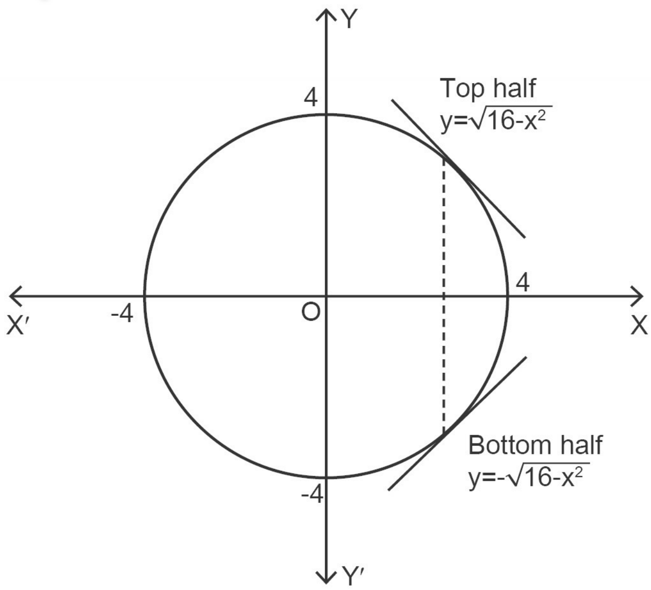

Fig. 6: Graph of $x^2 + y^2 = 16$

But let us consider the circle as a whole. The equation does represent a curve which has a tangent line at each point. The slope of this tangent can be found by differentiating the equation of the circle with respect to x.
That is

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(16)$$

$$\Rightarrow \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=0.$$

If we think that y is a function of x and apply the chain rule, we get

$$2x+2y\frac{dy}{dx}=0$$

On solving, we get $\frac{dy}{dx}=-\frac{x}{y}$.

Here, you see the derivative depends on both x and y instead of just on x.
This is because for each $x-$value (except for $x=\pm 4$) there are two $y-$values, and the curve has a different slope at each one.

If x and y both are positive, then we are in first quadrant and the slope is negative. Similarly the slope is negative when both x and y are negative. For x positive and y negative or x negative and y positive, the slope is positive. Also, the slope at $(4,0)$ and $(-4,0)$ has denominator zero, that means the tangents are vertical at these points.

In general, this process of implicit differentiation leads to a derivative whenever the expression for the derivative does not have a zero in the denominator.

Example 22: Find the values of y at $x-$values $6.8, 6.9, 7.0, 7.1$ and 7.2 for the equation $y^3-xy=-6$ near $x=7, y=2$.

Solution: To find the values of y for the given values of x, we would like to solve for y in terms of x, but we cannot isolate y by factoring. There is a formula for solving cubics, somewhat like the quadratic formula, but it is too complicated to be useful here. Instead, first observe that $x=7$ and $y=2$ satisfies the equation. Now, we shall use the implicit differentiation to find $\frac{dy}{dx}$.

That is $\frac{d}{dx}(y^3-xy)=\frac{d}{dx}(-6)$

$$3y^2.\frac{dy}{dx}-\left(1.y+x.\frac{dy}{dx}\right)=0$$

On simplifying, we get $\frac{dy}{dx}=\frac{y}{3y^2-x}$.

$$\left(\frac{dy}{dx}\right)_{\text{at }(7,2)}=\frac{2}{12-7}=\frac{2}{5}$$

Thus, the equation of the tangent at $(7,2)$ is

$$y-2=\frac{2}{5}(x-7)$$

$$\Rightarrow y=0.4x-0.8.$$

Fig.7 shows the graph of the curve, in which we see that the tangent is very close to the curve near the point $(7, 2)$. Therefore, we use the equation of the tangent line to calculate the approximate values of y at the given values of x.

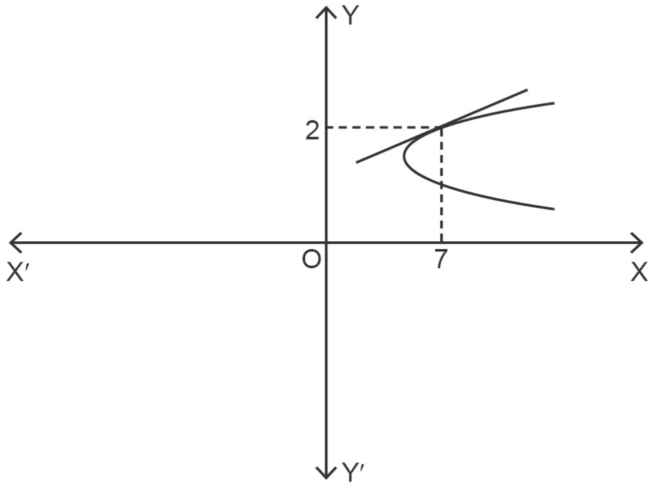

Fig. 7: Graph of $y^3 - xy = -6$

The approximate values of y are given in Table 3.

Table 3

x	6.8	6.9	7.0	7.1	7.2
y	1.92	1.96	2.00	2.04	2.08

***

See if you can solve these exercises now.

E11) Find all the points where the tangent line to $y^3 - xy = -6$ is either horizontal or vertical.

E12) Find the equations of the tangents of the following curves at the mentioned point.

i) $\ln(xy) = 2x$ at $(1, e^2)$

ii) $x^{2/3} + y^{2/3} = a^{2/3}$ at $(a, 0)$.

Let us apply implicit differentiation in some more equations given in the following examples.

Example 23: Find $\frac{dy}{dx}$ if x and y are related by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Solution: Differentiating throughout with respect to x, we get

$$a\frac{d}{dx}(x^2) + 2h\frac{d}{dx}(xy) + b\frac{d}{dx}(y^2) + 2g\frac{dx}{dx} + 2f\frac{dy}{dx} + \frac{dc}{dx} = 0$$

$$\Rightarrow 2ax + 2h.1.y + 2hx.\frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

or, $\frac{dy}{dx}(2hx + 2by + 2f) = -2ax - 2hy - 2g$

or, $\frac{dy}{dx} = \frac{-(ax + hy + g)}{(hx + by + f)}$.

***

Example 24: Find $\frac{dy}{dx}$ if $5y^2 + \sin y = x^2$.

Solution: $\frac{d}{dx}\left[5y^2 + \sin y\right] = \frac{d}{dx}[x^2]$

$\Rightarrow 5\frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] = 2x$

$\Rightarrow 5 \,.\, 2y.\frac{dy}{dx} + \cos y.\frac{dy}{dx} = 2x$

$\therefore \frac{dy}{dx} = \frac{2x}{10y + \cos y}$

***

Example 25: Find $\frac{dy}{dx}$, if $x^3 + y^3 = 4xy$. Find the tangent to $x^3 + y^3 = 4xy$ at the point $(2,2)$. Also, find the point in the first quadrant, where that tangent is horizontal.

Solution: Differentiating $x^3 + y^3 = 4xy$ both the sides w.r.t. x, we get

$3x^2 + 3y^2\frac{dy}{dx} = 4y + 4x\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$.

Slope of the tangent at $(2,2) = \left(\frac{dy}{dx}\right)$ at $(2,2)$

$$= \frac{4(2) - 3(2)^2}{3(2)^2 - 4(2)} = -1$$

Equation of the tangent $(2,2)$ with slope -1 is $(y - 2) = (-1)(x - 2)$ or $x + y - 4 = 0$. Now, the tangent line is horizontal when slope of the tangent is 0. Therefore, $\frac{dy}{dx} = 0$ gives $4y - 3x^2 = 0$ (provided $3y^2 - 4x \neq 0$). Solving $4y - 3x^2 = 0,$ we get $y = \frac{3x^2}{4}$.

Substituting the value of y in the given equation of the curve, we get

$x^3 + \left(\frac{3x^2}{4}\right)^3 = 4x\left(\frac{3x^2}{4}\right)$

$\Rightarrow x^3 + \frac{27x^6}{64} = 3x^3$

$\Rightarrow x^6 - \frac{128x^3}{27} = 0$

$\Rightarrow x = 0$ or $\frac{2^{7/3}}{3}$

Since $x = 0$ is not in first quadrant, thus $x = \frac{2^{7/3}}{3}$ and corresponding $y = 2^{8/3}$.

Hence $\left(\frac{2^{7/3}}{3}, 2^{8/3}\right)$ is the point in the first quadrant, where the tangent of the curve is horizontal.

***

Example 26: For the demand equation $x = \sqrt{100 - y^3}$, where y is the sale price (in thousand rupees), and x is sale, differentiate implicity to find dy/dx.

Solution: $\frac{d}{dx}x = \frac{d}{dx}\sqrt{100 - y^3}$

$\Rightarrow 1 = \frac{1}{2}(100 - y^3)^{-1/2}.(-3y^2).\frac{dy}{dx}$

$\Rightarrow 1 = \frac{-3y^2}{2\sqrt{100 - y^3}}.\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2\sqrt{100 - y^3}}{-3y^2}$

***

See, if you can find $\frac{dy}{dx}$ for the following implicit functions.

E13) Find $\frac{dy}{dx}$ if x and y are related as follows:

i) $x^2 + y^2 = 1$

ii) $y^2 = 4ax$

iii) $x^3y^3 + x^2y^2 + xy + 1 = 0$

iv) $\cos x \cos y - y^2 \sin^{-1} x + 2x^2 \tan x = 0$

v) $4x^2 - 2y^2 = 1$

vi) $xy^2 = 3 - y$

E14) Find $\frac{dy}{dx}$, if the equation of the curve is $y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$. Find the points, where this curve has horizontal tangents.

E15) Two cars start from the same point at the same time. One travels towards south at 40 km/h, and other travels towards west at a special of 30 km/h. How fast is the distance between them increasing at the end of 1 h?

In the following section, you will study various other techniques of differentiation.

10.7 OTHER DIFFERENTIATION TECHNIQUES

Till now we were concerned with functions which were expressed as $y = f(x)$.

We called x an independent variable, and y, a dependent variable. But sometimes the relationship between two variables x and y may be expressed in terms of another variable, say t. That is, we may have a pair of equations $x = \phi(t),\ y = \psi(t)$, where the functions ϕ and ψ have a common domain. For example, the circle $x^2 + y^2 = a^2$ is also described by the pair of equations, $x = a\cos t,\ y = a\sin t,\ 0 \le t \le 2\pi$.

In such cases, the auxiliary variable t is called a parameter and the equations $x = \phi(t),\ y = \psi(t)$ are called parametric equations. Such representation is called parametric representation. You may refer Appendix 1 for parametric representation of curves. Let $x = f(t)$ and $y = g(t)$. Suppose, we are able to eliminate the parameter from this parametric equation and can write the parametric equation in the form $y = F(x)$. Doing this gives $g(t) = F(f(t))$.

Now, differentiating with respect to t, we get

$$\frac{d}{dt}[g(t)] = \frac{d}{dt}[F(f(t))]$$

$$\Rightarrow g'(t) = F'(f(t)).f'(t)$$

$$\Rightarrow \frac{dy}{dt} = F'(x).\frac{dx}{dt}$$

$$\Rightarrow F'(x) = \frac{dy/dt}{dx/dt}, \text{provided } \frac{dx}{dt} \neq 0.$$

$$\Rightarrow \frac{dy}{du} = \frac{dy/dt}{dx/dt}.$$

Now, suppose a function is defined in terms of a parameter. To obtain its derivative, we need only to differentiate the relations in x and y separately with respect to the parameter. The following example illustrates this method.

Example 27: Find $\frac{dy}{dx}$ if $x = a\cos\theta$ and $y = b\sin\theta$, where θ is a parameter.

Solution: We differentiate the given equation w.r.t θ, and get

$$\frac{dy}{d\theta} = b\cos\theta, \text{ and } \frac{dx}{d\theta} = -a\sin\theta$$

Now, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$

***

Try to use this method now.

E16) Find $\frac{dy}{dx}$ if

i) $x = a\cos\theta,\ y = a\sin\theta$

ii) $x = at^2,\ y = 2at$

iii) $x = a\cos^3\theta,\ y = b\sin^3\theta$

iv) $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$

E17) Find the equation of the tangent line to the parametric curve given by $x = t^5 - 4t^3, y = t^2$ at $(0, 4)$.

Sometimes, the process of finding derivatives is simplified to a large extent by making use of some suitable transformations. We shall see some examples which will illustrate this fact.

Example 28: Find the derivative of $y = \cos^{-1}(4x^3 - 3x)$ using transformation.

Solution: As you know, we can differentiate this function by using the formula for the derivative of $\cos^{-1} x$ and the chain rule. But suppose we put $x = \cos\theta$, then we get

$$\begin{aligned} y &= \cos^{-1}(4\cos^3\theta - 3\cos\theta) \\ &= \cos^{-1}(\cos 3\theta) \qquad [\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta] \\ &= 3\theta \\ &= 3\cos^{-1} x . \end{aligned}$$

Now this is a much simpler expression, and can be differentiated easily as:

$$\frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}} .$$

***

Example 29: Using the transformation, differentiate $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to x.

Solution: Use the transformation $x = \tan\theta$. This gives us,

$$\begin{aligned} y &= \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) \\ &= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left[\frac{1-(1-2\sin^2\theta/2)}{2\sin\theta/2\cos\theta/2}\right] \quad \left[\begin{array}{l}\because \sin\theta = 2\sin\theta/2\cos\theta/2 \\ \cos\theta = 1-2\sin^2\theta/2\end{array}\right] \\ &= \tan^{-1}(\tan\theta/2) \\ &= \theta/2 = \frac{\tan^{-1} x}{2} . \end{aligned}$$

Now, we can write $\frac{dy}{dx} = \frac{1}{2(1+x^2)}$.

***

Let's tackle another problem.

Example 30: Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Solution: For this, let $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $z = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$. Our aim is to find dy/dz.

We shall use the transformation $x = \tan\theta$. This gives us

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta, \text{ and}$$

$$z = \sin^{-1}\left(\frac{2\tan 0}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta .$$

Now if we differentiate y and z with respect to θ, we get $dy/d\theta = 2$ and $dz/d\theta = 2$.

Therefore, $\frac{dy}{dz} = \frac{dy/d\theta}{dz/d\theta} = 1$.

***

So, you see, a variety of complex problems can be solved easily by using transformations. The key to a successfully solution is, however, the choice of a suitable transformation. We are giving some exercises below, which will give you the necessary practice in choosing the right transformation.

E18) Find the derivatives of the following functions using suitable transformation:

i) $\sin^{-1}(3x - 4x^3)$ ii) $\cos^{-1}(1 - 2x^2)$

iii) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ iv) $\tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

v) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Now let us summarise the points covered in this unit.

10.8 SUMMARY

In this unit we have covered the following points.

1. Obtained derivatives of the exponential and logarithmic functions, hyperbolic functions and their inverses. We give them in the following table.

Function	Derivative
e^x	e^x
$\ln x$	$\frac{1}{x}$
a^x	$a^x \ln a$
$\log_a x$	$\frac{1}{x}\log_a e$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\coth x$	$-\text{cosech}^2 x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\text{cosech}\, x$	$-\text{cosech}\, x \coth x$

2. The derivatives of inverse hyperbolic functions.

Function	Derivative
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}, \ x > 1$

$\tanh^{-1} x$	$\frac{1}{1-x^2}, \lvert x \rvert < 1$
$\coth^{-1} x$	$\frac{1}{1-x^2}, \lvert x \rvert > 1$
$\text{sech}^{-1} x$	$\frac{-1}{x\sqrt{1-x^2}}, 0 < x < 1$
$\text{cosech}^{-1} x$	$\frac{-1}{\lvert x \rvert \sqrt{1+x^2}}, x \neq 0$

3. Extended the result $\frac{d}{dx}(x^r) = rx^{r-1}$ to all $r \in \mathbb{R}$ and $x > 0$

4. Logarithm Differentiating:
   - Take modulus, if the function attains negative value for some x.
   - Take logarithm both the sides.
   - Differentiate using chain rule as $\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$.

5. Implicit Differentiation:
   - Differentiate both the sides of the equation with respect to x or whatever variable you are differentiating with respect to.
   - Apply the rules of differentiation as necessary.
   - Combine all the terms with $\frac{dy}{dx}$ as a factor on one side of the equation.
   - Find $\frac{dy}{dx}$.

6. Differentiation of Parametric Equations:

   If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

7. Various transformations can also be used to find derivatives.

10.9 SOLUTIONS/ANSWERS

E1) i) $\frac{d}{dx}(\log_2 2x) = \log_2 e . \frac{1}{2x} . \frac{d}{dx}(2x) = \frac{1}{x}\log_2 e$

ii) $\frac{d}{dx}(7\log_{11}(5x^2+2)) = 7\frac{d}{dx}(\log_{11}(5x^2+2))$

$$= 7\log_{11} e . \frac{1}{5x^2+2}\frac{d}{dx}(5x^2+2)$$

$$= \frac{70x}{5x^2+2}\log_{11} e.$$

iii) $\frac{d}{dx}(x^2 \ln x) = x^2 \frac{d}{dx}(\ln x) + \ln x . \frac{d}{dx}(x^2)$

$= x + (2x)\ln x$.

iv) $\frac{d}{dx}\left(\ln\left(\frac{1+x}{1-x}\right)\right), |x| < 1 = \frac{1-x}{1+x} . \frac{d}{dx}\left(\frac{1+x}{1-x}\right)$

$= \frac{1-x}{1+x} . \frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}$

$= \frac{2}{(1-x^2)}, |x| < 1.$

v) $\frac{d}{dx}(\ln(\sin^4 x)) = \frac{1}{\sin^4 x} . \frac{d}{dx}(\sin^4 x)$

$= \frac{1}{\sin^4 x} . 4\sin^3 x . \frac{d}{dx}\sin x$

$= \frac{4\cos x}{\sin x} = 4\cot x$.

vi) $\frac{d}{dx}(\log_{10}(2+\sin x)) = \frac{d}{dx}(\log_{10} t) . \frac{dt}{dx}$, where $t = 2+\sin x$

$= \frac{1}{t}\log_{10} e . \frac{d}{dx}(2+\sin x)$

$= \frac{\log_{10} e . \cos x}{(2+\sin x)}$

E2) Given is $w_o = 10^{-12}\, w/m^2$

$P = 10\log \frac{w}{10^{-12}}$

$= 10[\log w - 12\log 10]$

Now, using the formula for the derivative of a logarithm, and because $\log_{10}$ is a constant, we get

$\frac{dP}{dt} = 10\left[\frac{1}{w}\log_{10} e . \frac{dw}{dt} - 0\right]$

$= 10\left[\frac{1}{w}\log_{10} e . \frac{dw}{dt}\right]$

Now, we substitute our given values for w and $\frac{dw}{dt}$ for some t, we obtain

$\frac{dP}{dt} = 10\left[\frac{1}{7.2}\log_{10} e . (0.5)\right]$

$= 0.302\ dB/s$.

The unit of $\frac{dP}{dt}$ is dB/s because the sound pressure P is in dB is changing over time.

E3) i) $\frac{d}{dx}(5e^{(x^2-2)T}) = 5\frac{d}{dx}(e^{(x^2-2)T})$

$$= 5\frac{d}{dt}e^t.\frac{dt}{dx} \quad \text{where } t = (x^2-2)T$$
$$= 5e^{(x^2-2)T}.(2x)T$$
$$= 10x\,Te^{(x^2-2)T}.$$

ii) $$\frac{d}{dx}e^{(x+1)/x} = e^{(x+1)/x}.\frac{d}{dx}\left(\frac{(x+1)}{x}\right)$$
$$= e^{(x+1)/x}.\frac{x\frac{d}{dx}(x+1)-(x+1)\frac{dx}{dx}}{x^2}$$
$$= e^{(x+1)/x}.\frac{x.1-(x+1).1}{x^2} = -\frac{e^{(x+1)/x}}{x^2}$$

iii) $$\frac{d}{dx}((x+2)e^{\sqrt{x}}) = e^{\sqrt{x}}\frac{d}{dx}(x+2)+(x+2)\frac{d}{dx}e^{\sqrt{x}}$$
$$= e^{\sqrt{x}}.1+(x+2).\,e^{\sqrt{x}}.\frac{d}{dx}\sqrt{x}$$
$$= e^{\sqrt{x}}+(x+2).\,e^{\sqrt{x}}.\frac{1}{2\sqrt{x}}$$
$$= e^{\sqrt{x}}+(x+2).\frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

iv) $$\frac{d}{dx}(e^{-m\tan^{-1}x}) = \frac{d}{dt}(e^{-t}).\frac{dt}{dx} \quad \text{where } t = m\tan^{-1}x$$
$$= -e^t.\frac{d}{dx}(m\tan^{-1}x)$$
$$= -e^{-m\tan^{-1}x}.\frac{m}{1+x^2}$$
$$= \frac{-m\,e^{-m\tan^{-1}x}}{1+x^2}$$

v) $$\frac{d}{dx}(2^{2x}) = \frac{d}{dt}(2^t).\frac{dt}{dx} \quad \text{where } t = 2x$$
$$= 2^t\ln 2\,.\,2$$
$$= 2^{2x+1}\ln\,2$$

vi) $$\frac{d}{dx}(7^{\cos\,x}) = 7^{\cos x}.\ln 7.\frac{d}{dx}\cos x$$
$$= 7^{\cos x}.\ln 7.(-\sin x)$$
$$= -\sin x\ln 7.7^{\cos x}$$

E4) $f'(x) = 2^x\ln 2$

$f'(0) = \ln 2$

$f'(1/2) = 2^{1/2}\ln 2 = \sqrt{2}\ln 2$

Hence f increases $\sqrt{2}$ times faster at $x = 1/2$ than at $x = 0$.

E5) $f(t) = c\,e^{-3t}$

E6) i) $f'(x) = \sec h^2\left(\frac{4x+1}{5}\right).\frac{d}{dx}\left(\frac{4x+1}{5}\right) = \frac{4}{5}\sec h^2\left(\frac{4x+1}{5}\right)$

ii) $f'(x) = \cosh e^{2x}.\frac{d}{dx}e^{2x} = \cosh e^{2x}.e^{2x}.\frac{d}{dx}(2x) = 2e^{2x}\cosh e^{2x}$

iii) $f'(x) = -\text{cosech}^2(1/x).\frac{d}{dx}(1/x) = \frac{1}{x^2}\cos\text{ech}^2\left(\frac{1}{x}\right)$

iv) $f'(x) = -\sec h.(\ln x)\tanh(\ln x).\frac{d}{dx}(\ln x) = \frac{-\sec h(\ln x)\tanh(\ln x)}{x}$

v) $f'(x) = e^x.\sinh x + e^x.\cosh x = e^x(\sinh x + \cosh x)$

E7) $y = \cosh x$

$\frac{dy}{dx} = \sinh x = 1$

$\sinh x = 1$

$\Rightarrow \frac{e^x - e^{-x}}{2} = 1$

$\Rightarrow e^x - e^{-x} = 2$

$\Rightarrow e^x - e^{-x} - 2 = 0$

$\Rightarrow e^{2x} - 1 - 2e^x = 0$

$\Rightarrow e^x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

$e^x = 1+\sqrt{2}$ as it cannot be $-$ive .

