Indira Gandhi National Open University
School of Sciences

BMTC - 131
CALCULUS

Block

4

APPLICATIONS OF DIFFERENTIAL CALCULUS

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BLOCK 4 APPLICATIONS OF DIFFERENTIAL CALCULUS

In Block3, you have learnt some techniques of differentiation, and have differentiated a wide variety of functions. In this block, we shall use the derivative to explore various geometrical features of a curve, like maxima/minima, concavity/convexity, tangents, normals, asymptotes and so on. For this we have to make use of not only the first derivative, but also some higher order derivatives.

In Unit 12, we shall use the first derivative to find the limits involving an algebraic combination of functions in an independent variable, in which evaluation of limit gives form like $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, 0^{\infty}$, etc. Such a form is called indeterminate form.

In the next three units, Unit 13, Unit 14 and Unit 15, we shall illustrate how we can find the exact shape of a curve, when its equation is given to us. You will be surprised at the amount of information which is revealed by the first and second derivatives. We shall use this information to trace various standard curves in Unit 16. In Unit 16, we shall also tell you how the properties of some remarkable curves are put to use. We shall also ask you to trace some curves yourself. Do try and trace them by systematically following the procedure which we have outlined in Unit 16. We are sure, that after reading this block you will be aware of the presence of many of these curves in the objects around you, as also in nature.

We have also made a video programme, “Curves”, which you can watch after going through this block. This programme is available at your study center.

A word about some signs used in the unit! Throughout each unit, you will find theorems, examples and exercises. To signify the end of the proof of a theorem, we use the sign ■. To show the end of an example, we use ***. Further, equations that need to be referred to are numbered sequentially within a unit, as are exercises and figures. E1, E2 etc. denote the exercises and Fig. 1, Fig. 2, etc. denote the figures.

NOTATIONS AND SYMBOLS (used in Block 4)

See the list of notations and symbols in Block 1, Block 2 and Block 3.

UNIT 12

INDETERMINATE FORMS

Structure **Page No.**

12.1 INTRODUCTION

In Unit 7, we have discussed limit of a function. In this unit, we shall discuss a general method for using derivatives to find limits. We shall begin with the limits in which numerator and denominator both approach 0 as $x \to 0$. Limits which come in such forms are called **indeterminate forms**. We shall discuss such forms in Sec.12.2. The arguments we used in Unit 7, that is cancelling the common factor in numerator and denominator or sometimes using geometrical approach do not work for such limits. So in Sec.12.3 and Sec.12.4, we introduce a systematic method, known as L'Hôpital's rule, for evaluating indeterminate forms.

Fig. 1: L'Hôpital

L'Hôpital's rule, is named after the French mathematician L'Hôpital. In this rule, we use the derivative for evaluating limit. This is in contrast with what we have been doing so far, i.e., evaluating derivatives of functions by calculating certain limits. We shall discuss other indeterminate forms in Sec. 12.5.

In this unit, we will see that we are now able to find the limit of a wide variety of indeterminate forms that we were unable to deal with earlier.

And now we shall list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

Objectives

After reading this unit, you should be able to:

- identify the types of indeterminate forms;
- evaluate $\lim_{x\to a}\frac{f(x)}{g(x)}$ when $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$
- find $\lim_{x\to a}[f(x) - g(x)]$ when $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$
- evaluate $\lim_{x\to a}[f(x)]^{g(x)}$ when $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, or

  $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$, or

  $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, where $a \in \mathbb{R}$
- compute $\lim_{x\to a}[f(x)\ g(x)]$ when $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$
- obtain all the above limits when a is ∞ or $-\infty$.

12.2 INDETERMINATE FORMS

In Unit 7, we considered many limit problems, but deliberately avoided the forms $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty$, and a few others. In this section, we shall discuss these forms.

Consider, $\lim_{x\to a}\frac{f(x)}{g(x)}$, where $\lim_{x\to a} f(x) \to 0$ and $\lim_{x\to a} g(x) \to 0$. This is unlike the problems, say of the form $\frac{0}{5}$, all of which have the answer 0. The form $\frac{0}{0}$ can produce a variety of answers. Because of this unpredictability, the limit form $\frac{0}{0}$ is called **indeterminate**. In general, a limit form is indeterminate when different problems with the same form can have different answers. You may recall two special exceptions in the limits in Example 22 in Unit 7, where $\lim_{x\to 0}\frac{\sin x}{x} = 1$ was shown using the squeezing theorem and some careful manipulation of inequalities, and $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$ then followed using the identity $\sin^2 x + \cos^2 x = 1$. These limits are actually special cases of these derivatives, as can be seen by

$$\frac{d}{dx}(\sin x)\Big|_{x=0} = \lim_{x\to 0}\frac{\sin x - \sin 0}{x - 0} = \lim_{x\to 0}\frac{\sin x}{x} = 1$$

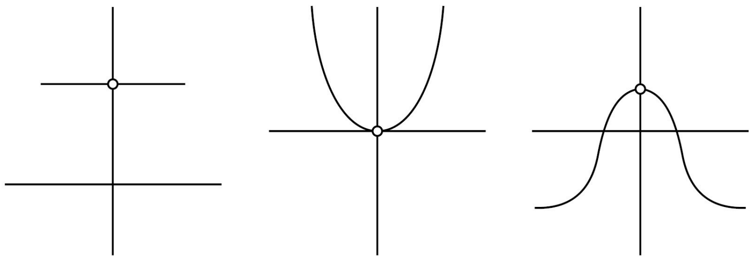
and

$$\frac{d}{dx}(\cos x)\Big|_{x=0} = \lim_{x\to 0}\frac{\cos x - \cos 0}{x - 0} = \lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

What makes these limits bothersome is the fact that the numerator and denominator both approach 0 as $x \to 0$. This means that the limit of the numerator is making the quotient very small, whereas the limit of the denominator is making the quotient very large. Such limits are called indeterminate forms of type $0/0$.

If $\lim_{x\to a} g(x) = 0 = \lim_{x\to a} h(x)$, then $\dfrac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)}$ is an expression of the form $\dfrac{0}{0}$. In this case, we say that $\dfrac{g(x)}{h(x)}$ is an indeterminate form of the type $\dfrac{0}{0}$ at $x = a$ or as $x \to a$.

Some other examples of $\dfrac{0}{0}$ form are $\lim_{x\to 0} \dfrac{x}{x} = 1$ (see Fig. 2 (a)), $\lim_{x\to 0} \dfrac{x^3}{x} = 0$ (see Fig. 2(b)), $\lim_{x\to 1} \dfrac{x^2-1}{x-1} = 2$ (see Fig. 2 (c)).



(a) **(b)** **(c)**

Fig. 2

There are other types of indeterminant forms also; for example, if the limit of the numerator is ∞ and is making the quotient very large and at the same time the limit of the denominator is also ∞, which makes the quotient very small. Such forms are indeterminate forms of type $\dfrac{\infty}{\infty}$.

If $\lim_{x\to a} g(x) = \pm\infty = \lim_{x\to a} h(x)$, then we say that $\lim_{x\to a} \dfrac{g(x)}{h(x)}$ is an indeterminate form of the type $\dfrac{\infty}{\infty}$ at $x = a$.

For example, $\lim_{x\to\infty} \dfrac{e^x}{\ln x} = \dfrac{\lim_{x\to\infty} e^x}{\lim_{x\to\infty} \ln x} = \dfrac{\infty}{\infty}$, and $\lim_{x\to-\infty} \left(\dfrac{x^2}{e^{-x}}\right) = \dfrac{\lim_{x\to-\infty} x^2}{\lim_{x\to-\infty} e^{-x}} = \dfrac{\infty}{\infty}$

Other examples of indeterminate form of ∞/∞ are $\lim_{x\to\infty} \dfrac{\ln\sqrt{x}}{x^2}$, $\lim_{x\to\infty} \dfrac{e^x}{x^3}$, $\lim_{x\to\infty} \dfrac{x^3}{e^{x^2}}$, etc.

You may think of many other such forms, in which the final answer of limit does not tend to a single value rather pulls the answer to two different and far away values. Due to which the limit is said to be in indeterminate form.
Other types such as, $0^o, 0\times\infty, \infty^o, 1^\infty$, and $\infty - \infty$ are also indeterminate forms.

Here we are giving various types of indeterminate forms in the following table:

Type of other Indeterminate forms	Example
$\infty - \infty$ **form:** Large value is subtracted from another large value	$\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x)$
1^∞ **form:** The behaviour of a large power of a number depends on whether the number is less than 1 or more than 1.	$\lim\limits_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$, $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$
0^0 **form:** The limit is pulled towards 0 by the base and towards 1 by the exponent.	$\lim\limits_{x \to 0} (e^x - 1)^{1 - \cos x}$
∞^0 **form:** The limit is pulled towards by the base and towards 1 by the exponent 0.	$\lim\limits_{x \to \infty} (e^x)^{1/x^2}$

We have said it before, and we repeat it once again that the methods developed by us so far do not enable us to calculate the limits in many situations mentioned above. In what follows we describe methods which would enable us to deal with almost all these situations. But first, see if you can do this exercise.

E1) Identify the types of indeterminate forms in the following cases:

i) $\lim\limits_{x \to \infty} \frac{e^{2x}}{x^n}, n \in \mathbb{N}$

ii) $\lim\limits_{x \to 0} \frac{\sin 3x}{x \cos x^2}$

iii) $\lim\limits_{x \to 0} \left(\operatorname{cosec} x - \frac{1}{x}\right)$

iv) $\lim\limits_{x \to 0} \left(\frac{\sin x}{1 - \cos x}\right)$

v) $\lim\limits_{x \to 0} x \ln|x|$.

In the following section, we will give a simple method for calculating the limits of functions which are in the $\frac{0}{0}$ form.

12.3 L'HÔPITAL'S RULE FOR $\frac{0}{0}$ FORM

Marquis de L'Hôpital, a French mathematician, was a student of Johann Bernoulli. He published the first textbook on calculus in 1696. This book was based on Bernoulli's lectures, and contained a method for evaluating $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$

when the limit is an indeterminate form of the type $\frac{0}{0}$ at $x = a$. This result is now universally known as L'Hôpital's rule, even though it was proved by Bernoulli. Before we state the rule, let us consider an example.

Consider the limit $\lim_{x\to 0} \frac{e^x - 1}{\sin x}$, this can be expressed as the ratio of two derivatives.

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x} = \lim_{x\to 0} \frac{(e^x - e^0)/(x-0)}{(\sin x - \sin 0)/(x-0)} = \frac{\left.\frac{d}{dx}(e^x)\right|_{x=0}}{\left.\frac{d}{dx}(\sin x)\right|_{x=0}} = \frac{e^0}{\cos 0} = 1$$

This method can be stated more generally. Suppose that f and g are differentiable functions at $x = a$ and that $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $0/0$, that is, $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.

In particular, the differentiability of f and g at $x = a$ implies that f and g are continuous at $x = a$, and hence $f(a) = \lim_{x\to a} f(x) = 0$ and $g(a) = \lim_{x\to a} g(x) = 0$.
Furthermore, since f and g are differentiable at $x = a$,
$\lim_{x\to a} \frac{f(x)}{x-a} = \lim_{x\to a} \frac{f(x) - f(a)}{x-a} = f'(a)$ and $\lim_{x\to a} \frac{g(x)}{x-a} = \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = g'(a)$
If $g'(a) \neq 0$, then, the indeterminate form can be evaluated as the ratio of derivative values, as given below

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)/(x-a)}{g(x)/(x-a)} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)} \qquad \ldots(1)$$

If $f'(x)$ and $g'(x)$ are continuous at $x = a$, the result in Eqn. (1) is a special case of L'Hôpital's rule, which converts an indeterminate form of type $0/0$ into a new limit involving derivatives. Moreover, L'Hôpital's rule is also true for limits at $-\infty$ and at $+\infty$. We state the result in the following theorem without proof.

Theorem 1 (L'Hôpital's rule for form $0/0$): Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.
If $\lim_{x\to a} [f'(x)/g'(x)]$ has a finite limit, or if this limit is $+\infty$ or $-\infty$, then
$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.
Moreover, this statement is also true in the case of a limit as $x \to a^-, x \to a^+, x \to -\infty,$ or as $x \to +\infty$.

Caution: You may note that when applying L'Hôpital's rule, we differentiate the numerator and denominator separately, which is not the same as differentiating $\frac{f(x)}{g(x)}$.

In the following examples we apply L'Hôpital's rule using the following three-step process:

Step 1: Check that the limit of $f(x)/g(x)$ is an indeterminate form of type $\frac{0}{0}$. If it is not, then L'Hôpital's rule cannot be used.

Step 2: Differentiate f and g separately.

Step 3: Find the limit of $f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $f(x)/g(x)$.

Here are examples which illustrate the utility of Theorem 1.

Example 1: Evaluate $\lim_{x\to 2}\frac{x^7-128}{x^3-8}$.

Solution: Note that this is an indeterminate form because $f(x)=x^7-128$ and $g(x)=x^3-8,$ both are differentiable around 2 and approach 0 as $x\to 2$. Therefore, we can apply L'Hôpital's rule. Applying L'Hôpital's rule, we get

$$\lim_{x\to 2}\frac{x^7-128}{x^3-8}=\lim_{x\to 2}\frac{\frac{d}{dx}(x^7-128)}{\frac{d}{dx}(x^3-8)}=\lim_{x\to 2}\frac{7x^6}{3x^2}$$

$$=\lim_{x\to 2}\frac{7x^4}{3}$$

$$=\frac{7(2)^4}{3}=\frac{112}{3}$$

***

Example 2: Find the following limits:

i) $\lim_{x\to 0}\frac{1-\cos x}{\sin x}$

ii) $\lim_{x\to \pi/2}\frac{(x-\pi/2)^2}{\cos x}$

Solution: i) Let $f(x)=1-\cos x,\ g(x)=\sin x$. Since, $\lim_{x\to 0}1-\cos x=1-\cos 0=0$ and $\lim_{x\to 0}\sin x=0$, therefore, they both are 0 and forms $\frac{0}{0}$ form. Hence, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to 0}\frac{1-\cos x}{\sin x}=\lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(\sin x)}=\lim_{x\to 0}\frac{0+\sin x}{\cos x}=\frac{\sin 0}{\cos 0}=\frac{0}{1}=0.$$

ii) Since, $\lim_{x\to \pi/2}(x-\pi/2)^2=0$ and $\lim_{x\to \pi/2}\cos x=0,$ therefore, the limit is in $\frac{0}{0}$ form. We can apply L'Hôpital's rule. We get

$$\lim_{x\to\frac{\pi}{2}}\frac{(x-\pi/2)^2}{\cos x}=\lim_{x\to\frac{\pi}{2}}\frac{\frac{d}{dx}\left(x-\frac{\pi}{2}\right)^2}{\frac{d}{dx}(\cos x)}=\lim_{x\to\frac{\pi}{2}}\frac{2(x-\pi/2)}{-\sin x}=\frac{2(0)}{-1}=0.$$

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Example 3: Evaluate $\lim\limits_{x\to0}\dfrac{1-\cos x}{\sec x}$.

Solution: You must always remember to check that you have an indeterminate form before applying L'Hôpital's rule. The limit is

$$\lim_{x\to0}\frac{1-\cos x}{\sec x}=\frac{\lim\limits_{x\to0}(1-\cos x)}{\lim\limits_{x\to0}\sec x}=\frac{0}{1}=0$$

Warning: If you blindly apply L'Hôpital's rule in Example 3, you obtain the WRONG answer:

$$\lim_{x\to0}\frac{1-\cos x}{\sec x}=\lim_{x\to0}\frac{\sin x}{\sec x\tan x}$$

$$=\lim_{x\to0}\frac{\cos x}{\sec x}=\frac{1}{1}=1$$

This answer is WRONG. Why?

If we find $\lim\limits_{x\to0}\dfrac{1-\cos x}{\sec x}$, we do not get indeterminate form. Therefore, L'Hôpital's rule cannot be applied.

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Example 4: Find $\lim\limits_{x\to0}\dfrac{x^2\sin(1/x)}{\sin x}$.

Solution: To find $\lim\limits_{x\to0}\dfrac{x^2\sin\frac{1}{x}}{\sin x}$, we first note that $\dfrac{x^2\sin\frac{1}{x}}{\sin x}$ is in the $\dfrac{0}{0}$ form as $x\to0$. But L'Hôpital's rule is not applicable, because,

$$\lim_{x\to0}\frac{\frac{d}{dx}\left(x^2\sin\frac{1}{x}\right)}{\frac{d}{dx}(\sin x)}=\lim_{x\to0}\frac{-\cos\frac{1}{x}+2x\sin\frac{1}{x}}{\cos x}\text{ does not exist.}$$

How can we be sure that this limit does not exist?

Note that if $\lim\limits_{x\to0}\dfrac{2x\sin\frac{1}{x}-\cos\frac{1}{x}}{\cos x}$ exists, then $\lim\limits_{x\to0}\left[2x\sin\frac{1}{x}-\cos\frac{1}{x}\right]$ would exist and consequently $\lim\limits_{x\to0}\cos\frac{1}{x}$ would exist, which is not true. As $x\to0,\cos\frac{1}{x}$ oscillates between -1 and 1, and does not tend to any limit.

$$f'(0)=\lim_{h\to0}\frac{f(h)-f(0)}{h}$$
$$=\lim_{h\to0}h\sin\frac{1}{h}$$
$$=0$$

However, we can still evaluate the limit of $\dfrac{x^2\sin\frac{1}{x}}{\sin x}$ as $x\to0$.

We have $\left|\dfrac{x^2\sin\frac{1}{x}}{\sin x}\right|<\left|\dfrac{x^2}{\sin x}\right|=\left|\dfrac{x}{\sin x}\right||x|$. [Since, $\left|h\sin\frac{1}{h}\right|\le|h|\to0$]

Since $\dfrac{x}{\sin x}\to1$ as $x\to0$, it follows that $\lim\limits_{x\to0}\dfrac{x^2}{\sin x}=\lim\limits_{x\to0}x\,.\lim\limits_{x\to0}\dfrac{x}{\sin x}=0$ and therefore, $\lim\limits_{x\to0}\dfrac{x^2\sin1/x}{\sin x}=0$.

***

The next example shows that L'Hôpital's rule $\left(\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}\right)$ is applicable only to those quotients which are in indeterminate form of the type $\frac{0}{0}$ at the given point.

Example 5: Find $\lim_{x\to 0}\frac{x^2}{\sin^2 x}$.

Solution: Let $f(x)=x^2$ and $g(x)=\sin^2 x$. It is clear that $\frac{x^2}{\sin^2 x}$ is in the $\frac{0}{0}$ form at $x=0$. Now,

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{2x}{2\sin x.\cos x}$$

$$=\lim_{x\to 0}\frac{1}{\frac{\sin x}{x}\cos x}$$

$$=\frac{1}{\lim\limits_{x\to 0}\frac{\sin x}{x}.\lim\limits_{x\to 0}\cos x}$$

$$=1$$

Therefore, using Theorem 1, $\lim_{x\to 0}\frac{x^2}{\sin^2 x}=1$.

You may note that here $g'(0)=0$, but $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists.

***

We, now, illustrate few examples, in which L'Hôpital's rule is applied for one-sided limits.

Example 6: Evaluate $\lim_{x\to \pi/2^-}\frac{1-\sin x}{\cos x}$.

Solution: It is clear that $\lim_{x\to \pi/2^-}\frac{1-\sin x}{\cos x}$ is in the $\frac{0}{0}$ form.
The functions $1-\sin x$ and $\cos x$ are differentiable.
Therefore, we can apply L'Hôpital's rule here. Thus,

$$\lim_{x\to \pi/2^-}\frac{1-\sin x}{\cos x}=\lim_{x\to \pi/2^-}\frac{-\cos x}{-\sin x}=\frac{\cos \pi/2}{\sin \pi/2}=0.$$

***

Example 7: Find $\lim_{x\to 1^+}\frac{\ln x}{x-\sqrt{x}}$.

Solution: Here $f(x)=\ln x$, and $g(x)=x-\sqrt{x}$. Clearly, $f(x)$ and $g(x)$ are differentiable and $\lim_{x\to 1^+}f(x)=0=\lim_{x\to 1^+}g(x)$. Also, $\lim_{x\to 1^+}\frac{f'(x)}{g'(x)}=\lim_{x\to 1^+}\frac{1/x}{1-\frac{1}{2\sqrt{x}}}=2$.

Thus, L'Hôpital's rule gives us $\lim_{x\to 1^+} \frac{\ln x}{x-\sqrt{x}} = 2$.

***

Example 8: Evaluate $\lim_{x\to 0^-} \frac{\tan x}{x^2}$.

Solution: The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type $\frac{0}{0}$. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^-} \frac{\tan x}{x^2} = \lim_{x\to 0^-} \frac{\sec^2 x}{2x} = -\infty$$

***

Now, let us discuss a few examples of the form $\frac{0}{0}$, in which $x \to \infty$ or $x \to -\infty$ and L'Hôpital's rule is applied.

Example 9: Evaluate $\lim_{x\to\infty} \frac{\tan(3/x)}{\sin(1/x)}$.

Solution: Let $f(x) = \tan\frac{3}{x}$ and $g(x) = \sin\frac{1}{x}$. Then $\frac{f(x)}{g(x)}$ is in an indeterminate form of the type $\frac{0}{0}$ as $x \to \infty$. Clearly, both $f(x)$ and $g(x)$ are differentiable for all $x \neq 0$, and $\lim_{x\to\infty} \frac{(-3/x^2)\sec^2(3/x)}{(-1/x^2)\cos(1/x)}$

$= \lim_{x\to\infty} \frac{3\sec^2(3/x)}{\cos(1/x)} = 3$.

Thus, L'Hôpital's rule for $\frac{0}{0}$ form at ∞ is applicable, and therefore,

$\lim_{x\to\infty} \frac{\tan(3/x)}{\sin(1/x)} = 3$.

***

Example 10: Evaluate $\lim_{x\to -\infty} x \sin\frac{5}{x}$.

Solution: Let $f(x) = \sin\left(\frac{5}{x}\right)$, $g(x) = \frac{1}{x}$. Then $\frac{f(x)}{g(x)}$ is $\frac{0}{0}$ form at $-\infty$, and

$$\lim_{x\to-\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to-\infty} \frac{\left(\frac{-5}{x^2}\right)\cos\left(\frac{5}{x}\right)}{-1/x^2} = 5$$

Therefore,

$$\lim_{x\to-\infty} \frac{\sin\left(\frac{5}{x}\right)}{1/x} = \lim_{x\to-\infty} \frac{f'(x)}{g'(x)} = 5.$$

***

Now try and evaluate the limits in the following exercise by applying Theorem 1.

E2) Find the following limits

i) $\lim_{x\to 0}\frac{x-\sin x}{\sin x}$ ii) $\lim_{x\to 1}\frac{x^4-1}{x^2-6x+5}$

iii) $\lim_{x\to 0}\frac{1-\cos^2 x}{\sin x^2}$ iv) $\lim_{x\to 1/2}\frac{2x^2-9x+4}{\cos\pi x}$

v) $\lim_{x\to\alpha}\frac{x^m-\alpha^m}{x^n-\alpha^n},\ \alpha>0$ vi) $\lim_{x\to 0}\frac{\ln(1+4x)}{4x}$

vii) $\lim_{x\to\pi/2}\frac{\ln\sin x}{1-\sin x}$ viii) $\lim_{x\to 0}\frac{3^x-2^x}{x}$

ix) $\lim_{x\to\pi/4}\frac{\sin x-\cos x}{x-\pi/4}$ x) $\lim_{x\to 2}\frac{\sqrt{x^2+12}-4}{x^2-4}$.

E3) Evaluate the following limits:

i) $\lim_{x\to 1^+}\frac{\ln x}{x-1}$

ii) $\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}$

E4) Evaluate the following limits:

i) $\lim_{x\to\infty}\frac{x^{-4/3}}{\sin(1/x)}$ ii) $\lim_{x\to\infty}x\tan^{-1}(1/x)$

iii) $\lim_{x\to-\infty}\frac{\sin(1/x)}{e^{1/x}-1}$ iv) $\lim_{x\to\infty}x(e^{1/x}-1)$

There are also situations where we need to apply L'Hôpital's rule multiple times. For an example, consider the functions $f(x)=1-\cos x$ and $g(x)=x^2$, which are differentiable on $\mathbb{R}$. Let us try to evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$. Here $f'(x)=\sin x$, and $g'(x)=2x$, and $\frac{f'(x)}{g'(x)}$ is again in an indeterminate form of the type $\frac{0}{0}$ as $x\to 0$. But let us now turn our attention to the functions $f'(x)$ and $g'(x)$. We find that the functions $f'(x)$ and $g'(x)$ are also differentiable functions, and $f''(x)=\cos x$, $g''(x)=2$. Clearly, $\lim_{x\to 0}\frac{f''(x)}{g''(x)}=\frac{1}{2}$. This means that we can apply L'Hôpital's rule to the quotient of $f'(x)$ and $g'(x)$ at $x=0$, and get $\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{f''(x)}{g''(x)}=\frac{1}{2}$.

Now since $\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\frac{1}{2}$, applying L'Hôpital's rule to $\frac{f(x)}{g(x)}$ we get

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\frac{1}{2}.$$

Thus, we can write,

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

We often come across similar situations where repeated use of L'Hôpital's rule enables us to evaluate the required limits. We now state the general result in the following theorem.

Theorem 2: Let $f(x)$ and $g(x)$ be two real-valued functions such that

$\lim\limits_{x \to a} f^{(k)}(x) = 0 = \lim\limits_{x \to a} g^{(k)}(x),\ 0 \le k \le n-1$, for some $n \in \mathbb{N}$. If $\lim\limits_{x \to a} \dfrac{f^{(n)}(x)}{g^{(n)}(x)}$ exists (may be equal to ∞ or $-\infty$), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

(Here, $f^{(0)} = f$, $g^{(0)} = g$, and $f^{(k)}$ denotes the k-th order derivative of f for $1 \le k \le n-1$.)

Now we give some general observations in the form of remarks.

Remark 1: Note that if for some n, $\lim\limits_{x \to a} \dfrac{f^{(n)}(x)}{g^{(n)}(x)}$ does not exist, and

$\lim\limits_{x \to a} f^{(k)}(x) = 0 = \lim\limits_{x \to a},\ g^{(k)}(x),\ 0 \le k \le n-1$, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ cannot be evaluated using L'Hôpital's rule.

Remark 2: We can now state the general L'Hôpital's rule for one sided limits. Let $f(x)$ and $g(x)$ be two real-valued functions such that

$\lim\limits_{x \to a^+} f^{(k)}(x) = 0 = \lim\limits_{x \to a^+} g^{(k)}(x),\ 0 \le k \le n-1$ for some $n \in \mathbb{N}$.

If $\lim\limits_{x \to a^+} \dfrac{f^{(n)}(x)}{g^{(n)}(x)}$ exists (may be equal to $+\infty$ or $-\infty$), then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

If we replace a^+ wherever it occurs by a^-, we get the statement for the left hand limit.

We now give examples to illustrate the above discussion.

Example 11: Evaluate $\lim\limits_{x \to 1} \dfrac{x^5 - 5x + 4}{x^3 - x^2 - x + 1}$.