$\Rightarrow x = \ln(1+\sqrt{2})$

$y = \frac{e^x - e^{-x}}{2} = \sqrt{2}$

Therefore, the points is $(\ln(1+\sqrt{2}), \sqrt{2})$.

E8) i) $\frac{-1}{5\sqrt{x}\sqrt{1+25x}}\left(\frac{5}{2\sqrt{x}}\right) = \frac{-1}{2x\sqrt{1+25x}}$

ii) $\frac{d}{dx}[\sec h^{-1}(\cos^2 x)]^{1/3}$

$= \frac{1}{3}[\sec h^{-1}(\cos^2 x)]^{-2/3}\left(\frac{-1}{\cos^2 x\sqrt{1-\cos^4 x}}\right)2\cos x(-\sin x)$

$= \frac{2\sin x[\sec h^{-1}(\cos^2 x)]^{-2/3}}{\cos x\sqrt{1-\cos^4 x}}.$

iii) $\frac{1}{1-e^{2(x^2+5x-6)}}e^{(x^2+5x-6)}(2x+5)$

iv) $\frac{-\text{cosech}^2 x}{1-\coth^2 x} + \frac{2}{(1-4x^2)}$

v) $\dfrac{1}{2\sqrt{x}\sqrt{1+x}}+\dfrac{4x}{\sqrt{4x^4-1}}$

E9) i) $\sqrt{2}x^{(\sqrt{2}-1)}$

ii) ex^{e-1}

E10) i) We take logarithm both the sides, and get

$\ln|y| = \ln|x^2-1| + 6\ln|x^2+2| + 5\ln|x^3-1|$

Differentiate both the sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2-1}+\frac{12x}{x^2+2}+\frac{15x^2}{x^3-1}$$

Thus, $\dfrac{dy}{dx} = (x^2-1)(x^2+2)^6(x^3-1)^5\left[\dfrac{2x}{x^2-1}+\dfrac{12x}{x^2+2}+\dfrac{15x^2}{x^3-1}\right]$

ii) $\ln|y| = -5\ln|x-1| - 6\ln|x-2| - 7\ln|x-3|$

$$\frac{dy}{dx} = \frac{-1}{(x-1)^5(x-2)^6(x-3)^7}\left(\frac{5}{x-1}+\frac{6}{x-2}+\frac{7}{x-3}\right)$$

iii) Let $f(x) = (\sin x)^x$ and $g(x) = (\cos x)^{\tan x}$

Then $f'(x) = \sin x^x(\ln \sin x + x\cot x)$ and

$g'(x) = \cos x^{\tan x}(\sec^2 x \ln \cos x - \tan^2 x)$

$$\frac{dy}{dx} = f'(x) + g'(x)$$

iv) Let $f(x) = (x^x)^x$, $g(x) = x^{(x^x)}$, $x > 0$

If $y = x^x$, $\ln y = x\ln x$

$\Rightarrow \dfrac{dy}{dx} = x^x(1+\ln x)$

$\ln f(x) = x\ln x^x$

$\Rightarrow \dfrac{1}{f(x)}f'(x) = \ln x^x + x(1+\ln x)$

$\Rightarrow f'(x) = (x^x)^x[\ln x^x + x(1+\ln x)]$

$\ln g(x) = x^x \ln x$

$\Rightarrow \dfrac{1}{g(x)}g'(x) = \dfrac{x^x}{x} + \ln x x^x(1+\ln x)$

$\Rightarrow g'(x) = x^{(x^x)}[x^{x-1} + x^x \ln x(1+\ln x)]$

Answer $= f'(x) + g'(x)$

$= (x^x)^x[\ln x^x + x(1+\ln x)] + x^{(x^x)}[x^{x-1} + x^x \ln x(1+\ln x)]$

v) $\dfrac{d}{dx}(\sin x)^{\ln x} = (\sin x)^{\ln x}\left(\ln x\cot x + \dfrac{\ln \sin x}{x}\right)$

$\dfrac{d}{dx}(x^x) = x^x(1+\ln x)$

Answer $= (\sin x)^{\ln x}\left(\ln x\cot x + \dfrac{\ln\sin x}{x}\right) + x^x(1+\ln x)$

E11) $\frac{d}{dx}[y^3 - xy] = \frac{d}{dx}[-6]$

$$\Rightarrow 3y^2 \frac{dy}{dx} - \left[1.\,y + x.\frac{dy}{dx}\right] = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{3y^2 - x}$$

For the horizontal tangent, numerator of $\frac{dy}{dx}$ equals 0, that is $y = 0$.

When we substitute $y = 0$ in the given equation of the curve, we get $0 = -6$, which is impossible. Therefore, there is no point on the curve where the tangent is horizontal.

For the vertical tangent, denominator of $\frac{dy}{dx}$ equals 0, that is

$3y^2 - x = 0$ giving $x = 3y^2$. Substituting this in the equation of the curve and solving, we get $y = \sqrt[3]{3}$ and $x = \frac{9}{\sqrt[3]{3}}$.

E12) i) $\frac{d}{dx}[\ln(xy)] = \frac{d}{dx}(2x)\, \frac{1}{xy}[x.\frac{dy}{dx} + 1.\,y] = 2$

$$\frac{dy}{dx} = \frac{2xy - y}{x}$$

$$\left.\frac{dy}{dx}\right|_{at\,(1,e^2)} = e^2$$

Thus, the equation of the tangent is $y - e^2 = e^2(x-1)$

ii) $\frac{d}{dx}[x^{2/3} + y^{2/3}] = \frac{d}{dx}(a^{2/3}) \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{at\,(a,0)} = \infty$$

Since, the slope at $(a,0)$ is undefined, therefore, the curve has a vertical tangent at $(a,0)$. The equation of this tangent is $x = a$.

E13) i) $2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

ii) $2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

iii) $3x^3y^2\frac{dy}{dx} + 3x^2y^3 + 2x^2y\frac{dy}{dx} + 2xy^2 + x\frac{dy}{dx} + y = 0$

$$\Rightarrow (3x^3y^2 + 2x^2y + x)\frac{dy}{dx} = -(3x^2y^3 + 2xy^2 + y)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(3x^2y^3 + 2xy^2 + y)}{(3x^3y^2 + 2x^2y + x)}$$

iv) $-\cos x \sin y \dfrac{dy}{dx} - \sin x \cos y - 2y\dfrac{dy}{dx}\sin^{-1} x - \dfrac{2}{\sqrt{(1-x^2)}} + 4x \tan x$

$+ 2x^2 \sec^2 x = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x \cos y + \dfrac{y^2}{\sqrt{(1-x^2)}} - 4x\tan x - 2x^2 \sec^2 x}{-(\cos x \sin y + 2y \sin^{-1} x)}$$

v) $\dfrac{d}{dx}(4x^2 - 2y^2) = \dfrac{d}{dx}(1)$

$\Rightarrow 8x - 4y\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{8x}{4y} = \dfrac{2x}{y}.$

vi) $\dfrac{d}{dx}(xy^2) = \dfrac{d}{dx}(3-y)$

$1.y^2 + x.2y\dfrac{dy}{dx} = -\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{y^2}{2xy+1}$

E14) $\dfrac{d}{dx}(y(y^2-1)(y-2)) = \dfrac{d}{dx}(x(x-1)(x-2))$

$\dfrac{dy}{dx}(y^2-1)(y-2) + 2y^2(y-2)\dfrac{dy}{dx} + y(y^2-1)\dfrac{dy}{dx}$

$= (x-1)(x-2) + x(x-1) + x(x-2)$

$\dfrac{dy}{dx} = -\dfrac{x(x-1)+x(x-2)+(x-1)(x-2)}{(y^2-1)(y-2)+2y^2(y-2)+y(y^2-1)}$. For horizontal tangent

$\dfrac{dy}{dx} = 0$

E15) Suppose the cars start at O and after time t, they reach A and B respectively as shown in Fig. 8.

Suppose $OA = x$ km and $OB = y$ km

$AB = d^2 = x^2 + y^2$

Also, after 1 h, $d^2 = 30^2 + 40^2$

$\Rightarrow d = 50$ km

Therefore, $(50)^2 = x^2 + y^2$

$\Rightarrow \dfrac{d}{dx}(2500) = \dfrac{d}{dx}[x^2 + y^2]$

$\Rightarrow 0 = 2x + 2y\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}.$

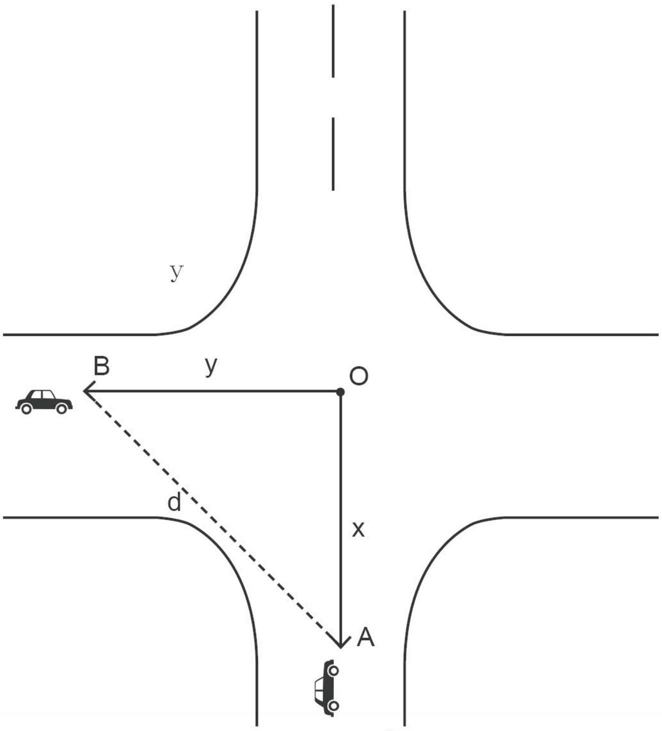

Fig. 8

E16) i) $\frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = a\cos\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\cot\theta$

ii) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$

iii) $\frac{dy}{dx} = \frac{-dy/d\theta}{dx/d\theta} = \frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{b}{a}\tan\theta$

iv) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{(1-\cos\theta)}$

E17) $\frac{dx}{dt} = 5t^4 - 12t^2, \frac{dy}{dt} = 2t$.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{2t}{5t^4 - 12t^2}$

$= \frac{2t}{5t^4 - 12t^2}$

Since $x = 0$ and $y = 4$, therefore $0 = t^5 - 4t^3 = t^3(t^2 - 4) \Rightarrow 4 = t^2$. Thus, $t = \pm 2$.

$\left.\frac{dy}{dx}\right|_{\text{at } t=2} = \frac{1}{8}$

$\therefore$ The tangent line is $y - 4 = \frac{1}{8}x$.

Also, $\left.\frac{dy}{dx}\right|_{\text{at } t=-2} = -\frac{1}{8}$

$\therefore$ The tangent line is $y-4=-\frac{1}{8}x$. The graph shown in Fig. 9 shows the tangents.

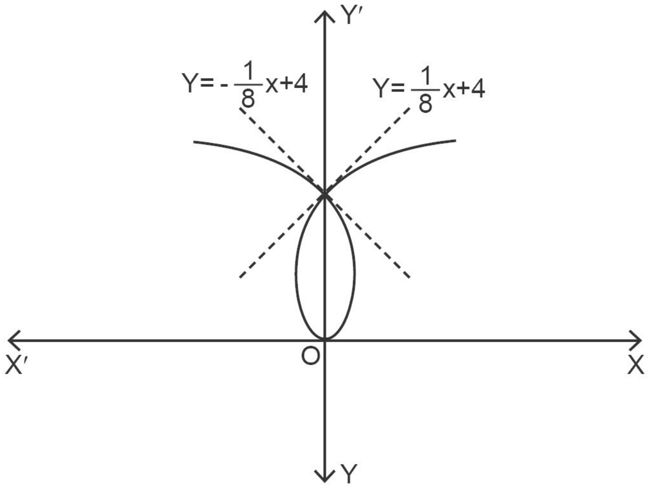

Fig. 9

E18) i) Put $x=\sin\theta$

$$\sin^{-1}(3x-4x^3)=\sin^{-1}(\sin 3\theta)=3\theta$$
$$=3\sin^{-1}x$$
$$\therefore \frac{d}{dx}(\sin^{-1}(3x-4x^3))=\frac{3}{\sqrt{1-x^2}}.$$

ii) Put $x=\sin\theta$

$$\frac{d}{dx}(\cos^{-1}(1-2x^2))=\frac{d}{dx}(\cos^{-1}\cos 2\theta)$$
$$=\frac{d}{dx}(2\theta)$$
$$=\frac{d}{dx}(2\sin^{-1}x)$$
$$=\frac{2}{\sqrt{1-x^2}}$$

iii) Put $x=\tan\theta, \frac{d}{dx}\left(\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right)=\frac{2}{\sqrt{1-x^2}}$

iv) Put $x=\tan\theta$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)\right)=\frac{3}{1+x^2}$$

v) Put $x=\tan\theta$

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right)=\frac{2}{1+x^2}.$$

UNIT 11

HIGHER ORDER DERIVATIVES

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11.1 INTRODUCTION

We have already seen that the concept of differentiation was motivated by some physical concepts (like the velocity of a moving particle) and also by geometrical notions (like the slope of a tangent to a curve). The second and higher order derivatives are also similarly motivated by some physical considerations (like the acceleration) and some geometrical ideas (like the curvature of a curve).

In Unit 9 and Unit 10, we studied about differentiation of functions. You know that the derivative f' of a function f is again a function, and is called the derived function of f. This new function f' may have a derivative of its own, which will again be a new function. In this unit, we will consider such functions.

We will find the second and third order derivatives in Sec. 11.2 and will continue the process of differentiation to find higher order derivatives in Sec. 11.3. Leibniz theorem, which is given in Sec. 11.4, gives us a formula for finding the higher derivatives of a product of two functions. Later in Sec. 11.5, we will consider how a polynomial is used to find the approximate value of a function at a point.

Now we shall list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

Objectives

After readying this unit, you should be able to:

- find the second and third order derivatives of differentiable function f;
- calculate the nth higher order derivatives of a given function f;
- use the Leibniz theorem to find the nth derivatives of product of two functions; and
- find the approximate value of a function at a given point using polynomial approximation.

11.2 SECOND ORDER DERIVATIVES

Let f be a given function which is differentiable. When we differentiate f, we get f'. We know that f' is a new function derived from f, so f' may or may not have its own derivative. If f' is differentiable, on differentiating f' again we get a new function.

For example, consider the function f defined by $f(x) = x^5$. We know that $f'(x) = 5x^4$. Now, this f' is again a polynomial function and hence, can also be differentiated. We can think of the derivative of f' as the rate of change of the slope of the tangent line of f. It can also be regarded as the rate at which f' is changing with respect to the independent variable. This rate of change of the slope of the tangent line develops curvature, which we shall study in Unit 14. Coming back to the derivative of f', we use the notation f'' for the derivative of f', that is, $(f')' = f''$. Thus, $f''(x) = 20x^3$. $f''(x)$ is called the **second derivative** of the function f at the point x.

Let $y = f(x)$, then we write the second derivative of y with respect to x as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ (read as d square y by d x square) or $f^{(2)}$ or D^2y or y'' or f''. Now let us find the second derivative in the following examples.

Example 1: Find f'', if $f(x) = x^2 + 2x$. Also draw the graphs of f, f' and f''. What is the relationship you see between the graphs, if any?

Solution: Given is that $f(x) = x^2 + 2x$. On differentiating $f(x)$ with respect to x, we get $f'(x) = 2x + 2$. Again on differentiating $f'(x)$ with respect to x, we get $f''(x) = 2$. You may note that f is a polynomial function, which is differentiable, f' is again a polynomial function which is differenatible. Fig. 1 shows the graphs of f, f' and f''.

Look at the graph of all three functions f, f' and f''. We see that f represents a parabola, f' represents a straight line. Next, f'' is the line showing a fixed value, this is parallel to $x-$axis. We can conclude that.
$f''(x)$ is the slope of the curve $y = f'(x)$ at the point $(x, f(x))$. You may note that f'' is positive, therefore f'' lies above the x-axis.

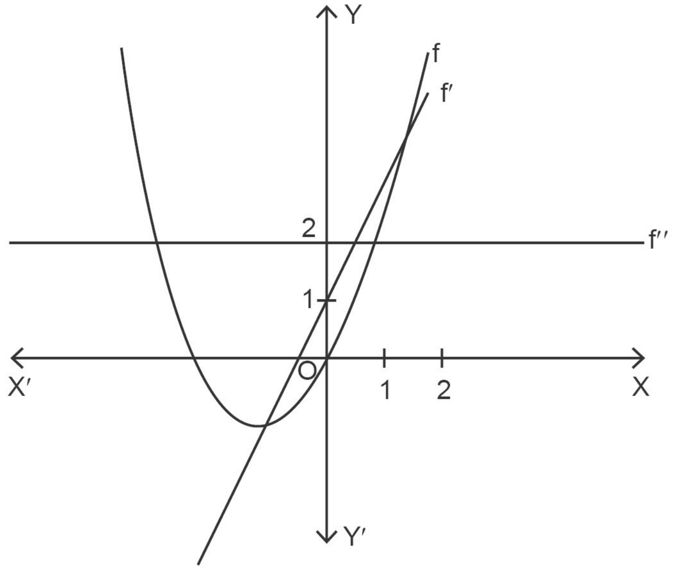

Fig. 1: Graph of f, f' and f''

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Example 2: If $y = 2\sin x + 3\cos x + 5$, then prove that $y'' + y = 5$.

Solution: Now, $y = 2\sin x + 3\cos x + 5$

Differentiating y with respect to x, we get

$$y' = 2\cos x - 3\sin x$$

Again differentiating w.r.t. x, we obtain

$$y'' = -2\sin x - 3\cos x$$

Now $y'' + y = -2\sin x - 3\cos x + 2\sin x + 3\cos x + 5 = 5$

***

In the introduction, we had mentioned that f' may not be differentiable. The next example gives a function f for which f' exists but f'' does not exist.

Example 3: Consider the function $f(x) = x|x|$ for all x in $\mathbb{R}$. Find the second derivative of f. Also, find the domain and range of the functions f' and f''.

Solution: The function $f(x)$ can be rewritten as $f(x) = \begin{cases} x^2, & \text{if } x \ge 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

$$\left[\because |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}\right]$$

At points other than 0, we have $f'(x) = 2x$ if $x > 0$ and $f'(x) = -2x$ if $x < 0$

At $x = 0$, the right derivative of f is given by

$Rf'(0) = \lim_{h\to 0^+} \frac{h^2 - 0^2}{h} = \lim_{h\to 0^+} h = 0$, and the left derivative of f is

$$Lf'(0) = \lim_{h\to 0^-} \frac{h^2 - 0^2}{h} = \lim_{h\to 0^-} h = 0.$$

Therefore, $f'(0) = 0$.

Thus, $f'(x) = 2|x|$ for all x in $\mathbb{R}$. The domain of f' is $\mathbb{R}$ and the range of f' is $[0, \infty[$.

We already know that the absolute value function $|x|$ fails to be differentiated at 0 [Refer Unit 9]. Therefore, f' is not differentiable at $x = 0$. Therefore, $f''(0)$ does not exist.

***

In the next example, we find the second order derivative, if both the variables are depending on a third variable.

Example 4: If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$.

Solution: Differentaiting x and y w.r.t t, we obtain

$$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t) = a\,t\cos t$$

and $$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = a\,t\sin t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \text{ provided } \frac{dx}{dt} \neq 0$$
$$= \tan t$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx} = \frac{d}{dt}(\tan t).\frac{dt}{dx} \qquad \text{[Using chain rule]}$$

$$= \sec^2 t.\frac{1}{a\,t\cos t} = \frac{1}{a\,t\cos^3 t}$$

***

In the next example, we find the second order derivative of an implicit function.

Example 5: If $a\,x^2 + 2hxy + by^2 = 1$, find $\frac{d^2y}{dx^2}$.

Solution: Differentiating the given equation w.r.t. x.

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2b\,y\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{ax + hy}{hx + by} \qquad \text{... (1)}$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{(hx + by)\left(a + h\frac{dy}{dx}\right) - (ax + hy)\left(h + b\frac{dy}{dx}\right)}{(hx + by)^2}$$

Substituting $\frac{dy}{dx}$ from (1) and solving it, we get $\frac{d^2y}{dx^2} = \frac{(h^2 - ab)}{(hx + by)^3}$.

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You may recall that the first derivative is the rate of change, therefore, the second derivative is the rate of change of the rate of change w.r.t. the **same variable**. For example, the rate of change of position of any object with respect to time is velocity. Velocity itself is a function of time. When a jet takes off or a vehicle comes to a sudden stop, the change in velocity is easily felt by the passengers. The rate at which velocity changes is called acceleration. Suppose that two cars start at rest. Car A reaches a speed of 60 km/h in 10 s and car B reaches a speed of 60 km/h in 8 s, then car B has a faster acceleration than A. We generally use the letter 'a' for acceleration. Let $s(t), v(t)$ and $a(t)$ be the position function, velocity and acceleration of an object respectively, then

$$v(t) = \frac{d}{dt}s(t) \text{ and } a(t) = \frac{d}{dt}(v(t)) = \frac{d}{dt}\left(\frac{d}{dt}s(t)\right) = \frac{d^2}{dt^2}(s(t)).$$

Let us find acceleration in the following examples.

Example 6: The equation of motion of a particle is $s(t) = t^3 - 3t$, where s is in meters and t is in seconds. Find

i) the velocity and acceleration as functions of t,

ii) the acceleration after 2 seconds, and

iii) the acceleration when the velocity is 0.

Solution: i) We have $s(t) = t^3 - 3t$.

$$\therefore v(t) = \frac{d}{dt}(s(t)) = \frac{d}{dt}(t^3 - 3t) = 3t^2 - 3$$

and $a(t) = \dfrac{d}{dt}(v(t)) = \dfrac{d}{dt}(3t^2 - 3) = 6t$.

ii) The acceleration after 2 seconds $= a(2) = 12\,m/s^2$.

iii) When velocity is 0,we have $v(t) = 0$, which gives $3t^2 - 3 = 0$ or $t = \pm 1$. As time cannot be negative, time $= 1$ second. Now acceleration after 1 second is $a(1) = 6\,m/s^2$.

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Example 7: A new product is placed on the market and becomes very popular. Its quantity sold N is given as a function of time t, where t is in weeks,

$$N(t) = \frac{250000\,t^2}{(2t+1)^2}, t > 0.$$

Find $N''(t)$, then use it to calculate $N''(52)$ and $N''(208)$ and interpret these results in the given situations.

Solution: To determine $N'(t)$ and $N''(t)$, we use the quotient rule.

$$N'(t) = \frac{d}{dt}\left[\frac{250000\ t^2}{(2t+1)^2}\right]$$

$$= \frac{(2t+1)^2 \dfrac{d}{dt}(250000\ t^2) - 250000\ t^2 . \dfrac{d}{dt}\left[(2t+1)^2\right]}{(2t+1)^4}$$

$$= \frac{500000\,t}{(2t+1)^3} \quad \text{[On simplification]}$$

$$N''(t) = \frac{d}{dt}[N'(t)]$$

$$= \frac{d}{dt}\left[\frac{500000\,t}{(2t+1)^3}\right]$$

$$= \frac{-2000000\,t + 500000}{(2t+1)^4} \quad \text{[On simplification]}$$

At $t = 52$, we have

$$N''(52) = \frac{-2000000(52) + 500000}{[2(52)+1]^4} \approx -0.852$$

Thus, after 52 weeks (1 year), the rate of the rate of sales is decreasing at -0.852 units per week per week.

$$N''(208) = \frac{-2000000\,(208) + 500000}{[2\,(208)+1]^4} \approx -0.014\,.$$

After 208 weeks (4 years), the rate of sales have slowed to almost zero nearly at a loss. Fig. 2 shows this.

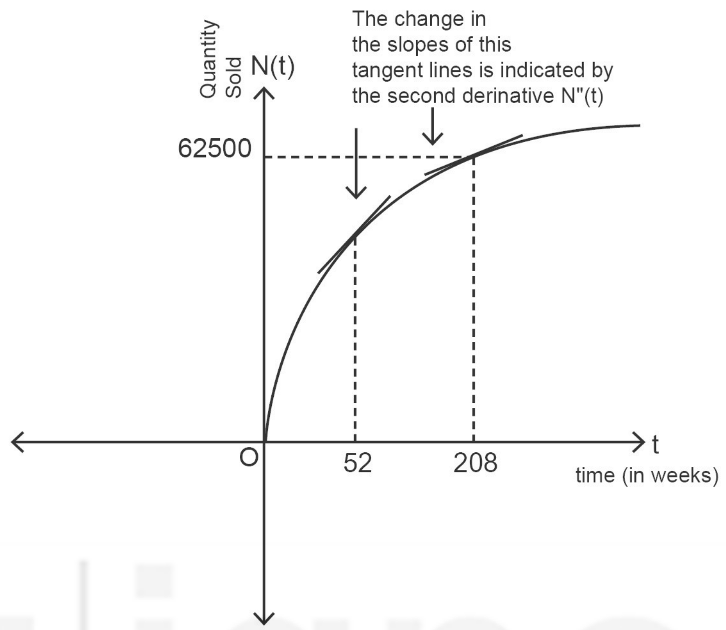

Fig. 2

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Try some exercises before going any further.

E1) Find the second derivatives of the following with respect to x.

i) $f(x) = x^3 - 4$

ii) $y = e^{2x}$

E2) If $y = e^{ax}\sin bx$, show that $y'' - 2ay' + (a^2 + b^2)y = 0$.

E3) Find the value of integer k in each of the following

i) $f(x) = \sin k\,x$ and $f^{(2)}(\pi/6) = 2\sqrt{3}$

ii) $f(x) = x^k + kx^2 + 1$ and $f^{(2)}(1) = 12$

E4) The position function of a particle is given by $s(t) = t^3 - 4.5t^2 - 7t,\ t \geq 0$.

i) When does the particle reach a velocity of $5\,m/s$?

ii) When is the acceleration 0? What is the significance of this value of t?

E5) The function $p(t) = \dfrac{2000\,t}{4t+75}$ models the population p in an area after t months.

i) Find $p'(10), p'(50)$ and $p'(100)$.

ii) Find $p''(10), p''(50)$ and $p''(100)$.

iii) Interpret the meaning of your answers to parts (i) and (ii). What is happening to this population in the long term?

As we saw, the first derivative is again a function and can be differentiable. Similarly, the second derivative of $y = f(x)$ is a function, which can further be differentiated. In the following section, we shall find higher order derivatives.

11.3 HIGHER ORDER DERIVATIVES

Consider a fucniton f defined by $f(x) = x^4$, we have $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Differentiating f'' again with respect to x, we get $f'''(x) = 24x$ where f''' denotes the derivative of f'', or the third derivative of f. Other notations for $f'''(x)$ are $\frac{d^3y}{dx^3}$ or $f^{(3)}$ or D^3y. Differentiating f''', we get the fourth derivative of f, $f^{(4)}(x) = \frac{d^4y}{dx^4} = 0$.

Thus, repeatedly differentiating **(if possible)** a given function f, we get the second, third, fourth, … derivatives of f. These are called the **higher order derivatives** of f.

If n is any positive integer, then the n^{th} derivative of f, a function of x, is denoted by $f^{(n)}$, or by $\frac{d^n f}{dx^n}$ (read as $d\ n\ f$ by $d\ x\ n$), or by y_n, or $D^n y$, where $y = f(x)$.

Note that in the notation $f^{(n)}$ the bracket is necessary to distinguish it from f^n, that is, f raised to the power n. This process of differentiating again and again, in succession, is called **successive differentiation**.

We have already seen that there are functions f that are not differentiable. In other words f' need not always exist. Similarly even when f' exists, it is possible that f'' does not exist. In general, for each positive integer n there are functions f such that $f^{(n)}$ exists, but $f^{(n+1)}$ does not exist. However, most functions that we consider in these sections possess all higher derivatives.

A twice differentiable function is a function f such that f'' exists. Let n be a positive integer. A function f such that $f^{(n)}$ exists is called an n-times differentiable function. If $f^{(n)}$ exists for every positive integer n, then f is said to be an infinitely differentiable function.

Now we give some simple examples of problems related to higher derivatives.

Example 8: If the third derivative of the function f, given by $f(x) = ax^3 + bx + c \quad a, b, c \in \mathbb{R}$ has the value 6 at the point $x = 1$, find the value of a.

Solution: Here, $f(x) = ax^3 + bx + c$

Differentiating this we get $f'(x) = 3ax^2 + b$. Differentiating this again, we get $f''(x) = 6ax$. Differentiating once again, we get $f^{(3)}(x) = 6a$.

Taking the value at $x = 1$,

$f^{(3)}(1) = 6$

Thus, $6a = 6$. Therefore, $a = 1$.

***

Example 9: If $f(x) = 2x^2 - x^3$, find $f'(x), f''(x), f'''(x)$ and $f''''(x)$. Compare the graphs of f, f', f'', f'''. Check whether the graphs are consistent with the geometric interpretation of these derivatives.

Solution: The function f is $f(x) = 2x^2 - x^3 = x^2(2-x)$

First derivative: $f'(x) = 4x - 3x^2 = x(4-3x)$

Second derivative: $f''(x) = 4 - 6x$

Third derivative: $f'''(x) = -6$

Fourth derivative: $f''''(x) = 0$

The graphs of f, f', f'', f''' are given in Fig. 3. We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$. You may note that f'' is negative when f' has negative slope and positive when f' has positive slope. Similarly, f''' is the slope of the curve $y = f''(x)$ at the point $(x, f''(x))$. So, the graphs in Fig. 3 serve as a check on our calculations.

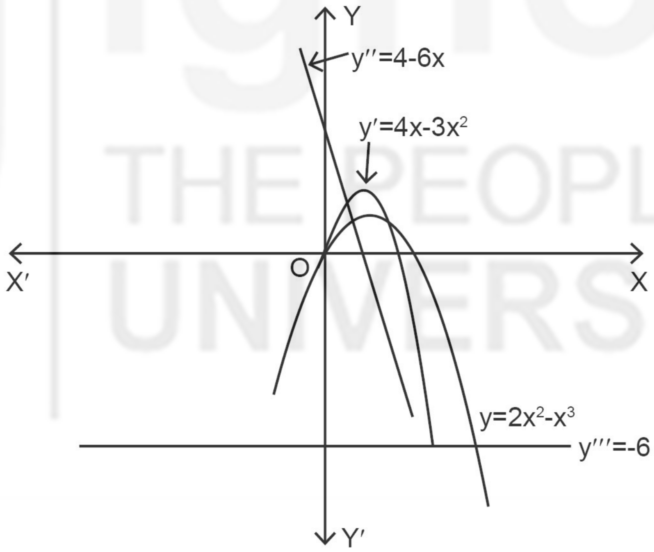

Fig. 3: Graph of y, y', y'' and y'''

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You may try the following exercises now.

E6) Find $f^{(3)}(\pi/4)$ for the following functions.

i) $f(x) = \sec x$

ii) $f(x) = \sin 2x + \cos 2x$.

E7) Find $\dfrac{d^{99}}{dx^{99}}(\sin x)$.

Now, we will find rules for obtaining the n^{th} derivatives of some functions. When a function f is given by a formula, it is often possible to express its n^{th} derivative also by a formula using f and n. Often, one can guess $f^{(n)}$ after working out $f^{(1)}$, $f^{(2)}$ and $f^{(3)}$ and seeing a pattern emerging. In fact, such formulas can be proved also, using the principle of mathematical induction. However, in this section, we will not prove any such formulae. We will derive the formulas for the n^{th} derivative of various functions by observing the first few derivatives. Study them carefully as we shall be using them in later sections.

Below, we obtain formulas for the n^{th} order derivatives of some standard functions.

I. n^{th} derivative of e^{ax} with respect to x.

Let $y = e^{ax}$, on differentiating y successively w.r.t. x, we obtain

$$y^{(1)} = a\,e^{ax},$$
$$y^{(2)} = a^2 e^{ax},$$
$$y^{(3)} = a^3 e^{ax},$$
$$-------$$
$$-------$$
$$y^{(n)} = a^n e^{ax}$$

Thus, we have $\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

In particular if $a = 1$, $\frac{d^n}{dx^n}(e^x) = e^x$

II. n^{th} derivative of the polynomial function $(ax+b)^m$ with respect to x.

Let $y = (ax+b)^m$, on differentiating y successively w.r.t. x, we get

$$y^{(1)} = m.a\,(ax+b)^{m-1}$$
$$y^{(2)} = m(m-1)\,a^2(ax+b)^{m-2}$$
$$y^{(3)} = m(m-1)(m-2)\,a^3(ax+b)^{m-3}$$
$$---------------------$$
$$---------------------$$
$$y^{(n)} = m(m-1)(m-2).......\{m-(n-1)\}\,a^n(ax+b)^{m-n}$$

Here, we see that n^{th} derivative depends on whether $m > n$, $m < n$ or $m = n$.
Let us consider these three cases one by one.

i) Suppose $m = n$ (a positive integer)

Then, $y^{(n)} = n(n-1)(n-2).....1.a^n(ax+b)^{n-n} = n!a^n$.

i.e, $\frac{d^n}{dx^n}(ax+b)^n = n!a^n$.

Here, you may note that when $m = n$, the n^{th} derivative is constant.

ii) Suppose m is a positive integer and $m > n$.

Then, $y^{(n)} = m(m-1).....\{m-(n-1)\}\,a^n(ax+b)^{m-n}$

$$= \frac{m!}{(m-n)!}.a^n (ax+b)^{m-n}.$$ [Multiplying and dividing by $(m-n)!$]

i.e. $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!}a^n (ax+b)^{m-n}.$

iii) Suppose m is a positive integer and $m < n$.

From case i), $\frac{d^m}{dx^m}(ax+b)^m = m!\, a^m$

As $n > m$, on differentiating it further, the right hand side is zero. This is because, since m^{th} derivative is constant, therefore, the $(m+1)^{th}$ derivative is zero.

Thus, $\frac{d^n}{dx^n}(ax+b)^m = 0$ if $n > m$.

In particular, if $a = 1$ and $b = 0$, then the n^{th} derivative of x^m is

$$\frac{d^n}{dx^n}(x^m) = \begin{cases} n! & \text{, if } m = n \\ \frac{m!x^{m-n}}{(m-n)!} & \text{, if } m > n \\ 0 & \text{, if } m < n \end{cases}$$

III. **n^{th} derivative of $\frac{1}{ax+b}$, when $ax+b \neq 0$, with respect to x.**

Let $y = \frac{1}{ax+b}$, on differentiating y successively w.r.t. x, we get

$$y^{(1)} = \frac{-a}{(ax+b)^2}$$

$$y^{(2)} = \frac{(-a)(-2a)}{(ax+b)^3} = \frac{(-1)^2.2a^2}{(ax+b)^3}$$

$$y^{(3)} = \frac{(-a)(-2a)(-3a)}{(ax+b)^4} = \frac{(-1)^3 3!\, a^3}{(ax+b)^4}$$

and $y^{(n)} = \frac{(-a)(-2a)(-3a).......(-na)}{(ax+b)^{n+1}} = \frac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$

Thus $\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right) = \frac{(-1)^n\, n!a^n}{(ax+b)^{n+1}}$

In particular, if $a = 1$ and $b = 0$, $\frac{d^n}{dx^n}\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}, x \neq 0.$

IV. **n^{th} derivative of $\ln(ax+b)$, with respect to x.**

Let $y = \ln\,(ax+b)$, we get $y^{(1)} = \frac{a}{ax+b}$.

Differentiating again, $y^{(2)} = -\frac{a^2}{(ax+b)^2}$.

Differentiating once again, $y^{(3)} = \frac{2a^3}{(ax+b)^3}$.

Can you guess $y^{(n)}$ now? If you have guessed correctly, you must have arrived at these conclusions.

i) The denominator of $y^{(n)}$ is $(ax+b)^n$.

ii) Its sign is positive or negative according as n is odd or even.

iii) Its numerator has $a^n(n-1)!$ Do not think that it is merely $(n-1)$. There is a factorial symbol too. To be convinced of this, calculate $y^{(4)}$ and see.

Therefore, our guess $y^{(n)} = \dfrac{d^n}{dx^n}\left(\ln(ax+b)\right) = \dfrac{(-1)^{n-1}\times(n-1)\,!\,a^n}{(ax+b)^n}$

In particular, if $a=1$ and $b=0$, $\dfrac{d^n}{dx^n}(\ln x) = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$.

V. n^{th} derivative of a^{mx}, with respect to x.

Let $y = a^{mx}$. Taking logarithm both the sides, we get

$\ln y = mx \ln a$ [Since $\ln\, m^n = n\ln m$].

Differentiating w.r.t. x, we get

$$\frac{1}{y}.\frac{dy}{dx} = m\ln a$$

or $y^{(1)} = y.m\ln a$

$y^{(2)} = (m\ln a).y_1 = (m\ln a)^2 y$

$y^{(3)} = (m\ln a)^3 y$

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$y^{(n)} = (m\ln a)^n y.$

or $y^{(n)} = \dfrac{d^n}{dx^n}(a^{mx}) = (m\ln a)^n a^{mx}$.

In particular, if $m=1$, $\dfrac{d^n}{dx^n}(a^x) = (\ln a)^n a^{mx}$.

Now let us use these results in the following examples.

Example 10: Find the n^{th} derivative of $\dfrac{x+3}{(x+1)(x+2)}$

Solution: Let $y = \dfrac{x+3}{(x-1)(x+2)}$. To find the n^{th} derivative of this rational function is difficult. So, we first break into sum of rational functions in which the denominator is expressed in linear polynomial. For this, we are giving method of doing partial fractions in the **Appendix 2** given at the end of this Block.
By partial fractions, we get

$$\frac{x+3}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} \quad \text{... (2)}$$

Then, $x+3 = A(x+2) + B(x-1)$

Taking $x=1 \Rightarrow A = 4/3$ and $x=-2 \Rightarrow B = -1/3$

Substituting values of A and B in eqn. (2), we get

$$y = \frac{4}{3(x-1)} - \frac{1}{3(x+2)}$$

Now, on differentiation with respect to x, we obtain

$$\frac{d^n}{dx^n}\left[\frac{x+3}{(x-1)(x+2)}\right]=\frac{4}{3}\frac{d^n}{dx^n}\left[\frac{1}{x-1}\right]-\frac{1}{3}\frac{d^n}{dx^n}\left[\frac{1}{x+2}\right]$$

$$=\frac{4}{3}\frac{(-1)^n n!}{(x-1)^{n+1}}-\frac{1}{3}\frac{(-1)^n n!}{(x+2)^{n+1}}$$

$$=\frac{(-1)^n n!}{3}\left[\frac{4}{(x-1)^{n+1}}-\frac{1}{(x+2)^{n+1}}\right]$$

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Now, try the following exercises.

E8) If $y=(1+x)^r$, where r is a real number, find $y^{(n)}$ where n is a natural number $(n<r)$.

E9) Find the n^{th} derivative of the following functions:

i)	$f(x)=(ax+b)^3$	iv)	$f(x)=e^{kx}$
ii)	$f(x)=(ax+b)^m$	v)	$\dfrac{x}{(x-1)(2x-3)}$
iii)	$f(x)=e^x$	vi)	$\dfrac{1}{x^2+a^2}$

E10) Prove that the n^{th} derivative of the polynomial function $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ is a constant.

Now, let us find n^{th} derivative of a tringonometric function in the following example:

Example 11: If $f(x)=\cos 2x$, find a formula for $f^{(n)}(0)$.

Solution: We first find $f^{(n)}(x)$ when $n=1, 2, 3, 4$.
We have $f(x)=\cos 2x$.
On differentiating this successively, we get

$$f^{(1)}(x)=-2\sin 2x$$
$$f^{(2)}(x)=-4\cos 2x$$
$$f^{(3)}(x)=8\sin 2x$$
$$f^{(4)}(x)=16\cos 2x$$

We see that in the formula for $f^{(n)}(x)$, we have to have

i) a sign (positive or negative),

ii) a coefficient (some power of 2), and

iii) a trigonometric function ($\sin 2x$ or $\cos 2x$)

Recall that
$\cos(\theta+\pi/2)=-\sin\theta$
$\cos(\theta+\pi)=-\cos\theta$
$\cos(\theta-\pi/2)=\sin\theta$

We observe that the first two terms carry negative sign, the next two carry positive sign, the next two negative and so on.
We also observe that sin and cos occur alternately. Therefore our guess is

$$f^{(n)}(x) = \begin{cases} -2^n \sin 2x & \text{if } n \text{ is of the form } 4k+1 \\ -2^n \cos 2x & \text{if } n \text{ is of the form } 4k+2 \\ 2^n \sin 2x & \text{if } n \text{ is of the form } 4k+3 \\ 2^n \cos 2x & \text{if } n \text{ is of the form } 4k \end{cases}$$

We can also write this in a compact form as

$$f^{(n)}(x) = 2^n \cos(2x + n\pi/2)$$

You can easily check that both the results of $f^{(n)}(x)$ are equivalent by putting $n = 4k+1,\ 4k+2,\ 4k+3$ and $4k$ in later $f^{(n)}(x)$.

Now substitute $x = 0$ in $f^{(n)}(x)$. We obtain

$$f^{(n)}(0) = 2^n \cos n\pi/2$$

This is the required answer.

We can also write a general result about the n^{th} derivative of a sum of two functions in the following theorem.

Theorem 1: If f and g are two functions from $\mathbb{R}$ to $\mathbb{R}$ and if both of them are differentiable n-times, then

i) $(f+g)^{(n)} = f^{(n)} + g^{(n)}$

ii) $(cf)^{(n)} = c.f^{(n)}$, where c is a constant.

Try to solve these exercises now.

E11) If $f(x) = \sin x$, find $f^{(n)}(x)$.

E12) If $y = \sin(ax+b)$, find $y^{(n)}$.

E13) If $y = \cos x$ and if n is any positive integer, prove that
$[y^{(n)}]^2 + [y^{(n+1)}]^2 = 1$.

In the following section, we shall state Leibniz theorem which states the nth derviation of the product of two functions.

11.4 LEIBNIZ THEOREM

In Unit 9, we have stated some rules regarding the derivatives of the sum, scalar multiple, product and quotient of two differentiable functions. These were

$$(f+g)' = f' + g'$$
$$(cf)' = cf'$$
$$(fg)' = fg' + gf'$$
$$(f/g)' = \frac{gf' - fg'}{g^2} \quad (g(x) \neq 0 \text{ anywhere in the domain})$$

In Theorem1, we have seen that the first two rules can be extended to the n^{th} derivatives if f and g are n-times differentiable functions. In this section, we are going to extend the product rule of differentiation. We shall give a formula for the n^{th} derivative of the product of two functions.

Let u and v be the functions of x, then the product rule for two functions u and v can also be written as $(uv)' = u'v + uv'$ [Where u' and v' are the first derivatives of u and v respectively with respect to x].

Now we look for a similar formula for $(uv)^{(2)}$, $(uv)^{(3)}$, etc.

To derive a formula for n^{th} derivative of product of two functions, we will need $C(n, r)$. You may recall the meaning of the notation $C(n, r)$, where n and $r \in \mathbb{Z}^+$ and $r \le n$. This $C(n, r)$ stands for the number of ways of choosing r objects from n objects. Sometimes it is also denoted by ${}^nC_r$ or $\binom{n}{r}$.

Also recall the formulas

i) $C(n, r) = \dfrac{n!}{r!(n-r)!}$

ii) $C(n, 0) = C(n, n) = 1$

iii) $C(n, r) = C(n, n-r)$

iv) $C(n, r) + C(n, r+1) = C(n+1, r+1)$

These are combinatorial identities, true for all positive integers r and n with $r \le n$. Now, we state Leibniz theorem, which is to find the n^{th} derivative of product of two functions.

Theorem 2 (Leibniz Theorem): Let n be a positive integer. If u and v are n times differentiable functions, then

$$(uv)^{(n)} = C(n, 0)u^{(n)}v + C(n, 1)u^{(n-1)}v^{(1)} + C(n,2)u^{(n-2)}v^{(2)} + \cdots + C(n, n)uv^{(n)}.$$

Where $u^{(n)}$ and $v^{(n)}$ are the nth derivatives of u and v respectively.

The pattern in the formula for $(uv)^n$ can be compared with the expansion of $(x+y)^n$.

i) The coefficients are binomial coefficients and they appear in the same order as those in the expansion of $(x+y)^n$.

ii) The order of the derivative of u goes on decreasing one at a time, and the order of the derivative of v goes on increasing one at a time.

iii) The number of terms is $n+1$.

Remark 1: We omit the proof of this theorem and merely indicate how this can be proved by mathematical induction on n. Firstly, when $n = 1$, the above formula is the same as the already known product formula, and therefore is true. Assuming that it is true for $n = m$, we can prove it for $n = m+1$, by applying the product rule for each term of the expansion of $(uv)^{(m)}$ and by using the combinatorial identities mentioned.

We start with a simple and direct application of the formula.

Example 12: If $f(x) = x \sin x$, find the fourth derivative of f, using Leibniz theorem.

Solution: We first observe that for $n = 4$, the Leibniz theorem states

$$(uv)^{(4)} = C(4, 0)u^{(4)}v + C(4, 1)u^{(3)}v^{(1)} + C(4,2)u^{(2)}v^{(2)} + C(4,3)u^{(1)}v^{(3)} + C(4, 4)uv^{(4)}$$
$$= u^{(4)}v + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)}.$$

Here, we take $u = x$ and $v = \sin x$, so that $f = uv$

We have $u = x$, $v = \sin x$

$u^{(1)} = 1$, $v^{(1)} = \cos x$

$u^{(2)} = 0 = u^{(3)} = u^{(4)}$, $v^{(2)} = -\sin x$

$v^{(3)} = -\cos x$

$v^{(4)} = \sin x$

Substituting these in the above formula, we get

$$f^{(4)} = (uv)^{(4)} = 0 + 0 + 0 + 4(1)(-\cos x) + x.\sin x$$
$$= x\sin x - 4\cos x$$

What happens if we attach the same problem directly without the use of Leibniz theorem? We have $f(x) = x\sin x$

Differentiating this once, we get $f^{(1)}(x) = x\cos x + \sin x$ (by product rule)

Differentiating once again, we get $f^{(2)}(x) = x(-\sin x) + 1.\cos x + \cos x$, that is $f^{(2)}(x) = 2\cos x - x\sin x$

Differentiating once again, we get $f^{(3)}(x) = -2\sin x - (x\cos x + \sin x)$, that is $f^{(3)}(x) = -3\sin x - x\cos x$

Differentiating once again, $f^{(4)}(x) = -3\cos x - [x(-\sin x) + \cos x]$, that is $f^{(4)}(x) = x\sin x - 4\cos x$

You may notice that we obtain the same answer. In this direct method, we had to apply the product formula four times, once for each differentiation.