Solution: If we take $f(x) = x^5 - 5x + 4$ and $g(x) = x^3 - x^2 - x + 1$, then

$$\lim_{x \to 1} f(x) = 0 = \lim_{x \to 1} g(x)$$

$$\lim_{x \to 1} f'(x) = \lim_{x \to 1} (5x^4 - 5) = 0$$

$$\lim_{x \to 1} g'(x) = \lim_{x \to 1} (3x^2 - 2x - 1) = 0$$

and $$\lim_{x \to 1} \frac{f''(x)}{g''(x)} = \lim_{x \to 1} \frac{20x^3}{6x-2} = 5$$

Therefore, by Theorem 2, i.e., by repeated use of L'Hôpital's rule, we obtain,

$$\lim_{x \to 1} \frac{x^5 - 5x + 4}{x^3 - x^2 - x + 1} = \lim_{x \to 1} \frac{5x^4 - 5}{3x^2 - 2x - 1} = \lim_{x \to 1} \frac{20x^3}{6x-2} = 5.$$

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Example 12: Evaluate $\lim_{x\to 0} \frac{e^{3x} - 3x - 1}{1 - \cos x}$.

Solution: If $f(x) = e^{3x} - 3x - 1$ and $g(x) = 1 - \cos x$, then

$f'(x) = 3e^{3x} - 3$ and $g'(x) = \sin x$. Also,

$\lim_{x\to 0} f(x) = \lim_{x\to 0} f'(x) = 0$, and

$\lim_{x\to 0} g(x) = \lim_{x\to 0} g'(x) = 0$.

Therefore, $\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{9e^{3x}}{\cos x} = 9$, which shows that $\lim_{x\to 0} \frac{e^{3x} - 3x - 1}{1 - \cos x} = 9$.

***

Example 13: Evaluate $\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

Solution: This is an indeterminate form of the type $0/0$, and we find that $\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1 - \cos x}{3x^2}$ after applying L'Hôpital's rule once.

This is still the indeterminate form of the type $0/0$, so L'Hôpital's rule can be applied once again and we obtain.

$$\lim_{x\to 0} \frac{1 - \cos x}{3x^2} = \lim_{x\to 0} \frac{-(-\sin x)}{6x} = \frac{1}{6} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{6}(1) = \frac{1}{6}$$

***

It may happen that even when L'Hôpital's rule applies to a limit, it is not the best way to proceed, as illustrated by the following example.

Example 14: Evaluate $\lim_{x\to 0} \frac{(1 - \cos x)\sin 4x}{x^3 \cos x}$.

Solution: This limit has the form $0/0$, but direct application of L'Hôpital's rule leads to a real mess (try it!). Instead, we compute the given limit by using the product rule for limits first, followed by two simple applications of L'Hôpital's rule. Specifically, using the product rule for limits (assuming the limits exist), we have

$$\lim_{x\to 0} \frac{(1 - \cos x)\sin 4x}{x^3 \cos x} = \left[\lim_{x\to 0} \frac{1 - \cos x}{x^2}\right]\left[\lim_{x\to 0} \frac{\sin 4x}{x}\right]\left[\lim_{x\to 0} \frac{1}{\cos x}\right]$$

$$= \left[\lim_{x\to 0} \frac{\sin x}{2x}\right]\left[\lim_{x\to 0} \frac{4\cos x}{1}\right]\left[\lim_{x\to 0} \frac{1}{\cos x}\right]$$

$$= \left(\frac{1}{2}\right)(4)(1) = 2$$

***

See if you can solve these exercises now.

E5) Evaluate the following limits:

i) $\lim_{x\to 0} \frac{(\tan^{-1} x)^2}{\ln(1 + x^2)}$ v) $\lim_{x\to 0} \frac{\sin 3x - 3x}{x^3}$

ii) $\lim_{x\to 0} \frac{x - \tan^{-1} x}{x - \sin x}$

iii) $\lim_{x\to 0} \frac{1-\cos x^2}{x^2 \sin^2 x}$

iv) $\lim_{x\to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x}$

vi) $\lim_{x\to \pi/2} \frac{1-\sin x}{1+\cos 2x}$

vii) $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

E6) Find the value of k for which the following limits are finite and hence evaluate the limit.

i) $\lim_{x\to 0} \frac{\sinh 2x + k\sin 2x}{x^3}$

ii) $\lim_{x\to 0} \frac{e^x + ke^{-x} - 2x}{1-\cos x}$

E7) Show that $\lim_{x\to 0} \frac{x^2 \sin 1/x}{5\tan x} = 0$. Also show that this limit cannot be evaluated by using L'Hôpital's rule.

In this section, we have seen how to evaluate $\lim_{x\to a} \frac{f(x)}{g(x)}$ by L'Hôpital's rule when $\frac{f(x)}{g(x)}$ is in the $\frac{0}{0}$ form at $x = a$. Now we shall study the rule for evaluating $\lim_{x\to a} \frac{f(x)}{g(x)}$ when $\frac{f(x)}{g(x)}$ is in the $\frac{\infty}{\infty}$ form at $x = a$.

12.4 L'HÔPITAL'S RULE FOR $\frac{\infty}{\infty}$ FORM

Consider the limit of $\frac{e^x}{x^n}$ as $x \to \infty$. As, you can see, this is of the form $\frac{\infty}{\infty}$. In order to evaluate $\lim_{x\to a} \frac{f(x)}{g(x)}$ when $\lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x)$, we have results similar to those proved in the last section. Here, we state these results without proofs, and then illustrate them.

Theorem 3 (L'Hôpital's rule for form ∞/∞): Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$.

If $\lim_{x\to a}[f'(x)/g'(x)]$ has a finite limit, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-, x \to a^+, x \to -\infty$, or as $x \to +\infty$.

We had also seen, in Theorem 2, how repeated use of L'Hôpital's rule sometimes helps us in evaluating the required limit. We now state an analogous result for indeterminate forms of the type $\frac{\infty}{\infty}$.

Theorem 4: Let $f(x)$ and $g(x)$ be two real-valued functions such that

i) $\lim_{x\to a} f^{(k)}(x) = \pm\infty = \lim_{x\to a} g^{(k)}(x)$,
where $0 \le k \le n-1$, n is a natural number and a is any real number, ∞ or $-\infty$, and

ii) If $\lim_{x\to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$ exists, and may even be infinite, then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

Here is another point that you should note.
If $\frac{f^{(k)}(x)}{g^{(k)}(x)}$ is an indeterminate form for $0 \le k \le n$ and $\frac{f^{(n)}(x)}{g^{(n)}(x)}$ fails to tend to a limit as $x \to a$, then this does not mean that $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist. It only means that we cannot apply L'Hôpital's rule, and that we have to adopt a different procedure to establish the existence or non-existence of the limit under consideration.

We shall bring out this and various other points with the help of a number of examples. Go through these carefully. They will help you to get a better understanding of the concepts involved.

Example 15: Show that $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty, n \ge 1$,

Solution: Let $f(x) = e^x$, $g(x) = x^n$, $n \ge 1$. Then

$\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$. If $n = 1$, then $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$ and therefore, by L'Hôpital's rule $\lim_{x\to\infty} \frac{e^x}{x} = \infty$.

If $n > 1$, then it is clear that, $\lim_{x\to\infty} f^{(k)}(x) = \infty = \lim_{x\to\infty} g^{(k)}(x), o \le k \le n$ and

$$\lim_{x\to\infty} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \lim_{x\to\infty} \frac{e^x}{n!} = \infty.$$

Consequently, $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ for all $n \ge 1$.

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Example 16: Find $\lim_{x\to\infty} \frac{\ln x}{x^n}$, $n > 0$.

Solution: Let $f(x) = \ln x$, $g(x) = x^n$, $n > 0$
The function $f(x)$ and $g(x)$ satisfy the requirements of Theorem 3.

Therefore, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1/x}{nx^{n-1}} = \lim_{x\to\infty} \frac{1}{nx^n} = 0$.

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Example 17: Evaluate $\lim_{x\to 0^+} \frac{\ln \tan 2x}{\ln \tan x}$.

Solution: Let $f(x) = \ln \tan 2x,\ g(x) = \ln \tan x$. Then

$\lim_{x\to 0^+} f(x) = -\infty = \lim_{x\to 0^+} g(x)$ and $\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{4\operatorname{cosec}4x}{2\operatorname{cosec}2x} = \lim_{x\to 0^+} \frac{1}{\cos 2x} = 1$.

Therefore, $\lim_{x\to 0^+} \frac{\ln \tan 2x}{\ln \tan x} = 1$.

Here, we cannot talk of $\lim_{x\to 0} \frac{\ln \tan 2x}{\ln \tan x}$ because $\tan 2x$ and $\tan x$ are negative for $x < 0$ and therefore, we cannot take their logarithms.

***

Example 18: Find the limit $\lim_{x\to\infty} \frac{(\ln x)^n}{x}$, where n is an integer, $n \ge 0$.

Solution: For $n = 0$, the result is clear. For $n = 1$, the result has been proved in Example 16. Let $f(x) = (\ln x)^n$ and $g(x) = x$. Then, the functions $f(x)$ and $g(x)$ are differentiable for $x > 0$, and $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$.

Therefore, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{x}$, provided the right-hand side limit exists. Considering the functions $(\ln x)^{n-1}$ and x instead of $(\ln x)^n$ and x, we get $\lim_{x\to\infty} \frac{(\ln x)^n}{x} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{x} = \lim_{x\to\infty} \frac{n(n-1)\,(\ln x)^{n-2}}{x}$, provided the right-hand side limit exists. Repeating the above process n times, we obtain $\lim_{x\to\infty} \frac{(\ln x)^n}{x} = \lim_{x\to\infty} \frac{n!}{x} = 0$.

***

Example 19: Evaluate $\lim_{x\to\infty} \frac{(\ln x)^m}{x^n}$, $m > 0,\ n > 0$ and m is an integer.

Solution: Let $f(x) = (\ln x)^m$ and $g(x) = x^n$. Then, $f(x)$ and $g(x)$ are differentiable for $x > 0$, and $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$.

Therefore, $\lim_{x\to\infty} \frac{(\ln x)^m}{x^n} = \lim_{x\to\infty} \frac{m(\ln x)^{m-1}}{nx^n}$, provided the right hand side limit exists. Considering the functions $(\ln x)^{m-1}$ and x^n instead of $(\ln x)^m$ and x^n respectively, we obtain $\lim_{x\to\infty} \frac{(\ln x)^m}{x^n} = \lim_{x\to\infty} \frac{m(m-1)\,(\ln x)^{m-2}}{n^2 x^n}$ provided the right-hand side limit exists. Thus, repeating the above process, we obtain,

$\lim_{x\to\infty} \frac{(\ln x)^m}{x^n} = \lim_{x\to\infty} \frac{m!}{n^m} \cdot \frac{1}{x^n} = 0$.

***

Example 20: Let $P(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$ and

$Q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0$ be two polynomials with real coefficients, $a_m \neq 0, b_n \neq 0$. Evaluate $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$.

Solution: Let us take the case when $m = n$.
If $0 \le k \le m, \lim_{x\to\infty} P^{(k)}(x)$ and $\lim_{x\to\infty} Q^{(k)}(x)$ are infinite, and

$$\lim_{x\to\infty} \frac{P^{(m)}(x)}{Q^{(m)}(x)} = \lim_{x\to\infty} \frac{a_m m!}{b_m m!} = \frac{a_m}{b_m}.$$

Therefore, $\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{f^{(m)}(x)}{g^{(m)}(x)} = \frac{a_m}{b_m}$ by Theorem 4.

Now, suppose $m < n$.
Again $\lim_{x\to\infty} P^{(k)}(x), \lim_{x\to\infty} Q^{(k)}(x)$ are infinite for $0 \le k \le m$, and

$$\lim_{x\to\infty} \frac{P^{(m)}(x)}{Q^{(m)}(x)} = \lim_{x\to\infty} \frac{m!\, a_m}{\sum_{r=m}^{n} b_r\ r(r-1)\ldots(r-m+1)x^{r-m}} = 0.$$

Thus, $\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{P^{(m)}(x)}{Q^{(m)}(x)} = 0$.

Now, if $m > n$, it is obvious that

$\lim_{x\to\infty} P^{(k)}(x), \lim_{x\to\infty} Q^{(k)}(x)$ are infinite for $0 \le k < n$, and

$$\lim_{x\to\infty} \frac{P^{(n)}(x)}{Q^{(n)}(x)} = \lim_{x\to\infty} \frac{\sum_{r=n}^{m} a_r r(r-1)\ldots(r-n+1)x^{r-n}}{n!\, b_n}$$

$= \pm\infty$, according as $\frac{a_m}{b_n} > 0$ or < 0.

Thus, $\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{P^{(n)}(x)}{Q^{(n)}(x)} = \infty$ or $-\infty$, according as $\frac{a_m}{b_n} > 0$ or < 0.

Therefore,

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \begin{cases} \frac{a_m}{b_n} \text{ if } m = n \\ 0, \text{ if } m < n \\ \pm\infty, \text{ if } m > n, \text{according to } \frac{a_m}{b_n} \text{ is positive or negative} \end{cases}$$

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Example 21: Evaluate $\lim_{x\to\infty} \frac{7x^3 + 5x^2 + 4x + 6}{5x^4 + 6x + 7}$.

Solution: By applying L'Hôpital's rule repeatedly, we get

$$\lim_{x\to\infty} \frac{7x^3 + 5x^2 + 4x + 6}{5x^4 + 6x + 7} = \lim_{x\to\infty} \frac{21x^2 + 10x + 4}{20x^3 + 6}$$

$$= \lim_{x\to\infty} \frac{42x + 10}{60x^2} = \lim_{x\to\infty} \frac{42}{120x} = 0.$$

Alternatively, using algebra of limits, we can find the above limit in a very simple way as follows:

$$\lim_{x\to\infty}\frac{7x^3+5x^2+4x+6}{5x^4+6x+7}=\lim_{x\to\infty}\frac{7+5/x+4/x^2+6/x^3}{5x+6/x^2+7/x^3}=0\text{, as } 1/x\to 0$$
when $x\to\infty$.

***

In the next example, you will find a situation where L'Hôpital's rule is not applicable.

Example 22: Evaluate $\lim_{x\to\infty}\frac{2x\sin x}{1+x^2}$.

Solution: Can we apply L'Hôpital's rule to evaluate this limit?
No. L'Hôpital's rule is not applicable because $\lim_{x\to\infty} 2x\sin x$ does not exist.

However, $\lim_{x\to\infty}\frac{2x\sin x}{1+x^2}=0$

because, $\left|\frac{2x\sin x}{1+x^2}\right|\le\left|\frac{2x}{1+x^2}\right|$, and $\lim_{x\to\infty}\frac{2x}{1+x^2}=0$.

***

We now give an example where L'Hôpital's rule is applicable but it yields no result. But such situations are very rare.

Example 23: Evaluate $\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}$.

Solution: Let us see what happens if L'Hôpital's rule is applied to evaluate its limit as $x\to\infty$. We get

$$\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

The right hand side is again in the $\frac{\infty}{\infty}$ form, but if we apply L'Hôpital's rule to evaluate it, we get back to where we started. Thus, it is useless to apply L'Hôpital's rule in this case. But we can still evaluate the limit as follows.

$\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to\infty}\frac{1-e^{-2x}}{1+e^{-2x}}=1$, because $\lim_{x\to\infty}e^{-x}=0$.

***

Example 24: Evaluate $\lim_{x\to+\infty}\frac{2x^2-3x+1}{3x^2+5x-2}$.

Solution: We could compute this limit by multiplying and dividing by $(1/x^2)$.
Instead, we note that this is of the form ∞/∞ and apply L'Hôpital's rule:

$$\lim_{x\to+\infty}\frac{2x^2-3x+1}{3x^2+5x-2}=\lim_{x\to+\infty}\frac{4x-3}{6x+5}=\lim_{x\to+\infty}\frac{4}{6}=\frac{2}{3}$$

***

Example 25: Evaluate $\lim_{x\to+\infty}\frac{x+\sin x}{x-\cos x}$.

Solution: The limit has the indeterminate form ∞/∞. If you try to apply L'Hôpital's rule, you find

$$\lim_{x\to+\infty}\frac{x+\sin x}{x-\cos x}=\lim_{x\to+\infty}\frac{1+\cos x}{1+\sin x}$$

The limit on the right does not exist, because both $\sin x$ and $\cos x$ oscillate between -1 and 1 as $x\to+\infty$. Recall that L'Hôpital's rule applies only if $\lim_{x\to c}\frac{f'(x)}{g'(x)}=L$ or is $\pm\infty$. This does not mean that the limit of the original expression does not exist or that we cannot find it. It simply means that we cannot apply L'Hôpital's rule. To find this limit, factor out an x from the numerator and denominator and proceed as follows:

$$\lim_{x\to+\infty}\frac{x+\sin x}{x-\cos x}=\lim_{x\to+\infty}\frac{x\left(1+\frac{\sin x}{x}\right)}{x\left(1-\frac{\cos x}{x}\right)}=\lim_{x\to+\infty}\frac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}}=\frac{1+0}{1-0}=1$$

***

After going through the above examples you should have no difficulty in solving these exercises.

E8) Evaluate the following limits:

i) $\lim_{x\to\infty}\frac{a_m x^m+a_{m-1}x^{m-1}+\cdots+a_0}{e^x}$, where $a_i\in\mathbb{R}, \forall\, i=0,1,\ldots,m$.

ii) $\lim_{x\to(\pi/2)}\frac{-\tan x}{\ln\cos x}$

iii) $\lim_{x\to\pi/2}\frac{\tan 3x}{\tan x}$

iv) $\lim_{x\to\infty}\frac{x^6+\ln x}{2x^6+5x^4+1}$

v) $\lim_{x\to-\infty}\frac{2x^8+5x^7+6x^3+1}{3x^8+5x^7+5x+1}$.

E9) Show that $\lim_{x\to\infty}\frac{x+\sin x}{x+\cos x}=1$ and that L'Hôpital's rule cannot be used to evaluate it.

E10) Evaluate the following limits and show that L'Hôpital's rule is not applicable in each case.

i) $\lim_{x\to\infty}\frac{x^2-\sin x^2}{x^2}$

ii) $\lim_{x\to\infty}\frac{x-\cos x}{x}$

iii) $\lim_{x\to\infty}\frac{|\sin x|+|\cos x|}{x}$

iv) $\lim_{x\to\infty}\frac{x\sin x+\cos x}{x^2}$

In the following section, we shall discuss other indeterminate forms.

12.5 OTHER INDETERMINATE FORMS

So far we have discussed indeterminate forms of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$. However, these are not the only possibilities. In general, the limit of an expression that has one of the forms $\frac{f(x)}{g(x)}, f(x)\cdot g(x), f(x)^{g(x)}, f(x)-g(x), f(x)+g(x)$ is called an indeterminate form if the limits of $f(x)$ and $g(x)$ individually exert conflicting influences on the limit of the entire expression. For example, the limit $\lim_{x\to 0^+} x\ln x$ is an indeterminate form of type $0\cdot\infty$ because the limit of the first factor is 0, the limit of the second factor is $-\infty$, and these two limits exert conflicting influences on the product. On the other hand, the limit $\lim_{x\to+\infty}[\sqrt{x}(1-x^2)]$ is not an indeterminate form because the first factor has a limit of $+\infty$, the second factor has a limit of $-\infty$, and these influences work together to produce a limit of $-\infty$ for the product.

Caution: It is tempting to argue that an indeterminate form of type $0\cdot\infty$ has value 0 since "zero times anything is zero". However, this is false, since $0\cdot\infty$ is not a product of numbers, but rather a statement about limits. For example, the following limits are of the form $0\cdot\infty$:

$$\lim_{x\to 0^+} x\cdot\frac{1}{x}=1,\ \lim_{x\to 0^+} x^2\cdot\frac{1}{x}=0,\ \lim_{x\to 0^+}\sqrt{x}\cdot\frac{1}{x}=+\infty$$

Indeterminate forms of type $0\cdot\infty$ can sometimes be evaluated by rewriting the product as a ratio, and then applying L'Hôpital's rule for indeterminate forms of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

You will understand this more clearly if you go though the following example.

Example 26: Evaluate $\lim_{x\to 1}\tan\left(\frac{\pi x}{2}\right)\ln x$.

Solution: Note that $\tan\frac{\pi x}{2}\ln x$ is a $0\ .\ \infty$ form at $x=1$. Now, we write

$$\tan\left(\frac{\pi x}{2}\right)\ln x=\frac{\sin(\pi x/2)}{\cos(\pi x/2)}\ln x$$

We know that $\lim_{x\to 1}\left(\sin\frac{\pi x}{2}\right)=1$. So, let us try to find $\lim_{x\to 1}\frac{\ln x}{\cos(\pi x/2)}$.

Now, $\frac{\ln x}{\cos(\pi x/2)}$ is $\frac{0}{0}$ form at $x=1$.

Therefore, by L'Hôpital's rule

$$\lim_{x\to 1}\frac{\ln x}{\cos(\pi x/2)}=\lim_{x\to 1}\frac{1/x}{-\sin(\pi x/2).\ \pi/2}=-\frac{2}{\pi}$$

Thus, $\lim_{x\to 1}\left(\tan\frac{\pi x}{2}\right)\ln x=\lim_{x\to 1}\frac{\sin(\pi x/2)\ \ln x}{\cos(\pi x/2)}$

$$= \lim_{x\to 1} \sin\frac{\pi x}{2} \lim_{x\to 1} \frac{\ln x}{\cos(\pi x/2)}$$

$$= 1.\left(-\frac{2}{\pi}\right) = -\frac{2}{\pi}.$$

***

Example 27: Find $\lim\limits_{x\to\infty} x^p e^{-qx}$ where p, q are positive integers.

Solution: We can write $\lim\limits_{x\to\infty} x^p e^{-qx} = \lim\limits_{x\to\infty} \frac{x^p}{e^{qx}}$

Now, $\frac{x^p}{e^{qx}}$ is in an indeterminate form of the type $\frac{\infty}{\infty}$ to which L'Hôpital's rule is applicable. Thus, we get,

$$\lim_{x\to\infty} \frac{x^p}{e^{qx}} = \lim_{x\to\infty} \frac{p!}{q^p e^{qx}} = 0 \text{, so that } \lim_{x\to\infty} x^p e^{-qx} = 0.$$

***

Example 28: Evaluate $\lim\limits_{x\to(\pi/2)^-} \left(x - \frac{\pi}{2}\right)\tan x$.

Solution: This limit has the form $0 \cdot \infty$, because

$$\lim_{x\to(\pi/2)^-} \left(x - \frac{\pi}{2}\right) = 0 \text{ and } \lim_{x\to(\pi/2)^-} \tan x = +\infty$$

Write $\tan x = \frac{1}{\cot x}$ to obtain

$$\lim_{x\to(\pi/2)^-} \left(x - \frac{\pi}{2}\right)\tan x = \lim_{x\to(\pi/2)^-} \frac{x - \frac{\pi}{2}}{\cot x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to(\pi/2)^-} \frac{1}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to(\pi/2)^-} (-\sin^2 x) = -1$$

***

Now, we shall discuss another indeterminate form. A limit problem that leads to one of the expressions; $(+\infty) - (+\infty), (-\infty) - (-\infty), (+\infty) + (-\infty), (-\infty) + (+\infty)$ is called an indeterminate form of type $\infty - \infty$. Such limits are indeterminate because the two terms exert conflicting influences on the expression: one pushes it in the positive direction and the other pushes it in the negative direction. However, limit problems that lead to one of the expressions; $(+\infty) + (+\infty),\ (+\infty) - (-\infty),\ (-\infty) + (-\infty),\ (-\infty) - (+\infty)$ are not indeterminate, since, the two terms work together.

Indeterminate forms of type $\infty - \infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, as you will see in the following examples.

Example 29: Evaluate limit $\lim_{x\to 0}\left(\operatorname{cosec} x - \frac{1}{x}\right)$.

Solution: Clearly the function is of the type $\infty - \infty$.

We can write $\operatorname{cosec} x - \frac{1}{x} = \frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$, so that the right hand side is in the $\frac{0}{0}$ form at $x = 0$, to which L'Hôpital's rule is applicable.

Thus, $\lim_{x\to 0}\left(\operatorname{cosec} x - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x - \sin x}{x \sin x}$

$$= \lim_{x\to 0}\frac{1 - \cos x}{\sin x + x\cos x} \quad \text{[by L'Hôpital's rule]}$$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x}, \text{ [by L'Hôpital's rule]}$$

$$= \frac{\lim_{x\to 0}\sin x}{\lim_{x\to 0}(2\cos x - x\sin x)} = \frac{0}{2} = 0.$$

***

Example 30: Find $\lim_{x\to(\pi/2)^-}\left[\sec x - \frac{1}{(1-\sin x)}\right]$.

Solution: Now, $\sec x - \frac{1}{1-\sin x} = \frac{1}{\cos x} - \frac{1}{1-\sin x} = \frac{1-\sin x - \cos x}{\cos x(1-\sin x)}$

and the right hand side is in the $\frac{0}{0}$ form as $x \to \left(\frac{\pi}{2}\right)^-$, to which L'Hôpital's rule is applicable. Thus,

$$\lim_{x\to(\pi/2)^-}\left(\sec x - \frac{1}{1-\sin x}\right) = \lim_{x\to(\pi/2)^-}\frac{1-\sin x - \cos x}{\cos x - \sin x\cos x}$$

$$= \lim_{x\to(\pi/2)^-}\frac{-\cos x + \sin x}{-\sin x - \cos 2x}, \text{ by L'Hôpital's rule}$$

$$= \lim_{x\to(\pi/2)^-}(-\cos x + \sin x).\lim_{x\to(\pi/2)^-}\left(\frac{1}{-\sin x - \cos 2x}\right)$$

$$= 1 \;.\; \infty = \infty.$$

***

Example 31: Evaluate $\lim_{x\to 0}\left[\frac{1}{x} - \frac{\ln(1+x)}{x^2}\right]$.

Solution: We can write $\lim_{x\to 0}\left[\frac{1}{x} - \frac{\ln(1+x)}{x^2}\right] = \lim_{x\to 0}\left[\frac{x - \ln(1+x)}{x^2}\right]$

And L'Hôpital's rule can be applied to evaluate the limit on the right hand side. Therefore,

$$\lim_{x\to 0}\left[\frac{x - \ln(1+x)}{x^2}\right] = \lim_{x\to 0}\left[\frac{1 - \frac{1}{1+x}}{2x}\right] = \lim_{x\to 0}\left[\frac{1/(1+x)^2}{2}\right] = \frac{1}{2}.$$

***

Let us now discuss the limits of the form $\lim f(x)^{g(x)}$, which give rise to indeterminate forms of type $0^0, \infty^0,$ and 1^∞. For example, the limit $\lim\limits_{x\to 0^+}(1+x)^{1/x}$ whose value we know to be e [Recall Unit 7] is an indeterminate form of type 1^∞. It is indeterminate because the expressions $1+x$ and $1/x$ exert two conflicting influences; the first approaches 1, which drives the expression toward 1, and the second approaches $+\infty$, which drives the expression toward $+\infty$.