***

It is clear that when we want the n^{th} derivative for bigger values of n, Leibniz theorem provides an easier method to write down the answer, avoiding the difficulty of repeatedly applying the product formula. Let us apply the theorem in more examples.

Example 14: If $y = (\sin^{-1} x)^2$, prove that $(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2 y^{(n)} = 0$ for each positive integer n.

Solution: Differentiating both sides of $y = (\sin^{-1} x)^2$, we get $y^{(1)} = \dfrac{2\sin^{-1} x}{\sqrt{1-x^2}}$

Squaring and cross multiplying, we get $(1-x^2)(y^{(1)})^2 = 4(\sin^{-1} x)^2 = 4y$

Differentiating once again, we get $\dfrac{d}{dx}[(1-x^2)(y^{(1)})^2] = \dfrac{d}{dx}(4y)$ [Since $y = (\sin^{-1} x)^2$]

$$\Rightarrow \left[\frac{d}{dx}(1-x^2)\right](y^{(1)})^2 + (1-x^2)\frac{d}{dx}(y^{(1)})^2 = 4\frac{d}{dx}(y)$$
$$\Rightarrow (-2x)(y^{(1)})^2 + (1-x^2).2y^{(1)}y^{(2)} = 4y^{(1)}$$
$$\Rightarrow 2(1-x^2)y^{(1)}y^{(2)} - 2x(y^{(1)})^2 - 4y^{(1)} = 0$$

Dividing throughout by $2\ y^{(1)}$ gives us $(1-x^2)y^{(2)} - xy^{(1)} - 2 = 0$

Differentiating n times, using Leibniz Theorem for each of the first two terms we get

$$D^n\left[\underbrace{(1-x^2)}_{v}\underbrace{y^{(2)}}_{u}\right] - D^n\left[\underbrace{x}_{v}\ \underbrace{y^{(1)}}_{u}\right] - D^n[2] = 0.$$

$$\Rightarrow [(1-x^2)D^n y^{(2)} + C(n,1)D(1-x^2)D^{n-1}y^{(2)} + C(n,2)D^2(1-x^2)D^{n-2}y^{(2)}]$$
$$-[x.D^n y^{(1)} + C(n,1)D(x)D^{n-1}y^{(1)}] - D^n(2) = 0$$

$$[\because D^n(1-x^2)=0, \text{if } n>2 \text{ and } D^n x = 0, \text{ if } n>1]$$

On solving this, we get

$$(1-x^2)y^{(n+2)} - C(n,1)2xy^{(n+1)} - C(n,\ 2)2y^{(n)} - \{xy^{(n+1)} + C(n,\ 1)y^{(n)}\} = 0$$

That is,

$$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2y^{(n)} = 0.$$

***

The following exercises will give you some practice in applying Leibniz theorem.

E14) State Leibniz Theorem when $n=5$. That is, $(uv)^{(5)}$.

E15) Prove that when $n=1$, Leibniz theorem reduces to the product rule of differentiation.

E16) Find the third derivative of $\sin^2 x$ using Leibniz theorem. Find the same directly also and verify that you obtain the same answer.

E17) If $f(x) = xe^x$, find the sixth derivative of f, using Leibniz formula.

E18) Find the n^{th} derivative $x^3 \ln x$.

E19) If $y = e^{ax}x^2$ prove that $y^{(n)} = e^{ax}[a^n x^2 + 2na^{n-1}x + n(n-1)a^{n-2}]$.

E20) i) Write down Leibniz formula for $(uv)^{(m)}$.

ii) Differentiate it term by term and obtain

$$(uv)^{(m+1)} = C(m,\ 0)u^{(m+1)}v + C(m,\ 1)\ (u^{(m)}v^{(1)} + u^{(m+1)}v^{(2)}) + \cdots + C(m,\ m)uv^{(m+1)}.$$

iii) Deduce that

$$(uv)^{(m+1)} = C(m,\ 0)u^{(m+1)}v + [C(m,0) + C(m,\ 1)]u^{(m)}v^{(1)} + [C(m,\ 1) + C(m,2)u^{(m-1)}v^{(2)} + \cdots + [C(m,m-1) + C(m,m)]u^{(1)}v^{(m)} + C(m,m)uv^{(m+1)}.$$

iv) Deduce from part (iii) the Leibniz formula for $(uv)^{(m+1)}$.

So far, we have discussed higher order derivatives. In the following section, we shall apply the concept of higher order derivatives to find approximations.

11.5 POLYNOMIAL APPROXIMATION

Recall Unit 9, where you have seen that a secant line can be approximated to find the slope of the tangent at any point x_0. In this section, we will approximate the given curves with the curves of the polynomial functions. Why do we use polynomial functions? It is because they have the simplified form of being built only with powers of x. The polynomials of degree 0 and 1 are straight lines and the polynomials of degree 2 or more have curvy graphs.

Suppose we find a polynomial which passes through a particular point, then there may be three possibilities that a curve is

i) a straight line parallel to $x-$axis that is polynomial of degree 0.

ii) a slant line that is a polynomial of degree 1.

iii) a curve that is a polynomial of degree 2 or more.

If the polynomial has its degree 0, then we do not have any choice that is the horizontal line passing through a given point is always unique. But if we find polynomials of degree 1, 2 or more, we can adjust constants to get different curves. So in this section, we will discuss approximation of linear polynomial (polynomial with degree 1), quadratic polynomial (polynomial with degree 2) and the Taylor polynomial (polynomial with degree n).

11.5.1 Linear Approximation

Suppose we want a linear polynomial which approximates the curve at a particular point. For this consider a point $(x_0, f(x_0))$. We know that there are several straight lines passing through the point as given in Fig. 4.

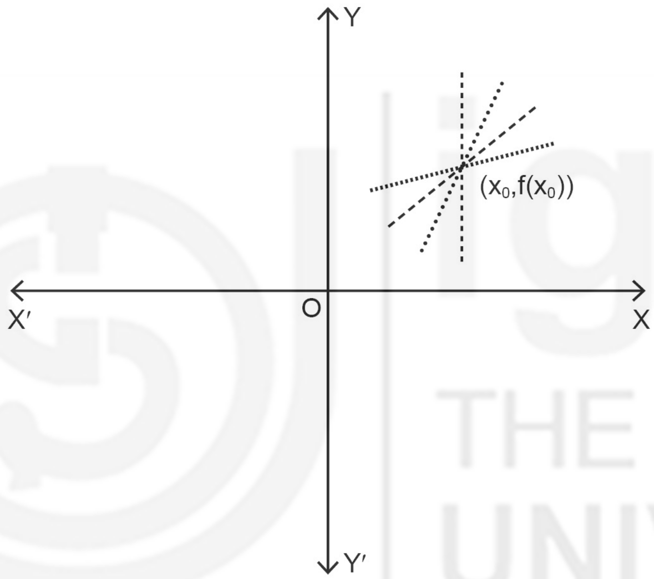

Fig. 4

We can adjust the slope which will in turn tilt the line any way we like. This is **linear approximation**. The best linear approximation is when the slope of the line passing through $(x_0, f(x_0))$ is same as the slope of the tangent to the curve at $(x_0, f(x_0))$.

Here, we are considering the slope of the given function f at a particular point $(x_0, f(x_0))$ and try to approximate the graph of f. This is called **linear approximation**. For this, suppose that a function f is differenatiable at x_0, and the slope of the tangent of f at x_0 is $f'(x_0)$. Then, the equation of the tangent at $(x_0, f(x_0))$ is $y - f(x_0) = f'(x_0)(x - x_0)$.

Since this tangent line closely approximates the graph of f for the values of x near x_0, we can write the **linear approximation of f at x_0** as $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$, provided x is close to x_0. If $x - x_0 = \delta x$, then $f(x_0 + \delta x) \approx f(x_0) + f'(x_0)\,\delta x$. Let us find the linear approximation in the following examples.

Example 12: Find the linear approximation of $f(x)=\sqrt{x+1}$ at $x_0=0$. Hence, find the approximate value of $\sqrt{1.1}$.

Solution: Given is $f(x)=\sqrt{x+1}$, on differentiating $f(x)$ with respect to x, we get $f'(x)=\dfrac{1}{2\sqrt{x+1}}$. Here $x_0=0$, thus $f(x_0)=f(0)=\sqrt{1}=1$ and $f'(x_0)=f'(x_0)=\dfrac{1}{2}$.

The linear approximation of f at x_0 is $f(x)\approx f(x_0)+f'(x_0)(x-x_0)$.
On substituting $x_0, f(x_0)$ and $f'(x_0)$, we get

$$\sqrt{x+1}\approx 1+\frac{1}{2}(x-0)=1+\frac{x}{2}.$$

We can say that as x is close to 0, $\sqrt{x+1}$ is about $1+\dfrac{x}{2}$. Fig. 5 shows the graph of f and the graph of local linear approximation of $\sqrt{x+1}$ at $x=0$.

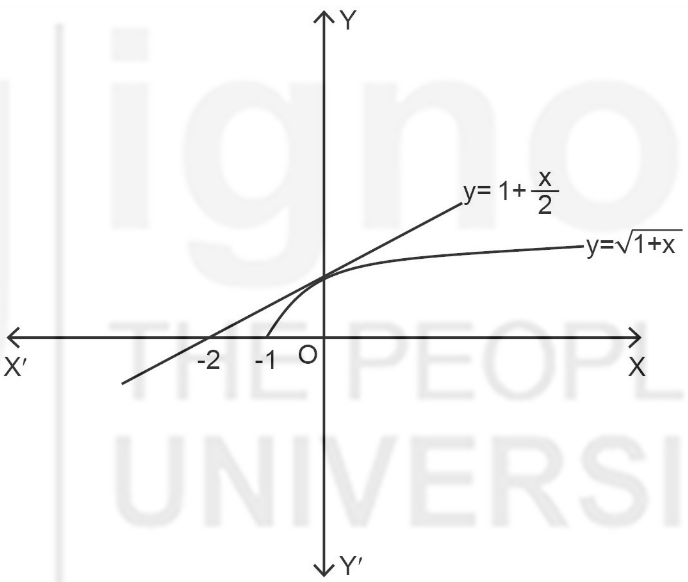

Fig. 5: Graph of $y=\sqrt{x+1}$ and $y=1+\dfrac{x}{2}$

To find the approximate value of $\sqrt{1.1}$, we put $x=0.1$. Thus, the approximate value of $\sqrt{1.1}$ is $1+\dfrac{0.1}{2}=1.05$.

***

Example 13: Show that $\cos x\approx 1$ if x is close to 0.

Solution: Let $f(x)=\cos x$, and $x_0=0$. Then $f'(x)=-\sin x$, and $f(x)$ and $f'(x)$ at x_0 are $f(x_0)=f(0)=1$ and $f'(x_0)=f'(0)=0$. Now the linear approximation of $\cos x$ is given by $\cos x\approx\cos 0+(-\sin 0)(x-0)$, that is $\cos x\approx 1+0$.
Thus, $\cos x\approx 1$

Fig. 6 shows the graph of $f(x)=\cos x$ near $x=0$.

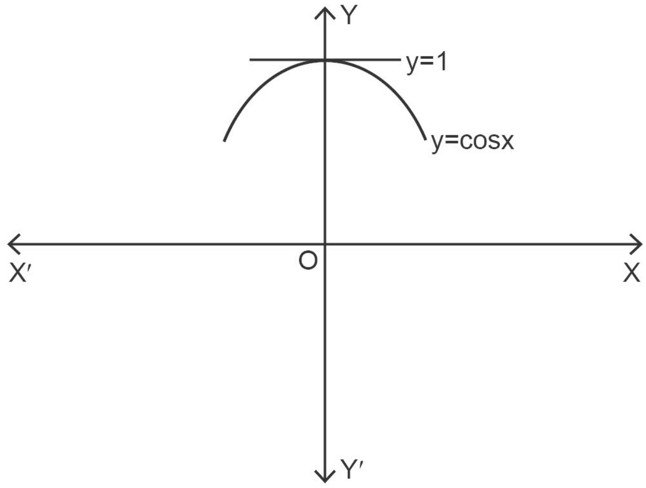

Fig. 6: Graph of $\cos x$ near 0

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Now try the following exercise.

E21) Find the local linear approximation of $\sin x$ at $x_0 = 0$. Hence, find the approximate value of $\sin 1^o$.

11.5.2 Quadratic Approximation

In the local linear approximation, we have used the tangent line at the point of tangent. Now, we will extend our discussion to improve the approximation using polynomials of degree more than 1. As the polynomial of degree two are called quadratic polynomial, the equation of degree two used for approximation is called the **quadratic approximation**. Let us begin with quadratic approximation of f at 0. Suppose this approximation has the quadratic polynomial form $p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2$ where a_0, a_1, and a_2 are to be taken such that the value of $p(x)$ and its first two derivatives at x_0 are equal to f and its first two derivatives at x_0 respectively. That is,

$p(x_0) = f(x_0), p'(x_0) = f'(x_0), p''(x_0) = f''(x_0)$.

Let $p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2$

$\therefore p'(x) = a_1 + 2a_2(x - x_0)$

and $p''(x) = 2a_2$

So, $p(x_0) = a_0 = f(x_0)$,

$p'(x_0) = a_1 = f'(x_0)$

and $p''(x_0) = 2a_2 = f''(x_0) \Rightarrow a_2 = \dfrac{f''(x_0)}{2}$.

Substituting a_0, a_1, a_2 in quadratic approximation of f, we get

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

Since we have taken $p(x)$ as quadratic approximation of $f(x)$, therefore $f(x) \approx p(x)$, which gives

$$f(x) \approx f(0) + f'(0)(x - x_0) + \frac{f''(0)}{2}(x - x_0)^2.$$

In particular, if we substitute x_0 as 0 we get $p(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

and the approximation of the function f is $f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2$.

Let us see how this quadratic approximation improves the linear approximation. In Example 12, we approximated $\sqrt{x+1}$ at $x_0 = 0$ using linear polynomial, now we will use quadratic approximation for the same function and compare the results.

For this, $\sqrt{x+1} \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

or $\sqrt{x+1} \approx 1 + \frac{1}{2}x - \frac{1}{4}x^2$

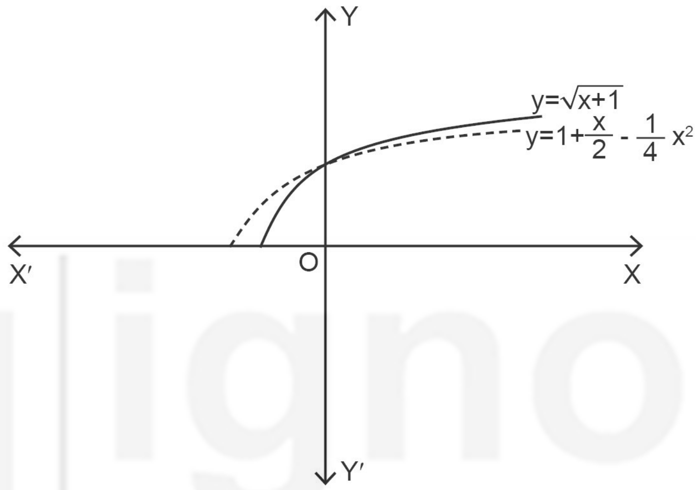

Fig. 7: Graph of f and p

Fig. 7 shows the quadratic approximation of $\sqrt{x+1}$ at $x = 0$. It also strengthens that the quadratic approximation is better than the linear approximation.

Hence $\sqrt{1.1} \approx 1 + \frac{0.1}{2} - \frac{0.01}{4} = 1.0475$

Therefore $\sqrt{1.1} \approx 1.0475$

If we compare the results obtained by linear approximation and quadratic approximation for $\sqrt{1.1}$, you will find that the quadratic approximation gives improved value.

In the following example, let us approximation the function about x_0 other than 0.

Example 14: Find the quadratic approximation of $f(x) = \frac{1}{x^2}$ at $x = 2$.

Solution: The quadratic approximation is

$$f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 \qquad (3)$$

Given that $f(x) = \frac{1}{x^2}$ and $f(2) = \frac{1}{4}$

Differentiating this with respect to x and putting $x = 2$, we get

$f'(x) = \frac{-2}{x^3}$ and $f'(2) = -\frac{1}{4}$; $f''(x) = \frac{6}{x^4}$ and $f''(2) = \frac{3}{8}$.

Substituing the values of $f(2)$, $f'(2)$ and $f''(2)$ in Eqn. (3), we get

$$f(x) = \frac{1}{x^2} \approx \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2.$$

Fig. 8 show this graphically. The dotted curve in Fig. 8 is the graph of the quadratic approximation of f at $x = 2$.

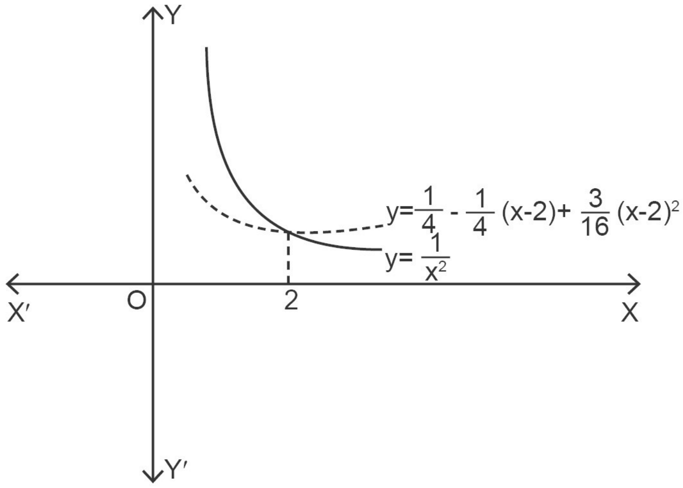

Fig. 8: Graph of f and $p(x)$ about $x = 2$

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Now, try the following exercises.

E22) Find the quadratic approximation of the function f defined as $f(x) = \sin(2x) + \cos(x)$ at $x = 0$.

E23) Find a quadratic approximation of f at $x = 9$ for the function f defined by $f(x) = \sqrt{x}$. Also, find the value of $\sqrt{9.1}$.

Now in the following subsection, we will generalize quadratic approximation to $n-$degree polynomial approximation.

11.5.3 Taylor Approximation

So far, we constructed polynomial of degree 0, degree 1 and degree 2 as approximation for a function f at the point $x = x_0$, which are summarised in Table 1.

Table 1

Degree	Polynomial
0	$f(x_0)$
1	$f(x_0) + f'(x_0)(x - x_0)$
2	$f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$

You may see a pattern in these formulations for the approximations so far. Similarly, we can find degree-three approximation called **cubic approximation**. The subsequent polynomial approximations are known as the **Taylor polynomials**. In this subsection, we will discuss Taylor and Maclaurin polynomials.

If we continue approximation by using a polynomial of degree n with the condition that the values of the polynomial and its first three derivatives at a point are same as those of f at the same point, then we will get improved approximations. If this process improves the accuracy, why not go on to the polynomial of higher degree? This leads us to consider the polynomial of degree n.

Let $p(x) = a_0 + a_1 x + ... + a_n x^n$ such that

$p(0) = f(0), p'(0) = f'(0), p''(0) = f''(0), ..., p^{(n)}(0) = f^{(n)}(0)$.

Differentiating $p(x)$ successively n-times, we get

$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + ... + n a_n x^{n-1}$.

$p''(x) = 2a_2 + 3a_3 x + ... + n(n-1)a_n x^{n-2}$

.

.

.

$p^{(n)}(x) = n(n-1)(n-2)....2.1.a_n$.

Substituting $x = 0$, the coefficients are

$a_0 = f(0),\ a_1 = f'(0),\ a_2 = \frac{f''(0)}{2},\ a_3 = \frac{f'''(0)}{3.2} = \frac{f'''(0)}{3!}$ and similarly,

$a_n = \frac{f^{(n)}(0)}{n!}$.

Fig. 9: Scottish Mathematician Colin Maclaurin(1698-1746)

If we substitute these values of coefficients a_i's where $i = 0,.1,...n$ in the polynomial approximation, we get

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n.$$

This is called the n^{th} **Maclaurin's polynomial** for f.
This discussion leads to the following definition.

Definition: Let f be a function whose derivatives upto n times exist at 0, then $p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n$, is called the n^{th} Maclaurin's polynomial for f provided that the values of the polynomial and its first n derivatives are same as the values of f and its first n derivatives at $x = 0$.

Accordingly $p_1(x)$ and $p_2(x)$ are the linear and quadratic Maclaurin's polynomials of f at $x = 0$ respectively.

From this, we get the polynomial approximation of $f(x)$ at $x = 0$, which is more accurate than linear or quadratic approximation.

Hence $f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n$.

You may note that we are using $p_n(x)$ in the definition of Maclaurin polynomial. This is just to represent a polynomial of degree n.

Now we will expand few functions in the following examples.

Example 15: Find the n^{th} Maclaurin's polynomial for $\sin x$. Hence deduce linear, quadratic and cubic polynomials for $\sin x$.

Solution: Let $f(x) = \sin x$, thus $f(0) = 0$

On differentiating $f(x)$ and putting $x=0$, the results we get, are

$f'(x)=\cos x,\ \ f'(0)=1$

$f''(x)=-\sin x,\ \ f''(0)=0$

$f'''(x)=-\cos x,\ \ f'''(0)=-1$

$f^{iv}(x)=\sin x,\ \ f^{iv}(0)=0$

. . .

. . .

. . .

$$f^{(n)}(x)=\sin\left(x+\frac{n\pi}{2}\right),\ \text{if}\ \ n=2k$$

$$f^{(n)}(0)=\begin{cases}0 & ,\text{if}\ \ n=2k\\(-1)^k & ,\text{if}\ \ n=2k+1,\end{cases}\ \text{where}\ k=0,1,2,....$$

The n^{th} Maclaurin's polynomial is

$$p_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+...+\frac{f^{(n)}(0)}{n!}x^n.$$

$$p_n(x)=0+1.x+\frac{0}{2!}.x^2+\frac{(-1)}{3!}.x^3+\frac{0}{4!}.x^4+...+\frac{(1)^k}{(2k+1)!}x^{2k+1},\ \text{where}\ k=0,1,2,....$$

or $p_n(x)=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-....+\dfrac{(-1)^k}{(2k+1)!}x^{2k+1}.$

or $\sin x\approx x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-...+\dfrac{(-1)^k}{(2k+1)!}x^{2k+1}$, where $k=0,1,2$.