Indeterminate forms of types $0^0, \infty^0$, and 1^∞ can sometimes be evaluated by first introducing a dependent variable $y = f(x)^{g(x)}$ and then calculating the limit of $\ln y$ by expressing it as $\lim \ln y = \lim [\ln (f(x)^{g(x)})] = \lim [g(x) \ln f(x)]$. Once the limit of $\ln y$ is known, the limit of $y = f(x)^{g(x)}$ itself can generally be obtained by a method that we shall illustrate in the next example.

Example 32: Evaluate $\lim\limits_{x\to 1^+} x^{1/(x-1)}$.

Solution: It is clear that $x^{1/(x-1)}$ is in an indeterminate form of the type 1^∞ as $x \to 1^+$. Let $y = x^{1/(x-1)}$. Then, $\ln y = \dfrac{1}{x-1}\ln x$

Now, $\dfrac{\ln x}{x-1}$ is in the $\dfrac{0}{0}$ form as $x \to 1^+$, and L'Hôpital's rule is applicable to it.

Therefore, $\lim\limits_{x\to 1^+} \dfrac{\ln x}{x-1} = \lim\limits_{x\to 1^+} \dfrac{1/x}{1} = 1$.

Hence, $\lim\limits_{x\to 1^+} x^{1/(x-1)} = \lim\limits_{x\to 1^+} e^{\ln y} = e^{\lim\limits_{x\to 1^+} \ln y} = e^0 = 1$.

***

Example 33: Evaluate $\lim\limits_{x\to 0^+}\left(\ln \dfrac{1}{x}\right)^x$.

Solution: Let $y = \left(\ln \dfrac{1}{x}\right)^x$, so that y is in the form ∞^0 as $x \to 0^+$.

Then, $\lim\limits_{x\to 0^+}\left(\ln \dfrac{1}{x}\right)^x = \lim\limits_{x\to 0^+} y = \lim\limits_{x\to 0^+} e^{\ln y} = e^{\lim\limits_{x\to 0^+} \ln y}$ …(2)

But, $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} x \ln\left(\ln \dfrac{1}{x}\right) = \lim\limits_{x\to 0^+} \dfrac{\ln\left(\ln \dfrac{1}{x}\right)}{1/x}$

Now $\dfrac{\ln\left(\ln \dfrac{1}{x}\right)}{1/x}$ is in the $\dfrac{\infty}{\infty}$ form as $x \to 0^+$. Therefore, L'Hôpital's rule gives

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\dfrac{x}{\ln(1/x)} \cdot \dfrac{-1}{x^2}}{-1/x^2} = 0.$$

Substituting this in Eqn. (2) we get

$$\lim_{x\to 0^+}\left(\ln \frac{1}{x}\right)^x = e^0 = 1.$$

***

Example 34: Find $\lim_{x\to\pi/2^-}(\cos x)^{\cos x}$.

Solution: Let $y=(\cos x)^{\cos x}$, $0<x<\pi/2$. Then,

$\ln y=\cos x\ln\cos x=\dfrac{\ln\cos x}{\sec x}$, and therefore by applying L'Hôpital's rule we obtain

$$\lim_{x\to\pi/2^-}\ln y=\lim_{x\to\pi/2^-}\frac{\ln\cos x}{\sec x}=\lim_{x\to\pi/2^-}\frac{-\tan x}{\sec x\tan x}=0.$$

Thus,

$$\lim_{x\to\pi/2^-}(\cos x)^{\cos x}=e^{\lim\limits_{x\to\pi/2^-}\ln y}=e^0=1.$$

***

Example 35: Show that $\lim_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^x=e$.

Solution: Note that this limit is indeed of the indeterminate form 1^∞. Let

$$L=\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$$

Take the logarithm both the sides, we get

$$\ln L=\ln\left[\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x\right]$$

$$=\lim_{x\to+\infty}\ln\left(1+\frac{1}{x}\right)^x \qquad [\ln x \text{ is continuous}]$$

$$=\lim_{x\to+\infty}x\ln\left(1+\frac{1}{x}\right) \qquad [\text{Property of logarithms}]$$

$$=\lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \qquad \left[\text{Form }\frac{0}{0}\right]$$

$$=\lim_{x\to+\infty}\frac{\frac{1}{1+1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \qquad [\text{L'Hôpital's rule}]$$

$$=\lim_{x\to+\infty}\frac{1}{1+\frac{1}{x}} \qquad [\text{Simplifying}]$$

$$=\frac{1}{1+0}$$

$$=1$$

Thus, $\ln L=1$ and $L=e^1=e$.

***

Example 36: Find $\lim_{x\to0^+}x^{\sin x}$.

Solution: This is a 0^0 indeterminate form. We begin by using properties of logarithm.

Let $L=\lim_{x\to0^+}x^{\sin x}$

$$\ln L = \ln \lim_{x\to 0^+} \ln x^{\sin x}$$

$= \lim_{x\to 0^+} \ln x^{\sin x}$ [ln is continuous]

$= \lim_{x\to 0^+} (\sin x)\ln x$ [This is $0 \cdot \infty$ form.]

$= \lim_{x\to 0^+} \dfrac{\ln x}{\operatorname{cosec} x}$ [This is $\dfrac{\infty}{\infty}$ form.]

$= \lim_{x\to 0^+} \dfrac{1/x}{-\operatorname{cosec} x \cot x}$ [L'Hôpital's rule]

$= \lim_{x\to 0^+} \dfrac{-\sin^2 x}{x\cos x}$

$= \lim_{x\to 0^+} \left(\dfrac{\sin x}{x}\right)\left(\dfrac{-\sin x}{\cos x}\right)$

$= (1)(0) = 0$

Thus, $L = e^0 = 1$.

***

Example 37: Find $\lim_{x\to +\infty} x^{1/x}$.

Solution: This is a limit of the indeterminate form ∞^0.

If $L = \lim_{x\to +\infty} x^{1/x}$, then $\ln L = \ln \lim_{x\to +\infty} x^{1/x} = \lim_{x\to +\infty} \dfrac{1}{x}\ln x$

$= \lim_{x\to +\infty} \dfrac{\ln x}{x}$ [This is $\dfrac{\infty}{\infty}$ form.]

$= \lim_{x\to +\infty} \dfrac{1/x}{1}$ [L'Hôpital's rule]

$= 0$

Thus, we have $\ln L = 0$. Hence, $L = e^0 = 1$.

***

Now, try the following exercise.

E11) Evaluate

i) $\lim_{x\to\infty} x \sin \dfrac{1}{x}$

ii) $\lim_{x\to 0^+} x \ln x$

iii) $\lim_{x\to \pi/4} (1-\tan x)\sec 2x$

E12) Evaluate the following limits

i) $\lim_{x\to \pi/2^-} (\sec x - \tan x)$

ii) $\lim_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

E13) Evaluate the following limits. In each case, you will have to first identify the type of indeterminate form, and then decide upon the procedure.

i) $\lim_{x \to 0^+} (1+x)^{1/x}$

ii) $\lim_{x \to 0^+} (\cos x)^{1/x^2}$

iii) $\lim_{x \to 0^+} x^x$

iv) $\lim_{x \to 0^+} (\tan x)^{\sin 2x}$

v) $\lim_{x \to (\pi/2)^+} (\tan x)^{\sin 2x}$

E14) Show that $\lim_{x \to 0} (1+x)^{1/x} = e$.

That brings us to the end of this unit. Let us summarise all that we have learnt in it.

12.6 SUMMARY

In this unit we have, covered the following points:

1. A limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ where $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are either both 0 or both ∞, such limits are called $\frac{0}{0}$ indeterminate forms and $\frac{\infty}{\infty}$ indeterminate forms respectively.

2. The other indeterminate forms are $0^\infty, \infty - \infty, 0 . \infty,$ etc.

3. A rule to evaluate such indeterminate forms known as L'Hôpital's rule, which relates the evaluation to a computation of $\lim_{x \to a} \frac{f'(x)}{g'(x)}$, if limit exists.

4. We described how to reduce indeterminate forms of the types $\infty - \infty, 1^\infty, 0^\infty$ and 0^0, to the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

12.9 SOLUTIONS/ANSWERS

E1) i) $\frac{\infty}{\infty}$.

ii) $\frac{0}{0}$

iii) $\infty - \infty$

iv) $\frac{0}{0}$

v) $0 \times -\infty$.

E2) i) $\lim_{x\to 0} \frac{x - \sin x}{\sin x} = \lim_{x\to 0} \frac{1-\cos x}{\cos x} = \frac{1-1}{1} = 0$.

ii) $\lim_{x\to 1} \frac{x^4 - 1}{x^2 - 6x + 5}$

$= \lim_{x\to 1} \frac{x^3 + x^2 + x + 1}{x - 5} = \frac{4}{-4} = -1$.

iii) By L'Hôpital's rule

$\lim_{x\to 0} \frac{1-\cos^2 x}{\sin x^2} = \lim_{x\to 0} \frac{2\sin x \cos x}{2x \cos x^2}$

Now, $\lim_{x\to 0} \frac{2\sin x \cos x}{2x \cos x^2} = \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{\cos x}{\cos x^2} = 1$

Therefore, the required limit is 1.

iv) $7/\pi$

v) $\left(\frac{m}{n}\right)\alpha^{m-n}$

vi) 1

vii) -1

viii) $\ln 3/2$

ix) $\sqrt{2}$

x) $1/8$

E3) i) $\lim_{x\to 1^+} \ln x = \ln 1 = 0$ and $\lim_{x\to 1^+} (x-1) = 0$

According to L'Hôpital's rule,

$\lim_{x\to 1^+} \frac{\ln x}{x-1} = \lim_{x\to 1^+} \frac{1/x}{1}$

$= \lim_{x\to 1^+} \frac{1}{x} = 1$

ii) $\lim_{x\to \pi^-} \frac{\sin x}{1-\cos x}$,

Here $\sin x \to 0$ as $x \to \pi^-$ but $(1-\cos x)$ does not approach 0 as $x \to \pi^-$.

$\lim_{x\to \pi^-} \frac{\sin x}{1-\cos x} = \frac{0}{1-(-1)} = 0$.

E4) i) The $\lim_{x\to\infty} \frac{x^{\frac{-4}{3}}}{\sin\left(\frac{1}{x}\right)}$ is in $\frac{0}{0}$ form. Therefore, L'Hôpital's rule can be applied. $\lim_{x\to\infty} \frac{x^{-4/3}}{\sin(1/x)} = \lim_{x\to\infty} \frac{\frac{-4}{3}x^{-7/3}}{\left(-\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right)}$

$$= \lim_{x\to\infty} \frac{\frac{4}{3}x^{-1/3}}{\cos(1/x)}$$

$$= \frac{0}{1} = 0.$$

ii) 1

iii) 1

iv) 1

E5) i) 1

ii) 2

iii) The given limit is of $\frac{0}{0}$ form. Thus, we can apply L'Hôpital's rule.

$$\lim_{x\to 0} \frac{1-\cos x^2}{x^2 \sin^2 x} = \lim_{x\to 0}\left(\frac{1-\cos x^2}{x^4}\,\frac{x^2}{\sin^2 x}\right)$$

$$\lim_{x\to 0} \frac{1-\cos x^2}{x^4} = \lim_{x\to 0} \frac{2x\sin x^2}{4x^3} = \lim_{x\to 0} \frac{1}{2}\left(\frac{\sin x^2}{x^2}\right) = \frac{1}{2}$$

$$\therefore \lim_{x\to 0} \frac{1-\cos x^2}{x^2 \sin^2 x} = \lim_{x\to 0} \frac{1-\cos x^2}{x^4}.\lim_{x\to 0}\left(\frac{x^2}{\sin^2 x}\right) = \frac{1}{2}$$

A direct application of L'Hôpital's rule will also yield the result.

iv) The given limit is of $\frac{0}{0}$ form. Thus, we can apply L'Hôpital's rule.

$$\lim_{x\to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x} = \lim_{x\to 0}\left(\frac{\tan^2 x - x^2}{x^4}.\frac{x^2}{\tan^2 x}\right)$$

$$\lim_{x\to 0} \frac{\tan^2 x - x^2}{x^4} = \lim_{x\to 0} \frac{2\tan x \sec^2 x - 2x}{4x^3}$$

$$= \lim_{x\to 0} \frac{\sec^4 x + 2\sec^2 x \tan^2 x - 1}{6x^2} = \frac{1}{3}$$

For,

$$\lim_{x\to 0} \frac{\sec^4 x - 1}{6x^2} = \lim_{x\to 0} \frac{3\sec^4 x \tan x}{12x}$$

$$= \lim_{x\to 0} \frac{1}{3}\sec^4 x.\lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3},$$

and $\lim_{x\to 0} \frac{2\sec^2 x \tan^2 x}{6x^2} = \frac{1}{3}$

v) $-\frac{9}{2}$

vi) $\frac{1}{4}$

vii) 2.

E6) i) $k = -1,$ and the limit is $\frac{8}{3}$.

ii) $k = -1,$ and the limit is 0.

E7) $\lim_{x\to 0} \frac{x^2 \sin 1/x}{5\tan x} = \lim_{x\to 0} \frac{x \sin \frac{1}{x}}{5\frac{\tan x}{x}}$

$$= \frac{\lim_{x\to 0} x \sin \frac{1}{x}}{5\lim_{x\to 0} \frac{\tan x}{x}}$$

$= \frac{0}{5}$, since $\left|\sin \frac{1}{x}\right| \le 1$, and since $\lim_{x\to 0} \frac{\tan x}{x} = 1$

$= 0$

L'Hôpital's rule cannot be applied as $\lim_{x\to 0} \frac{2x\sin\frac{1}{x} - \cos\frac{1}{x}}{5\sec^2 x}$ does not exist,

because $\lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$ does not exist.

E8) i) $\lim_{x\to\infty} \frac{a_m x^m + \cdots + a_0}{e^x} = \lim_{x\to\infty} \frac{m!\, a_m}{e^x} = 0$.

ii) ∞

iii) By L'Hôpital's rule

$$\lim_{x\to\pi/2} \frac{\tan 3x}{\tan x} = \lim_{x\to\pi/2} \frac{3\sec^2 3x}{\sec^2 x}$$

which is again $\frac{\infty}{\infty}$ form. It can be handled more easily by

converting it into $\frac{0}{0}$ form.

$$\lim_{x\to\pi/2} \frac{3\sec^2 3x}{\sec^2 x} = \lim_{x\to\pi/2} \frac{3\cos^2 x}{\cos^2 3x}$$

$$= \lim_{x\to\pi/2} \frac{-6\cos x \sin x}{-6\cos 3x \sin 3x}$$

$$= \lim_{x\to\pi/2} \frac{\sin 2x}{\sin 6x}$$

$$= \lim_{x\to\pi/2} \frac{2\cos 2x}{6\cos 6x} = \frac{1}{3}$$

iv) $\frac{1}{2}$

v) $\frac{2}{3}$

E9) $\lim\limits_{x\to\infty}\frac{x+\sin x}{x+\cos x}=\lim\limits_{x\to\infty}\frac{1+\frac{\sin x}{x}}{1+\frac{\cos x}{x}}=\frac{1}{1}=1$.

Since, $\sin x$ and $\cos x$ are bounded functions, and $\frac{1}{x}\to 0$ as $x\to\infty$.

You may note that the given limit is an $\frac{\infty}{\infty}$ form. Therefore, L'Hôpital's rule can be applied. But, when we take derivatives after applying L' Hôpital's rule, we get $\frac{1+\cos x}{1-\sin x}$ and that does not have a limit.

E10) i) $\lim\limits_{x\to\infty}\frac{x^2-\sin x^2}{x^2}=\lim\limits_{x\to\infty}\frac{1-\frac{\sin x^2}{x^2}}{1}$

$=1$, since $\sin x^2$ is bounded, and $\frac{1}{x^2}\to 0$ as $x\to\infty$

ii) 1. L'Hôpital's rule cannot be applied by the argument similar to the one in i).

iii) 0. L'Hôpital's rule is not applicable as $\frac{f(x)}{g(x)}$ is not in an indeterminate form as $x\to\infty$.

iv) $\lim\limits_{x\to\infty}\frac{\sin x}{x}+\frac{\cos x}{x^2}=0$.

L'Hôpital's rule is not applicable since $\lim\limits_{x\to\infty}(x\sin x+\cos x)$ does not exist.

E11) i) This is $0\cdot\infty$ Form is

Therefore, $\lim\limits_{x\to\infty}x\sin\frac{1}{x}=\lim\limits_{x\to\infty}\frac{\sin 1/x}{1/x}$

$=\lim\limits_{x\to\infty}\cos\frac{1}{x}=1$

ii) The factor x has a limit of 0 and the factor $\ln x$ has a limit of $-\infty$, so the stated problem is an indeterminate form of type $0\cdot\infty$. There are two possible approaches: we can rewrite the limit as $\lim\limits_{x\to 0^+}\frac{\ln x}{1/x}$ or $\lim\limits_{x\to 0^+}\frac{x}{1/\ln x}$ the first being an indeterminate form of type $\frac{\infty}{\infty}$ and the second an indeterminate form of type $0/0$. However, the first form is the preferred initial choice because the derivative of $1/x$ is less complicated than the derivative of $1/\ln x$. That choice yields

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$$

iii) The stated problem is an indeterminate form of type $0\cdot\infty$. We will convert it to an indeterminate form of type $0/0$:

$$\lim_{x\to\pi/4}(1-\tan x)\sec 2x = \lim_{x\to\pi/4}\frac{1-\tan x}{1/\sec 2x} = \lim_{x\to\pi/4}\frac{1-\tan x}{\cos 2x}$$

$$= \lim_{x\to\pi/4}\frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1$$

E12) i) $\lim\limits_{x\to(\pi/2)^-}(\sec x - \tan x)$

$$= \lim_{x\to(\pi/2)^-}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to(\pi/2)^-}\left(\frac{1-\sin x}{\cos x}\right)$$

$$= \lim_{x\to(\pi/2)^-}\left(\frac{-\cos x}{-\sin x}\right)$$

$= 0$

The use of L'Hôpital's rule is justified because $1-\sin x \to 0$ and $\cos x \to 0$ as $x \to (\pi/2)^-$.

ii) As it stands, this has the form $\infty - \infty$, because $\frac{1}{x} \to +\infty$ and $\frac{1}{\sin x} \to +\infty$ as $x \to 0$ from the right. However, using a little algebra, we find

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+}\frac{\sin x - x}{x\sin x}$$

This limit is now of the form $0/0$, so the hypothesis of L'Hôpital's rule are satisfied. Thus,

$$\lim_{x\to 0^+}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0^+}\frac{\cos x - 1}{\sin x + x\cos x}$$

Again, the form $\frac{0}{0}$, therefore

$$\lim_{x\to 0^+}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0^+}\frac{-\sin x}{\cos x + x(-\sin x) + \cos x}$$

$$= \frac{0}{2} = 0.$$

E13) i) 1^∞. If $y = (1+x)^{1/x}$, then $\ln y = \frac{\ln(1+x)}{x}$ is in the $\frac{0}{0}$ form

$$\therefore \lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+x)}{x} = \lim_{x\to 0^+}\frac{1}{1+x} = 1.$$

$\therefore \lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^+} y = e^1 = e$.

ii) 1^∞. If $y = \cos x^{1/x^2}$, $\ln y = \dfrac{\ln\cos x}{x^2}$, which is in the $\dfrac{0}{0}$ form as $x \to 0$.

$$\therefore \lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln\cos x}{x^2} = \lim_{x\to 0}\frac{-\tan x}{2x} = -\frac{1}{2}$$

$$\therefore \lim_{x\to 0}(\cos x)^{1/x^2} = e^{-1/2} = \frac{1}{\sqrt{e}}.$$

iii) 0^0 form, If $y = x^x$, then $\ln y = x\ln x$, which is $0\cdot -\infty$ form as $x \to 0^+$.

$$\therefore \lim_{x\to 0^+}\ln y = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x}$$

$$= \lim_{x\to 0^+}\frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} -x = 0.$$

$$\therefore \lim_{x\to 0^+} x^x = e^0 = 1.$$

iv) $0^0. \lim_{x\to 0^+}(\tan x)^{\sin 2x} = 1$.

v) ∞^0. If $y = (\tan x)^{\sin 2x}$, $\ln y = \sin 2x \ln\tan x$

$$\therefore \lim_{x\to \pi/2^-}\ln y = \lim_{x\to \pi/2^-}\frac{\ln\tan x}{\operatorname{cosec} 2x}$$

$$= \lim_{x\to \pi/2^-}\frac{\frac{1}{\tan x}\sec^2 x}{2\operatorname{cosec} 2x\cot 2x}$$

$$= \lim_{x\to \pi/2^-}\frac{\frac{2}{\sin 2x}}{\frac{-2\cos 2x}{\sin^2 2x}}$$

$$= \lim_{x\to \pi/2^-}\frac{-\sin 2x}{\cos 2x} = 0$$

$$\therefore \lim_{x\to \pi/2^-}(\tan x)^{\sin 2x} = e^0 = 1.$$

E14) We begin by introducing a dependent variable $y = (1+x)^{1/x}$ and taking the natural logarithm of both sides:

$$\ln y = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$$

Thus, $\lim_{x\to 0}\ln y = \lim_{x\to 0}\dfrac{\ln(1+x)}{x}$ which is an indeterminate form of type $\dfrac{0}{0}$, so L'Hôpital's rule $\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1/(1+x)}{1} = 1$. Since we have shown that $\ln y \to 1$ as $x \to 0$, the continuity of the exponential function implies that $e^{\ln y} \to e^1$ as $x \to 0$, and this implies that $y \to e$ as $x \to 0$. Thus, $\lim_{x\to 0}(1+x)^{1/x} = e$.

UNIT 13

THE UPS AND DOWNS

Structure **Page No.**

13.1 INTRODUCTION

One of the principal goals of calculus is to investigate the behaviour of various functions. As a part of the investigation, we will be laying the groundwork for solving a large class of problems that involve finding the maximum or minimum value of a function, if one exists. Such problems are called **optimisation problems**. For instance, honeycombs have hexagonal cells because this shape enables bees to store a fixed amount of honey by using the minimum of wax for sealing, the drops of oil on the surface of water coalesce so as to minimise the total surface tension, a drop of water is spherical due to minimum surface tension.

In Secs. 13.2 and 13.3 we shall discuss an important technique involved in solving the problem of maximising or minimising various functions. This technique, as you will soon see, involves the use of derivatives. In Sec. 13.4, we apply the derivatives to find if a function is increasing, or decreasing, or neither in a given interval.

In Secs. 13.5 and 13.6, we will see other applications of derivatives using the first derivative test and second derivative test respectively. In Sec. 13.7, we shall discuss Rolle's theorem and Lagrange's mean value theorem, which have a very important role in your study of calculus.

Now we shall list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

Objectives

After going through this unit, you should be able to:

- obtain the relative and absolute maximum and minimum values of some functions, if they exist;
- apply maxima and minima in real life situations;
- state and apply Rolle's theorem and Lagrange's mean value theorem;
- find whether a function is monotonic or not using its derivatives.

13.2 RELATIVE EXTREMA

Maxima and minima are the respective plurals of maximum and minimum.

In this section, we will discuss methods for finding the high and the low points on the graph of a function. If we imagine the graph of a function f to be a two-dimensional mountain range with hills and valleys, then the tops of the hills are called **relative maxima**, and the bottoms of the valleys are called **relative minima**. Relative maxima and relative minima are the high and low points respectively in their immediate vicinity as shown in Fig. 1. Maxima and minima are collectively known as **extrema**, which is the plural of **extremum**.

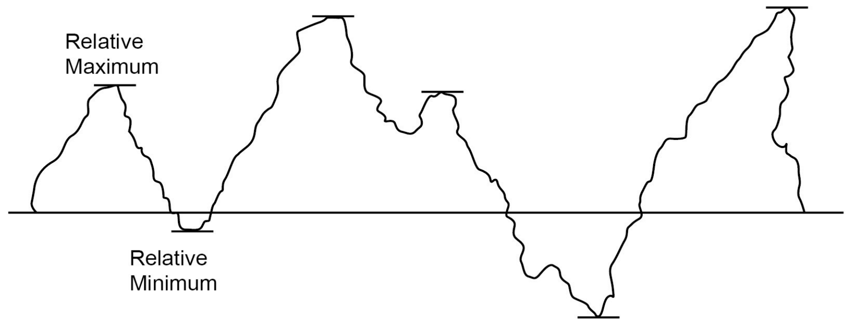

Fig. 1: Hills and Valleys.

The extrema of a continous function occur either at endpoints of the interval or at points where the graph has a "peak" or a "valley" (point where the graph is higher or lower than all nearby points). For example, the function f in Fig. 2 (a) and Fig. 2 (b) has "peaks" at Q, R and S and "valleys" at P and T. Such peaks and valleys are what we call **relative extrema**.

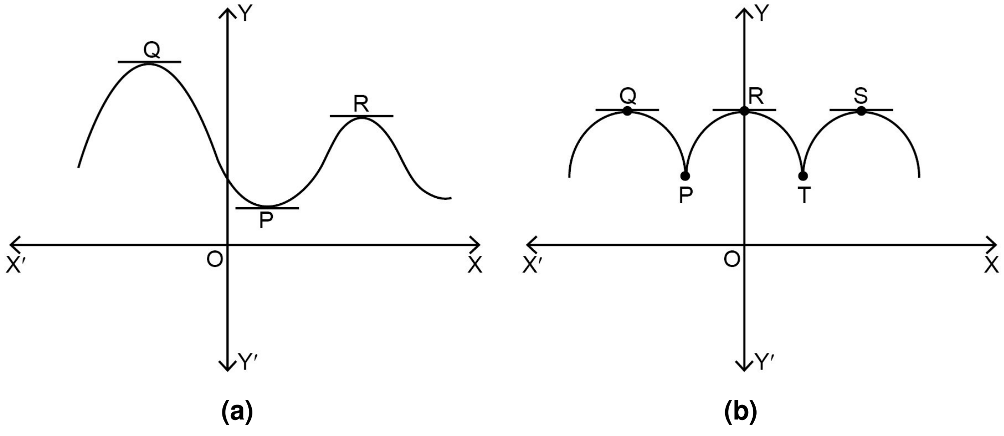

Fig. 2: Peaks and valleys shown in graphs of functions.

So, let us formally define the terms we have been using in the following definitions:

Definition: A function f is said to have a **relative maximum** at a point x_0 if $f(x_0) \geq f(x)$ for all x in an open interval containing x_0. Similarly, f is said to have a **relative minimum** at x_0 if $f(x_0) \leq f(x)$ for all x in an open interval containing x_0. Relative maximum and relative minimum are called **relative extremum**.

Next, we will formulate a procedure for finding relative extrema. By looking at Fig. 2 (a), we see that there are horizontal tangents at P, Q and R, which means that the derivative of the function is zero at P, Q and R, while in Fig. 2 (b), the derivative does not exist at the point P. This suggests that the relative extrema of f occur either where the derivative is zero or where the derivative does not exist. The point at which $f' = 0$ or f' does not exist, has a special name, critical point, which we define in the following definition:

Definition: If f is defined at x_0 and either $f'(x_0) = 0$ or $f'(x_0)$ does not exist, then the number x_0 is called **a critical number** of f, and the point $P(x_0, f(x_0))$ on the graph of f is called **a critical point**.