You may observe that the Maclaurin polynomial for $\sin x$ has only odd powers of x because if you see the pattern of the derivatives at $x=0$, you find it is $1,0,-1,0,......$ That is every alternate derivative is 0 at $x=0$. Therefore, all the coefficients of even powers of x in the Maclaurin's polynomial are zero.
If we deduce the polynomials of order one, two and three, we get

$$p_1(x)=0+1.x=x$$

$$p_2(x)=0+1.x+\frac{0}{2!}.x^2=x$$

$$p_3(x)=0+1.x+\frac{0}{2!}.x^2+\frac{(-1)}{3!}.x^3=x-\frac{x^3}{3!}.$$

Fig.10 shows the graphs of $f(x), p_1(x)$ and $p_3(x)$.

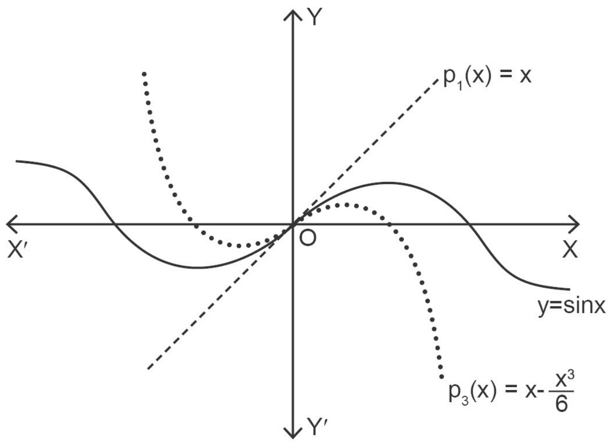

Fig. 10: Graph of $\sin x, p_1(x)$ and $p_3(x)$

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Example 16: Find the Maclaurin's polynomial for e^x. Also, draw the graphs of p_0, p_1, p_2, p_3. Hence find the approximate value of $e^{0.1}$ correct to 3 places of decimals.

Solution: Let $f(x) = e^x,\ f(0) = e^0 = 1$

Differentiating $f(x)$ successively with respect to x and putting $x = 0$ in the results, we get

$f'(x) = e^x, f'(0) = 1$

$f''(x) = e^x, f''(0) = 1$

$f'''(x) = e^x, f'''0) = 1$

$f^{iv}(x) = e^x, f^{iv}(0) = 1$

$\vdots$

$f^{(n)}(x) = e^x, f^{(n)}(0) = 1$

Therefore,

$$p_n(x) = 1 + 1.x + \frac{1}{2!}.x^2 + \frac{1}{3!}.x^3 + ... + \frac{1}{n!}.x^n$$

or $p_n(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ... + \dfrac{x^n}{n!}$

Substituting $x = 0.1$, we get

$$e^{0.1} \approx 1 + 0.1 + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!}$$ [all other terms will not contiribute to the value upto three places of decimal, therefore, ignore all other]

$$= 1 + 0.1 + 0.005 + 0.0002$$
$$= 1.1052$$

Thus, $e^{0.1} \approx 1.105$

Now finding p_0, p_1, p_2, p_3,

$p_0(x) = 1$,

$p_1(x) = 1 + x$,

$p_2(x) = 1 + x + \dfrac{x^2}{2!}$,

and $p_3(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$.

The graph of these polynomials are given in Fig. 11.

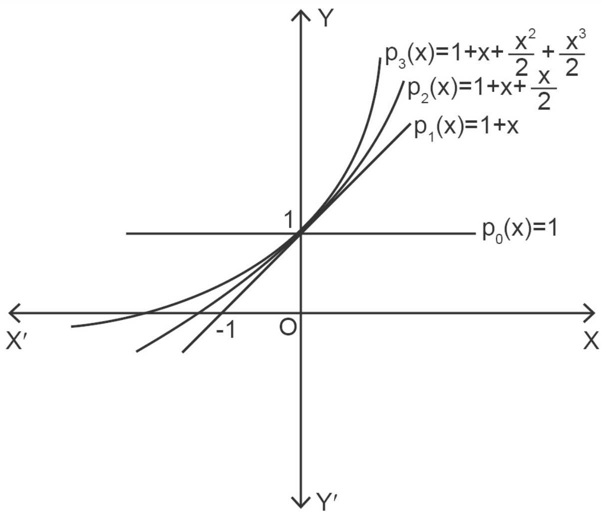

Fig. 11: Graphs of p_0, p_1, p_2, p_3

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Example 17: Write down Maclaurin polynomial for $(1+x)^r$ around zero, where r is a fixed real number.

Solution: Let $f(x)=(1+x)^r$. Then $f(0)=1$,

$$f'(x)=r(1+x)^{r-1};\ f'(0)=r$$

$$f^{(2)}(x)=r(r-1)(1+x)^{r-2};\ f^{(2)}(0)=r(r-1)$$

$$f^{(n)}(x)=r(r-1)\ldots(r-n+1)\ (1+x)^{r-n};\ f^{(n)}(10)=r(r-1)\ldots(r-n+1)$$

Therefore Maclaurin polynomial around zero is

$$(1+x)^r \approx 1+\frac{r}{1!}x+\frac{r(r-1)}{2!}x^2+\cdots+\frac{r(r-1)\ldots(r-n+1)}{n!}x^n+\cdots$$

***

Example 18: Write down Maclaurin polynomial for the function $\sin x$.

Solution: Let $f(x)=\sin x$. Then successive derivatives of $\sin x$ at 0 are given by

$$f^{(n)}(0)=\begin{cases}0 \text{ if } n \text{ is even}\\ 1 \text{ if } n \text{ divisible by } 4\\ -1 \text{ otherwise}\end{cases}$$

$$f^{(n)}(0)=\sin\frac{n\pi}{2}$$

We see that, as n varies over $0, 1, 2, 3, 4, 5, 6, 7\ldots$, $f^{(n)}(0)$ takes the values $0, 1, 0, -1, 0, 1, 0, -1, \ldots$

Therefore, Maclaurin polynomial for $\sin x$ is

$$\sin x=0+\frac{1}{1!}x+\frac{0}{2!}x^2+\frac{(-1)}{3!}x^3+\frac{0}{4!}x^4+\cdots+$$

$$=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

***

Example 19: Find Maclaurin polynomial for $\cos 2x$ around zero.

Solution: Let us write $f(x)=\cos 2x$. We can find the n^{th} derivative of f at 0, these are given as

$$f^{(n)}(x)=2^n\cos\frac{n\pi}{2} \text{ and } f^n(0)=\begin{cases}2^n(-1)^{n/2} & n \text{ is even}\\ 0 & n \text{ is odd}\end{cases}$$

Therefore, Maclaurin's polynomial around zero is

$$\cos 2x=\begin{cases}1+\frac{0.x}{1!}-\frac{2^2x^2}{2!}-\frac{0.x^3}{3!}+\frac{2^4x^4}{4!}-\frac{0.x^5}{5!}+\cdots+\frac{2^n(-1)^{n/2}}{n!}, & \text{if } n \text{ is even}\\ 0+\frac{0.x}{1!}-\frac{2^2x^2}{2!}-\frac{0.x^3}{3!}+\frac{2^4x^4}{4!}-\frac{0.x^5}{5!}+\cdots+\frac{2^{n-1}(-1)^{\frac{n-1}{2}}}{(n-1)!}, & \text{if } n \text{ is odd}\end{cases}$$

***

Example 20: Write down the following:

i) the first four terms of Maclaurin polynomial for $\tan x$.

ii) the first three non-zero terms of this polynomial.

Solution: i) Let $f(x)=\tan x$. Then $f(0)=0$

$$f'(x)=\sec^2 x,\ f'(0)=1$$

$$f''(x)=2\sec^2 x\tan x,\ f''(0)=0$$

$$f^{(3)}(x)=2\sec^4 x+4\sec^2 x\tan^2 x$$

$$=2\sec^2 x(\sec^2 x+2\tan^2 x)$$

$$f^{(3)}(0)=2(1+0)=2$$

Therefore the first four terms of Maclaurin polynomial for $\tan x$ are given by

$$0,\ \frac{1}{1!}x,\ \frac{0}{2!}x^2,\ \frac{2}{3!}x^3,$$

b) Here, Maclaurin polynomial for $\tan x$ is $x+\frac{x^3}{3}+\cdots+\frac{x^{2n+1}}{2n+1}$.

Now, we want the next non-zero term.

We have $f^{(4)}(x)=16\sec^4 x\tan x+8\sec^2 x\tan^3 x$ and $f^{(4)}(0)=0$

Next

$f^{(5)}(x)=16\sec^6 x+64\sec^4 x\tan^2 x+24\sec^4 x\tan^2 x+16\sec^2 x\tan^4 x$

This means $f^{(5)}(0)=16$.

Thus, required first three non-zero terms are

$\frac{1}{1!}x,\ \frac{2}{3!}x^3$ and $\frac{16}{5!}x^5$, that is they are $x,\ \frac{x^3}{3}$ and $\frac{2x^5}{15}$.

Thus, we have $\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots$

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Example 21: If in Maclaurin polynomial for $\sin kx$, the coefficient of x^3 is given to be $-6k$, find all possible values of k.

Solution: Since $f(x)=\sin kx$, therefore $f'(x)=k\cos kx$

$f^{(2)}(x)=-k^2\sin kx$ and $f^{(3)}(x)=-k^3\cos kx$

$\therefore$ Maclaurin polynomial for $\sin kx$ is given by

$$\sin kx=\frac{kx}{1!}-\frac{k^3x^3}{3!}+\cdots$$

The coefficient of x^3 in the expansion is $-(k^3/6)$

Therefore $-(k^3/6)=-6k$

This gives the equation $k(k^2-36)=0$

The roots are $k=0,\ 6$ or -6

Thus, $0, 6$ and -6 are all the possible values of k such that the coefficient of x^3 in Maclaurin polynomial for $\sin kx$ is $-6k$.

***

Now, try the following exercises.

E24) Find the Maclaurin's polynomial p_0, p_1, p_2, p_3, p_4 and p_n for

i) $\cos x$ iii) $\ln(1+x)$

ii) $\frac{1}{1-x}$ iv) $x e^x$

E25) Find Maclaurin's polynomial of degree three for $f(x) = 1 + 2x - x^2 + x^3$.

E26) Write down Maclaurin polynomial for the following

i) $\frac{1}{(1+x)^2}$

ii) $(x-2)^2 + 1$

iii) $1/(1-2x)$

E27) Write down the first three non-zero terms in Maclaurin polynomial of the following

i) $\sin 3x$

ii) $\ln(1-x)$

So far, we have been focussing on approximating a function in the vicinity of $x = 0$. Now, we will consider the more general case of approximating f in the vicinity of an arbitrary value x_0. Again we will be using the same idea which we used for Maclaurin's polynomial, that is the valueS of the polynomial and the value of its successive derivatives at x_0 (if exists) are same as those of f at x_0.

Now consider the n^{th} degree polynomial in ascending powers of $(x - x_0)$, we have $p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + ... + a_n(x-x_0)^n$.
Again assuming the property
$p_n(x_0) = f(x_0), p_n'(x_0) = f'(x_0), p_n''(x_0) = f''(x_0) ... \; p_n^{(n)}(x_0) = f^{(n)}(x_0)$, we get

$$p_n(x_0) = f(x_0) = a_0$$

$$p_n'(x_0) = f'(x_0) = a_1$$

$$p_n''(x_0) = f''(x_0) = 2!a_2 \Rightarrow a_2 = \frac{f''(x_0)}{2!}$$

$$p_n'''(x_0) = f'''(x_0) = 3!a_3 \Rightarrow a_3 = \frac{f'''(x_0)}{3!}$$

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.

$$p_n^{(n)}(x_0) = f^{(n)}(x_0) = n!a_n \Rightarrow a_n = \frac{f^{(n)}(x_0)}{n!}$$

Substituting these values in $p_n(x)$, we obtain

$$p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$
$$+ .... + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Fig. 12: Brook Taylor (1685-1731)

This polynomial is called the n^{th} **Taylor polynomial** of f about $x = x_0$. It is defined in the following definition.

Definition: Let the function f be differentiable n times at x_0, then n^{th} Taylor polynomial of f at x_0 is

$$p_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

In sigma notation, it can be written as

$$p_n(x)=\sum_{i=0}^{n}\frac{f^{(i)}(x_0)}{i!}(x-x_0)^i.$$

You may observe that n^{th} Maclaurin polynomial is a particular case of n^{th} Taylor polynomial about $x_0=0$. Since Taylor polynomial is used to approximate $f(x)$, therefore $p_n(x)\approx f(x)$.

Let us find Taylor polynomial in the following examples.

Example 22: Find the first five Taylor polynomials for $\ln x$ about $x=5$.

Solution: Let $f(x)=\ln x$, then $f(5)=\ln 5$. Differentiating f successively and putting $x=5$, we get

$$f'(x)=\frac{1}{x},\ f'(5)=\frac{1}{5}$$

$$f''(x)=-\frac{1}{x^2},\ f''(5)=-\frac{1}{25}$$

$$f'''(x)=\frac{1.2}{x^3},\ f''(5)=\frac{2!}{125}$$

$$f^{IV}(x)=-\frac{1.2.3}{x^4},\ f^{IV}(5)=-\frac{3!}{625}$$

The Taylor polynomials are

$$p_0(x)=f(5)=\ln 5$$

$$p_1(x)=f(5)+f'(5)(x-5)=\ln 5+\frac{1}{5}(x-5)$$

$$p_2(x)=f(5)+f'(5)(x-5)+\frac{f''(5)}{2!}(x-5)^2=\ln 5+\frac{1}{5}(x-5)-\frac{1}{50}(x-5)^2$$

$$p_3(x)=\ln 5+\frac{1}{5}(x-5)-\frac{1}{50}(x-5)^2+\frac{1}{375}(x-5)^3$$

$$p_4(x)=\ln 5+\frac{1}{5}(x-5)-\frac{1}{50}(x-5)^2+\frac{1}{375}(x-5)^3-\frac{1}{2500}(x-5)^4.$$

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Example 23: Find the n^{th} Taylor's polynomial for $1/x$ about $x=1$. Also, write the polynomial in sigma notation.

Solution: Let $f(x)=1/x$. There $f(1)=1$

Differentiating f successively and putting $x=1$, we obtain

$$f'(x)=-\frac{1}{x^2},f'(1)=-1$$

$$f''(x)=\frac{2!}{x^3},f''(1)=2!$$

$$f'''(x)=-\frac{3!}{x^4},f'''(1)=-3!$$

$f^{IV}(x) = \frac{4!}{x^5}$, $f^{IV}(1) = 4!$

. . .

. . .

. . .

$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}, f^{(n)}(1) = (-1)^n n!$

Thus, Taylor's polynomial is

$$p_n(x) = 1 + (-1)(x-1) + \frac{2!}{2!}(x-1)^2 + \frac{-3!}{3!}(x-1)^3$$

$$+\frac{4!}{4!}(x-1)^4 + \ldots + \frac{(1)^n n!}{n!}(x-1)^n .$$

That is $p_n(x) = 1-(x-1)+(x-1)^2-(x-1)^3+\ldots+(1)^n(x-1)^n$

In sigma notation,

$$p_n(x) = \sum_{i=0}^{n} (-1)^i (x-1)^i$$

***

Try the following exercises.

E28) Consider the function $y = a + \tan^{-1} bx$ where a and b are fixed real numbers. We are given that its Taylor polynomial around zero is $2 + 3x - 9x^3 + \cdots$. Find the values of a and b.

E29) Find the coefficient of x^3 in Taylor polynomial around zero for the function $\sin^{-1} x$.

E30) i) Fig. 13 shows a sector of radius π and central angle θ is small, use local quadratic approximation of $\cos\theta$ at $\theta = 0$ and show that $x \approx \frac{\theta^2}{8}$.

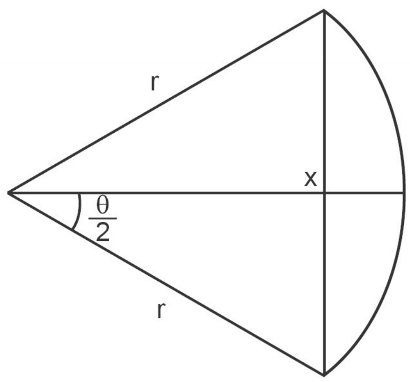

Fig. 13

ii) Assuming that the Earth is a sphere of radius 3000 km, use the result obtain part i) to approximate the maximum amount by which a 100 km arc along the equator will diverge from its chord.

Now, let us summarize the unit.

11.6 SUMMARY

In this unit, we have studied following points:

1. Introduced higher order derivatives.

2. Derived a formula (Leibniz'z Theorem) for the n^{th} derivative of a product of two functions

$$(uv)^{(n)} = C(n,\,0)u^{(n)}v + C(n,\,1)u^{(n-1)}v^{(1)} + C(n,2)u^{(n-2)}v^{(2)} + \cdots + C(n,n)uv^{(n)}.$$

3. The linear approximation of a function f about any point x_0 is $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.

4. The quadratic approximation of a function f about the point x_0 is
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2.$$

5. Written Maclaurin's polynomial of a function by using the formula
$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

6. Written Taylor polynomial for a function f about the point x_0 as
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots. + \frac{f^{(x)}(x_0)}{n!}(x - x_0)^n.$$

11.7 SOLUTIONS/ANSWERS

E1) i) $f'(x) = 3x^2$

$f''(x) = 6x$

ii) $y' = 2e^{2x}$

$y'' = 4.e^{2x}$

E2) $y = e^{ax}\sin bx,$

$y_1 = e^{ax}(\cos bx\,.b) + \sin bx(e^{ax}a)$

$= be^{ax}\cos bx + ay$

or $y_1 - ay = be^{ax}\cos bx$

Again differentiating both sides,

$y_2 - ay_1 = be^{ax}(-\sin bx\,.b) + b\cos bx(e^{ax}.a)$

$= -b^2y + a(y_1 - ay)$

or $y_2 - 2ay_1 + (a^2 + b^2)y = 0$

E3) i) $f(x) = \sin kx \Rightarrow f^{(2)}(x) = -k^2\sin kx$

$\Rightarrow f^{(2)}(\pi/6) = -k^2\sin k\pi/6$

Now, $-k^2\sin k\pi/6 = 2\sqrt{3} \Rightarrow \sin k\pi/6 = -2\sqrt{3}/k^2$

Since $-1 < \sin k\pi/6 < 0, -\pi < k\pi/6 < 0$

$\Rightarrow k = -1$ or -2

$k \neq 0$ since $k = 0 \Rightarrow 0 = 2\sqrt{3}$, which is impossible.

Out of these, $k=-2$ is the value which satisfies
$\sin k\pi/6=-2\sqrt{3}/k^2$

ii) $f(x)=x^k+kx^2+1 \Rightarrow f'(x)=kx^{k-1}+2kx$
$f''(1)=k(k-1)+2k=12 \Rightarrow k=3$ or -4

E4) Given is $s(t)=t^3-4.5t^2-7t$

i) Velocity $v(t)=3t^2-9t-7$

$v(t)=5 \Rightarrow 3t^2-9t-7=5$
$\Rightarrow 3t^2-9t-12=0$
$\Rightarrow 3t^2-12t+3t-12=0$
$\Rightarrow 3t(t-4)+3(t-4)=0$
$\Rightarrow (t-4)(3t+3)=0$
$\Rightarrow t=4$, as t cannot be –ive.

ii) acceleration $a(t)=6t-9$

$a(t)=0,$ when $6t-9=0$ or $t=1.5$ sec.
The acceleration 0 means that the rate of change of velocity is 0, which means further that the velocity is constant.

E5) i) $$p'(t)=\frac{2000\,(4t+75)-2000\,t\,(4)}{(4t+75)^2}$$

$$p'(10)=\frac{150000}{(4t+75)^2}$$
$$=11.34$$
$p'(50)=6.12$
$p'(100)=0.665$

ii) $$p''(t)=\frac{-300000}{(4t+75)^3}$$

$p''(10)=-0.197$
$p''(50)=-0.0144$
$p''(100)=-0.003$

iii) $p'(t)$ represents the rate of change in population in t years. This rate of change in 10 years, 50 years and 100 years is $11.34, 6.12, 0.665$ respectively. Similarly the rate of change of population is represented by $p''(t)$. It is also decreasing.

E6) i) $f'(x)=\sec x\tan x$

$f^{(2)}(x)=\sec x\tan^2 x+\sec^3 x$
$f^{(3)}(x)=\sec x\tan^3 x+2\sec^3 x\ \tan x+3\sec^3 x\tan x$
$f^{(3)}(\pi/4)=\sqrt{2}.1+2.\ 2\sqrt{2}.1+3.\ 2\sqrt{2}=11\sqrt{2}$

ii) $f'(x)=2\cos 2x-2\sin 2x$

$f''(x)=-4\sin 2x-4\cos 2x$
$f^{(3)}(x)=-8\cos 2x+8\sin 2x$
$f^{(3)}(\pi/4)=8$

E7) Let $y = \sin x$

$y_1 = \cos x$

$y_2 = -\sin x$

$y_3 = -\cos x$

$y_4 = \sin x$

Fourth derivative is $\sin x$, therefore, afterwards derivative of every multiple of four order will be $\sin x$.

Thus, $\frac{d^{96}}{dx^{96}}(\sin x) = \sin x$

$$\frac{d^{97}}{dx^{97}}(\sin x) = \cos x$$

$$\frac{d^{98}}{dx^{98}}(\sin x) = -\sin x$$

$$\frac{d^{99}}{dx^{99}}(\sin x) = -\cos x$$

E8) $y_n = r(r-1)\ldots(r-n+1)\ (1+x)^{r-n},\ n < r$

E9) i) $f^{(n)}(x) = \begin{cases} \frac{3!a^n}{(3-n)!}(ax+b)^{3-n}; & \text{if } n \le 3 \\ 0, \text{ if } n > 3 \end{cases}$

ii) $f^{(n)}(x) = \begin{cases} \frac{m!a^n}{(m-n)!}(ax+b)^{m-n} \text{ if } n \le m \\ 0, \text{ if } n > m \end{cases}$

iii) $f^{(n)}(x) = e^x$

iv) $f^{(n)}(x) = k^n e^{kx}$

v) $\frac{x}{(x-1)(2x+3)} = \frac{1}{5}.\frac{1}{x-1} + \frac{3}{5}.\frac{1}{2x+3}$ [using partial fractions]

E10) $f'(x) = a_1 + 2a_2x + .. + n\,a_n x^{n-1}$

$f''(x) = 2a_2 + ... + n\,(n-1)a_n x^{n-2}$

$\vdots$

$f^{(n)}(x) = n\,(n-1).......1.a_n x^{n-n}$

$= n!a_n.$

E11) $f(x) = \sin x \Rightarrow f'(x) = \cos x,\ f''(x) = -\sin x$,

$f^{(3)}(x) = -\cos x,\ f^{(4)}(x) = \sin x$ and so on. So our guess is that

$$f^{(n)}(x) = \begin{cases} \cos x & \text{if } n = 4k+1 \\ -\sin x & \text{if } n = 4k+2 \\ -\cos x & \text{if } n = 4k+3 \\ \sin x & \text{if } n = 4k \end{cases}$$

or $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$.