You may note that if $f(x_0)$ is not defined, then x_0 cannot be a critical number.

Now, let us understand this in the following example.

Example 1: Find the critical points for the function f, defined by $f(x) = 5x^3 - 24x^2 - 21x + 23$.

Solution: Let $f(x) = 5x^3 - 24x^2 - 21x + 23$, then $f'(x) = 15x^2 - 48x - 21$ is defined for all values of x.

Now, $f'(x) = 0$ gives $15x^2 - 48x - 21 = 0$

$\Rightarrow 3(5x - 7)(3x + 1) = 0$

On solving it, we get $x = \frac{7}{5}, -\frac{1}{3}$, which are the critical numbers.

Accordingly, $f\left(\frac{7}{5}\right) = -\frac{1003}{25}$ and $f\left(-\frac{1}{3}\right) = \frac{733}{27}$

Therefore, the critical points are $\left(\frac{7}{5}, \frac{-1003}{25}\right)$ and $\left(-\frac{1}{3}, \frac{733}{27}\right)$.

***

Example 2: Find the critical points of the function f defined by $f(x) = \frac{e^{2x}}{x - 5}$.

Solution: Note that f is defined for $\mathbb{R}$ except at $x = 5$.

Here, $f'(x) = \frac{2(x-5)e^{2x} - e^{2x}(1)}{(x-5)^2} = \frac{e^{2x}(2x - 11)}{(x-5)^2}$ [Note that $x \neq 5$.]

The derivative is not defined at $x = 5$ and f is not defined at $x = 5$ either, so $x = 5$ is not a critical number.

When $f'(x) = 0$, it gives $\frac{e^{2x}(2x-11)}{(x-5)^2} = 0$, which gives $x = \frac{11}{2}$, which is the only critical number because $e^{2x} > 0$ and cannot be zero. Therefore, the critical point is $\left(\frac{11}{2}, f\left(\frac{11}{2}\right)\right)$, that is $\left(\frac{11}{2}, 2e^{11}\right)$.

***

Example 3: Find the critical numbers and the critical points for the function f defined by $f(x) = (x-2)^2 (x+1)$. Also, show these points on the graph of f.

Solution: Given function f is a polynomial function and we know that it is continuous and its derivative exists for all x. Thus, we find the critical numbers by using the equation $f'(x) = 0$. We obtain

$$\begin{aligned} f'(x) &= (x-2)^2 (1) + 2(x-2)(1)(x+1) \\ &= (x-2)[(x-2) + 2(x+1)] \\ &= 3x(x-2) \end{aligned}$$

The critical numbers are $x = 0, 2$. To find the critical points, we need to find the y-coordinate for each critical number.

$$f(0) = (0-2)^2 (0+1) = 4$$

$$f(2) = (2-2)^2 (2+1) = 0$$

Thus, the critical points are $(0, 4)$ and $(2, 0)$. The graph of $f(x) = (x-2)^2 (x+1)$ is shown in Fig. 3, in which P and Q are the critical points.

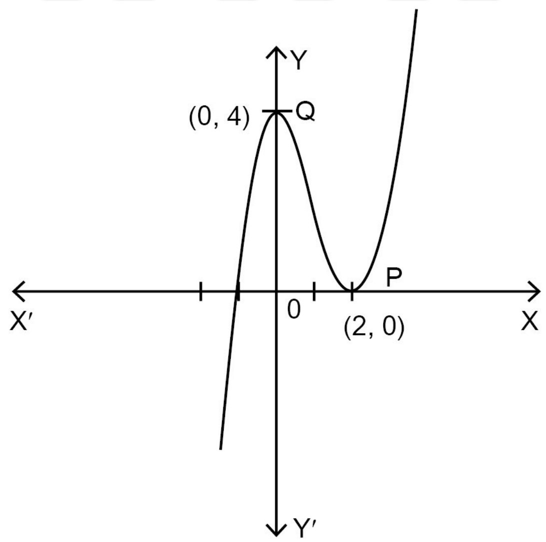

Fig. 3: Graph of f.

You may observe how the relative extrema occur at the critical points. You can see in Fig. 3 that the relative extrema occur only at points on a graph where there is a horizontal tangent line.

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In the examples above, you have found the critical points. In the following theorem, you will see how critical numbers are used to find relative extrema.

Theorem 1(Critical number theorem): If a continuous function f has a relative extremum at a point x_0 in its domain, then x_0 must be a critical number of f.

We are not giving the proof of the theorem here, but let us look at an application.

Now, Theorem 1 states that a relative extremum of a continuous function f can occur only at a critical number, but it does not say that a relative extremum must occur at each critical number.

For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$, so 0 is a critical number. But there is no relative extremum at $x_0 = 0$ on the graph of f because the graph is rising for $x < 0$ and also for $x > 0$, as shown in Fig. 4. Thus, the graph of $f(x) = x^3$ has no relative extremum at $x_0 = 0$ even though $f'(0) = 0$.

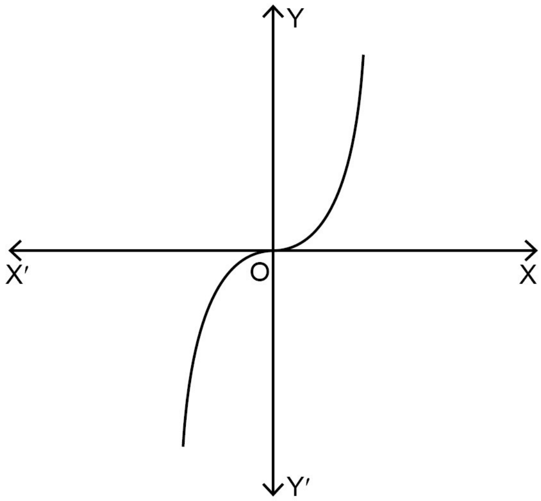

Fig. 4: Graph of $y = x^3$.

Similarly, it is also quite possible for a continuous function g to have no relative extremum at a point x_0 where $g'(x_0)$ does not exist. Fig. 5 shows the graph of a function, for which tangent at the point P does not exist. Therefore, although $g'(-1)$ does not exist, no relative extremum occurs at $x_0 = -1$.

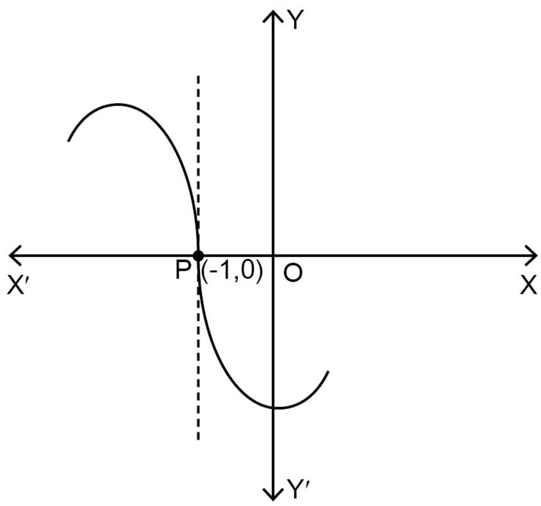

Fig. 5

So, we can say that the critical number theorem is only a necessary condition. Let us find relative extrema in the following example:

Example 4: Find the relative maxima or minima, if any, for the function f defined by $f(x) = |x-1|$ on the interval $]-2, 2[$.

Solution: Fig. 6 shows the graph of f. If $x > 1$, then $f(x) = x - 1$ and $f'(x) = 1$. However, if $x < 1$, then $f(x) = -(x-1)$ and $f'(x) = -1$. Therefore, neither $f'(x) = 0$ nor $f'(x)$ does not exist on $]-2,2[$ if $x < 1$ and $x > 1$. Now, we need to check what happens at $x = 1$.

$$f'(1) = \lim_{\Delta x \to 0} \frac{f(1+\Delta x - 1) - f(1-1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{|\Delta x| - |0|}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$$

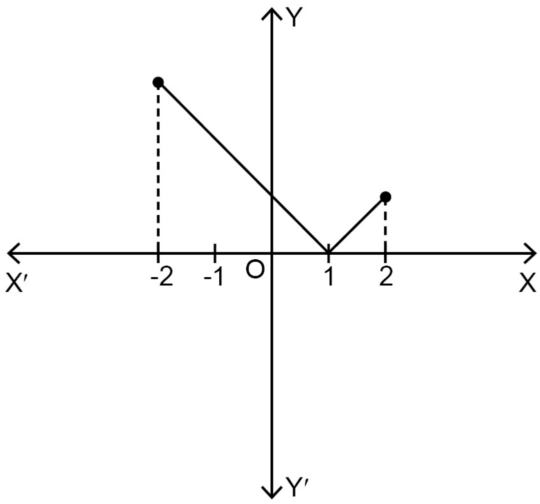

Fig. 6: Graph of f.

We consider the left hand limit and right hand limit:

$$\lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1$$

and

$$\lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$$

Since these limits are not equal, therefore, the derivative of f does not exist at $x = 1$. Since, $f(1)$ is defined and $f(1) \le f(x) \, \forall \, x \in]-2,2[$, therefore, 1 is the only critical number, at which f has relative minima. Fig. 6 also verifies this.

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Now, try to solve the following exercise:

E1) Find the relative maxima and minima for f, where f is defined as follows:

i) $f(x) = 2$ for all $x \in \mathbb{R}$ ii) $f(x) = x$ for all $x \in \mathbb{R}$

iii) $f(x) = x$ for $0 < x < 4$ iv) $f(x) = x^2$ for all $x \in \mathbb{R}$

v) $f(x) = \sqrt{x}$ for all $x \in [9, 25]$. vi) $f(x) = |x - 3|$ on $[-4, 4]$

vii) $f(x) = \dfrac{\ln \sqrt{x}}{x}$ on $[1, 3]$ viii) $f(x) = x\,e^{-x}$ on $[0, 2]$

ix) $f(x) = \sin^2 x + \cos x$ on $\left]0, \frac{\pi}{2}\right[$ x) $f(x) = |x|$ on $[-1,1]$

So far, we were concerned with the values of the function only in an open interval around the extreme point. Thus, the concept of relative maxima-minima is essentially a local phenomenon. What happens globally, or elsewhere? Let us discuss that now in the following section:

13.3 ABSOLUTE EXTREMA

So far, we have discussed relative extrema. In Fig. 1, there exist the highest peaks as well as deepest valley. If we talk about finding the maximum and minimum values of a function, we find the highest point and the lowest point. These highest and lowest values of the function are absolute maximum and absolute minimum, respectively, and are given in the following definition:

Definition: If f is a function defined on an interval I that contains the number x_0, then $f(x_0)$ is an **absolute maximum** of f on the interval I if $f(x_0) \geq f(x)$ for all x in I. Similarly $f(x_0)$ is an **absolute minimum** of f on I if $f(x_0) \leq f(x)$ for all x in I.

Sometimes we drop the word 'absolute' and just use the terms maximum and/or minimum. Together, the absolute maximum and minimum of f on the interval I are called the **extreme values**, or the **absolute extrema**, of f on I. An absolute maximum or absolute minimum is sometimes called **global maximum or global minimum.** A function does not necessarily have extreme values on a given interval. For instance, the continuous function $f(x) = x$ has neither a maximum nor a minimum on the open interval $]-1,1[$, as shown in Fig. 7. This is because there exists no x_0 for which either $f(x_0) \geq f(x)$ or $f(x_0) \leq f(x)$ for all $x \in]-1,1[$.

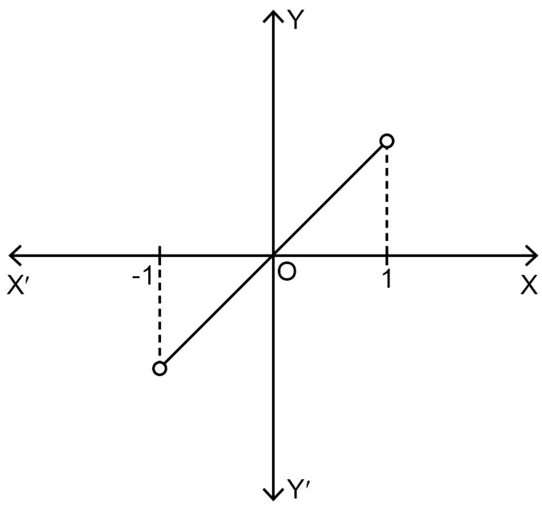

Fig. 7: Graph of $f(x) = x$ on $]-1,1[$.

The function f defined by $f(x) = \begin{cases} -x^2 & \text{for } x \neq 0 \\ -25 & \text{for } x = 0 \end{cases}$, which is discontinuous at $x = 0$, has a minimum in the closed interval $[-5,5]$, but no maximum, as shown in Fig. 8.

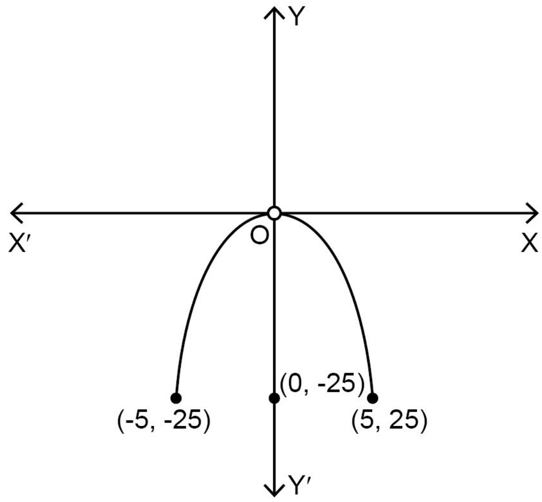

Fig. 8

This graph also illustrates the fact that a function may assume an absolute extremum at more than one point. In this case, the minimum occurs at the points $(-5,-25),(0,-25),$ and $(5,-25)$. If a function f is continuous and the interval I is closed and bounded, it can be shown that both an absolute maximum and an absolute minimum must occur. This result plays an important role in finding maxima and minima and is called **the extreme value theorem**, and is given as follows:

Theorem 2 (The extreme value theorem): A continuous function f has both an absolute maximum and an absolute minimum on any closed interval $[a,b]$. We are not proving this theorem, but giving a geometrical interpretation, which is shown in Fig. 9. If a continuous function f has no peaks it would be increasing throughout or decreasing throughout its domain. In this case, the maximum and minimum occur at the endpoints of $[a,b]$. If it does have peaks, the maximum would correspond to the highest peak or to an endpoint.

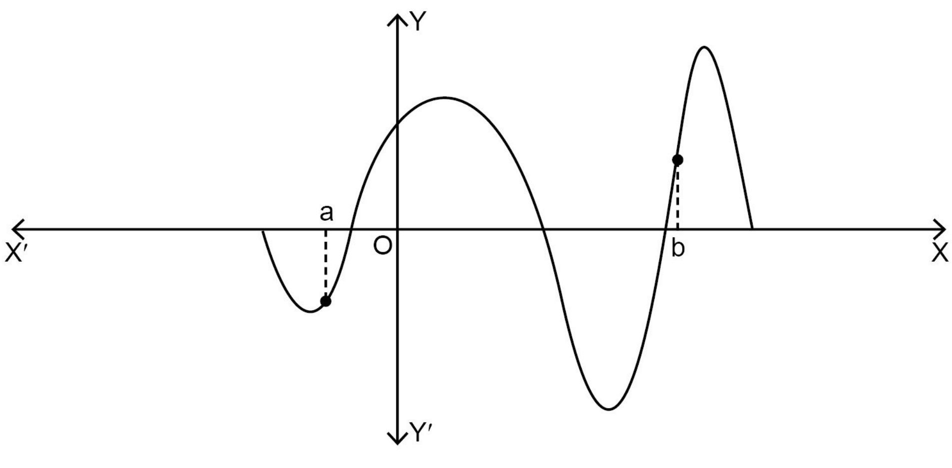

Fig. 9: Peaks and depths of a continuous function in a closed interval.

If f is not continuous in the closed interval or the interval is not closed, you cannot conclude that f has both a largest and smallest value. Sometimes, there are extreme values even when the conditions of the theorem are not satisfied, but if the conditions hold, the extreme values exist.

You may note that the maximum of a function occurs at the highest point on its graph and the minimum occurs at the lowest point. For example, consider the function f whose graph is shown in Fig. 10.

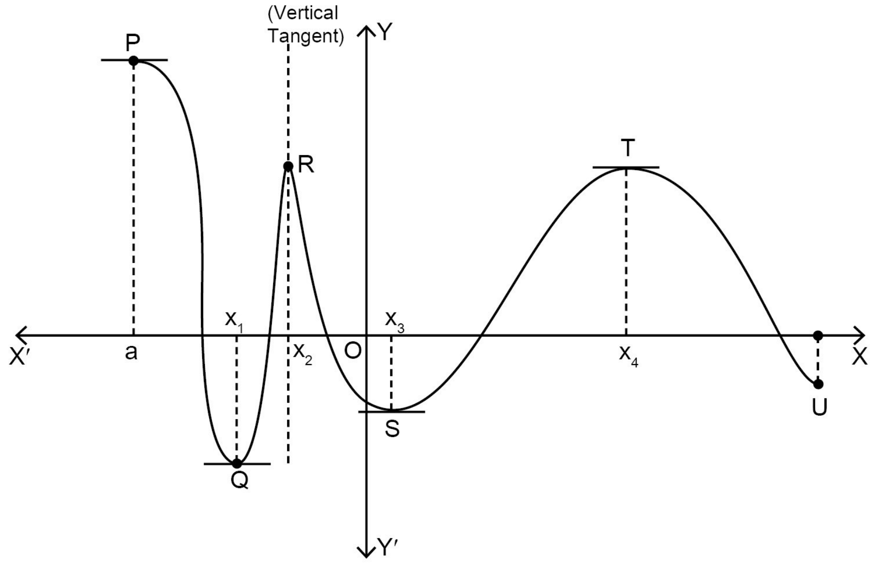

Fig. 10: A continuous function on $[a,b]$.

The highest point on the graph occurs at the left endpoint that is at P, and the lowest point at Q. Thus, the absolute maximum is $f(a)$, and the absolute minimum is $f(x_1)$. Here, the existence of maxima and minima as required by the extreme value theorem can be seen clearly , but there are times when it seems that the extreme value theorem fails. Let us illustrate another instance, where you will see that any of the conditions of the extreme value theorem is not satisfied.

Example 5: Verify the extreme value theorem for the function f defined by $f(x)=\begin{cases} 3x, & \text{if } 0 \le x < 2 \\ 2, & \text{if } 2 \le x \le 4 \end{cases}$

Solution: From the graph of the function f shown in Fig. 11, you can see that the function f has no maximum. It takes on all values less than, but arbitrarily close to 6. However, it never reaches the value 6. This function does not contradict the extreme value theorem because f is not continuous on $[0,4]$.

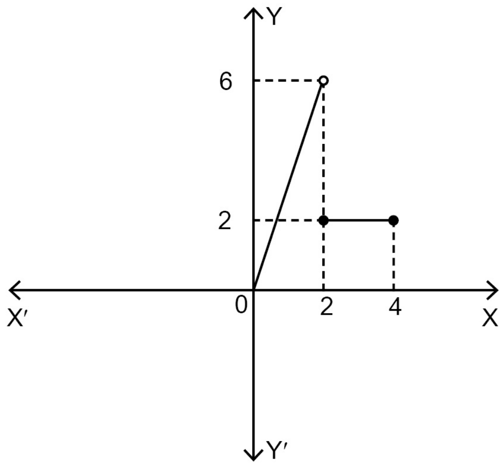

Fig. 11: Graph of f.

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Example 6: Verify the extreme value theorem for the function f, defined by $f(x)=x^2$ on $0 < x \le 5$.

Solution: The graph of f given in Fig. 12 shows that the values of the function f become arbitrarily small as x approaches 0. $f(x)$ never reaches the value 0. So, f has no minimum. The function f is continuous on the interval $]0,5]$, but the extreme value theorem is not contradicted as the interval is closed only at one end.

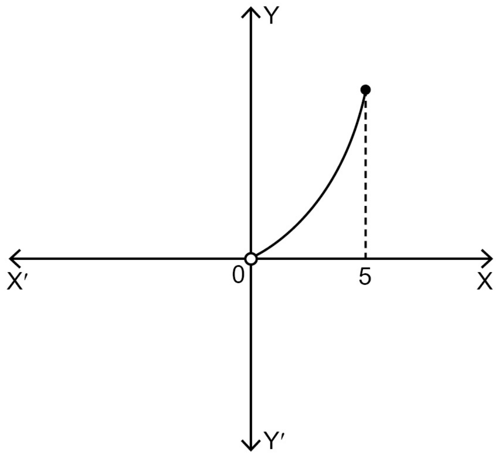

Fig. 12: Graph of x^2 on $]0,5]$.

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Now, we are giving a procedure to find the absolute extrema of a continuous function f on $[a,b]$ in the following steps:

- Find all the critical numbers of f on $[a,b]$ using $f'(x)=0$ or $f'(x)$ does not exist.
- Evaluate f at the critical numbers as well as at the end points of the interval.
- Of these values, the largest value of f is the absolute maximum of f on $[a,b]$, and the smallest value of f is the absolute minimum of f on $[a,b]$.

Let us understand this with the help of the following examples.

Example 7: Find the absolute extrema of the function f defined by $f(x)=x^4-8x^2+5$ on $[-3,3]$. Show these values on the graph of f.

Solution: Here, f is a polynomial function, it is continuous on the closed interval $[-3,3]$. Theorem 2 states that there must be an absolute maximum and an absolute minimum on the given closed interval.
For this, we differentiate it and find the critical numbers.

$$f'(x)=4x^3-8(2x)=4x(x^2-4)=4x(x-2)(x+2)$$

Thus, the critical numbers are $x=0,2$ and -2.
Now, let us find the values of the function at end points of the interval and at the critical numbers. We get $f(3)=14$, $f(-3)=14$, $f(0)=5$, $f(2)=-11$, and $f(-2)=-11$.

We can say that the absolute maximum of f occurs at $x=3$ and $x=-3$. The absolute maximum value of f is $f(3)=f(-3)=14$. The absolute minimum of f occurs at $x=2$ and $x=-2$. Thus, the absolute minimum value of f is $f(2)=f(-2)=-11$. The graph of f is shown in Fig. 13, which verifies that there are four absolute extrema.

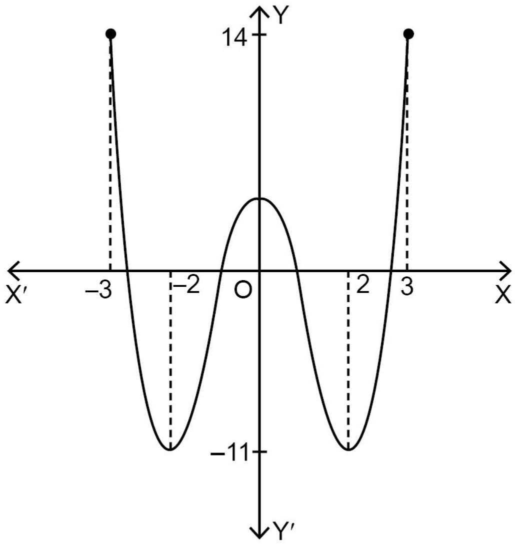

Fig. 13: Graph of f.

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In the following example, we shall find the absolute extrema when the derivative does not exist.

Example 8: Find the absolute extrema of the function f defined by $f(x)=2x^{2/3}(5-3x)$ on the interval $[-1,1]$. Show these values on the graph of f.

Solution: To find the derivative, we rewrite the given function as

$f(x)=10x^{2/3}-6x^{5/3}$, then $f'(x)=\frac{20}{3}x^{-1/3}-10x^{2/3}=\frac{10}{3}x^{-1/3}(2-3x)$.

We find critical numbers by solving $f'(x)=0$. This gives $x=2/3$.

Here, $f(0)$ exists, you may note that $f'(0)$ does not exist. Thus, the critical numbers are $x=0$ and $x=2/3$.

Let us now find the values of the function at the endpoints and at the critical numbers, we get $f(-1)=16$, $f(1)=4$, $f(0)=0$, $f(2/3)=2^{5/3}\cdot 3^{1/3}\approx 4.579$.

It is clear that the absolute maximum of f occurs at $x=-1$ and the maximum value of f is $f(-1)=16$. The absolute minimum of f occurs at $x=0$ and the minimum value of f is $f(0)=0$. The graph of f, shown in Fig. 14, verifies these values.

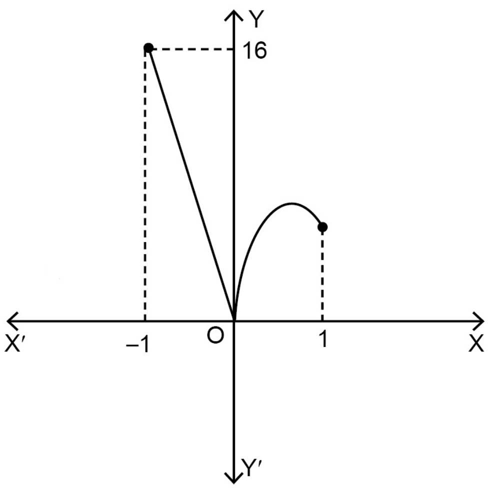

Fig. 14: Graph of f.

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In the next example, we shall find the absolute extrema of a trigonometric function.

Example 9: Find the absolute extrema of the function f defined by $f(x) = -x - 2\cos x - \frac{1}{2}(\cos^2 x + \sin x)$ on the interval $[0, \pi]$.

Solution: To find the critical numbers, we find $f'(x)$.

$$\begin{aligned} f'(x) &= -1 + 2\sin x + \frac{1}{2}(2\cos x \sin x - \cos x) \\ &= \frac{1}{2}(-2 + 4\sin x + 2\cos x \sin x - \cos x) \\ &= \frac{1}{2}[2(\sin x)(\cos x + 2) - (\cos x + 2)] \\ &= \frac{1}{2}[(\cos x + 2)(2\sin x - 1)] \end{aligned}$$

Since, the factor $(\cos x + 2)$ is never zero on $[0, \pi]$, therefore, $f'(x) = 0$ when $2\sin x - 1 = 0$, which gives $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ in $[0, \pi]$.

Let us evaluate the function at the endpoints and the critical numbers. We get $f(0) = \frac{-5}{2}$, $f(\pi) = -\pi + \frac{3}{2} \approx -1.641$, $f(\pi/6) \approx -2.881$ and $f\left(\frac{5\pi}{6}\right) \approx -1.511.$

The absolute maximum of f is at $x = \frac{5\pi}{6}$ and the absolute minimum of f is at $x = \frac{\pi}{6}$. The maximum and minimum values of f are -1.511 and -2.881, respectively.

***

Now, try the following exercises.

E2) Find the critical numbers of the following:

i) $f(x) = 2\sqrt{x}\,(6 - x)$ on $\mathbb{R}$.

ii) $f(x) = x^3$ on $\left[\frac{-1}{2}, 1\right]$

iii) $f(x) = \frac{\ln\sqrt{x}}{x}$ on $[1, 3]$

iv) $f(x) = \sin^2 x + \cos x$ on $\left[0, \frac{\pi}{2}\right]$

E3) Find the critical points, if they exist, for the functions given in E2).

E4) Find the absolute extrema of each of the functions f defined as follows:

i) $f(x) = (x - 5)(x - 3)\ \forall\, x \in [-4, 4]$

ii) $f(x) = x^3 + 13x^2 + 5x + 7\ \forall\, x \in [-10, 10]$

iii) $f(x) = \sin x + 3\ \forall\, x \in \mathbb{R}$

iv) $f(x) = 2|x|\ \forall\, x \in [-1, 1]$

v) $f(x) = |x| + 2 \forall x \in [-2, 2]$

vi) $f(x) = |x| + |x-1| \forall x \in [-5, 5]$

vii) $f(x) = x + \frac{1}{x} \forall x > 0$

In our next few examples, we shall apply some maximisation or minimisation in real life situtaitons.

Example 10: A rectangular playground is to be fenced. What is the maximum area for this playground if it is to be fitted into a right-triangular plot having perpendicular sides 8m and 15m?

Solution: Fig. 15 shows a right triangular plot with perpendicular sides 8m and 15m. Let x and y denote the length and width of the inscribed rectangular playground. The area of the rectangle is, $A = xy$.

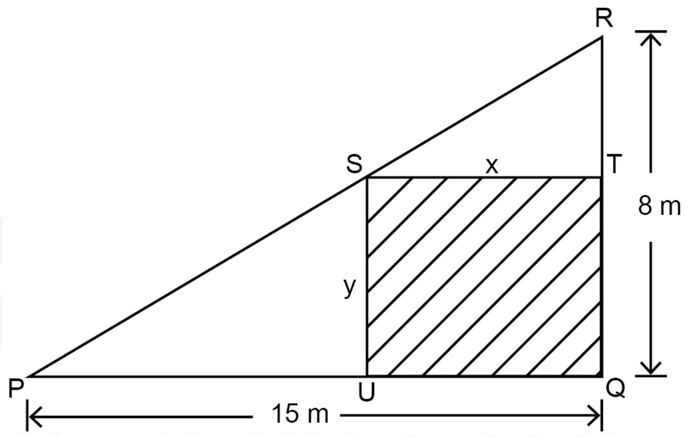


Fig. 15: Area of a rectangle.

The function A is of two variables and the method we discussed here deals with the functions of one variable. Therefore, first we must express A as a function of a single variable. To do this, we need to substitute either x in terms of y or y in terms of x with the given conditions of the right-triangular plot. Since, ΔPQR is similar to ΔSTR, therefore, the corresponding sides of these triangles are proportional. Thus, we have $\frac{8-y}{8} = \frac{x}{15}$, which gives

$$y = 8 - \frac{8x}{15}$$

Therefore, area A can be rewritten as $A(x) = x\left(8 - \frac{8}{15}x\right) = 8x - \frac{8}{15}x^2$

The domain of the function A is $0 \le x \le 15$. The critical numbers for A are the values of x such that $A'(x) = 0$ (since $A'(x)$ exists for all x). Since, $A'(x) = 8 - \frac{16}{15}x$, therefore, the only critical number is $x = \frac{15}{2}$. Now, let us evaluate $A(x)$ at the endpoints and the critical number, we get

$$A(15) = 0,\ A(0) = 0,\ A\left(\frac{15}{2}\right) = 30$$

The area is maximum when $x = \frac{15}{2}$m,. This gives $y = 8 - \frac{8}{15} \times \frac{15}{2} = 4$m.

Thus, the largest rectangular playground that can be built in the triangular plot is a rectangle with dimensions $\frac{15}{2}\text{m}$ and 4m along the sides 15m and 8m respectively. The maximum area of the rectangular playground is $30\,\text{m}^2$.

***

Example 11: An open-topped cuboidal box is to be made out of a square sheet of tin, with each side a cm. For this, squares out of each corner of the sheet are to be cut and then the edges of the sheet are bent upward to form the sides of the box. What should be the height of the box, so that the volume of the box is maximum?

Solution: Fig. 16 shows the square sheet and the corners cut from it. Let the sides of the squares to be cut on corners be x cm. The box will be x cm deep, $(a-2x)$ cm long, and $(a-2x)$ cm wide. The volume of the box $V(x) = x(a-2x)(a-2x)\,\text{cm}^3$ and this is the quantity to be maximized. To find the domain, we note that the dimensions must all be non-negative; therefore, $x \geq 0, a-2x \geq 0$ (or $x \leq a/2$). This implies that the domain of the function V is $\left[0, \frac{a}{2}\right]$.

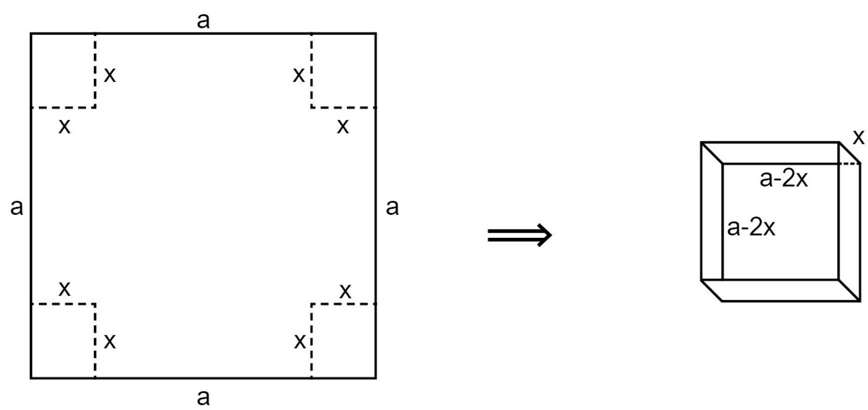


Fig. 16: Square of side a.

To find the critical numbers (the derivative is defined everywhere in the domain), we find values for which the derivative is 0.

Here, $V'(x) = (a-2x)(a-2x-4x) = 0 \Rightarrow x = a/2, a/6$. Evaluating $V(x)$ at the critical numbers and the endpoints, we get

$$V(0) = 0, V(a/2) = 0, V\left(\frac{a}{6}\right) = \frac{2a^3}{27}.$$

Thus, the volume of the box is maximum when $x = \frac{a}{6}$. Such a box has dimensions $\frac{2a}{3}, \frac{2a}{3}, \frac{a}{6}$. Hence, the height of the box will be $\frac{a}{6}$ cm, so that the volume of the box is maximum.

***

You will note in Example 11, that we did not have to test whether the critical point was a maximum or a minimum, but knew that it was a maximum because the continuous function V is non-negative on the interval $[0, a/2]$ and is zero at the endpoints. Since, there is only one critical number $x = \frac{a}{6}$ in between, it must have the maximum. Similar reasoning can be used in many examples.

In the following example, we will see how we can maximize the illumination by adjusting the height of a lamp just above the center of a circular table.

Example 12: A lamp with adjustable height hangs directly above the center of a circular table of radius 1m. The illumination I at the edge of the table to be directly proportional to the cosine of the angle θ and inversely proportional to the square of the distance d, where θ and d are as shown in Fig. 17. How close to the table should the lamp be lowered to maximise the illumination at the edge of the table?

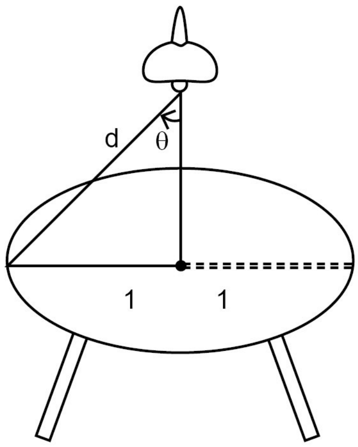


Fig. 17

Solution: According to the question, illumination $I \propto \cos\theta$ and $I \propto \frac{1}{d^2}$, so we have $I = \frac{k\cos\theta}{d^2}$, where k is a (positive) constant of proportionality. Again here, I is a function of two variables θ and d. We will substitute d in terms of θ. From Fig. 17, $\sin\theta = \frac{1}{d}$ or $d = \frac{1}{\sin\theta}$.

'$\propto$' is used as the sign of proportionality.

Hence, $I = k\cos\theta\left[\frac{1}{(1/\sin\theta)^2}\right] = k\cos\theta\sin^2\theta$, where θ may vary from 0 to $\pi/2$ only. Hence, we need to find the absolute maximum of the function on $\left[0, \frac{\pi}{2}\right]$.

Let us now differentiate I w.r.t. θ,

$$I'(\theta) = k[\cos\theta(2\sin\theta\cos\theta) + \sin^2\theta(-\sin\theta)]$$
$$= k(2\cos^2\theta\sin\theta - \sin^3\theta)$$
$$= k\sin\theta(2\cos^2\theta - \sin^2\theta)$$

$I'(\theta) = 0$, when, either $\sin\theta = 0$ or $2\cos^2\theta - \sin^2\theta = 0$. If $\sin\theta = 0$, that means $\theta = 0$ (since, $\sin 0 = 0$) and if $2\cos^2\theta - \sin^2\theta = 0$, then $\tan^2\theta = 2$ and $\theta = \tan^{-1}\sqrt{2}$, ignoring other value as $\tan\theta$ is positive in $\left[0, \frac{\pi}{2}\right]$.

The critical numbers are 0 and $\tan^{-1}\sqrt{2}$. Evaluating $I(\theta)$ at the endpoints, and critical numbers we get

$I(0) = k(\cos 0)(\sin^2 0) = 0$,

$I\left(\frac{\pi}{2}\right) = k\left(\cos\frac{\pi}{2}\right)\left(\sin^2\frac{\pi}{2}\right) = 0$, $I(\tan^{-1}\sqrt{2}) \approx I(0.9553)$

$$\approx k(\cos 0.9553)\ (\sin^2 0.9553)$$
$$\approx 0.3856\,k.$$

Also, from Fig. 17, height $= \dfrac{1}{\tan\theta}$.

Since, $\tan\theta = \sqrt{2}$ when I is maximum, therefore, the height $= \dfrac{1}{\tan\theta} = \dfrac{1}{\sqrt{2}} \approx 0.7075\,m$.

To maximise the illumination, the lamp should be placed about $0.7075\,m$ above the center of the table.

***

Now, try the following exercises.

E5) A manufacturer estimates that when x units of a particular item are produced each month, the total cost (in thousand rupees) will be $C(x) = \frac{1}{8}x^2 + 4x + 200$ and all the items can be sold at a price of $p(x) = 49 - x$ rupee per item when $0 \le x \le 49$. Determine the selling price so that profit is maximum.

E6) Consider an asset whose market price after t years from now is given by $V(t) = 10000e^{\sqrt{t}}$. If the prevailing rate of interest is 8% per annum compounded, when should the asset be sold?

E7) Let $C(t)$ denote the concentration of a drug injected into the body intramuscularly in the blood at time t. In a study, it was observed that the concentration may be modelled by $C(t) = \frac{k}{b-a}(e^{-at} - e^{-bt}); t \ge 0$ where a, b (with $b > a$), and k are positive constants that depend on the drug. At what time does the largest concentration occur? What happens to the concentration as $t \to +\infty$?

So far, we have discussed the necessary condition for the existence of an extreme point. We have also seen that the condition is not a sufficient one. In the following section, we shall show how derivatives are used to find whether the function is increasing or decreasing.

13.4 INCREASING AND DECREASING FUNCTIONS

In this section, we shall see how information about the derivative f' can be used to determine the shape of the graph of f. We begin by showing how the sign of f' is related to whether the graph of f is rising or falling, that is, whether f is increasing or decreasing which you have studied in Unit 6, but here the difference is that we are making use of derivatives to identify whether a function is monotonic or not.

Let us begin with an example. Consider a population of a certain species as a function f of the time t (months) as shown in Fig. 18. From the graph shown in

Fig. 18, we can say that the population is increasing between $t = 0$ and $t = 24$. If the graph represents a fuctions f, we shall say that f is increasing on the interval $[0, 24]$. Similarly, we say that it is decreasing on the interval $[24,36]$.

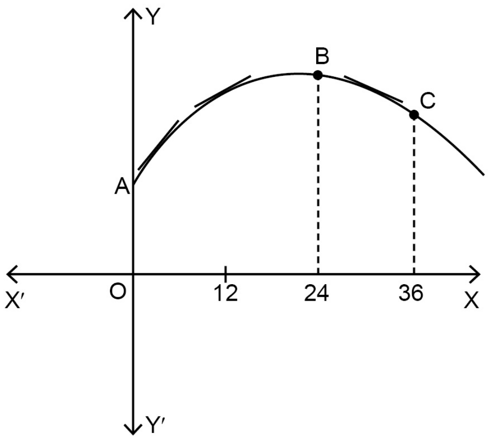
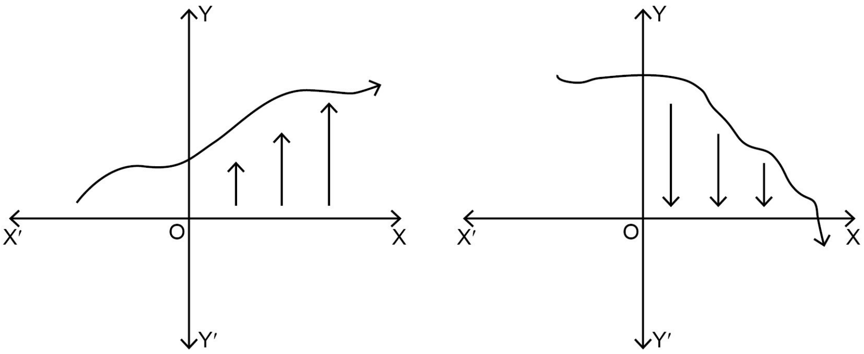


Fig. 18

In Fig. 18, the curve between A and B has tangent lines with positive slopes and so $f'(t) > 0$. But between B and C, the curve has tangent lines with negative slopes and so $f'(t) < 0$. Thus, it appears that f increases when the derivative of f that is f' is positive and decreases when f' is negative. We can also say that the graph of a function f is rising when $f' > 0$ [see Fig. 19 (a)] and falling when $f' < 0$ [see Fig. 19 (b)].



(a) Rising Graph **(b) Falling Graph**

Fig. 19

This leads to the following theorem:

Theorem3: Let f be differentiable on the open interval $]a, b[$. If $f'(x) > 0$ on $]a, b[$, then f is increasing on $]a, b[$. If $f'(x) < 0$ on $]a, b[$, then f is decreasing on $]a, b[$.

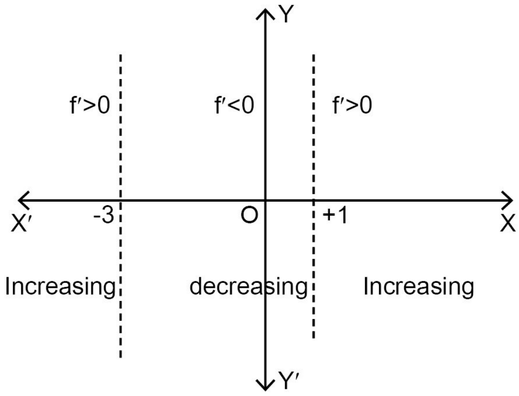
We now apply this theorem to find whether a function is increasing or decreasing in the following examples:

Example 13: Determine where the function f defined by $f(x) = x^3 + 3x^2 - 9x + 2$ is increasing and where it is decreasing. Also, compare the graphs of f and f'.

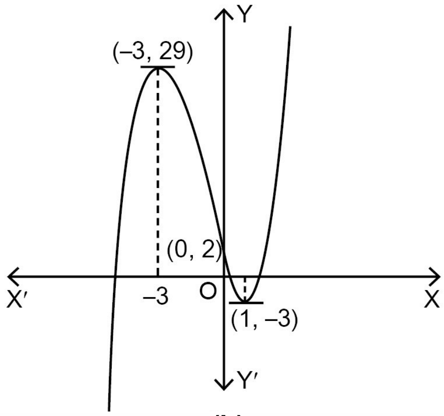
Solution: Given that $f(x) = x^3 + 3x^2 - 9x + 2$, we differentiate f w.r.t. x and get

$$f'(x) = 3x^2 + 6x - 9 = 3(x-1)(x+3)$$

To find whether f is increasing or decreasing, we have to find where $f'(x) > 0$ and where $f'(x) < 0$. This depends on the signs of the two factors of f', namely, $(x-1)$ and $(x+3)$. We divide the real line into intervals whose endpoints are critical numbers -3 and 1 as shown in Fig. 20 (a). Then, in these intervals, we find the sign of f' and finally mark each interval as increasing or decreasing according to whether the derivative is positive or negative, respectively. This is shown in Fig. 20 (b). When $x < -3$, both the factors in f' are negative, therefore, $f' > 0$, and f is increasing. Similarly, when $-3 < x < 1, (x-1)$ is negative and $(x+3)$ is positive, therefore, $f' < 0$, and f is decreasing. Also, when $x > 1$, both the factors $(x-1)$ and $(x+3)$ are positive, therefore, f is increasing.



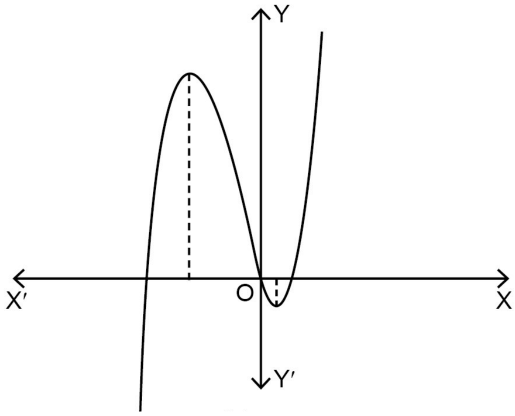
(a)



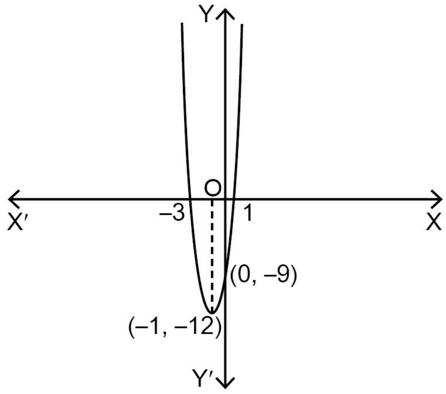
(b)

Fig. 20

The graph of f and f' are shown in Fig. 21 (a) and Fig. 21(b) respectively. From the graph, it is clear that when $x < -3$ and when $x > 1$, the graph of f' is above the $x-$axis, and when $-3 < x < 1$, the graph of f' is below the x-axis. The critical numbers of f are where $f'(x) = 0$ that is, at $x = -3$ and $x = 1$, so they are the x-intercepts of the graph of f'.



(a) Graph of f



(b) Graph of f'

Fig. 21

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Now, try the following exercises:

E8) Find where the function f defined by $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

E9) Find the sub intervals on which f is increasing or decreasing.

i) $f(x) = \sin x + \cos x$ on $[0, 2\pi]$

ii) $f(x) = e^{2x} + e^{-x}$ on $\mathbb{R}$

iii) $f(x) = x^2 - x - \ln x$ on $\mathbb{R}^+$

Going back to the extrema, you may recall that if f has a maximum or minimum at any point, then the point must be a critical number of f. But not every critical number gives rise to a maximum or minimum. So, is there a test that will tell us whether or not f has a maximum or minimum at a critical number? Let's look into this in the following sections.

13.5 THE FIRST DERIVATIVE TEST

Every relative extremum is a critical point. However, as you saw in Sec.13.3, not every critical point of a continuous function is necessarily a relative extremum. Look at the graphs given in Fig. 22 (a), you can see that both the graphs have a maxima at A and B respectively. Now consider the derivative at these points after this. What do you notice? If the derivative is positive to the immediate left of a critical number and negative to its immediate right, the graph changes from increasing to decreasing and the critical point must be a relative maximum, as shown in Fig. 22 (a). If the derivative is negative to the immediate left of a critical number and positive to its immediate right, the graph changes from decreasing to increasing and the critical point is a relative minimum as shown in Fig. 22 (b) on the points C and D. However, if the sign of the derivative is the same on both immediate sides of the critical number, then it is neither a relative maximum nor a relative minimum as shown by points E and F in Fig. 22 (c).

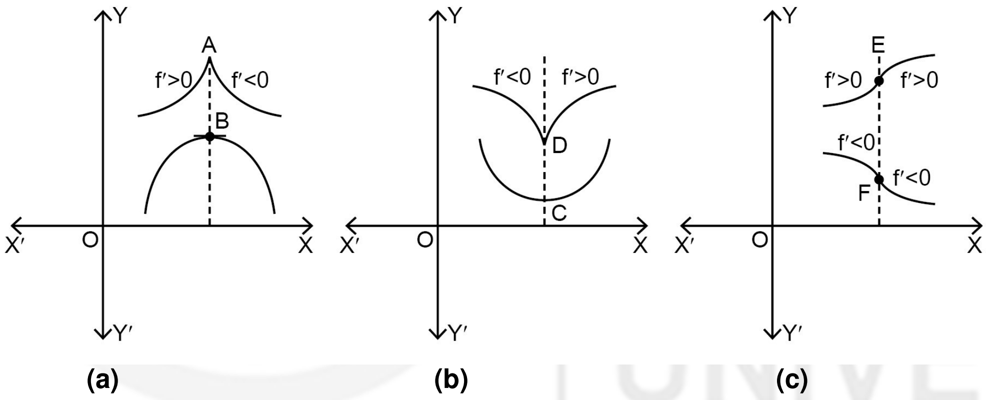

Fig. 22: Three patterns of behaviour of f near a critical point.

The following steps are done to apply the first derivative test to find relative extrema:

1. First of all, find all the critical numbers of a continuous function f. That is, find all the numbers x_0 such that $f(x_0)$ is defined and either $f'(x_0)=0$ or $f'(x_0)$ does not exist.

2. Classify each critical point $(x_0, f(x_0))$ as given in Table 1.

Table 1

Point	Nature of the point	Observation
$(x_0, f(x_0))$	Relative maximum	$f'(x)>0$ (graph rising) for all x in an open interval $]a, x_0[$ to the left of x_0, and $f'(x)<0$ (graph falling) for all x in an open interval $]x_0, b[$ to the right of x_0.

$(x_0, f(x_0))$	Relative minimum	$f'(x) < 0$ (graph falling) for all x in an open interval $]a, x_0[$ to the left of x_0, and $f'(x) > 0$ (graph rising) for all x in an open interval $]x_0, b[$ to the right of x_0.
$(x_0, f(x_0))$	Not an extremum	If the derivative $f'(x)$ has the same sign for all x in open intervals $]a, x_0[$ and $]x_0, b[$ on each side of x_0.

Suppose we apply this first-derivative test to the polynomial $f(x) = x^3 - 3x^2 - 9x + 2$. We find that the critical numbers of f are -3 and 1. Also, f is increasing when $x < -3$ and $x > 1$ and decreasing when $-3 < x < 1$. The first-derivative test tells us that there is a relative maximum at -3 and a relative minimum at 1. To understand this let us solve few more examples.

Example 14: Find all critical numbers of $f(x) = x - 2\sin x$ for $0 \le x \le 2\pi$, and determine whether each corresponds to a relative maximum, a relative minimum, or neither. Sketch the graph of f.

Solution: Here, $f(x) = x - 2\sin x$, differentiating it w.r.t. x, we get $f'(x) = 1 - 2\cos x$, which exists for all x, therefore, the critical number occurs when $f'(x) = 0$. This gives $\cos x = \frac{1}{2}$. On solving, we find that the critical numbers for $f(x)$ on the interval $[0, 2\pi]$, these critical numbers are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. Next, we examine the sign of $f'(x)$, which is given in Table 2.

Table 2

Interval	Sign of $f'(x)$	Monotonicity of f
$0 < x < \frac{\pi}{3}$	$-$ive	decreasing
$\frac{\pi}{3} < x < \frac{5\pi}{3}$	$+$ive	increasing
$\frac{5\pi}{3} < x < 2\pi$	$-$ive	decreasing

According to the first-derivative test, we can say that the sign of f' is changing from –ive to +ive at $x = \frac{\pi}{3}$, therefore, there is a relative minimum at $x = \frac{\pi}{3}$.

Similarly, the sign of f' is changing from +ive to –ive at $x = \frac{5\pi}{3}$, therefore, there is a relative maximum at $x = \frac{5\pi}{3}$. The graph of f is shown in Fig. 23, which also verifies this.

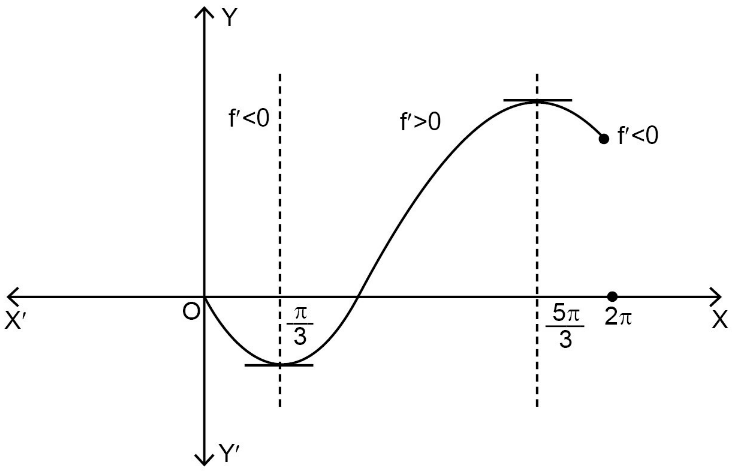

Fig. 23: Graph of f .

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Now try the following exercises:

E10) Find all possible relative extreme values of each of the following functions by applying the first derivative test.

i) $f(x) = x^5 - 5x^4 + 5x^3 - 1$ for all $x \in [0,3]$.

ii) $f(x) = 2x^4 + 8x^3 - 4x^2 - 24x + 15$ for all $x \in \mathbb{R}$

iii) $f(x) = (x-1)^2(x+1)$ for all $x \in \mathbb{R}$.

iv) $f(x) = x\sqrt{6-x}$ on $[0,6]$

v) $f(x) = e^{-x^2}$ on $[-1,1]$.

E11) Find the local maximum and minimum values of the function f defined as $f(x) = x + 2\sin x$ on $[0, 2\pi]$.

You may recall Unit 11, where in we discussed the second derivative. One of the applications of the second derivative is to test for maximum and minimum values. We discuss the second derivative test in the following section:

13.6 SECOND DERIVATIVE TEST

We now investigate another condition which, if satisfied, does away with the need to examine the sign of $f'(x)$ as in the first derivative test. This condition is also only sufficient, but very useful.

The Second-Derivative Test for Relative Extrema

Let f be a function such that $f'(x_0) = 0$ and the second derivative exists on an open interval containing x_0.

If $f''(x_0) > 0$, there is a relative minimum at $x = x_0$.

If $f''(x_0) < 0$, there is a relative maximum at $x = x_0$.