E12) $y = \sin(ax + b)$

$y_1 = a\cos(ax + b)$

$y_2 = -a^2 \sin(ax + b)$

$y_3 = -a^3 \cos(ax + b)$

$y_4 = a^4 \sin(ax + b)$

Thus,

$$y^{(n)}(x) = \begin{cases} a^n \cos(ax+b), \text{if } n = 4k+1 \\ -a^n \sin(ax+b), \text{if } n = 4k+2 \\ -a^n \cos(ax+b), \text{if } n = 4k+3 \\ a^n \sin(ax+b), \text{if } n = 4k \end{cases}$$

Thus, $y^{(n)}(x) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$

E13) $y = \cos x \Rightarrow y_1 = -\sin x,\ y_2 = -\cos x,\ y_3 = \sin x,\ y_4 = \cos x$ and so on.

$y_n = \cos(x + n\pi/2)$

$\Rightarrow y_{n+1} = -\sin(x + n\pi/2)$

$\Rightarrow y_n^2 + y_{n+1}^2 = \cos^2(n\pi/2) + \sin^2(n\pi/2) = 1$.

E14) $(uv)^{(5)} = u^{(5)}v + 5u^{(4)}v^{(1)} + 10u^{(3)}v^{(2)} + 10u^{(2)}v^{(3)} + 5u^{(1)}v^{(4)} + v^{(5)}$

E15) $(uv)^{(1)} = u^{(1)}v + uv^{(1)}$ which is the product rule of differentiation.

E16) $\frac{d^3(\sin^2 x)}{dx^3} = \frac{d^3}{dx^3}(\sin x.\sin x) = -\cos x \sin x - 3\sin x \cos x$

$-3\sin x\cos x - \cos x \sin x = -8\sin x \cos x$

Now, differentiating directly, we get

$\frac{d}{dx}(\sin^2 x) = 2\sin x \cos x$

$\Rightarrow \frac{d^2}{dx^2}(\sin^2 x) = 2(\cos^2 x - \sin^2 x)$

$\Rightarrow \frac{d^3}{dx^3}(\sin^2 x) = -8\sin x \cos x$

Both the results are same.

E17) $u = x,\quad v = e^x$

$(uv)^{(6)} = vu^{(6)} + 6v^{(1)}u^{(5)} + 15v^{(2)}u^{(4)} + 20v^{(3)}u^{(3)} + 15v^{(4)}u^{(2)} + 6v^{(5)}u^{(1)} + v^{(6)}u$

$= 0 + 0 + 0 + 0 + 0 + 6.e^x.1 + e^x.x = (x+6)e^x$

E18) $D^n(x^3 \ln x) = \frac{(-1)^{n+1}x^3}{x^n}[(n-1)! - 3C(n, 1)\ (n-2)! + 6C(n, 2)\,(n-3)!$

$- 6C(n, 3)\ (n-4)!]$

E19) $y^{(n)} = D^n[e^{ax}x^2]$ [where $u = x^2$ and $v = e^{ax}$]

$$= x^2.D^n(e^{ax}) + C(n,1)D(x^2).D^{n-1}(e^{ax}) + C(n,2)D^2(x^2)D^{n-2}(e^{ax})$$

[$\because D^3(x^2)$ onwards all derivatives are zero]

$$= a^n x^2 e^{ax} + n.2x.a^{n-1}e^{ax} + \frac{n(n-1)}{2!}.2.(a)^{n-2}e^{ax}$$

$$= a^{(n-2)}e^{ax}[a^2x^2 + 2nax + n(n-1)a^2]$$

E20) i) $(uv)_m = C(m,0)u_m v + C(m,1)u_{m-1}v_1 + C(m,2)u_{m-2}v_2 + \ldots + C(m,m)u\,v_m$.

ii) On differentiating again, we get

$$(uv)_{m+1} = C(m,0)u_{m+1}v + C(m,0)u_m v_1 + C(m,1)u_m v_1 + C(m,1)u_{m-1}v_2 + \ldots + C(m,m)u_1 v_m + C(m,m)u\,v_{m+1}$$

iii) $$(uv)_{m+1} = [C(m+1,0)u_{m+1}v + C(m+1,1)u_m v_1 + C(m+1,2)u_{m-1}v_2 + \ldots + C(m+1,m+1)u\,v_{m+1}]$$

E21) Let $f(x) = \sin x$, $f'(x) = \cos x$

$x_0 = 0 \Rightarrow f(x_0) = 0$ and $f'(x_0) = 1$

$\sin x \approx f(0) + f'(0)(x-0)$

$\sin x \approx 0 + 1(x)$

$\sin x \approx x$

The Fig. 14 shows the graph

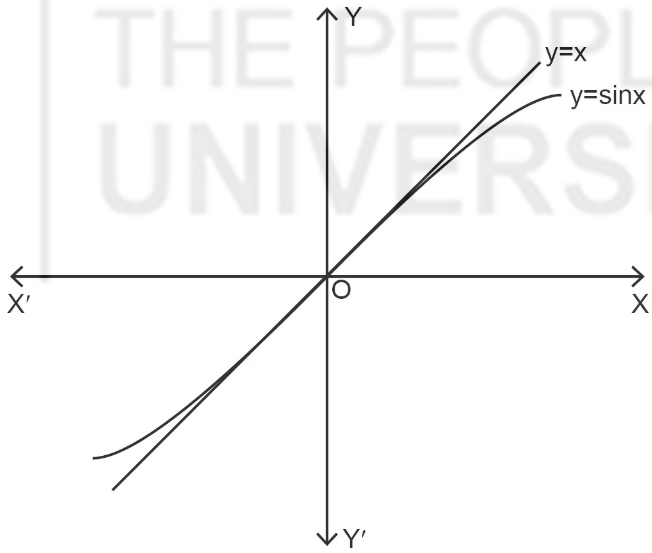

Fig. 14: Graph of $\sin x$ near 0

$$\sin 1^\circ = \sin\left(1 \times \frac{\pi}{180}\right) \text{(in radians)}$$

$$= \sin\frac{\pi}{180} = \sin(0.01744) \approx 0.01744$$

Therefore, $\sin 1^\circ \approx 0.01744$.

E22) $f(x) = \sin(2x) + \cos x$ and $f(0) = 1$

$f'(x) = 2\cos(2x) - \sin x$ and $f'(0) = 2$

$f''(x) = -4\sin(2x) - \cos x$ and $f''(0) = -1$

Thus, the quadratic approximatin of $f(x)$ is

$f(x) = f(0) + f'(0)(x-0) + \dfrac{f''(0)}{2}(x-0)^2$ which gives us

$\sin(2x) + \cos(x) = 1 + 2x - \dfrac{1}{2}x^2$. Fig. 13 shows this graphically.

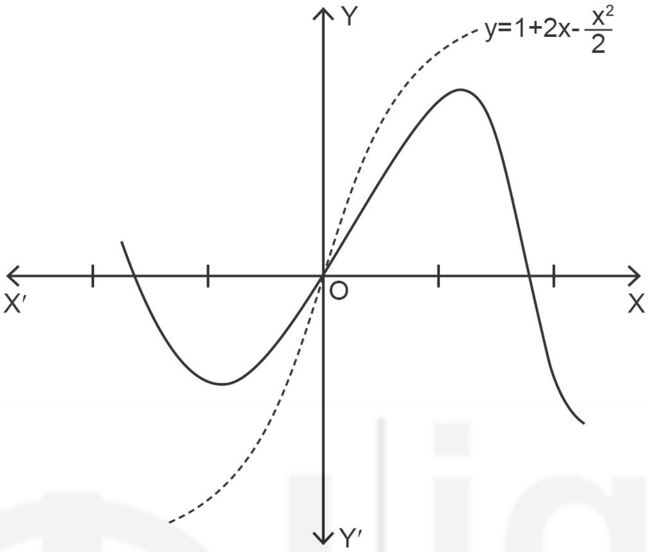

Fig. 15: Graph of $\sin(2x) + \cos x$ **and** $1 + 2x - \dfrac{1}{2}x^2$

E23) $f(x) = \sqrt{x}$, $f(9) = 3$

$f'(x) = \dfrac{1}{2\sqrt{x}}$, $f'(9) = \dfrac{1}{6}$

$f''(x) = \dfrac{-1}{4x^{3/2}}, f''(9) = \dfrac{-1}{108}$

$$p_2(x) = f(9) + \frac{f'(9)}{1!}(x-9) + \frac{f''(9)}{2!}(x-9)^2$$

$$= 3 + \frac{1}{6}(x-9) + \frac{(-1/108)}{2!}(x-9)^2$$

$$= 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2$$

Now, $\sqrt{9.1} \approx 3 + \dfrac{1}{6}(0.1) - \dfrac{1}{216}(0.1)^2 = 3.015$

E24) i) $f(x) = \cos x, f(0) = 1$

$f'(x) = -\sin x, f'(0) = 0$

$f''(x) = -\cos x, f''(0) = -1$

$f'''(x) = \sin x, f'''(0) = 0$

$f^{iv}(x) = \cos x, f^{iv}(0) = 1$

$$p_0(x) = 1$$

$$p_1(x) = 1 + 0.x = 1$$

$$p_2(x) = 1 + 0.x + \frac{(-1)}{2!}.x^2 = 1 - \frac{x^2}{2!}$$

$$p_3(x) = 1 + 0.x + \frac{(-1)}{2!}.x^2 + \frac{0}{3!}.x^3 = 1 - \frac{x^2}{2!}$$

$$p_4(x) = 1 + 0.x + \frac{(-1)}{2!}.x^2 + \frac{0}{3!}.x^3 + \frac{1}{4!}.x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

and

$$p_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^k \frac{x^{2k}}{(2k)!}, \text{where } k = 0,1,2,...$$

ii) $f(x) = \ln(1+x),\ f(0) = \ln 1 = 0$

$$f'(x) = \frac{1}{1+x},\ f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2},\ f''(0) = -1$$

$$f'''(x) = \frac{+1.2}{(1+x)^3},\ f'''(0) = 2!$$

$$f^{iv}(x) = \frac{-1.2.3}{(1+x)^4},\ f^{iv}(x) = -3!$$

Now

$$p_0(x) = 0$$

$$p_1(x) = 0 + 1.x = x$$

$$p_2(x) = 0 + 1.x + \frac{(-1)}{2!}.x^2 = x - \frac{x^2}{2}$$

$$p_3(x) = 0 + 1.x + \frac{(-1)}{2!}.x^2 + \frac{2!}{3!}.x^3 = x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$p_4(x) = 0 + 1.x + \frac{(-1)}{2!}.x^2 + \frac{2!}{3!}.x^2 + \frac{2!}{3!}.x^3 + \frac{(-1)3!}{4!}.x^4$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

and

$$p_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + (-1)^{n-1}\frac{x^n}{n}, \text{where } n = 1,2,...$$

iii) $f(x) = \dfrac{1}{1.x}$

$$p_n(x) = 1 + x + x^2 + ... + x^n,\ n = 0,1,2,...$$

and $p_0(x) = 1$

$p_1(x) = 1 + x$

$p_2(x) = 1 + x + x^2$

$p_3(x) = 1 + x + x^2 + x^3$

$p_4(x) = 1 + x + x^2 + x^3 + x^4$.

iv) $f(x) = x\,e^x, f(0) = 0$

$f'(x) = e^x + x\,e^x, f'(0) = 1$

$f''(x) = e^x + e^x + x\,e^x, f''(0) = 2$

$f'''(x) = e^x + e^x + e^x + x\,e^x, f'''(0) = 3$

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$f^{(n)}(x) = n.$

$p_0(x) = 0$

$p_1(x) = 0 + 1.x = x$

$p_2(x) = x + x^2$

$$p_3(x) = x + x^2 + \frac{x^3}{2!}$$

$$p_4(x) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!}$$

.

.

.

$$p_n(x) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + ... + \frac{x^n}{(n-1)!}, \text{where } n = 0,1,2...$$

E25) $f(x) = 1 + 2x - x^2 + x^3, f(0) = 1$

$f'(x) = 2 - 2x + 3x^2, f'(0) = 2$

$f''(x) = -2 + 6x, f''(0) = -2$

$f'''(x) = 6,\ f'''(0) = 6$

All other higher order derivatives will be zero.

$$p_3(x) = 1 + 2.x + \frac{-2}{2!}.x^2 + \frac{6}{3!}.x^3$$

$$= 1 + 2x - x^2 + x^3$$

E26) i) $1 - 2x + 3x^2 - 4x^3 + \cdots$

ii) $5 - 4x + x^2 + 0.x^3 + 0.x^4 + \cdots$

iii) $1+2x+2^2x^2+2^3x^3+\cdots$

E27) i) $3x-\frac{3^3x^3}{3!}+\frac{3^5x^5}{5!}$

ii) $-x-\frac{x^2}{2}-\frac{x^3}{3}$

E28) We take that $\tan^{-1}bx$ always takes value between $-\pi/2$ and $\pi/2$. Then, $a=2, b=3$

E29) $1/6$

MISCELLANEOUS EXAMPLES AND EXERCISES

The examples and exercises given below cover the concepts and processes you have studied in this block. Doing them will give you a better understanding of the concepts concerned, as well as practice in solving such problems.

Example 1: Show that if $f(x) = x^2$, the function f is derivable at $x = 1$.

Solution: Now $f(x) = x^2 \Rightarrow f(1) = 1^2 = 1$.

To find the derivative of f at $x = 1$, we use the definition of derivative at $x = 1$ and get

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} \\ &= \lim_{h \to 0} 2 + h, \text{[as } h \neq 0] \\ &= 2 \end{aligned}$$

Hence, f is derivable at $x = 1$ and its derivative $f'(1)$ is 2.

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Example 2: For $y = f(x) = x^2$, find the average rate of change as:

i) x changes from 2 to 4.

ii) x changes from 2 to 3.

iii) x changes from 3 to 4.

Solution: The graph shown in Fig. 1 gives a look at two of the secant lines AB and AC. We are computing slopes of these secant lines.

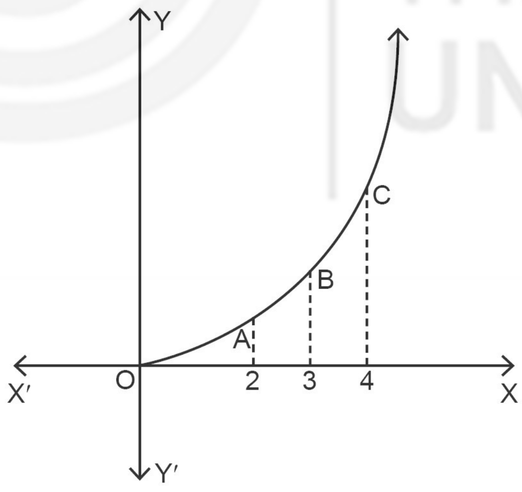

Fig. 1

i) When $x_1 = 2$, $y_1 = f(2) = 4$ and when $x_2 = 4$, $y_2 = f(4) = 16$.

The average rate of change is $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{16 - 4}{4 - 2} = \dfrac{12}{2} = 6$.

ii) When $x_1 = 2, y_1 = f(x_1) = 4$; and when $x_2 = 3$, $y_2 = f(3) = 9$.

The average rate of change $= \dfrac{9 - 4}{3 - 2} = 5$.

iii) When $x_1 = 3, y_1 = f(3) = 9$; and when $x_2 = 4$, $y_2 = f(4) = 16$.

The average rate of change $= \dfrac{16-9}{4-3} = 7$.

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Example 3: For $f(x) = 3x^2$, find the difference quotient for the following:

i) $x = 1$ and $\delta x = 0.1$

ii) $x = 1$ and $\delta x = 0.01$.

Solution: i) We substitute $x = 1$ and $\delta x = 0.1$ into the formula, we get

$$\frac{f(x+\delta x)-f(x)}{\delta x} = \frac{f(1+0.1)-f(1)}{0.1} = \frac{f(1.1)-f(1)}{0.1}$$

Now $f(1.1) = (1.1)^2 = 3.63$ and $f(1) = 3$, and we have

$$\frac{f(1.1)-f(1)}{0.1} = \frac{3.63-3}{0.1} = \frac{0.63}{0.1} = 6.3.$$

ii) We substitute $x = 1$ and $\delta x = 0.01$ into the formula, we get

$$\frac{f(x+\delta x)-f(x)}{\delta x} = \frac{f(1+0.01)-f(1)}{0.01} = \frac{f(1.01)-f(1)}{0.01}.$$

Now $f(1.01) = (1.01)^2 = 31.0603$ and $f(1) = 3$, and we have

$$\frac{f(1.01)-f(1)}{0.01} = \frac{3.0603-3}{0.01} = \frac{0.0603}{0.01} = 6.03.$$

You may note the trend in the average rate of change as δx gets closer to 0.

***

Example 4: Find the derivative of f if $f(x) = x^3$ using first principle.

Solution: The domain of the function f is the entire set $\mathbb{R}$ of real numbers. Let $x \in \mathbb{R}$. When $\delta x \neq 0$, we have

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{(x+\delta x)^3 - x^3}{\delta x} = \lim_{\delta x \to 0} (\delta x^2 + 3\delta x x + 3x^2) = 2x$$

Thus, $f'(x) = 3x^2 \; \forall x \in \mathbb{R}$.

***

Example 5: Find the derivative of f if $f(x) = \sqrt{x}$ using first principle.

Solution: The domain of the function f is the set of all non-negative real numbers i.e. the interval $[0, \infty[$.

Let $x > 0$. We have, when $\delta x \neq 0$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\sqrt{(x+\delta x)} - \sqrt{x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{\sqrt{(x+\delta x)} - \sqrt{x}}{\delta x}\right]\left[\frac{\sqrt{(x+\delta x)} + \sqrt{x}}{\sqrt{(x+\delta x)} + \sqrt{x}}\right]$$

$$= \lim_{\delta x \to 0} \left(\frac{1}{\sqrt{(x+\delta x)} + \sqrt{x}}\right)$$

Therefore, $\displaystyle\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \frac{1}{2\sqrt{x}}; x > 0$

Thus, for $x > 0$, $f(x) = \sqrt{x} \Rightarrow f'(x) = 1/2\sqrt{x}$.

We start afresh to examine the existence of the derivative at 0. We have

$$\frac{f(0+\delta x) - f(0)}{\delta x} = \frac{\sqrt{\delta x}}{\delta x} = \frac{1}{\sqrt{\delta x}} \to \infty \text{ when } \delta x \to 0 \text{ through positive values.}$$

Thus, the derivative of f at $x = 0$ does not exist. We know that $\sqrt{h}$ is not defined for negative values of h. Thus, $f'(x) = 1/2\sqrt{x},\ \forall\, x \in]0, \infty[$.

***

Example 6: Show that the tangent to the hyperbola $y = 1/x$ at $(1,1)$ makes an angle $3\pi/4$ with $x -$axis.

Solution: The hyperbola $y = 1/x$ gives $\frac{dy}{dx} = -\frac{1}{x^2}$.

Now $\left(\frac{dy}{dx}\right)_{\text{at }(1,1)} = -1$.

Slope of any line is always equal to the tangent of the angle it makes from $x -$axis. Therefore, $\tan\theta = -1$, where θ is the angle of the tangent to the hyperbola $y = 1/x$ at $(1,1)$ with $x -$axis. Thus, $\theta = 3\pi/4$.

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Example 7: Find the differential dy and the increment δy when $y = x^3$ for

i) aribitrary values of x and δx

ii) at $x = 10$ and $\delta x = 0.1$.

Solution: i) We have $y + \delta y = (x + \delta x)^3$

$$= x^3 + 3x^2\,\delta x + 3x(\delta x)^2 + (\delta x)^3$$

$$\delta y = 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3$$

$$= 3x^2\delta x + [3x\,\delta x + (\delta x)^2]\delta x.$$

Also, we have $dy = 3x^2\,dx$.

ii) For $x = 10$ and $\delta x = 0.1$, we have

$$3x^2dx = 300(0.1) = 30 = dy$$

$$\delta y = 30 + [30 \times 0.1 + (0.1)^2](0.1)$$

$$= 30 + \cdot 301 = 30 \cdot 301$$

Thus, considering dy in place of δy, we have an error of 0.301.

***

Example 8: A particle is moving in a straight line and the positions (in meter) in t seconds. Find the velocity and acceleration (i) at the end of 3 seconds, (ii) initially, in each of the following cases.

i) $s(t) = t^2 + 2t + 3$

ii) $s(t) = 1/(t+1)$

iii) $s(t) = \sqrt{(t+1)}$.

Solution: i) We have $s(t) = t^2 + 2t + 3$

The velocity at time t is $\frac{ds}{dt} = 2t + 2$

The velocity at the end of 3 sec. is $\left(\frac{ds}{dt}\right)_{\text{at } t=3} = [2t+2]_{\text{at } t=3} = 8m/s$

The initial velocity is $\left(\frac{ds}{dt}\right)_{\text{at } t=0} = [2t+2]_{\text{at } t=0} = 2\,m/s$

The acceleration at time $t = 3$ is $\left[\frac{d^2s}{dt^2}\right]_{\text{at } t=3} = [2]_{\text{at } t=3} = 2m/s^2$

The acceleration at the end of $3s$ is $\left(\frac{d^2s}{dt^2}\right)_{\text{at } t=3} = 2\,m/s^2$.

The initial acceleration is $\left(\frac{d^2s}{dt^2}\right)_{\text{at } t=0} = 2\,m/s^2$.

ii) Let $s(t) = \frac{1}{t+1}$

The first derivative is $\frac{ds}{dt} = -\frac{1}{(t+1)^2}$

Velocity at the end of $3s$ is $\left(\frac{ds}{dt}\right)_{\text{at } t=3} = -\frac{1}{16}\,m/s$.

Initial velocity is $\left(\frac{ds}{dt}\right)_{\text{at } t=0} = -1\,m/s$.

The second derivative is $\frac{d^2s}{dt^2} = \frac{2}{(t+1)^3}$

Acceleration at the end of $3s = \left(\frac{d^2s}{dt^2}\right)_{\text{at } t=3} = \frac{1}{32}\,m/s^2$

Initial acceleration $= \left(\frac{d^2s}{dt^2}\right)_{\text{at } t=0} = 2\,m/s^2$

iii) Let $s(t) = \sqrt{t+1}$

$\frac{ds}{dt} = \frac{1}{2\sqrt{t+1}}$

The velocity at $t = 3$ is $\left(\frac{ds}{dt}\right)_{\text{at } t=3} = \frac{1}{4}\,m/s$

The initial velocity $= \left(\frac{ds}{dt}\right)_{\text{at } t=0} = \frac{1}{2}\,m/s$

The acceleration at $t = 3$ is $= \frac{d^2s}{dt^2} = -\frac{1}{4(t+1)^{3/2}}$

The acceleration at $t = 3$ is $= \left(\frac{d^2s}{dt^2}\right)_{\text{at } t=3} = -\frac{1}{32}\,m/s^2$

The initial acceleration $= \left(\frac{d^2s}{dt^2}\right)_{\text{at } t=0} = -\frac{1}{4}\,m/s^2$.