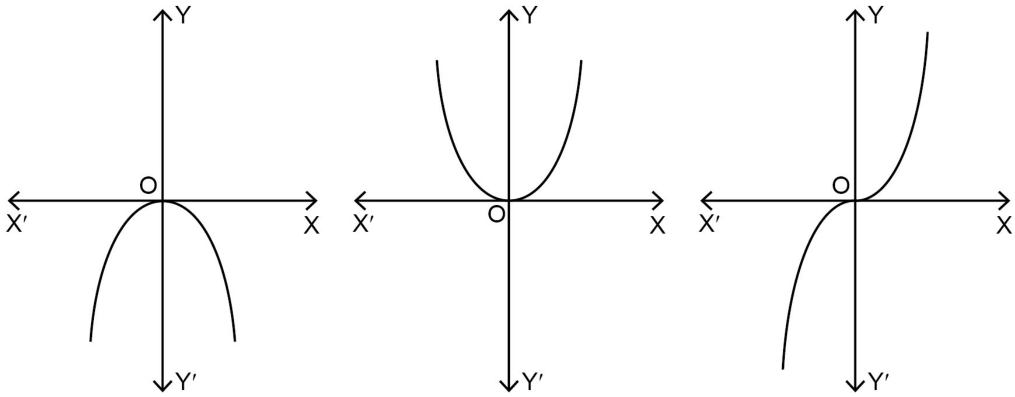
Remark: You must have observed that this test says nothing about the case when $f''(x_0)$ is zero. In this case, the function may have a maximum or a minimum value or neither as shown in the following examples:

i) $f(x) = -x^4$, for all $x \in \mathbb{R}$.
Here $f'(0) = 0 = f''(0)$, but the function has a maximum at 0. (see Fig 24(a)).

ii) $f(x) = x^4$, for all $x \in \mathbb{R}$.
Here $f'(0) = 0 = f''(0)$, but the function has a minimum at 0. (see Fig 24 (b)).

iii) $f(x) = x^3$, for all $x \in \mathbb{R}$.

Here $f'(0) = 0 = f''(0)$ and the function has neither a maximum nor a minimum at 0 (see Fig 24(c)).
Thus, the first derivative test does have some merit.



(a) **(b)** **(c)**

Fig. 24

Let us apply second derivative test in the following examples:

Example 15: Find the extreme values of the function f defined by $f(x) = 2x + \frac{3}{x}$, for all $x \neq 0$, using the first derivative and second derivative tests. What do you conclude about ease of procedure here?

Solution: Here, $f'(x) = 2 - \frac{3}{x^2}$, and therefore $f'(x) = 0 \Rightarrow x = \pm\sqrt{3/2}$.

Using the first derivative test, we can say that the sign of f' is changing from –ive to +ive at $x = \sqrt{\frac{3}{2}}$, therefore, f has a minima and the minimum value is $2\sqrt{6}$. Similarly, the sign of f' is changing from +ive to –ive at $x = -\sqrt{\frac{3}{2}}$, therefore, f has a maxima and the maximum value of f is $-2\sqrt{6}$.

Also, $f''(x) = 6x^{-3}$. This means $f''(\sqrt{3/2}) > 0$ and $f''(-\sqrt{3/2}) < 0$. Thus, using the second derivative test, we can say that f has a minimum at $\sqrt{3/2}$ and a maximum at $-\sqrt{3/2}$. The minimum value is $f(\sqrt{3/2}) = 2\sqrt{6}$, and the maximum value is $f(-\sqrt{3/2}) = -2\sqrt{6}$.

Sometimes, the second derivative test is inconclusive when $f''(x_0) = 0$. In other words, at such a point there might be a maximum or minimum or neither. The second derivative test also fails when $f''(x_0)$ does not exist. In such cases, the first derivative test must be used. You can see in this example that even when both tests apply, the first derivative test is the easier one to use.

***

Let us apply second derivative test in the following example:

Example 16: From each corner of a square paper of side 24 cm, suppose we remove a square of side x cm and fold the edges upward to form an open cuboid box. Find that value of x which will give us a box with maximum capacity.

Solution: Clearly, $0 \le x \le 12$ for a box to be formed. Also, the box thus formed has dimensions $(24-2x), (24-2x)$ and x (see Fig 25).

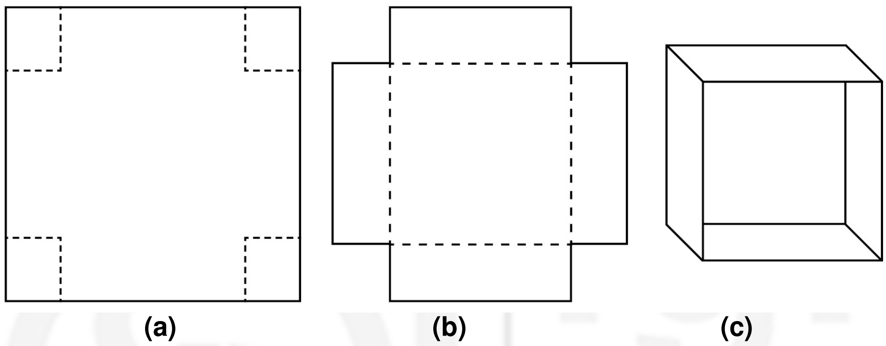


Fig. 25

The volume $f(x)$ is a function of x given by

$$f(x) = (24-2x)^2 x,\ 0 \le x \le 12,$$

$$= 4x^3 - 96x^2 + 576x.$$

$$f'(x) = 12x^2 - 192x + 576 = 12(x-4)(x-12)$$

Now, $f'(x) = 0 \Rightarrow x = 12$ or $x = 4$.
Here, $f''(x) = 24x - 192$, $f''(4) = 96 - 192 < 0$ and $f''(12) = 288 - 192 > 0$.
Hence, $x = 4$ is a maximum point of f. The maximum value $f(4)$ of f (that is, the maximum capacity of the box) is $1024\ cm^3$.

Are you surprised that the box is not a cube for maximum capacity? But, had it been a cube, four squares each of side $8\,cm$ (the removed portions) would have been wasted, whereas now four squares each of side only $4\ cm$ have been thrown away. There had to be a compromise between the waste material and making the box as near a cube as possible!

***

Here are some exercises for you to solve.

E12) Find the extremum points for each of the following functions. Using the second derivative test, point out which of them are maximum, which are minimum and which are neither. Also, find the extremum values of f.

i) $f(x) = x^2,\ x \in \mathbb{R}$.

ii) $f(x) = -x^3,\ x \in \mathbb{R}$.

iii) $f(x) = 3x^2 + 7x + 1,\ x \in \mathbb{R}$.

iv) $f(x) = a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n}$, where $x \in \mathbb{R}$ and each a_i is positive.

v) $f(x) = x/(x^2 + 1),\ 0 < x < \infty$.

E13) Show that $\pi/3$ is a critical number of f, where $f(x) = \sin x(1 + \cos x),\ x \in \mathbb{R}$. Does f have a maximum or a minimum at this point?

E14) Show that the rectangle of maximum area which can be inscribed in a circle, is a square.

E15) Reena wants a name-plate with display area equal to 48cm^2 bordered by a white strip 2cm along top and bottom and 1cm along each of the two remaining sides. What dimensions should the plate have so that the total area of the plate is a minimum?

In the following section, we shall discuss Rolle's theorem and Lagrange's mean value theorem.

13.7 MEAN VALUE THEOREMS

Let us begin with an example. For example, suppose the average speed of a vehicle was 60 Kmph. Then the instantaneous speed cannot always have been more than 60, for then the average would also be more than 60. Similary, the instantaneous speed cannot always have been less than 60. Hence, at least at one point of time the instantaneous speed must have been 60 as well. This is called a mean value result as it relates mean values to actual values. In this section, we shall study the mean value theorems. These theorems have proved to be very handy tools in proving other theorems not only in calculus, but also in other branches of mathematics, such as Numerical Analysis. Their importance lies in their wide applicability and tremendous usefulness.

13.7.1 Rolle's Theorem

We shall first consider Rolle's Theorem, which is a special case of Lagrange's mean value theorem. We shall not attempt the proofs of these theorems here, but you will agree that both are intuitively obvious. We shall discuss their geometrical significance and illustrate their usefulness through some examples.

Rolle's Theorem was not actually proved by Rolle. He had only stated it as a remark. In fact, Michel Rolle (1652-1719) was known to be a critic of the newly found theory of Newton and Leibniz. It is ironical, then, that one of the most important theorems of this theory is known after him. Now, let us see what this theorem is.

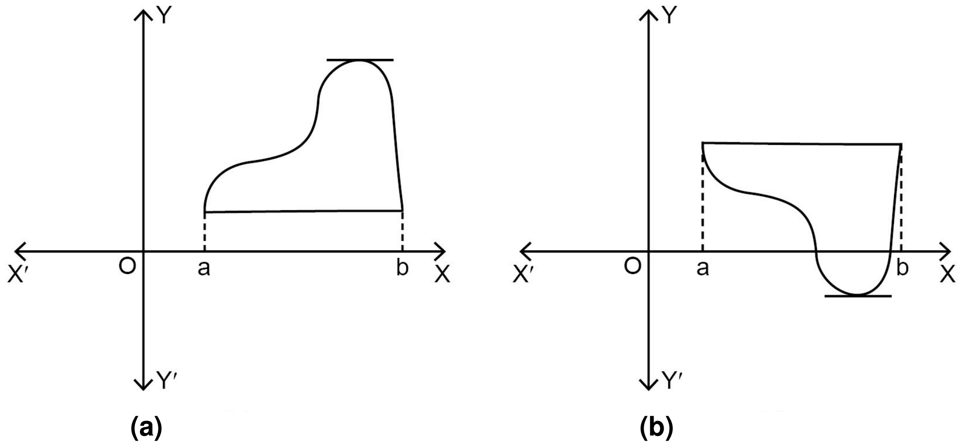


Fig. 26

In Fig. 26 (a) and Fig. 26 (b), we see the graphs of two continuous functions defined on the closed interval $[a, b]$. Here, we observe few features common to both of them as given in Table 3.

Table 3

S.No.	Rough Statement	Precise Statement
1.	The curve is drawn without breaks or gaps.	The function f is continuous on $[a, b]$.
2.	There are no corners in the curve.	The function is differentiable in the open interval $]a, b[$.
3.	The two end points of the curve lie on the same horizontal line.	$f(a) = f(b)$
4.	The curve admits a horizontal tangent (drawn as a dotted line) at some point.	$f'(c) = 0$ for some c in $]a, b[$.

The line joining the two end points may be imagined to be pushed upward or downward, keeping it always horizontal, and keeping the curve unmoved. Then, there is a position, shown by the dotted line, where it touches the curve. This makes us believe that the fourth property holds for all the functions satisfying the first three properties. This is what Rolle's Theorem asserts.

Theorem 4 (Rolle's Theorem): Let f be a function continuous on the closed interval $[a, b]$ and differentiable in the open interval $]a, b[$. Further, let $f(a) = f(b)$. Then, there is some c in $]a, b[$ such that $f'(c) = 0$.

For example, if a train on a straight track is at the same location at both 1 PM and 5 PM, then at some time between 1 PM and 5 PM it was not moving, that is, to return to the same position, it would have needed to stop and reverse at some point. Here, let the position of the train at time t be $f(t)$. If $f(1) = f(5)$, then the train is at the same place at both $t = 1$ and $t = 5$. Rolle's theorem states that the derivative should be zero somewhere between 1 and 5. Now, we give an example to illustrate this theorem.

Example 17: Consider $f(x) = \sin x$ on the interval $[0, 2\pi]$. Check whether Rolle's theorem is verified.

Solution: The function f, given by $f(x) = \sin x$ is continuous on $[0, 2\pi]$ and is also differentiable on $]0, 2\pi[$. Thus, all assumptions of Rolle's theorem are satisfied here.
Now, $f(0) = 0 = f(2\pi)$.

Therefore, according to Rolle's theorem, there should exist c in $]0, 2\pi[$, such that, $f'(c) = 0$. Here, $f'(x) = \cos x$ and $f'(c) = \cos c$.

Can we find an element c such that $\cos c = 0$?

Yes. In fact, there are two such points c in $]0, 2\pi[$, namely $\pi/2$ and $3\pi/2$.

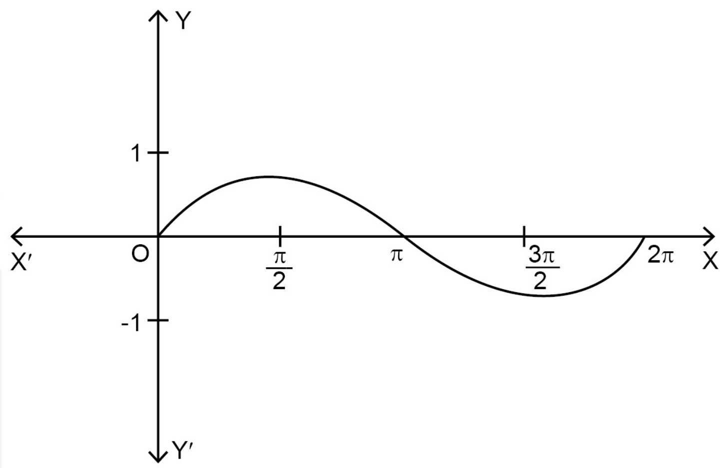


Fig. 27

At $\pi/2$, the function $\sin x$ attains its maximum value.
At $3\pi/2$, the function $\sin x$ attains its minimum value.
Both these belongs to the interval $]0, 2\pi[$. Fig. 27 also verifies this.

***

Rolle's theorem asserts that there is at least one c in $]a, b[$ such that $f'(c) = 0$. Example 17 shows us that there may be more than one point in $]a, b[$ at which $f'(x) = 0$.

In Rolle's theorem, a function f on $[a, b]$ has to satisfy three conditions.

i) f is continuous on $[a, b]$

ii) f is differentiable on $]a, b[$

iii) $f(a) = f(b)$

Now, we shall see through some more examples that each of these conditions is essential. We cannot drop any one of them and still prove the theorem.

Example 18: Check whether the function f defined by $f(x) = x - [x] =$ fractional part of x, on $[0, 1]$ verifies Rolle's theorem.

Solution: The function f can be rewritten as $f(x) = \begin{cases} x, & \text{if } 0 \le x < 1 \\ 0, & \text{if } x = 1 \end{cases}$

Here, $f(0) = f(1) = 0$. f is differentiable in the open interval $]0,1[$. Thus, two of the three conditions of Rolle's theorem are satisfied by f. The derivative of

f is 1 at every point of $]0, 1[$. There is no point in $]0, 1[$, where the derivative is zero. What happens to Rolle's Theorem in this example? Obviously, its conclusion does not hold here. The graph of f is shown in Fig. 28.

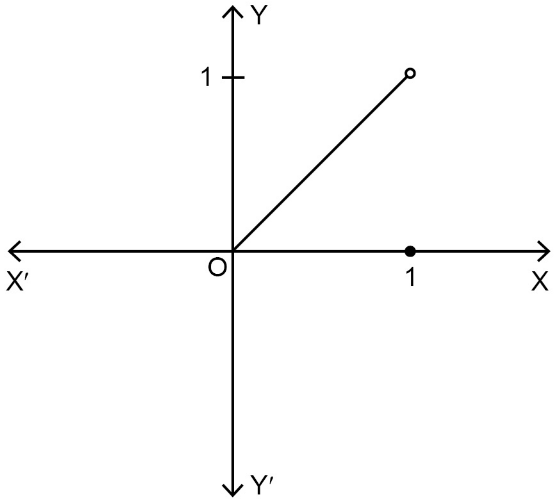


Fig. 28: Graph of f.

The reason is that f is not continuous on the closed interval $[0, 1]$, since it fails to be continuous at 1.

***

In the next example, we shall see that the assumption of differentiability in $]a, b[$ cannot be omitted.

Example 19: Consider the function f defined by $f(x) = |x|$ on $[-1, 1]$, and check whether Rolle's theorem holds or not.

Solution: There is no c in $]-1,1[$ such that $f'(c) = 0$. Actual computation shows that

$$f' = \begin{cases} -1, & \text{if } -1 < x < 0 \\ 1, & \text{if } 0 < x < 1 \\ \text{does not exist at } x = 0 \end{cases}$$

f is continuous on $[-1, 1]$. Also, $f(-1) = f(1)$.

But f is not differentiable in $]-1, 1[$. Therefore, Rolle's theorem does not hold. Fig. 29 shows graph of f.

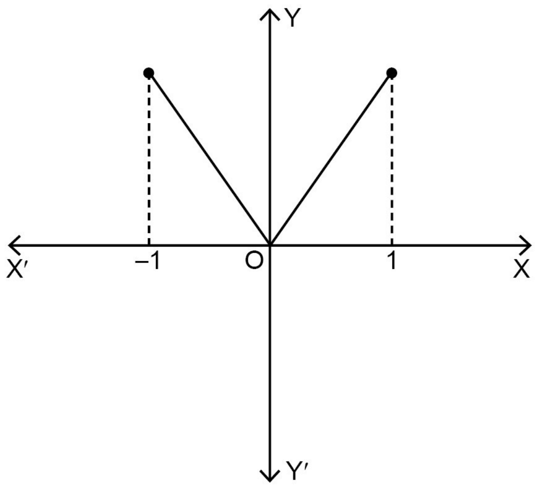


Fig. 29: Graph of f.

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Our next example shows that the assumption $f(a) = f(b)$ is essential in Rolle's Theorem.

Example 20: Check Rolle's theorem for $f(x) = x^3$ on $[0, 1]$.

Solution: f is continuous on $[0, 1]$, and is differentiable in $]0, 1[$. But $f(0) \neq f(1)$.

In this case, $f'(x) = 3x^2 \neq 0$ for any $x \in]0, 1[$. Thus, we see that the conclusions of Rolle's theorem may not hold when $f(a) \neq f(b)$.

***

Lastly, we give an example where Rolle's Theorem is applicable and yields a unique c.

Example 21: Consider $f(x) = x^2$ on $[-1, 1]$. Verify Rolle's theorem.

Solution: Let $f(x) = x^2$ on $[-1,1]$, then $f'(x) = 2x$.

Here all the three conditions of Rolle's Theorem are satisfied. There is only one c, namely $c = 0$, such that $f'(c) = 0$. Fig. 30 shows the corresponding graph.

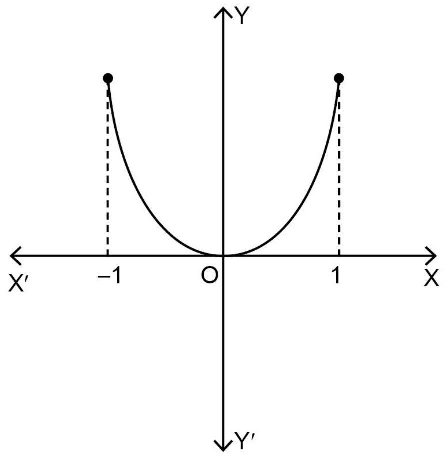

Fig. 30: Graph of f.

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You will now be able to solve these exercises.

E16) Can Rolle's theorem be applied to each of the following function? Find 'c' in case it can be applied.

i) $y = \sin^2 x$ on interval $[0, \pi]$.

ii) $f(x) = x^2 + 1$ on $[-2, 2]$.

iii) $f(x) = x^3 + x$ on $[0, 1]$.

iv) $f(x) = \sin x + \cos x$ on $[0, \pi/2]$.

v) $f(x) = \sin x - \cos x$ on $[0, 2\pi]$.

E17) Consider the function f given by $f(x) = x^2 - 3x + 2$. Prove that $f(-1) = f(4)$. Find a point c between -1 and 4, such that the derivative of f vanishes at c. Is this point the midpoint of -1 and 4?

E18) Let $f(x) = ax^2 + bx + c$ be the given function. If p and q are two real numbers such that $f(p) = f(q)$, prove that $f'\left[\frac{p+q}{2}\right] = 0$.

E19) Consider the curve $y = ax^2 + bx + c$. Let x_0 be the unique real number such that the tangent at (x_0, y_0) to this curve is horizontal. Prove that the function y is one-one on the interval $[x_0, \infty[$.

E20) Let I be an open interval of $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be a differentiable function such that f does not vanish on I. Prove that f is one-one on I.

Now, we shall discuss the mean value theorem. It was proved by Joseph Louis Lagrange, a mathematician of the eighteenth century.

13.7.2 Lagrange's Mean Value Theorem

We have already mentioned that Rolle's theorem is a special case of the mean value theorem. Let us recall the statement of Rolle's theorem in the following form:

Fig. 31: Lagrange

Let f be a continuous function on the closed interval $[a, b]$, and differentiable in the open interval $]a, b[$. The graph of f is a curve in the plane. If the endpoints of this curve lie in the same horizontal line, (that is, $f(a) = f(b)$), then, there is a point c on the curve where the tangent to the curve is horizontal $(f'(c) = 0)$.

The last sentence can be restated as follows.

If the endpoints of this curve lie in the same horizontal line, then, there is a point on the curve, where the tangent to the curve is parallel to the line joining its endpoints.

The mean value theorem asserts the same conclusion, even without the assumption of horizontality of the line joining the endpoints of the curve. Fig. 32 illustrate this difference. Here P and Q are the end points of the curve. The line PQ is horizontal in Fig. 32 (a), but not in Fig. 32 (b). But in both the cases the point R on the curve has the property that the tangent to the curve at R is parallel to the line PQ. The number c is the x-coordinate of the point R.

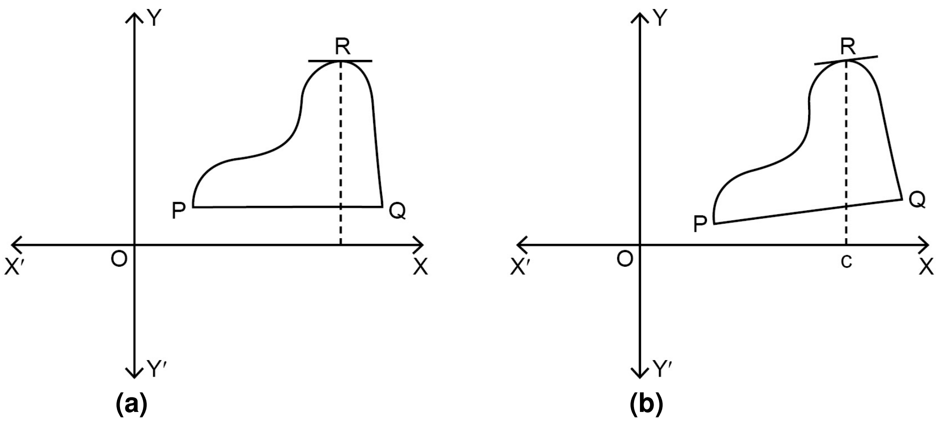

Fig 32

Fig. 32 (a) illustrates Rolle's theorem, whereas Fig. 32 (b) illustrates Lagrange's mean value theorem.

The two end points of the curve are $(a, f(a))$ and $(b, f(b))$. The line joining these two points has the slope $[f(b)-f(a)]/(b-a)$. Any line parallel to this line will also have the same slope. Therefore, the conclusion of the mean value theorem is $f'(c)=\frac{[f(b)-f(a)]}{(b-a)}$ for some $a<c<b$.

This is because, we already know that $f'(c)$ is the slope of the tangent to the curve at $(c, f(c))$. Now, we are ready to give the precise statement of the theorem.

Theorem 5 (Lagrange's Mean Value Theorem): Let f be a continuous function on a closed interval $[a, b]$. Let f be differentiable in the open interval $]a, b[$. Then, there is a point c in the open interval $]a, b[$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

Rolle's Theorem has three assumptions namely, a continuity assumption, a differentiability assumption, and the assumption $f(a)=f(b)$.

The mean value theorem has only two assumptions. These are the same as the first two assumptions of Rolle's Theorem.

Suppose in addition to the two assumptions of the mean value theorem, $f(a)=f(b)$ also holds. Then what does the mean value theorem yield? It says that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \text{ for some } a<c<b. \text{ But } f(b)-f(a)=0.$$

Therefore, we get $f'(c)=0$ for some $a<c<b$. This is the same as the conclusion of Rolle's theorem. This proves our contention that Rolle's theorem can be deduced from the mean value theorem.

But why the name mean value theorem? What is the mean value here? $f(a)$ is the initial value of f. $f(b)$ is the final value of f. Therefore, $f(b)-f(a)$ is the total change in the value of f. This change has occurred when the x-coordinate has changed from a to b. For a change of $(b-a)$ in the domain, there is a change of $f(b)-f(a)$ in the value of f. Therefore, the mean value, that is, the average value of the rate of change is $[f(b)-f(a)]/(b-a)$. The mean value theorem asserts that this average value of the rate of change of f is assumed at some point c by derivative f'.

We shall illustrate the same thing by means of an example. Consider a car moving from time a to time b. Let $f(t)$ be the position of the car at time t. Then, the average speed of the car is distance/time$=\frac{[f(b)-f(a)]}{(b-a)}$. According to mean value theorem, the speedometer of the car would have shown this $\frac{[f(b)-f(a)]}{(b-a)}$ at some time between a and b. For instance, if the car has travelled 100 kms in two hours, then at some point of time, its speed would

have been actually 50 kmph (which is its average speed over the span of two hours).
Now, let us see how to verify the theorem in the following examples.

Example 22: Verify Lagrange's mean value theorem for the function $f(x) = x^2 - 2x$ on the interval $[1, 2]$.

Solution: This is a polynomial function. Therefore, it is continuous on $[1, 2]$ and differentiable in $]1, 2[$. Here, $a = 1,\ b = 2, f(a) = 1 - 2 = -1$, $f(b) = 2^2 - 2\times 2 = 0$, and $\frac{f(b) - f(a)}{b - a} = \frac{0 - (-1)}{2 - 1} = 1$
We want to check that $f'(c) = 1$ for some c such that $1 < c < 2$.
Now $f'(x) = 2x - 2$. For what value of x will it be 1?
Now, $2x - 2 = 1$, when, $x = 3/2$, and $\frac{3}{2} \in]1, 2[$. Thus, we see that

$$f'(3/2) = \frac{f(2) - f(1)}{2 - 1}.$$

Now, consider the function $f : [a, b] \to \mathbb{R}$ which satisfies the assumptions of mean value theorem. Let p and q be any two points such that $a \le p < q \le b$. Is there some c between p and q such that $f'(c) = \frac{[f(q) - f(p)]}{(q - p)}$? To answer this, consider the restriction of f to the interval $[p, q]$. It satisfies the assumptions of the mean-value-theorem. Therefore, such a point c exists.
This result can be geometrically interpreted as follows. $(p, f(p))$ and $(q, f(q))$ are two points on the curve $y = f(x)$. The line joining them is called a chord of the curve and $\frac{f(q) - f(p)}{(q - p)}$ is the slope of this chord. What we have shown is that the slope of this chord is the same as the slope of the tangent at the point $(c, f(c))$. This means, that the tangent at $(c, f(c))$ is parallel to the chord (see Fig. 33). Thus, for any chord of the curve, there is a point on the curve where the tangent is parallel to the chord.

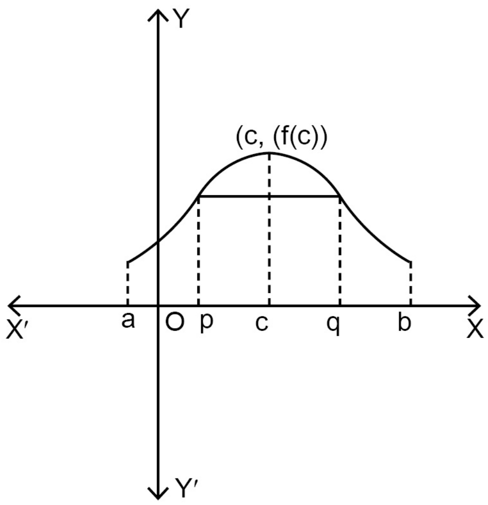

Fig. 33

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Let us solve few examples:

Example 23: i) Find the point c in $]-\pi/4, \pi/4[$ such that the tangent to $f(x) = \cos x$ at $(c, f(c))$ is parallel to the chord joining $(-\pi/4, f(-\pi/4))$ and $(\pi/4, f(\pi/4))$.

ii) We shall further prove that for the same c, the tangent at $(c, g(c))$ to the curve $g(x) = \cos x + x^2 + x$ is parallel to the chord joining $(-\pi/4, g(-\pi/4))$ and $(\pi/4, g(\pi/4))$.

Solution: i) The slope of the chord joining $(-\pi/4, f(-\pi/4))$ and $(\pi/4, f(\pi/4))$ is $\dfrac{f(\pi/4) - f(-\pi/4)}{\pi/4 - (-\pi/4)} = \dfrac{1/\sqrt{2} - 1/\sqrt{2}}{\pi/2} = 0$.

Therefore, we seek c that $f'(c) = 0$. We have $f'(x) = -\sin x$.

The only point in $]-\pi/4, \pi/4[$, where this vanishes is at $c = 0$.

The corresponding point on the curve is $(0, f(0)) = (0, 1)$.

ii) $g(-\pi/4) = (1/\sqrt{2}) + (\pi^2/16) - (\pi/4)$

$g(\pi/4) = (1/\sqrt{2}) + (\pi^2/16) + (\pi/4)$.

The slope of the chord joining $((-\pi/4), g(-\pi/4))$ and $(\pi/4, g(\pi/4))$ is

$$\frac{g(\pi/4) - g(-\pi/4)}{\pi/4 - (-\pi/4)} = \frac{(\pi/4) + (\pi/4)}{(\pi/4) + (\pi/4)} = 1.$$

When $c = 0$, we want to prove that the tangent at $(c, g(c))$ to the curve $g(x)$ also has the same slope 1. In other words, we must prove that $g'(0) = 1$.

Now $g'(x) = -\sin x + 2x + 1$

$\therefore \quad g'(0) = -0 + 0 + 1 = 1$.

This proves that, for both the functions $f(x)$ and $g(x)$ over $]-\pi/4, \pi/4[$, it is the same point c where the conclusion of the mean value theorem holds.

***

Example 24: For the curve $y = \ln x$, find a point on the curve where the tangent is parallel to the chord joining the points $(1, 0)$ and $(e, 1)$.

Solution: Since, $\ln 1 = 0$ and $\ln e = 1$, therefore these two points $(1, 0)$ and $(e, 1)$ lie on the curve $y = \ln x$. Consider this function on the closed interval $[1, e]$ (see Fig 34). It is continuous there. It is also differentiable on $]1, e[$. Therefore, by the mean value theorem, there is a point c between 1 and e such that the tangent at $(c, \ln c)$ is parallel to the chord joining $(1, 0)$ and $(e, 1)$. We have to find this point. Now $y' = 1/x$. Its value at c is $1/c$

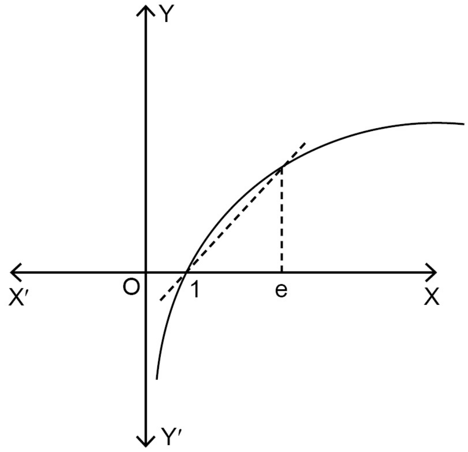


Fig. 34

The required point is given by

$$\frac{1}{c} = \frac{\ln e - \ln 1}{e-1} = \frac{1-0}{e-1} = \frac{1}{e-1}$$

$\therefore \quad c = e-1$

The required point on the curve is $(e-1, \ln\ (e-1))$.

***

Remark: Let $f:[a, b] \to \mathbb{R}$ satisfy the assumptions of the mean value theorem. Then $\theta,\ 0<\theta<1$, such that $f(b) = f(a) + (b-a)f'(a+\theta(b-a))$. This is because any point c between a and b is of the form $a+\theta(b-a)$ for some $0<\theta<1$. Note that $a = a+0(b-a)$ and $b = a+1(b-a)$.

Just as in the case of Rolle's theorem, there may be more than one points at which the tangents may be parallel to the chord joining the end points of a curve represented by a function which is continuous at every point in the closed interval and is differentiable at every point in the open interval (see Fig 35).

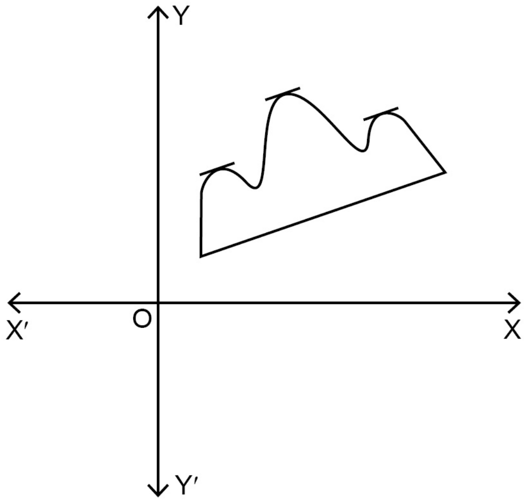


Fig. 35

Both Rolles' theorem and Lagrange's mean value theorem are existence theorems. They tell us that there exists at least one point where the tangent is parallel to the chord joining the end points. But they do not tell us how many such points are there, nor how to find these points.

For example, consider the function $f(x) = x^3 - \sin x$ on $[0, 5\pi]$. It satisfies the conditions of the mean value theorem. So, there is at least one value c at which $3c^2 - \cos c = 25\pi^2$. The mean value theorem assures us that the equation $3x^2 - \cos x = 25\pi^2$ has at least one solution, c. But it does not enable us to find the value or values of c. You can study methods of solving such equations in the course on numerical analysis.

Now, try the following exercises:

E21) Verify the mean value theorem for $f(x) = x^2 + 1$ on the following intervals

i) $[-1, 1]$ ii) $[-1, 2]$

E22) Verify the mean value theorem on the interval $[0, 2]$ for the following functions.

i) $f(x) = \sin \pi x$ ii) $f(x) = 2x^2 + 3$

E23) i) Let $f(x) = x^3$ on $[0, 1]$. Find a point c in $]0,1[$ as in the mean value theorem.

ii) Let $f(x) = x^3$ on $[-1, 0]$. Find a point c in $]-1, 0[$ as in the mean value theorem.

iii) Let $f(x) = x^3$ on $[-1, 1]$. Show that there are two points c in $]-1,1[$ such that $f'(c) = \frac{f(1) - f(-1)}{1-(-1)}$.

E24) Let f be a function on $[a, b]$ satisfying the assumptions of the mean value theorem. Let c be a point guaranteed by the mean value theorem. Prove that if $g_1(x) = f(x) + 1$ and $g_2(x) = f(x) + x$ for all x in $[a, b]$, then the same point c satisfies $\frac{g_1(b) - g_1(a)}{b-a} = g_1'(c)$ and $\frac{g_2(b) - g_2(a)}{b-a} = g_2'(c)$ also.

E25) At what point is the tangent to the curve $y = x^n$ parallel to the chord from

i) $(0, 0)$ to $(2, 2^n)$?

ii) $(0, 0)$ to $(1, 1)$?

That brings us to the end of this unit. Let us summarise all that we have done in it.

13.6 SUMMARY

In this unit, we have discussed the following points:

1. A function f is said to have a maximum/ minimum value at a point x_0 of its domain if $f(x) \le f(x_0) / f(x) \ge f(x_0)$ for all x in the domain. Maximum and minimum values are known as the extreme values of the function.

2. If the derivative of a function f at x_0, does not exist or is zero, then x_0 may not be an extreme point.

3. Critical points for a function are those where either the derivative does not exist or else has the value zero. A critical point may fail to be an extreme point.

4. A sufficient condition for a function f to have an extreme value at $x = x_0$ is that f is continuous at x_0 and the derivative f' changes sign in passing through x_0. If the change is from positive to negative, x_0 is a

maximum point. In the other event, x_0 is a minimum point. This test is known as the first derivative test.

5. Second derivative test, another sufficient condition for the existence of extreme points asserts that if $f'(x_0) = 0$ then $f''(x_0) > 0$ implies f has a minimum at $x = x_0$ and $f''(x_0) < 0$ guarantees a maximum value at x_0.

6. In Rolle's theorem, if a function f is continuous on $[a, b]$, differentiable on $]a, b[$ and $f(a) = f(b)$, then there is some c in $]a, b[$ such that $f'(c) = 0$.

7. In Lagrange's mean value theorem, if a function f is continuous on $[a, b]$, and differentiable on $]a, b[$, then, there is a point c in the open interval $]a, b[$ such that $f'(c) = \frac{f(b) - f(a)}{(b - a)}$.

13.7 SOLUTIONS/ANSWERS

E1) i) All points of $\mathbb{R}$ are maxima as well as minima.

ii) no maxima or minima on $\mathbb{R}$.

iii) no maxima or minima on $]0, 4[$

iv) $0 \in \mathbb{R}$ has a minimum. No maxima.

v) $x = 9$ has a minimum, $x = 25$ has a maximum.

vi) $x = 3$ has a minimum and $x = -4$ has a maximum.

vii) $x = 1$ has a minimum and $x = e$ has a maximum.

viii) $x = 1$ has a maximum and $x = 0$ has a minimum.

ix) $x = \frac{\pi}{3}$ has maximum.

x) $x = 0$ is a minimum. $x = 1$ and $x = -1$ are maxima.

E2) i) $f(x) = 2\sqrt{x}(6 - x)$

$$f'(x) = \frac{2 \cdot \frac{1}{2}}{\sqrt{x}}(6 - x) + 2\sqrt{x}(-1)$$

$$= \frac{1}{\sqrt{x}}(6 - x) - 2\sqrt{x} = \frac{6}{\sqrt{x}} - 3\sqrt{x} = \frac{6 - 3x}{\sqrt{x}}$$

$f'(x) = 0$, gives $x = 2$ and $f'(x)$ does not exist, gives $x = 0$.
Thus, $x = 0, 2$ are the critical numbers.

ii) $f'(x) = 3x^2$, here $x = 0$ is the only critical number.

iii) $f'(x) = \dfrac{1-2\ln\sqrt{x}}{2x^2}$

$f'(x) = 0 \Rightarrow x = e$ and $f'(x)$ does not exist $\Rightarrow x = 0$

The critical number is e, as $0 \notin [1,3]$.

iv) $f'(x) = 0$

$\Rightarrow \sin x = 0$ or $\cos x = 1/2$

$\Rightarrow x = 0, \pi/3$ are the critical numbers.

E3) i) $(0,0), (2, 8\sqrt{2})$ are the critical points.

ii) $x = 0$ is critical number but no critical point as there is no relative extrema at $x = 0$.

iii) $\left(e, \dfrac{1}{2e}\right)$ is the only critical point

iv) $(0,1)$ and $\left(\dfrac{\pi}{3}, \dfrac{5}{4}\right)$ are the critical points.

E4) i) $f'(x) = (x-3) + (x-5) = 2x - 8$

$f'(x) = 0 \Rightarrow x = 4$

$\therefore x = 4$ is a critical number.

Now, $f(-4) = 63, f(4) = -1$. Thus, f has absolute maxima at $x = -4$ and absolute minima at $x = 4$. The maximum and minimum values are 63 and -1 respectively.

ii) $f'(x) = 3x^2 + 26x + 5 = 0 \Rightarrow x = \dfrac{1}{3}(-13 \pm \sqrt{154}) = -0.197, -8.47$

$f(-10) = 257$

$f(10) = 2357$

$f(-0.197) \approx 6.512$

$f(-8.47) \approx 289.636$

Thus, maxima is at $x = 10$ and minima is at $x = -0.197$.

iii) Absolute minima is at $x = n\pi, n \in \mathbb{Z}$ and absolute maxima is at $x = (2n+1)\dfrac{\pi}{2}, n \in \mathbb{Z}$.

iv) Absolute maxima is at $x = 1, -1$ and absolute minima is at $x = 0$.

v) Absolute maxima is at $x = -2, 2$ and absolute minima is at $x = 0$.

vi) All points x for which $0 \le x \le 1$ are critical points because the function is defined as

$$f(x) = \begin{cases} 1-2x, & \text{if} \quad -5 \le x < 0 \\ 1 & \text{if} \quad 0 \le x \le 1 \\ 2x-1 & \text{if} \quad 1 < x \le 5 \end{cases}$$

$f'(x) = 0$ if $0 < x < 1$ and f is not derivative at $x = 0$ and at $x = 1$.

Absolute maxima is at $x = -5$ and absolute minima is at $0 \le x \le 1$.

vii) Absolute minima is at $x = 1$ and absolute maxima cannot be found as the interval is not closed.

E5) The marginal cost is $C'(x) = \frac{1}{4}x + 4$.

The revenue is $R(x) = x\,p(x) = x(49 - x) = 49x - x^2$

The marginal revenue is $R'(x) = 49 - 2x$.

The profit is maximised when $R'(x) = C'(x)$.

Thus, we get $49 - 2x = \frac{1}{4}x + 4$ and $x = 20$

Hence, the price that corresponds to the maximum profit is $p(20) = 49 - 20 =$ ₹29

E6) The present value of the asset in t years is given by the function $P(t) = V(t)e^{-rt}$, where r is the annual interest rate and t is the time in years. Thus, $P(t) = 10000e^{\sqrt{t}}e^{-0.08t} = 10000e^{(\sqrt{t}-0.08t)}$

Differentiating $P(t)$ w.r.t. t, we obtain

$$P'(t) = 10000e^{(\sqrt{t}-0.08t)}\left(\frac{1}{2}\cdot\frac{1}{\sqrt{t}} - 0.08\right)$$

$P'(t) = 0$ when $\frac{1}{2\sqrt{t}} - 0.08 = 0$ or $t \approx 39.06$ years. Thus, the asset should be held for 39 years and then sold.

E7) To locate the extrema, we solve $C'(t) = 0$.

$$C'(t) = \frac{d}{dt}\left[\frac{k}{b-a}(e^{-at} - e^{-bt})\right]$$

$$= \frac{k}{b-a}[(-a)e^{-at} - (-b)e^{-bt}] = \frac{k}{b-a}(be^{-bt} - ae^{-at})$$

We see that $C'(t) = 0$ when $be^{-bt} = ae^{-at}$

which gives $\frac{b}{a} = e^{bt}e^{-at} = e^{bt-at}$, $bt - at = \ln\frac{b}{a}$ and $t = \frac{1}{b-a}\ln\frac{b}{a}$

Therefore, the largest concentration would occur when $t = \frac{1}{b-a}\ln\frac{b}{a}; b > a$. Let us now find the concentration as $t \to +\infty$.

$$\lim_{t\to+\infty} C(t) = \lim_{t\to+\infty}\frac{k}{b-a}[e^{-at} - e^{-bt}]$$

$$= \frac{k}{b-a}\left[\lim_{t\to+\infty}\frac{1}{e^{at}} - \lim_{t\to+\infty}\frac{1}{e^{bt}}\right]$$

$$= \frac{k}{b-a}[0-0]$$

$$= 0$$

This shows that the longer the drug is in the blood, the closer the concentration is to 0. The graph of C is shown in Fig. 36. Intuitively, we would expect the concentration function to begin at 0, increase to a maximum, and then gradually drop off to 0 in a finite amount of time. Fig. 36 indicates that $C(t)$ does not have these characteristics, because it does not quite get back to 0 in finite time.

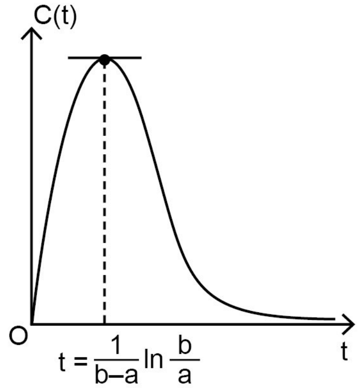


Fig. 36: Graph of $C(t)$.

E8) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$f'(x) = 12x^3 - 12x^2 - 24x$

$= 12x(x+1)(x-2)$

We divide the real line into intervals whose endpoints are the critical numbers $-1, 0$ and 2 as given in the Table 4.

Table 4

Interval	$12x$	$x+1$	$x - 2$	$f'(x)$	Monotonicity of f
$x < -1$	$-$ive	$-$ive	$-$ive	$-$ive	decreasing on $]-\infty, -1[$
$-1 < x < 0$	$-$ive	$+$ive	$-$ive	$+$ive	increasing on $]-1, 0[$
$0 < x < 2$	$+$ive	$+$ive	$-$ive	$-$ive	decreasing on $]0, 2[$
$x > 2$	$+$ive	$+$ive	$+$ive	$+$ive	increasing on $]2, \infty[$

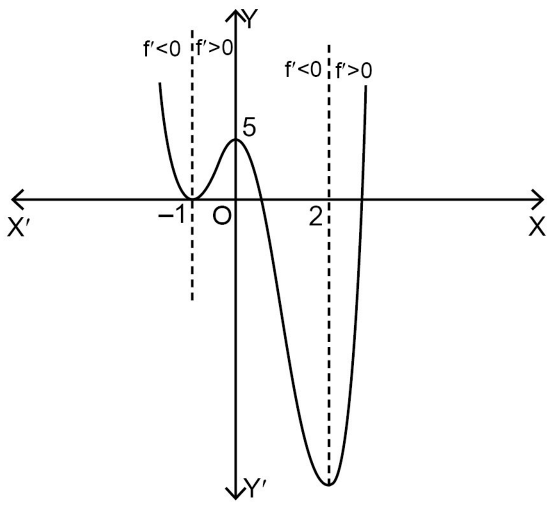


Fig. 37

The graph of f is shown in Fig. 37, which verifies the monotonicity of f given in the last column of Table 4.

E9) i) Increasing on $\left]0, \frac{\pi}{4}\right[$ and $\left]\frac{5\pi}{4}, 2\pi\right[$

Decreasing on $\left]\frac{\pi}{4}, \frac{5\pi}{4}\right[$.

ii) Increasing on $\left]-\frac{1}{3}\ln 2, \infty\right[$

Decreasing on $\left]-\infty, -\frac{1}{3}\ln 2\right[$

iii) Increasing on $]1, \infty[$
Decreasing on $]0, 1[$.

E10) i) Maximum at $x = 1$, minimum at $x = 3$, and the maximum and minimum values are 0 and -28 respectively. $x = 0$ is not an extremum because there is a neighbourhood of 0 in which $f'(x)$ has the same sign on either side of 0.

ii) Minima is at $x = 1,\ x = -3$.
Maxima is at $x = -1$
Extreme values are -3 and 29.

iv) Minima is at $x = 1$ and maxima is at $x = -1/3$, extreme values are 0 and $32/27$.

v) Maxima at $x = 4$, and the maximum value is $4\sqrt{2}$.

vi) Maxima at $x = 0$, and the maximum value is 1.

E11) $f(x) = x + 2\sin x$
$f'(x) = 1 + 2\cos x$
$f'(x) = 0$ gives $\cos x = \frac{-1}{2}$. The solution of this equation are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Table 5

Interval	**$f'(x)$**	**Monotonicity of f**
$0 < x < \frac{2\pi}{3}$	$+$ive	increasing
$\frac{2\pi}{3} < x < \frac{4\pi}{3}$	$-$ive	decreasing
$\frac{4\pi}{3} < x < 2\pi$	$+$ive	increasing

From Table 5, it is clear that $f'(x)$ changes sign from positive to negative at $\frac{2\pi}{3}$, the first derivative test tells us that there is a local maximum at $\frac{2\pi}{3}$, and the maximum value is $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sqrt{3} \approx 3.83$.

Similarly, $f'(x)$ changes sign from negative to positive at $\frac{4\pi}{3}$, therefore f has a local minima and the minimum value is

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3} \approx 2.46.$$

E12) i) At $x = 0$, minima, extreme value $= 0$

ii) There are no extreme points.

iii) Minimum at $x = -7/6$ and the minimum value is $-37/12$.

iv) $f'(x) = xg(x)$ where $g(x)$ is polynomial in x^2 with all co-efficients positive. Hence, $g(x) > 0$ for all $x \neq 0$. Therefore, the only extreme point of f is $x = 0$. Clearly, $f(0) = a_0$ and $f(x) > a_0, x \neq 0$. Hence, $x = 0$ is minima and the minimum value is a_0.

v) $g(x) = \dfrac{1}{f(x)} = x + \dfrac{1}{x}$. Extreme points of g are $x = \pm 1$. Minima at $x = 1$ and maxima at $x = -1$. Hence, the extreme value of f are $\pm 1/2$.

E13) $f'(x) = \cos x(1 + \cos x) + \sin x(-\sin x)$

$= \cos x + \cos 2x = 0$

$f'(x) = 0 \Rightarrow 2\cos^2 x + \cos x - 1 = 0$

$\Rightarrow (\cos x + 1)(2\cos x - 1) = 0$

$\Rightarrow \cos x = -1, \dfrac{1}{2}$

$\Rightarrow x = \dfrac{\pi}{3}$ is a critical number

f has maxima at $x = \dfrac{\pi}{3}$.

E14) If a and b are the sides of the inscribed rectangle.

$a^2 + b^2 = d^2 \Rightarrow b = \sqrt{d^2 - a^2}$

Area $= A = ab = a\sqrt{d^2 - a^2}$

$A' = 0$ if $a = d/\sqrt{2}$

$A'' < 0$ for $a = d/\sqrt{2} \Rightarrow a = d/\sqrt{2}$ is max.

$a = d/\sqrt{2} \Rightarrow b = d/\sqrt{2} \Rightarrow$ the rectangle is a square.

E15) Suppose the display area is a rectangle with dimensions a cm and b cm. Then the dimensions of the name plate are $(a+2)$ cm and $(b+4)$ cm.

$ab = 48 \Rightarrow b = 48/a$.

$A = (a+2)(b+4) = (a+2)\ (48/a + 4)$

$\dfrac{dA}{da} = 4 - \dfrac{96}{a^2} = 0 \Rightarrow a^2 = \dfrac{96}{4} = 24 \Rightarrow a = 2\sqrt{6}$

$\Rightarrow b = 4\sqrt{6}$

$\dfrac{d^2A}{da^2} = \dfrac{198}{a^3} > 0$ for $a = 2\sqrt{6} \Rightarrow$ This is a minimum.

Dimensions of the plate: $2(1+\sqrt{6}), 4(1+\sqrt{6})$.

E16) i) Yes. $y' = 2\sin x \cos x = \sin 2x = 0$ if $x = \pi/2 \in [0, \pi]$

ii) Yes. $f'(x) = 2x = 0$ if $x = 0 \in [-2, 2]$

iii) No. $f'(x) = 3x^2 + 1 \neq 0$ for $x \in [0, 1]$. Rolle's theorem does not hold as $f(0) \neq f(1)$.

iv) Yes. $f'(x) = \cos x - \sin x = 0$ if $x = \pi/4 \in [0, \pi/2]$.

v) Yes. $f'(x) = \cos x + \sin x = 0$ if $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4} \in [0, 2\pi]$

E17) $f(x) = x^2 - 3x + 2$, f is continuous on $[-1, 4]$ and is differentiable on $]-1, 4[$.
$f(-1) = 6 = f(4)$.
$f'(c) = 2c - 3 = 0$ gives $c = \frac{3}{2} \in]-1, 4[$
Rolle's theorem is verified.

E18) $ap^2 + bp + c = aq^2 + bq + c \Rightarrow ap^2 + bp - aq^2 - bq = 0$
$\Rightarrow a(p^2 - q^2) + b(p - q) = 0$
$\Rightarrow a(p + q) + b = 0$ (since, $p \neq q$)
$f'(x) = 2ax + b$
$f'\left(\frac{p+q}{2}\right) = a(p+q) + b = 0$.

E19) Suppose $f(x)$ is not one-one on $[x_0, \infty[$
$\Rightarrow$ $p, q \in [x_0, \infty[$, such that $p \neq q$ and $f(p) = f(q)$
$f'\left(\frac{p+q}{2}\right) = 0$ by E18).
$\frac{p+q}{2} = x_0$, as x_0 is the unique point with $f'(x_0) = 0$
Therefore either $p < x_0$ or $q < x_0$, since p and q both cannot be equal to x_0. This is a contradiction as we have taken $p, q \in [x_0, \infty[$.

E20) Suppose $p, q \in I$ s.t. $p \neq q$ and $f(p) = f(q)$
If $p < q$ we have $[p, q] \subset I$, f is differentiable on $[p, q]$ and $f(p) = f(q)$.
Thus, f satisfies the conditions of Rolle's theorem on $[p, q]$
$\Rightarrow$ $f'(x_0) = 0$ for some $x_0 \in [p, q] \subset I$.
But, this is a contradiction.
Therefore, f is one-one.

E21) i) $f(-1) = 2 = f(1) \Rightarrow \frac{f(1) - f(-1)}{1 - (-1)} = 0$
$f'(x) = 2x = 0$ if $x = 0$
$\therefore\ \exists\ 0 \in [-1, 1]$ s.t. $f'(0) = \frac{f(1) - f(-1)}{1 - (-1)}$

ii) You may like to try it yourself.

E22) i) $f(0) = 0 = f(2)$

$f'(x) = \pi\cos\pi x \Rightarrow f'(1/2) = 0$.

ii) $f(0) = 3,\ f(2) = 11 \Rightarrow \frac{f(2) - f(0)}{2 - 0} = 4$

$f'(x) = 4x \Rightarrow f'(1) = 4$

$\exists\, 1 \in [0,\ 2]$ s.t. $f'(1) = \frac{f(2) - f(0)}{2 - 0}$

E23) i) $f(0) = 0,\ f(1) = 1 \Rightarrow \frac{f(1) - f(0)}{1 - 0} = 1$

$f'(x) = 3x^2 = 1 \Rightarrow x = 1/\sqrt{3} \in [0,\ 1]$

$\therefore\ c = 1/\sqrt{3}$

ii) $f(-1) = -1,\ f(0) = 0 \Rightarrow \frac{f(0) - f(-1)}{0 - (-1)} = 1$

$f'(x) = 3x^2 = 1 \Rightarrow x = -1/\sqrt{3} \in [-1,\ 0]$

$c = -1/\sqrt{3}$.

iii) $\frac{f(1) - f(-1)}{1 - (-1)} = 1$

$c = 1/\sqrt{3},\ -1/\sqrt{3}$ are two points in $[-1,\ 1]$

such that $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$

E24) $f'(c) = \frac{f(b) - f(a)}{b - a} = 1$

$\frac{g_1(b) - g_1(a)}{b - a} = \frac{f(b) - f(a)}{b - a} = f'(c) = g_1'(c)$

$\frac{g_2(b) - g_2(a)}{b - a} = \frac{f(b) - f(a)}{b - a} + 1 = f'(c) + 1 = g_2'(c)$.