***

Example 9: Give the derivatives of functions with following values.

i) $(x^2-2x)\,(3x^2+4)$

ii) $(x^2-1)^2$

iii) $(x^3+3)(x^2-4)$

Solution: i) Let $f(x)=(x^2-2x)(3x^2+4)$

$$f'(x)=\frac{d}{dx}(x^2-2x).(3x^2+4)+(x^2-2x).\frac{d}{dx}(3x^2+4)$$
$$=(2x-2)(3x^4+4)+(x^2-2x)(6x)$$

ii) Let $f(x)=(x^2-1)^2$

$$f'(x)=\frac{d}{dx}(x^2-1)^2$$
$$=\frac{d(t^2)}{dt}\cdot\frac{dt}{dx}, \text{ where } t=x^2-1$$
$$=2(x^2-1).(2x)$$

iii) Let $f(x)=(x^3+3)(x^2-4)$

$$f'(x)=3x^2(x^2-4)+(x^3+3)(2x)$$

***

Example 10: Find the derivatives of the functions defined by the following expressions.

i) $\sqrt{(1+x^2)}$

ii) $\sqrt{[(1+x)/(1-x)]}$

Solution: i) We write $u=1+x^2, y=\sqrt{u}$ so that $y=\sqrt{(1+x^2)}$.

We have $\frac{du}{dx}=2x, \frac{dy}{du}=\frac{1}{2}u^{-1/2}=\frac{1}{2}(1+x^2)^{-1/2}$

Hence $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{2}(1+x^2)^{-1/2}\,2x=\frac{x}{\sqrt{(1+x^2)}}$.

ii) We have $y=\sqrt{[(1+x)/(1-x)]}$.

Let $u=\frac{1+x}{1-x},\ y=u^{1/2}$

So that $y=\sqrt{\left(\frac{1+x}{1-x}\right)}$.

We have $$\frac{du}{dx}=\frac{(1-x)\frac{d(1+x)}{dx}-(1+x)\frac{d(1-x)}{dx}}{(1-x)^2}$$
$$=\frac{(1-x)1-(1+x)(-1)}{(1-x)^2}=\frac{2}{(1-x)^2}.$$

$$\frac{dy}{du}=\frac{1}{2}u^{-1/2}=\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2}$$

Hence $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2} \frac{2}{(1-x)^2}$

$= \frac{1}{(1+x)^{1/2}(1-x)^{3/2}}$

***

Example 11: Find the derivatives of the functions defined by the following values:

i) $\sin 2x$

ii) $\cos^3 x$

iii) $\sqrt{(\sin\sqrt{x})}$

Solution: i) Let $y = \sin 2x$. We write $u = 2x$ so that $y = \sin u$.

Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u.2 = 2\cos 2x$

Or briefly

$\frac{d(\sin 2x)}{dx} = \frac{d(\sin 2x)}{d(2x)} \cdot \frac{d(2x)}{dx} = \cos 2x.2 = 2\cos 2x$.

ii) Let $y = \cos^3 x = (\cos x)^3$. We write

$u = \cos x$ so that $y = u^3$.

Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2(-\sin x) = -3\cos^2 x \sin x$.

iii) Let $y = \sqrt{(\sin\sqrt{x})}$. We write

$u = \sqrt{x}, v = \sin u$ so that $y = \sqrt{v}$.

Now $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$

$= \frac{1}{2} v^{-\frac{1}{2}} \cos u . \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{4} \frac{\cos\sqrt{x}}{\sqrt{(\sin\sqrt{x})}} \cdot \frac{1}{\sqrt{x}}$.

***

Example 12: Find the equation of the tangent to the following curves at the specified points.

i) $y = \sin^{-1} x$ at $(0,0)$

ii) $y = \cos^{-1} x$ at $(1,0)$

Solution: i) The first derivative $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\left(\frac{dy}{dx}\right)_{\text{at}\,(0,0)} = 1$

Thus, the equation of the tangent is $y - 0 = 1(x - 0)$, which is $y = x$.

ii) $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$$\left(\frac{dx}{dy}\right)_{\text{at }(1,0)} = \infty$$

Thus, the equation of the tangent is $x = 1$.

***

Example 13: Find the derivatives of the functions defined by the following expressions.

i) $\ln(\sec x + \tan x)$

ii) $\sqrt{(a^{\sqrt{x}})}$

Solution: i) Let $u = \sec x + \tan x$

$$\frac{d}{dx}[\ln(\sec x + \tan x) = \frac{d}{du}(\ln u).\frac{du}{dx}$$
$$= \frac{1}{u}\cdot\frac{d}{dx}(\sec x + \tan x)$$
$$= \frac{1}{(\sec x + \tan x)}\cdot(\sec x \tan x + \sec^2 x)$$
$$= \sec x.$$

ii) Let $a^{\sqrt{x}} = u$ and $\sqrt{x} = v$.

$$\frac{d}{dx}\left[\sqrt{(a^{\sqrt{x}})}\right] = \frac{d\sqrt{u}}{du}\cdot\frac{d}{dv}(a^{v})\cdot\frac{d}{dx}(\sqrt{x})$$
$$= \frac{1}{2u^{-1/2}}\cdot a^{v}\ln a\cdot\frac{1}{2\sqrt{x}}$$
$$= \frac{\sqrt{a^{\sqrt{x}}}}{4\sqrt{x}}\cdot a^{\sqrt{x}}\ln a$$

***

Example 14: Find $\frac{dy}{dx}$, when

i) $x = a(\cos t + t\sin t)$,
$y = a(\sin t - t\cos t)$,

ii) $x = 3\cos t - 2\cos^3 t$,
$y = 3\sin t - 2\sin^3 t$

Solution: i) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided $\frac{dx}{dt} \neq 0$.

$$= \frac{a(\cos t + t\sin t - \cos t)}{a(-\sin t + \sin t + t\cos t)}$$
$$= \tan t$$

ii) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided $\frac{dx}{dt} \neq 0$.

$$= \frac{3\cos t - 6\sin^2 t.\cos t}{-3\sin t + 6\cos^2 t\sin t}$$

***

Example 15: Differentiate $y = [x^{\tan x} + (\sin x)^{\cos x}]$.

Solution: Let $u = x^{\tan x}$, $v = (\sin x)^{\cos x}$

Since $y = u + v$, therefore, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

By taking logarithms and then on differentiating, we get

$$\frac{du}{dx} = x^{\tan x}\left(\sec^2 x \log x + \frac{\tan x}{x}\right) \qquad \text{... (1)}$$

$$\frac{dv}{dx} = (\sin x)^{\cos x}\left(-\sin x \log \sin x + \frac{\cos^2 x}{\sin x}\right) \qquad \text{... (2)}$$

Adding (1) and (2), we obtain $\frac{dy}{dx}$.

***

Example 16: Differentiate $y = \dfrac{x^{\frac{1}{2}}(1-2x)^{\frac{2}{3}}}{(2-3x)^{\frac{3}{4}}(3-4x)^{\frac{4}{5}}}$.

Solution: All the factors given in the quotient of y are positive. Therefore, taking logarithms, we obtain

$$\ln y = \frac{1}{2}\ln x + \frac{2}{3}\ln(1-2x) - \frac{3}{4}\ln(2-3x) - \frac{4}{5}\ln(3-4x).$$

On differentiating, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x} + \frac{2}{3}\cdot\frac{-2}{1-2x} - \frac{3}{4}\cdot\frac{-3}{2-3x} - \frac{4}{5}\cdot\frac{-4}{3-4x}$$

$$= \frac{1}{2x} - \frac{4}{(3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}\right].$$

***

Example 17: If $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, show that f is derivable for every value of x but the derivative is not continuous for $x = 0$.

Solution: For $x \neq 0$, $f'(x) = 2x\sin\frac{1}{x} + x^2\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$

$$= 2x\sin\frac{1}{x} - \cos\frac{1}{x}.$$

For $x = 0$, we have

$$\frac{f(x)-f(0)}{x-0} = \frac{x^2\sin\frac{1}{x}}{x} = x\sin\frac{1}{x} \Rightarrow f'(0) = 0.$$

Thus, the function possesses a derivative for every value of x given by

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} \text{ when } x \neq 0,\ f'(0) = 0.$$

We now show that f' is not continuous for $x = 0$.

We write

$$\cos\frac{1}{x} = 2x\sin\frac{1}{x} - \left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$$

Here, $\lim\limits_{x\to 0}\left(2x\sin\frac{1}{x}\right)=0$.

In case $\lim\limits_{x\to 0} f'(x)$, i.e., $\lim\limits_{x\to 0}\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)$ had existed, it would follow that

$\lim\limits_{x\to 0}\left(\cos\frac{1}{x}\right)$ would also exist.

But, this is not the case. Hence $\lim\limits_{x\to 0} f'(x)$ does not exist so that f' is not continuous for $x=0$.

***

Example 18: Examine the continuity and derivability for the function defined as follows:

$$f(x)=\begin{cases} 1 & \text{, if } x\in\,]-\infty,0[, \\ 1+\sin x & \text{, if } x\in\left[0,\frac{1}{2}\pi\right[, \\ 2+\left(x-\frac{1}{2}\pi\right)^2 & \text{, if } x\in\left[\frac{1}{2}\pi,\infty\right[\end{cases}$$

Solution: The function f is derivable for every value of x except perhaps for $x=0$ and $x=\pi/2$.

i) Firstly, we check for the continuity of f at $x=0$.

Now, $f(0)=1+\sin 0=1$.

$\lim\limits_{x\to 0^-} f(x)=1$ and $\lim\limits_{x\to 0^+} f(x)=1$

Thus, $\lim\limits_{x\to 0} f(x)=1=f(0)$.

Hence, f is continuous for $x=0$.

Now, let us find left hand derivative and right hand derivative at $x=0$.

$$\text{LHD}=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{1-1}{x}=0$$

again $x>0 \Rightarrow \dfrac{f(x)-f(0)}{x-0}=\dfrac{1+\sin x-1}{x-0}=\dfrac{\sin x}{x}$

$$\text{RHD}=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{\sin x}{x}=1.$$

Thus, $\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}\neq\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}$.

Hence, the function is not derivable at $x=0$.

ii) Now, to check whether the function is differentiable at $\pi/2$ or not, letus first check the continuity at $x=\pi/2$. We have,

$$f(\pi/2)=2+\left(\frac{1}{2}\pi-\frac{1}{2}\pi\right)^2=2$$

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} f(x)=\lim_{x\to\left(\frac{\pi}{2}\right)^-}(1+\sin x)=1+1=2,$$

$$\lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x)=\lim_{x\to\left(\frac{\pi}{2}\right)^+}\left[2+\left(x-\frac{1}{2}\right)^2\right]=2.$$

$$\therefore\ \lim_{x\to\pi/2} f(x)=2=f(\pi/2).$$

Hence, f is continuous for $x = \pi/2$.

$$\text{LHD} = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{f(x) - f\left(\frac{1}{2}\pi\right)}{x - \frac{1}{2}\pi} = \frac{(1+\sin x) - 2}{x - \frac{1}{2}\pi} = \frac{1-\sin x}{\frac{1}{2}\pi - x}.$$

Putting $\frac{1}{2}\pi - x = t$, we see that

$$\frac{1-\sin x}{\frac{1}{2}\pi - x} = \frac{1 - \sin\left(\frac{1}{2}\pi - t\right)}{t}$$

$$= \frac{1-\cos t}{t} = \frac{2\sin^2 \frac{1}{2}t}{t} = \sin\frac{1}{2}t \frac{\sin\frac{1}{2}t}{\frac{1}{2}t}.$$

Thus, $\lim\limits_{x \to \left(\frac{\pi}{2}\right)^-} \dfrac{1-\sin x}{\frac{1}{2}\pi - x} = 0$.

$$\text{RHD} = \lim_{x \to \left(\frac{\pi}{2}\right)^+} \frac{f(x) - f\left(\frac{1}{2}\pi\right)}{x - \frac{1}{2}\pi} = \lim_{x \to \left(\frac{\pi}{2}\right)^+} \frac{2 + \left(x - \frac{1}{2}\pi\right)^2 - 2}{x - \frac{1}{2}\pi} = \lim_{x \to \left(\frac{\pi}{2}\right)^+} \left(x - \frac{1}{2}\pi\right)$$

Thus, $\lim\limits_{x \to \left(\frac{\pi}{2}\right)^+} \dfrac{f(x) - f\left(\frac{1}{2}\pi\right)}{x - \frac{1}{2}\pi} = 0$.

Hence, $f'\left(\frac{1}{2}\pi\right)$ exists and is equal to 0.

***

Now, you may try the following exercises.

E1) Find the derivatives of f at the given values of x using first principle for the following:

i) $f(x) = 2x^2 + 3x - 4$ at $x = 5/2$.

ii) $f(x) = 1/x$ at $x = 5$.

iii) $f(x) = \sqrt{x}$ at $x = 1$.

iv) $f(x) = 1/\sqrt{x}$ at $x = 1$.

E2) Find the derived function f' if $f(x)$ is given as follows:

i) $1/(x^2 + 3)$ ii) $1/\sqrt{x}$

iii) x^3 iv) $ax^2 + bx + c$

E3) Show that if the function f defined by $f(x) = |x| + |x-1|$ is continuous for every real value of x, but not derivable for $x = 0$ and $x = 1$.

E4) Construct a function which is continuous in the interval $[1, 5]$, but not derivable at the points $2, 3,$ and 4.

E5) For $f(x) = x^3$, find $f'(x)$. Then, find $f'(-1)$ and $f'(1.5)$ and interpret these results.

E6) The population of a city grows from an initial size of $100{,}000$ to a size P given by $P(t) = 100{,}000 + 2000t^2$, where t is in years.

i) Find the growth rate, dP/dt.

ii) Find the population after 10 years.

iii) Find the growth rate at $t = 10$.

iv) Explain the meaning of your answer to part (iii).

E7) The circular area A, in square centimetres, of a healing wound is approximated by $A(r) = 3.14r^2$, where r is the wound's radius, in centimetres.

i) Find the rate of change of the area with respect to the radius.

ii) Explain the meaning of your answer to part (i).

E8) A particle moves along a straight line such that the position at any time t is a quadratic function of t. Prove that its acceleration remains constant.

E9) If $s(t) = t^3 - 2t^2 + 3t - 4$, gives the position, find the velocity and acceleration of the particle at the end of $0, 1, 2$ seconds.

E10) Find $\frac{dy}{dx}$ in each of the following cases.

i) $y = \dfrac{x^2 - 1}{(x-1)^2}$

ii) $y = \dfrac{(x+4)^2}{(x-3)}$

iii) $y = \dfrac{x^2 + 1}{x^2 - 3x + 2}$

E11) Find the derivatives of the functions defined by the following expressions:

i) $\sqrt{(ax^2 + 2bx + c)}$

ii) $\dfrac{\sqrt{(x^2+1)} - \sqrt{(x^2-1)}}{\sqrt{(x^2+1)} + \sqrt{(x^2-1)}}$

iii) $\frac{2x^2-1}{x\sqrt{(1+x^2)}}$

E12) Suppose that the demand function for a product is given by $D(p)=\frac{80,000}{p}$, and that price p is a function of time given by $p=1.6t+9,$ where t is in days.

i) Find the demand as a function of time t.

ii) Find the rate of change of the quantity demanded when $t=100$ days.

E13) Differentiate $y=\sqrt{(1-3x)^{2/3}(1+3x)^{1/3}}$.

E14) Draw the graph of the function f given by $f(x)=5x^3-30x^2+45x+5\sqrt{x}$ and draw the graph of its derivative f' over the interval $[0,5]$. Then, estimate points at which the tangent line to f is horizontal.

E15) Find the equation of the tangent line to the graph of the function f defined by $f(x)=e^{-x}$ at the point $(0,1)$.

E16) Find the $\frac{dy}{dx}$, when $x=a\left(\frac{1-t^2}{1+t^2}\right)$, $y=b\left(\frac{2t}{1+t^2}\right)$.

E17) Find the n^{th} Taylor polynomial for the following at the specified x_0.

i) e^x at $x_0=1$

ii) $\frac{1}{x}$ at $x_0=-1$

iii) $\sin\pi x$ at $x_0=\frac{1}{2}$

iv) $\ln x$ at $x_0=1$

E18) Find the coefficient of x^9 in Maclaurin polynomial for the functions

i) $\cos 2x$

ii) $\sin\left(x+\frac{\pi}{4}\right)$.

E19) If Maclaurin polynomial for $\sin x$ is differentiated term by term, do you get Maclaurin polynomial for $\cos x$?

E20) If Maclaurin polynomial for e^x is differentiated term by term, we get the same polynomial again. Prove this.

SOLUTIONS/ANSWERS

E1) i) $f'(5/2)=\lim\limits_{h\to 0}\frac{f(5/2+h)-f(5/2)}{h}$

$$= \lim_{h\to 0} \frac{\left[2\left(\frac{5}{2}+h\right)^2 + 3\left(\frac{5}{2}+h\right) - 4\right] - \left[2\left(\frac{5}{2}\right)^2 + 3\left(\frac{5}{2}\right) - 4\right]}{h}$$

$$= \lim_{h\to 0} \frac{2\left(\frac{25}{4} + h^2 + 5h\right) + \frac{15}{2} + 3h - 4 - \frac{25}{2} - \frac{15}{2} + 4}{h}$$

$$= \lim_{h\to 0} \frac{2h^2 + 13h}{h} = \lim_{h\to 0}(2h + 13)$$

$$= 13$$

ii) $f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h}$

$$= \lim_{h\to 0} \frac{\left(\frac{1}{5+h}\right) - \left(\frac{1}{5}\right)}{h}$$

$$= \lim_{h\to 0} \frac{-h}{5(5+h)(h)}$$

$$= \lim_{h\to 0} \frac{-1}{5(5+h)} = -\frac{1}{25}$$

iii) $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h}$$

$$= \lim_{h\to 0} \frac{\left[1 + \frac{1}{2}h + \frac{(1/2)(-1/2)}{2}h^2 + \frac{(1/2)(-1/2)(-3/2)}{3\times 2}h^3 + \ldots\right] - 1}{h}$$

$$= \lim_{h\to 0} \left[\frac{1}{2} + \frac{(1/2)(-1/2)}{2}h + \ldots\right]$$

$$= \frac{1}{2}$$

iv) $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+h}} - \frac{1}{\sqrt{1}}}{h}$$

$$= \lim_{h\to 0} \frac{1 - \sqrt{1+h}}{h\sqrt{1-h}} \cdot \frac{1 + \sqrt{1+h}}{1 + \sqrt{1+h}}$$

$$= \lim_{h\to 0} \frac{1 - (1+h)}{h\sqrt{1-h}\,(1 + \sqrt{1+h})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{1-h}\,(1 + \sqrt{1+h})}$$

$$= -\frac{1}{\sqrt{2}}.$$

E2) i) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{(x+h)^2 + 3} - \frac{1}{x^2 + 3} \right)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 3 - x^2 - h^2 - 2xh - 3}{h(x^2+3)\{(x+h)^2 + 3\}}$$

$$= \lim_{h \to 0} \frac{-h - 2x}{(x^2+3)\{(x+h)^2+3\}} = \frac{-2x}{(x^2+3)^2}$$

ii) The domain of the function f defined by $f(x) = \frac{1}{\sqrt{x}}$ is the set of all positive real numbers that is $]0, \infty[$.

Let $x > 0$, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}$$

$= \frac{-1}{2x\sqrt{x}}$ when $x > 0$.

iii) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} [h^2 + 2x(x+h)]$$

$= 2x^2$, where $x \in \mathbb{R}$

iv) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{a(x+h)^2 + b(x+h) + c - ax^2 - bx - c}{h}$$

$= \lim_{h \to 0} [ah + 2ax + b] = 2ax + 3$, when $x \in \mathbb{R}$.

E3) $f(x) = |x| + |x-1|$ at $x = 0, f(0) = 1$. Also $\lim_{x \to 0^-} f(x) = 1$ and $\lim_{x \to 0^+} f(x) = 1$. Thus, f is continuous at $x = 0$. Similarly, $f(1) = 1$, $f(1) = 1$ and $\lim_{x \to 1^+} f(x) = 1$.

Thus, f is continuous at $x = 1$.

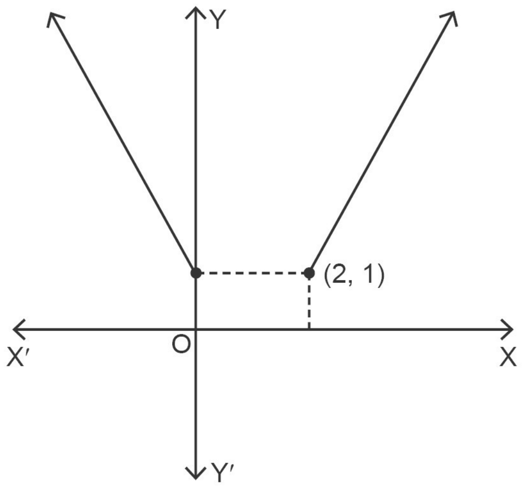

Fig. 2

Fig. 2 shows that at $x = 0$ and $x = 1$, the graph has corners. So, no tangent exists at these points. Also, we can rewrite

$$f(x) = \begin{cases} -2x+1, & \text{if } x < 0 \\ 1 & , \text{if } 0 \le x < 1 \\ 2x-1, & \text{if } x \ge 1 \end{cases}$$

We can find $f'(0^-) = -2$ and $f'(0^+) = 0$ and $f'(1^-) = 0$ and $f'(1^+) = 2$.
Thus, f is not differentiable at $x = 0$ and $x = 1$.

E4) One of such function is $f(x) = |x-2| + |x-3| + |x-4|$.

E5) We have $\dfrac{f(x+\delta x) - f(x)}{\delta x} = \dfrac{(x+\delta x)^3 - x^3}{\delta x} = 3x^2 + 3x\delta x + \delta x^2, \quad \delta x \neq 0$.

Then, $f'(x) = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x} = \lim\limits_{\delta x \to 0} (3x^2 + 3x\delta x + \delta x^2) = 3x^2$.

Thus, $f'(-1) = 3(-1)^2 = 3$ and $f'(1.5) = 3(1.5)^2 = 6.75$.

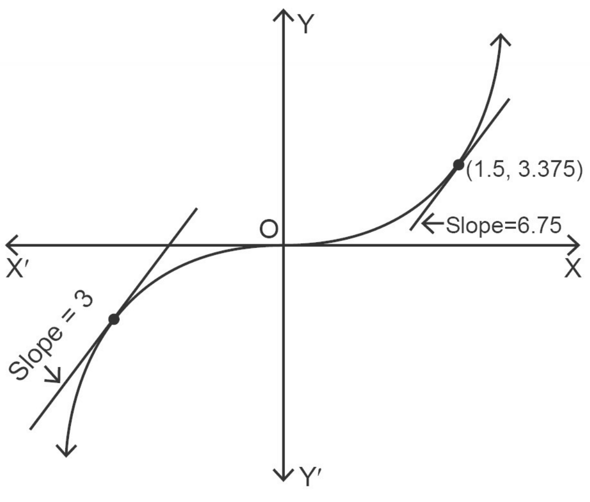

Fig. 3

E6) i) $\dfrac{dP}{dt} = 4000t$

ii) 300, 000 people

iii) 40,000 people/year

iv) The growth rate shows the change in population with respect to the time.

E7) i) $A'(r) = 6.28r$

ii) The rate of change of area with respect to the radius is 6.28 times the radius.

E8) Let $s(t) = at^2 + bt + c$

$$\frac{ds}{dt} = 2at + 3$$

$$\frac{d^2s}{dt^2} = 2a$$

We can see that the second derivative of the distance w.r.t. time which is acceleration is a constant function. Thus, the acceleration will always remain constant.

E9) The velocity $\frac{ds}{dt} = 3t^2 - 4t + 3$

$$\left(\frac{ds}{dt}\right)_{t=0} = 3\,m/s$$

$$\left(\frac{ds}{dt}\right)_{\text{at } t=1} = 2\,m/s$$

$$\left(\frac{ds}{dt}\right)_{\text{at } t=2} = 7\,m/s$$

The acceleration $\frac{d^2s}{dt^2} = 6t - 4$

$$\left(\frac{d^2s}{dt^2}\right)_{\text{at } t=0} = -4\,m/s^2$$

$$\left(\frac{d^2s}{dt^2}\right)_{\text{at } t=1} = 2\,m/s^2$$

$$\left(\frac{d^2s}{dt^2}\right)_{\text{at } t=2} = 8\,m/s^2$$

E10) i) $$\frac{dy}{dx} = \frac{2x(x-1)^2 - (x^2-1).2(x-1)}{(x-1)^4}$$

$$= \frac{2(x-1)^2[x - x - 1]}{(x-1)^4}$$

$$= \frac{-2}{(x-1)^2}$$

ii) $$\frac{dy}{dt} = \frac{2(x+4)(x-3)-(x+4)^2.(1)}{(x-3)^2}$$

$$= \frac{(x+4)(2x-6-x-4)}{(x-3)^2}$$

$$= \frac{(x+4)(x-10)}{(x-3)^2}$$

iii) $$\frac{dy}{dt} = \frac{2x(x^2-3x+2)-(x^2+1)(2x-3)}{(x^2-3x+2)^2}$$

E11) i) Let $y = \sqrt{(ax^2+2bx+c)}$

$$\frac{dy}{dx} = 2(ax^2+2bx+c)^{-1/2}.(2ax+2b)$$

$$= \frac{4(ax+b)}{\sqrt{ax^2+2bx+c}}, \text{ when } ax^2+2bx+c \neq 0.$$

ii) Let $y = \dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}} \cdot \dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$

$$= \frac{(x^2+1)+(x^2-1)-2\sqrt{x^4-1}}{x^2+1-x^2+1}$$

$$= x^2-\sqrt{x^4-1}$$

$$\frac{dy}{dx} = 2x - \frac{4x^3}{2\sqrt{x^4-1}}$$

$$= 2x - \frac{2x^3}{\sqrt{x^4-1}}, \text{where } x^4-1 \neq 0$$

iii) Let $y = \dfrac{2x^2-1}{x\sqrt{1+x^2}}$

$$\frac{dy}{dx} = \frac{4x.x\sqrt{1+x^2}-(2x^2-1)\left[\sqrt{1+x^2}+\dfrac{2x^2}{\sqrt{1+x^2}}\right]}{x^2(1+x^2)}$$

E12) i) $$D(t) = \frac{80000}{1.6t+9}$$

ii) $$\left(\frac{dD}{dt}\right)_{\text{at } t=100} = -4.482 \text{ units/day}$$

E13) $$\frac{dy}{dx} = \frac{-1-9x}{2(1-3x)^{2/3}(1+3x)^{5/6}}$$

E14)

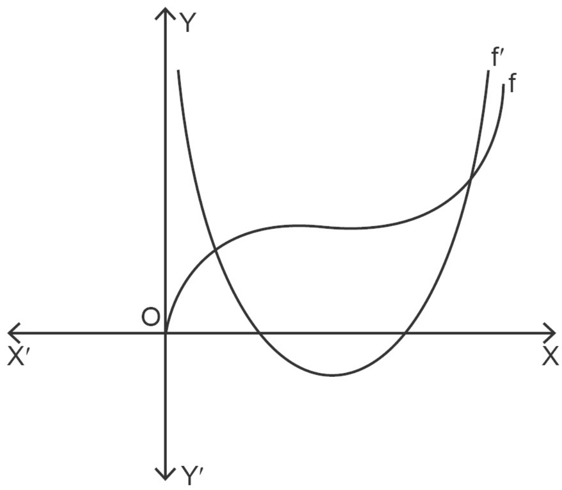

Fig. 4 shows the graph of f and f' on $[0,5]$.

E15) $y = -x + 1$

E16) $\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}$

$\frac{dy}{dt} = \frac{2a(1-t^2)}{(1+t^2)^2}$

$\frac{dy}{dx} = -\frac{1-t^2}{2t}$

E17) i) $f(x) = e^x$ and $f(1) = e$

$p_0(x) = e$

$p_1(x) = e + e(x-1)$

$p_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2$

$p_3(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{3!}(x-1)^3$

$\vdots$

$p_n(x) = \sum_{i=0}^{n} \frac{e}{i!}(x-1)^i$

ii) $p_n(x) = \sum_{i=0}^{n} (-1)(x+1)^i$

iii) $p_n(x) = \sum_{i=0}^{(n/2)} \frac{(-1)^i \pi^{2i}}{(2i)!}\left(x - \frac{1}{2}\right)^{2i}$

iv) $p_n(x) = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i}(x-1)^i$

E18) i) 0 ii) $\frac{1}{9!}\sqrt{2}$

E19) Yes

E20) Yes, you may try to prove it yourself.

APPENDIX 1: PARAMETRIC REPRESENTATION OF CURVES

So far, most of the curves we have considered have been graphs of functions of the form $y = f(x)$. However, there are curves where the coordinates x and y of each point $P(x, y)$ on the curve are themselves functions of a third variable, called a **parameter**. We will begin by examining parametric representations in $\mathbb{R}^2$.

Let f and g be continuous functions of t on an interval I, then the equations $x = f(t)$ and $y = g(t)$ are called parametric equations with parameter t. As t varies over the parametric set I, the points $(x, y) = (f(t), g(t))$ trace out a parametric curve.

You may note that the letter "t" used for the parameter does not necessarily denote time, although in many applications, time is a suitable parameter. Indeed, any letter or symbol may be used to denote a parameter. Let us go through the following examples to understand more.

Example 1: Sketch the curve $x = t^2 - 9, y = \frac{1}{3}t$ for $-3 \le t \le 2$.

Solution: Values of x and y corresponding to various choices of the parameter t are shown in the Table 1.

Table 1

t	x	y	
-3	0	-1	(Starting or initial point)
-2	-5	$-\frac{2}{3}$	
-1	-8	$-\frac{1}{3}$	
0	-9	0	
1	-8	$\frac{1}{3}$	
2	-5	$\frac{2}{3}$	(Ending or terminal point)

The graph is shown in Fig. 1. You may observe how the arrows show the orientation as t increases from -3 to 2.

Sometimes, however, we wish to eliminate the parameter to obtain a cartesian equation. For instance, here, we have $y = \frac{1}{3}t$, so $t = 3y$, and by substituting into the equation $x = t^2 - 9$, we obtain

$$x = (3y)^2 - 9 = 9y^2 - 9$$

which is the cartesian equation for a parabola that opens to the right. Because of the domain of the parameter t, we see that the parametric curve in Fig. 1 is a subset of the set of points that satisfy the equation $x = 9y^2 - 9$.

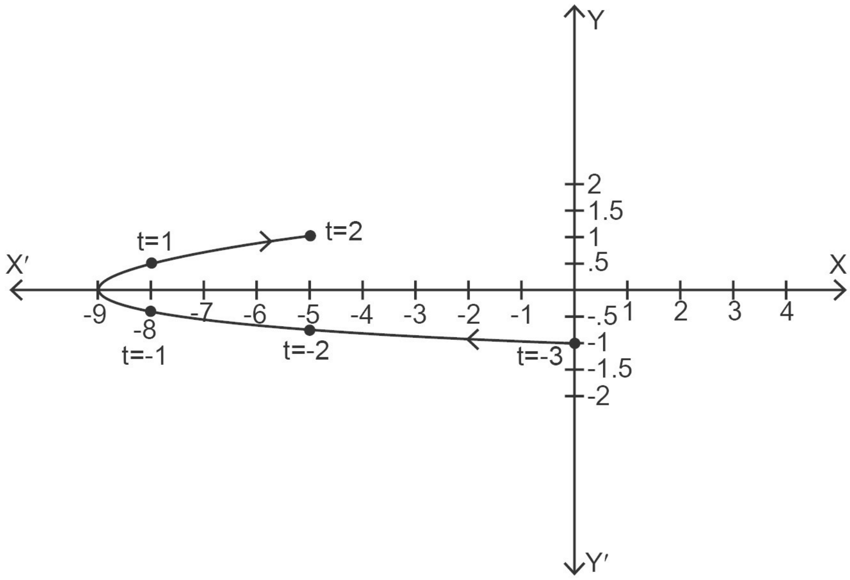

Fig. 1: Graph of $x = t^2 - 9, y = \frac{1}{3}t,$ **for** $-3 \le t \le 2$

Remark: Parameterizations are not unique. For example, the curve with parametric equations given in example 1 can also be represented by $x = 9(9t^2 - 1),\ y = 3t$ for $-\frac{1}{3} \le t \le \frac{2}{9}$. It is the same as the curve in Fig. 1.

***

Example 2: Describe the path $x = \sin \pi t, y = \cos 2\pi t$ for $0 \le t \le 0.5$.

Solution: We know that $\cos 2\pi t = 1 - 2\sin^2 \pi t$.

So that $y = 1 - 2x^2$

We recognize this as a cartesian equation for a parabola.

Since t is restricted to the interval $0 \le t \le 0.5$, therefore, the parametric representation involves only part of the right side of parabola $y = 1 - 2x^2$. The curve is oriented from the point $(0,1)$, where $t = 0$, to the point $(1,-1)$, where $t = 0.5$, and is the portion of the parabola shown in Fig. 2.

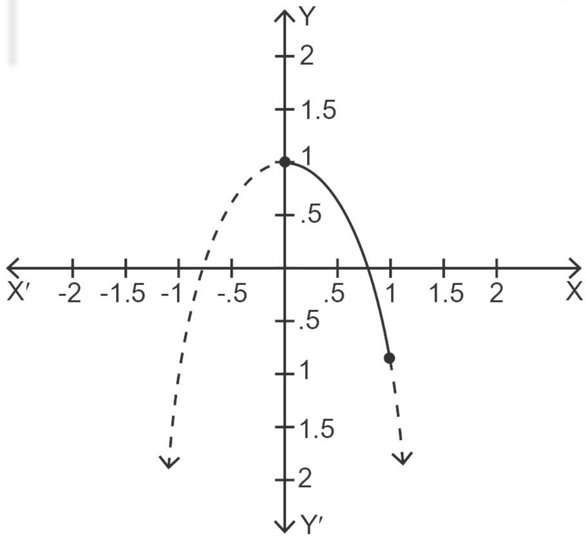

Fig. 2: The parabolaic arc $x = \sin \pi t, y = \cos 2\pi t$ **for** $0 \le t \le 0.5$

When it is difficult to eliminate the parameter from a given parametric representation, we can sometimes get a good picture of the parametric curve by plotting points.

***

Example 3: Discuss the path of the curve described by the parametric equations $x = e^{-t}\cos t, \quad y = e^{-t}\sin t$ for $t \geq 0$

Solution: We have no convenient way of eliminating the parameter so we write the values of (x, y) corresponding to various values of t in Table 2. The curve is obtained by plotting these points in a cartesian plane and passing a smooth curve through the plotted points, as shown in Fig. 3.

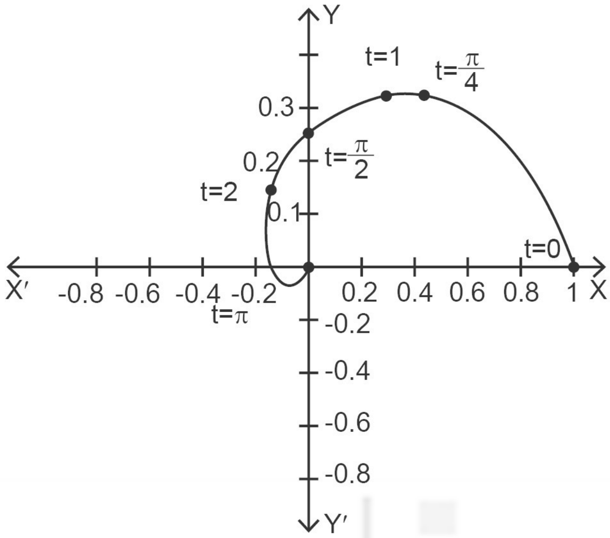

Fig. 3: Graph of $x = e^{-t}\cos t, y = e^{-t}\sin t$ **for** $t \geq 0$

Table 2

t	x	y
0	1	0
$\frac{\pi}{4}$	0.32	0.32
1	0.20	0.31
$\frac{\pi}{2}$	0	0.21
2	-0.06	0.12
π	-0.04	0
$\frac{3\pi}{2}$	0	-0.01
2π	0.00	0

Note that for each value of t, the distance from $P(x, y)$ on the curve to the origin is $\sqrt{x^2 + y^2} = \sqrt{(e^{-t}\cos t)^2 + (e^{-t}\sin t)^2} = \sqrt{e^{-2t}(1)} = e^{-t}$.

Because e^{-t} decreases as t increases, it follows that P gets closer and closer to the origin as t increases. However, because $\cos t$ and $\sin t$ vary between -1 and $+1$, the approach is not direct but takes place along a spiral.

***

So far, our examples have dealt with sketching a parametric curve given the parametric equations. In general, this process may be tedious. However, the reverse process, finding a suitable set of parametric equations for a given curve, is an art for which there is no simple procedure. Indeed, a given curve can have many different parameterizations and there are curves for which no simple parameterization can be given. The following example illustrates various methods for paramererizing a given curve.

Example 4: In each of the following cases, write the parameteric form of the curve:

i) $y = 9x^2$ ii) $r = 5\cos^3\theta$ in polar coordinates.

Solution: i) The usual parameterization for a parabola is to let the parameter t be the variable that is squared: $x = t, y = 9t^2$. However, another parameterization is to let $t = 3x$ so that $x = \frac{1}{3}t$ and $y = t^2$.

ii) In polar coordinates we have $x = r\cos\theta, y = r\sin\theta$, so we can parameterize x and y in terms of the parameter θ:

$$x = r\cos\theta = (5\cos^3\theta)\cos\theta = 5\cos^4\theta$$

$$y = r\sin\theta = (5\cos^3\theta)\sin\theta$$

Therefore the parametric form of the curve $r = 5\cos^2\theta$ is $x = 5\cos^4\theta$ and $y = 5\cos^3\theta\sin\theta$, where θ is the parameter.

***

APPENDIX 2: PARTIAL FRACTIONS

Recall the rational functions you studied in Unit 6. Rational function is the ratio of the polynomials provided the polynomial in denominator is non-zero. Here, we will show how a rational function is expressed as a sum of simpler fractions, called **partial fractions**.

Let us consider a rational function f defined by $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. If the degree of $p(x)$ is less than the degree of $q(x)$, then the rational function is said to be **proper rational function.** If the degree of $p(x)$ is greater than or equal to the degree of $q(x)$, then the rational function is called **improper rational function.**

Let us first consider that f is improper rational function that is degree of $p(x) \geq$ degree of $q(x)$. Then we divide $p(x)$ by $q(x)$ using long division method until the degree of $r(x)$ is less than the degree of $q(x)$, where $r(x)$ is the remainder polynomial.

That is $f(x) = \dfrac{p(x)}{q(x)} = s(x) + \dfrac{r(x)}{q(x)}$, where $s(x)$ and $r(x)$ are the polynomials and are quotient and remainder respectively.

After this form $f(x)$ is the sum of a polynomial and a proper rational function. Following examples illustrate this.

Example 1: Express $\dfrac{x^4+1}{x-5}$ as a sum of polynomial and proper rational function.

Solution: Here $p(x) = x^4+1, q(x) = x-5$. Clearly degree of $p(x) >$ degree of $q(x)$. Therefore, we divide $p(x)$ by $q(x)$.

$$
\begin{array}{r|l}
 & x^3 + 5x^2 + 24x + 120 \\
\hline
x-5 & x^4 + 1 \\
 & \underline{x^4 - 5x^3} \\
 & \quad 5x^3 - 1 \\
 & \quad \underline{5x^3 - 25x^2} \\
 & \qquad 24x^2 - 1 \\
 & \qquad \underline{24x^2 - 120x} \\
 & \qquad\quad 120x - 1 \\
 & \qquad\quad \underline{120x - 600} \\
 & \qquad\qquad 599
\end{array}
$$

Therfore, we can write $\dfrac{x^{4+1}}{x-5} = x^3 + 5x^2 + 24x + 120 + \dfrac{599}{x-5}$.

***

In school, you must have studied the factorisation of polynomials. For example, we know that $x^2 - 4x + 3 = (x-1)(x-3)$. Here $(x-1)$ and $(x-3)$ are two linear factors of $x^2 - 4x + 3$. You must have also come across polynomials like x^2+1, which cannot be factorised into real linear factors. Thus, it is not always possible to factorise a given polynomial into linear factors. But any polynomial can, in principle, be factored into linear and quadratic factors. We shall not prove this statement here. But we shall consider both the cases that is when the denominator can be factorised into linear factors and when the denominator can be factorised in linear and irreducible real quadratic factors. To find partial fractions, we shal consider the following examples.

Example 2: Write the rational function f defined by $f(x)=\dfrac{6x-8}{x^2-x-6}$ into partial fractions.

Solution: Here $x^2-x-6=(x-3)(x+2)$. Now

$$\frac{6x-8}{x^2-x-6}=\frac{6x-8}{(x-3)(x+2)}=\frac{A}{(x-3)}+\frac{B}{(x+2)} \quad (1)$$

One way to find the constants A and B is to multiply Eqn. (1) by $(x-3)(x+2)$ to clear fractions. This gives

$$6x-8=A(x+2)+B(x-3) \quad (2)$$

This relationship holds for all x, so it holds in particular for $x=-2$ and $x=3$. Putting $x=-2$ in Eqn. (2), we get $-12-8=B(-2-3)$ or $B=4$. Similarly putting $x=3$ in Eqn. (2), we get $18-8=A(5)$ or $A=2$.

Now substituting these values in Eqn. (1), we get

$$\frac{6x-8}{x^2-x-6}=\frac{2}{(x-3)}+\frac{4}{(x+2)} \quad (3)$$

The other way to find the constants A and B from Eqn. (1) is that we collect and equate coefficients of like powers of x from left hand side as well as right hand side. Equating the coefficients of x and constant terms both the sides of Eqn. (2), we obtain

$$6=A+B \text{ and } -8=2A-3B$$

Solving the system of equations for A and B, we get $A=2$ and $B=4$ as before. Therefore, the partial fractions decomposition is

$$\frac{6x-8}{x^2-x-6}=\frac{2}{(x-3)}+\frac{4}{(x-2)}.$$

***

Let us consider another example where the denominator has repeated linear factors.

Example 3: Write $\dfrac{2x+4}{x^3-2x^2}$ as a sum of partial fractions.

Solution: $\dfrac{2x+4}{x^3-2x^2}=\dfrac{2x+4}{x^2(x-2)}$. Here x^2 is a quadratic factor, but it is not irreducible, since $x^2=x.x$. Thus the factor x^2 introduces two partial fractions of the form $\dfrac{A}{x}+\dfrac{B}{x^2}$ and the factor $(x-2)$ introduces one term that is $\dfrac{C}{(x-2)}$.

So, the partial fractions are

$$\frac{2x+4}{x^3-2x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x-2)} \quad (4)$$

Multiplying Eqn (4) by $x^2(x-2)$, we get

$$2x+4=Ax(x-2)+B(x-2)+Cx^2$$

Equating the coefficients of x^2, x and constant both the sides, we get

Coefficient of $x^2 \Rightarrow 0 = A + C$

Coefficient of $x \Rightarrow 2 = -2A + B$

Constant term $\Rightarrow 4 = -2B$

Solving the system of equations for A, B and C, we obtain $A = -2, B = -2$ and $C = 2$.

Now, the partial fractions are $\dfrac{2x+4}{x^3 - 2x^2} = \dfrac{-2}{x} + \dfrac{-2}{x^2} + \dfrac{2}{(x-2)}$

***

Example 4: Reduce $\dfrac{x}{x^3 - 3x + 2}$ into partial fractions.

Solution: The denominator $x^3 - 3x + 2$ factors into $(x-1)^2(x+2)$. The linear factor $(x-1)$ is repeated twice in $x^3 - 3x + 2$. In this case, we write

$$\frac{x}{x^3 - 3x + 2} = \frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

From this point we proceed as before to find A, B and C. We get

$x = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$

We put $x = 1, x = -2$ and get $C = \dfrac{1}{3}$ and $A = -\dfrac{2}{9}$. Then to find B, let us put any other convenient value, say $x = 0$. This gives us $0 = A - 2B + 2C$ or

$0 = \dfrac{-2}{9} - 2B + \dfrac{2}{3}$

This implies $B = \dfrac{2}{9}$. Thus

$$\frac{x}{x^3 - 3x + 2} = -\frac{2}{9(x+2)} + \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2}$$

***

In our next example, we shall consider the case when the denominator contains an irreducible quadratic factor that is a quadratic factor which cannot be further factored into linear factors.

Example 5: Write $\dfrac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x}$ as a sum of partial fractions.

Solution: We first factorise $x^4 - 2x^3 + x^2 - 2x$ as $x(x-2)(x^2+1)$.

Then we write

$$\frac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x} = \frac{A}{x} + \frac{B}{x-2} + \frac{Cx + D}{x^2 + 1} \qquad (5)$$

You may note that the irreducible quadratic factor $(x^2 + 1)$ introduces the terms $\dfrac{Cx + D}{x^2 + 1}$.

We multiply Eqn. (5) by $x(x-2)(x^2+1)$, and get

$6x^3 - 11x^2 + 5x - 4 = A(x-2)(x^2+1) + Bx(x^2+1) + (Cx + D).x.(x-2)$.

Next, we substitute $x = 0$ and $x = 2$ to get $A = 2$ and $B = 1$.

Then we put $x = 1$ and $x = -1$ (some convenient values) to get $C = 3$ and $D = -1$.

Thus $\dfrac{6x^3-11x+5x-4}{x^4-2x^3+x^2-2x}=\dfrac{2}{x}+\dfrac{1}{x-2}+\dfrac{3x-1}{x^2+1}$

***

In the following example, we will consider the repeated irreducible quadric factors.

Example 6: Reduce $\dfrac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}$ into partial fractions.

Solution: Observe that since the numerator has degree 4 and the denominator has degree 5, therefore the given rational function is a proper rational function. Thus

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}=\frac{A}{(x+2)}+\frac{Bx+C}{x^2+3}+\frac{Dx+E}{(x^2+3)^2} \qquad (6)$$

We multiply by $(x+2)(x^2+3)^2$ both the sides and get

$$3x^4+4x^3+16x^2+20x+9=A(x^2+3)^2+(Bx+c)(x^2+3)+(Dx+E)(x+2)$$

We now equate the corresponding coefficients and get the following system of equations,

$A+B=3$

$2B+C=4$

$6A+3B+2C+D=16$

$6B+3C+2D+E=20$

$9A+6C+2E=9$.

The solution of this system of equations gives $A=1,\ B=2,\ C=0,\ D=4,$ $E=0$.

Thus, the required partial fractions are

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}=\frac{1}{(x+2)}+\frac{2x}{(x^2+3)}+\frac{4x}{(x^2+3)^2}.$$

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