E25) i) $y' = nx^{n-1}$. Slope of the chord from $(0,\ 0)$ to $(2,\ 2^n)$ is

$\frac{y_2 - y_1}{x_2 - x_1} = 2^{n-1}$

$nx^{n-1} = 2^{n-1} \Rightarrow x^{n-1} = \frac{2^{n-1}}{n} \Rightarrow x = \frac{2}{n^{1/(n-1)}}$

Point: $\left(\frac{2}{n^{1/(n-1)}},\ \frac{2^n}{n^{n/(n-1)}}\right)$.

ii) Slope of the chord from $(0,\ 0)$ to $(1,\ 1)$ is 1

$\therefore\ nx^{n-1} = 1 \Rightarrow = x^{n-1} = 1/n \Rightarrow x = \frac{1}{n^{\frac{1}{n-1}}}$

Point: $\left(\frac{1}{n^{1/(n-1)}},\ \frac{1}{n^{n/(n-1)}}\right)$.

UNIT 14

CURVATURE

Structure **Page No.**

14.1 INTRODUCTION

In the last unit, we discussed some geometrical features of functions, like maxima, minima, monotonicity, etc. We continue this discussion in this unit to find what the second derivative f'' says about f in Sec.14.2. We started our study of Calculus by stating two problems. One of them was the problem of finding a tangent to a curve at a given point. In Unit 9, we have seen that the solution of this problem was instrumental in the development of differential calculus. Now having studied various techniques of differentiation, we shall once again take up this problem. We shall study the tangents to a curve and normals in Sec.14.3.

If two curves intersect at any point, then the tangents to both the curves at that point form an angle. This angle is the angle of intersection of the two curves, which we shall discuss in Sec.14.4. What happens if the curve passes through a point twice or more? Such points, where the curve shows different behaviour are not ordinary points, they are called singular points, which we shall study in Sec.14.5. You will see that all these will prove very useful when we tackle curve tracing in Unit 16.

Now, we shall list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

Objectives

After going through this unit, you should be able to:

- determine whether a curve is concave, or convex, or neither in a given interval;
- find the points of inflection and the curvature of a curve;
- obtain the equations of the tangent and the normal to the curve at a given point;
- calculate the angle of intersection of two curves at a given point of intersection;
- define, and identify, a singular point.

Recall the increasing and decreasing functions in Unit 13. There, you got an idea whether a certain graph is increasing, or decreasing on the basis of a visual of the graph. Sometimes, rising or falling of a graph does not give the complete picture of the graph, it gives only a partial picture of the graph. To get a more clear picture of the graph, we shall discuss concavity and convexity of a curve using the second derivative in the following section.

14.2 CONCAVITY

In this section, we shall use the second derivative to get a better picture of the graph of a function f. Let us begin with an example. Consider the function f defined by $f(x) = x^2 + 6$. We get $f'(x) = 2x$ and $f''(x) = 2$. Observe that $f'(-1) = -2$ and $f''(-1) = 2$. Here, $f'' > 0$ for all x around -1. Since, $f'(-1) = -2 < 0$, therefore, we can say that f is decreasing at -1 at the rate of -2. At the right of $x = -1$, since, $f''(-1)$ is positive, therefore, $f'(x)$ is larger than $f'(-1)$. That is, $f'(x)$ is less negative than $f'(-1)$. If we further move to the right, $f'(x)$ is still less negative. If we continue, then there would be some slope $f'(x)$, which may become positive. In Fig. 1, the arrow starting from -1 shows that the slope of $f'(x)$ is decreasing.

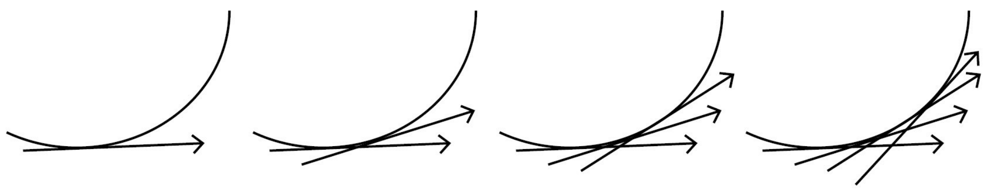


Fig. 1: Graph Bending Upwards.

We can say that when $f''(a) > 0$, the graph of f near the point a, is bending upward, whether $f'(a) < 0$ or $f'(a) > 0$. On the other hand, when $f''(a) < 0$, the graph of f near the point a is bending downward, whether $f'(a) < 0$ or $f''(a) > 0$. The "**bending**" behaviour of a graph is called its **concavity**. This leads to the following definition.

Definition: When the chord lies above the graph in an interval I, the graph is concave upward and when it lies below the graph in an interval I, the graph is concave downward.

Fig. 2 (a) and Fig. 2(b) shows this respectively.

(a) Concave upward **(b) Concave downward**

Fig. 2

We can also say that, if f is twice differentiable on an interval I and f' is increasing on that interval, the graph of f is **concave upward** on I. If f' is decreasing on I, the graph of f is **concave downward** on I.

We know that the slope of the tangent line is computed through derivative. Therefore, from the discussion above, we conclude that the graph of a function f is concave upward if f' is strictly increasing and the graph of a function f is concave downward if f' is strictly decreasing. This can further be said that if f' is increasing, then $(f')' > 0$ (recall from Unit 13), which means that the graph of f is concave upward where the second derivative f'' satisfies $f'' > 0$. Similarly, the graph is concave downward where $f'' < 0$. We can use this observation to test concavity of a graph of a function, which is as follows.

Concavity Test:

i) If $f''(x) > 0$ for all x on I, then the graph of f is concave upward on I.

ii) If $f''(x) < 0$ for all x on I, then the graph of f is concave downward on I.

You may note that a function can be concave upward and increasing or concave upward and decreasing or concave downward and increasing or concave downward and decreasing or concave upward/downward but neither increasing nor decreasing, like $y = x^2$. This means that concavity and monotonicity are independent.

Some books use the term '**convex downward'** for concave upward and '**convex upward'** for concave downward. Sometimes, we drop upward and downward and simply write **concave** (for **concave downward** or **convex upward**) and **convex** (for **concave upward** or **convex downward**).

We also say that a function f is concave at a point, if it is concave in a small open interval around that point. Similarly, a function f is convex at a point, if it is convex in a small open interval around that point.

Remark: i) Only concave and convex functions have the property that each of their tangents intersects their graph exactly once in the interval of concavity.

ii) If f is concave on I, then $-f$ is convex on I. Similarly, if f is convex on I, then $-f$ is concave on I.

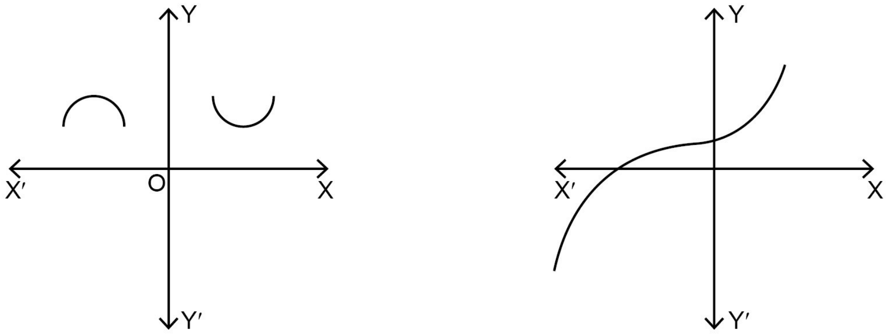
Let us investigate concavity in the following examples:

Example 1: Find the concavity of the graph of the function f defined by $f(x) = 3x^3 + 2x + 5$. Also, draw its rough sketch.

Solution: We have $f(x) = 3x^3 + 2x + 5$, and we get $f'(x) = 9x^2 + 2$ and $f''(x) = 18x$.

Here, $f''(x) < 0$, when $x < 0$, and $f''(x) > 0$, when $x > 0$.

So, the graph of f is concave downward when $x < 0$ and the graph of f is concave upward when $x > 0$ as shown in Fig. 3 (a). The graph of f is shown in Fig. 3 (b).



(a) (b)

Fig. 3

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Example 2: Sketch the graph of the function f defined by $f(x) = x^3 + 3x^2 - 9x - 13$. State, where it is concave upward or concave downward along with monotonicity.

Solution: To find the concavity of f, we determine f' and f''.

$f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1)$

$f''(x) = 6x + 6 = 6(x+1)$

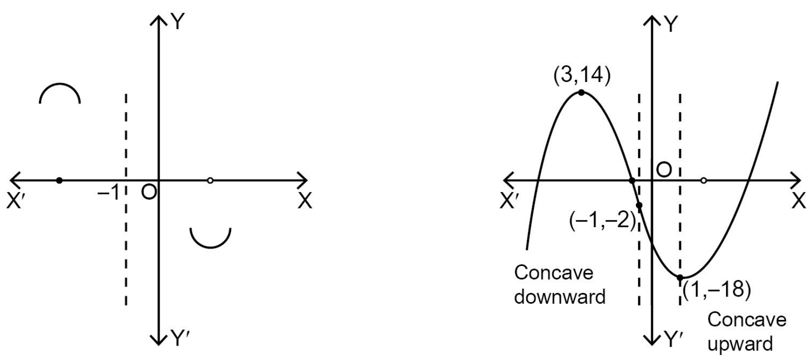
$f''(x) > 0,$ when $x > -1$ and $f''(x) < 0,$ when $x < -1$. Thus, the graph of f is concave upward when $x > -1$ and is concave downward when $x < -1$. For the rough sketch of f, let us see where f is increasing or decreasing. For this, we solve $f'(x) = 0$

$3x^2 + 6x - 9 = 0$

$3(x+3)(x-1) = 0$

$x = -3$ or $x = 1$

When $x < -3, f' > 0,$ therefore, f is increasing. When $-3 < x < 1, f' < 0$, therefore, f is decreasing. When $x > 1, f' > 0$, therefore, f is increasing. We can now plot the points $(-3, 14)$ and $(1, -18)$, including short arcs at each point to indicate the concavity of the graph as shown in Fig. 4(a). Then, by calculating and plotting a few more points, we can make a graph, as shown in Fig. 4 (b).



(a) (b)

Fig. 4: Graph of f.

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Example 3: From the graph given in Fig. 5, find the intervals in which f is concave upward and concave downward.

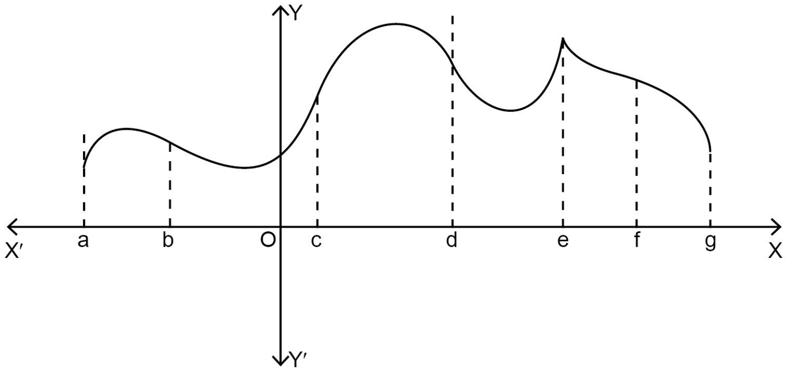

Fig. 5

Solution: Fig. 5 shows that the function f is concave upward on the intervals $]b, c[,]d, e[$ and $]e, f[$ and concave downward on the intervals $]a, b[,]c, d[$ and $]f, g[$.

***

Example 4: Suppose that water is poured in the vase as shown in Fig. 6, at a constant rate, measured in volume per unit time. The height of water at time t is $f(t)$. Comment on the concavity of the graph of depth of water in the vase.

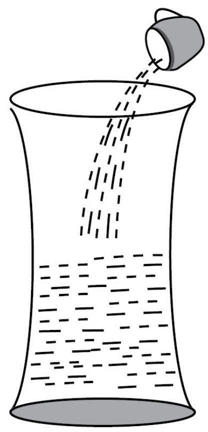

Fig. 6

Solution: At the bottom of the vase, the water level would rise slowly, because, the base of the vase is wide and, so it would take a lot of water to make the height increase. However, as the vase narrows, the rate at which the water is rising, increases. This means that initially f' is increasing at an increasing rate and the graph is concave upward. The rate of increase in the water level is at a maximum when the water reaches the middle of the vase, where the diameter is smallest. After that, the rate at which f' increases starts to decrease again, and so the graph is concave downward. The graph of the height of water in the vase f against the time t is shown in Fig. 7.

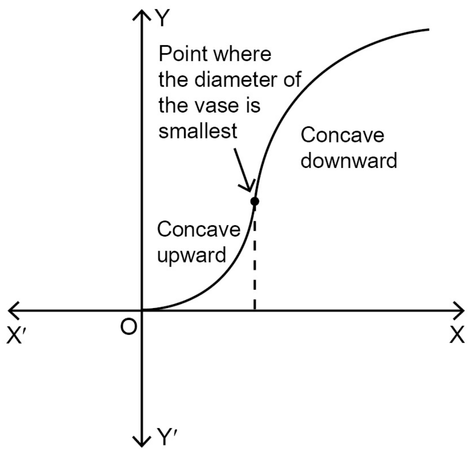

Fig. 7: Graph of f.

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Try the following exercises.

E1) Sketch a possible graph of the function f that satisfies the following conditions:

i) $f'(x) > 0$ on $]-\infty, 2[$ and $f'(x) < 0$ on $]2, \infty[$.

ii) $f''(x) > 0$ on $]-\infty, -3[$ and $]3, \infty[$ and $f''(x) < 0$ on $]-3, 3[$.

iii) $\lim_{x \to -\infty} f(x) = -4$ and $\lim_{x \to \infty} f(x) = 1$

E2) Find the intervals in which the curve $y = x^4 - 4x^3$ is concave or convex.

So far, we were concerned only with the manner of bending of a graph. Now, let us discuss about that transition point which changes the graph from concave upward to concave downward. Look at the graph of the function in Example 2. The concavity of f changes from downward to upward at the point $(-1, -2)$. The point across which the direction of concavity changes is called a **point of inflection** or **an inflection point**. Fig. 8 shows the transition from concave upward to concave downward at the points P and R, and the transition from concave downward to concave upward at Q. The points P, Q and R are points of inflection as the sign of f'' changes.

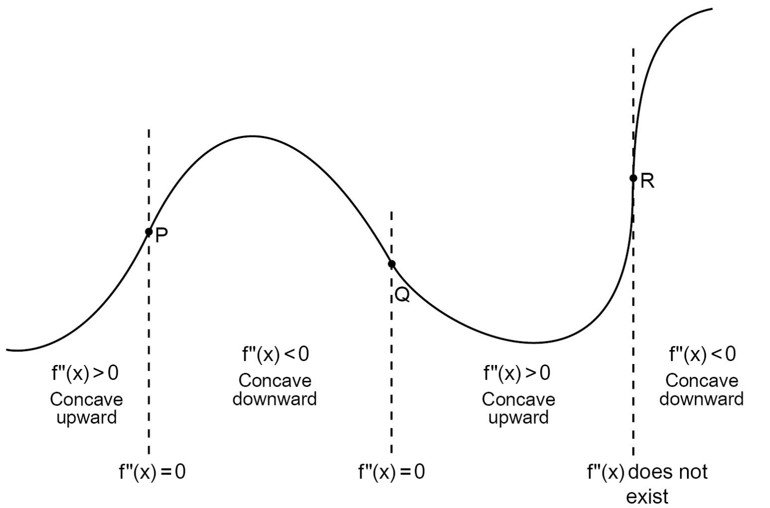

Fig. 8: Point of inflection.

In Fig. 8, as we move from left to right, we see that the concavity changes at P, Q and R and either the value of $f''(x_0)$ at P and Q must be 0 or $f''(x)$ must not exist at R. This leads to the following definition.

Definition: A function f has a point of inflection at a point x_0, if the concavity changes at x_0.

Therefore, we can say that for a curve to locate a point of inflection at x_0, the necessary condition is either $f''(x_0) = 0$ or $f''(x_0)$ does not exist. You may note that an inflection point must be on the graph, meaning $f(x_0)$ must be defined if there is an inflection point at $x = x_0$. You may also note that, this condition is only necessary. If $f''(x_0) = 0$ or $f''(x_0)$ does not exist, then, there may or may not be a point of inflection at x_0. There must be a change in the direction of concavity on either side of x_0 for $(x_0, f(x_0))$ to be a point of inflection.

Now, let us understand this through the following examples.

Example 5: Find the point of inflection of the function f defined by $f(x) = x^4$, if it exists.

Solution: We have $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Also, $f''(0) = 0$. Here, f does not have a point of inflection at $x = 0$ even though $f''(0) = 0$ (See Fig. 9). This happens because $f''(x) = 12x^2 > 0$ for all $x \neq 0$, and accordingly, it does not change sign in passing through 0. Thus, $f''(x_0) = 0$ is not sufficient, for f to have a point of inflection at $x = x_0$.

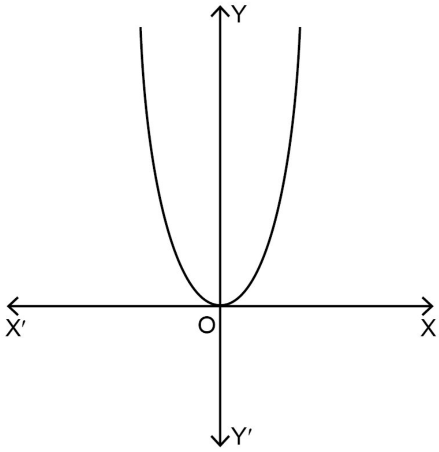

Fig. 9: Graph of $y = x^4$.

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Example 6: Find the values of x for which the graph of the function f defined by $f(x) = 1/x, x \in \mathbb{R} \setminus \{0\}$ is concave upward and concave downward. Also, find the points of inflection, if any.

Solution: We shall find if the graph (See Fig. 10) has any point of inflection.
Here, $f'(x) = -1/x^2$, and $f''(x) = 2/x^3$.
Clearly, i) $f''(x) > 0$ if $x > 0$

ii) $f''(x) < 0$ if $x < 0$

It follows from (i) that the graph of f is concave upward in $]0, \infty[$. From (ii) we deduce that the graph is concave downward in $]-\infty, 0[$. From the fact that f'' exists for all x in the domain of f and not at $x = 0$, we conclude that the graph has no point of inflection.

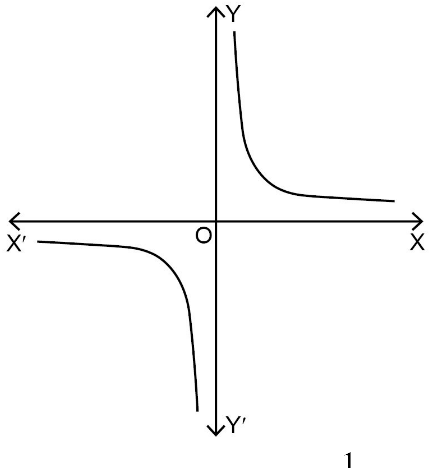

Fig. 10: Graph of $y = \frac{1}{x}$.

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Example 7: Find the point(s) of inflection for $f(x) = x^{2n+1}$, for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Solution: We have $f'(x) = (2n+1)\,x^{2n}$ and $f''(x) = (2n+1).2n.x^{2n-1}$
Here, $f''(x) = 0 \Rightarrow x = 0$. Also, when $x < 0, f''(x) < 0$ and when $x > 0, f''(x) > 0$. Therefore, the sign of f'' changes and direction of concavity changes accordingly, that is, f'' changes sign (from negative to positive) while passing through the origin. Thus, the origin is a point of inflection on the graph of $f(x) = x^{2n+1}$.

***

Example 8: Determine the point(s) of inflection for the function f given by $f(x) = 2x^5 - 15x^3$ and sketch the graph.

Solution: The function f is given by $f(x) = 2x^5 - 15x^3$ and its first and second derivatives are $f'(x) = 10x^4 - 45x^2$ and $f''(x) = 40x^3 - 90x$, respectively. We set the second derivative equal to 0 and solve for x.

$$40x^3 - 90x = 0$$

$$40x\left(x^2 - \frac{9}{4}\right) = 0$$

$x = 0$ or $x = +\frac{3}{2}$ or $x = -\frac{3}{2}$

Next, we check the sign of $f''(x)$ over the intervals bounded by these three x-values, the change in sign of f'' is given in Table 2.

Table 2

Interval	Sign of f''	Concavity of f
$x < -\frac{3}{2}$	$-$	Concave downward
$-\frac{3}{2} < x < 0$	$+$	Concave upward
$0 < x < \frac{3}{2}$	$-$	Concave downward
$x > \frac{3}{2}$	$+$	Concave upward

The graph of f changes from concave downward to concave upward at $x = -\frac{3}{2}$, concave upward to concave downward at $x = 0$ and concave downward to concave upward at $x = \frac{3}{2}$. Therefore,

$\left(-\frac{3}{2}, f\left(-\frac{3}{2}\right)\right), (0, f(0))$ and $\left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right)$ are points of inflection.

Since, $f\left(-\frac{3}{2}\right) = \frac{567}{16}, f(0) = 0$ and $f\left(\frac{3}{2}\right) = -\frac{567}{16}$, therefore, the points $p\left(-\frac{3}{2}, \frac{567}{16}\right), O(0,0)$ and $Q\left(\frac{3}{2}, -\frac{567}{16}\right)$ are the points of inflection and these points are shown in Fig. 11.

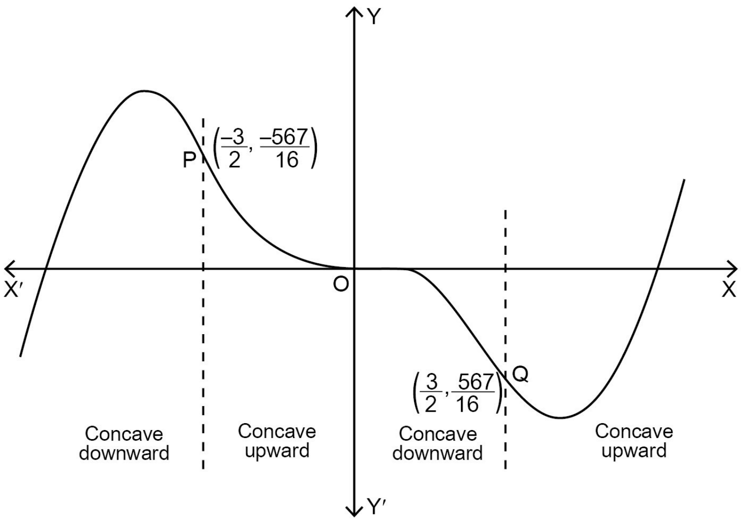

Fig. 11: Graph of $y = 2x^5 - 15x^3$.

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Example 9: Determine the concavity and point(s) of inflection for the function f given by $f(x) = (2x-5)^{1/3} + 1$.

Solution: Derivaties of f are

$$f'(x) = \frac{1}{3}(2x-5)^{-2/3} \cdot 2 = \frac{2}{3}(2x-5)^{-2/3}$$

$$f''(x) = -\frac{4}{9}(2x-5)^{-5/3} \cdot 2 = -\frac{8}{9}(2x-5)^{-5/3}$$

When $x < \frac{5}{2}, f''(x) > 0$, so, f is concave upward on $\left]-\infty, \frac{5}{2}\right[$. When $x > \frac{5}{2}, f''(x) < 0,$ so, f is concave downward on $\left]\frac{5}{2}, \infty\right[$. To find the point of inflection, we find where $f''(x) = 0$ and where $f''(x)$ does not exist. Since, $f''(x)$ is never 0, we only need to find where $f''(x)$ does not exist. Thus, the possible inflection point is $\left(\frac{5}{2}, f\left(\frac{5}{2}\right)\right)$, that is $\left(\frac{5}{2}, 1\right)$. The graph is shown in Fig. 12.

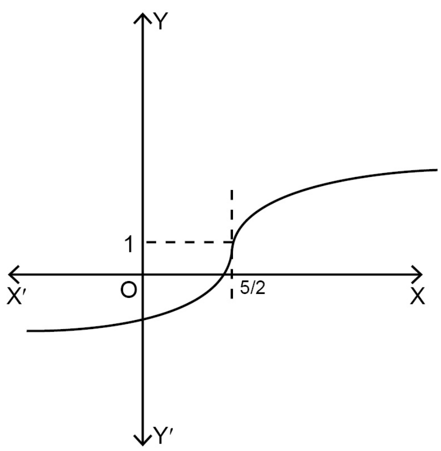

Fig. 12: Graph of $y = (2x-5)^{\frac{1}{3}} + 1$.

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Example 10: An efficiency study of the morning shift at a factory indicates that the number of units produced by an average worker in t hours after 8:00 am is given by $Q(t) = -t^3 + 9t^2 + 12t$. At what time in the morning is the worker performing most efficiently?

Solution: We assume that the morning shift runs from 8:00 am until noon and that worker's efficiency is maximized when the rate of production $Q'(t) = -3t^2 + 18t + 12$ is as large as possible for $0 \le t \le 4$. The second derivative of Q is $Q''(t) = -6t + 18$, which is zero, when $t = 3$. This is the point of inflection, we can say that the rate of production $Q(t)$ is greatest and the worker is performing most efficiently, when $t = 3$, that is, at 11:00 am. The graphs of the production function Q, its derivative and the rate of production function are shown in Fig. 13. Notice that the production curve is steepest and the rate of production is greatest when $t = 3$.

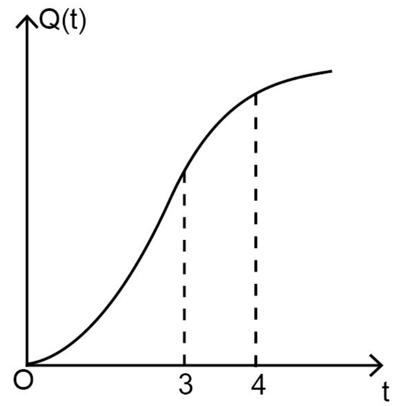

Fig. 13: Graph of a production function curve showing the point of inflection.

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Try the following exercise.

E3) Determine the concavity, convexity and points of inflection of the following functions.

i) $f(x) = x^3, \forall x \in \mathbb{R}$

ii) $f(x) = x^{1/3}, \forall x \in \mathbb{R}$

iii) $f(x) = x^4 - 2x^3 - 12x^2 + 1, \forall x \in \mathbb{R}$

iv) $f(x) = (x-2)/(x-3), \forall x \in \mathbb{R} - \{3\}$

v) $f(x) = \ln x, x > 0$

vi) $f(x) = \cos x, 0 < x < 2\pi$

Let us now discuss the measure of the bending of a graph at a point, which is known as the **curvature** at that point. For this, we draw a circle that closely fits nearby points on a local section of a curve, as given in Fig. 14.

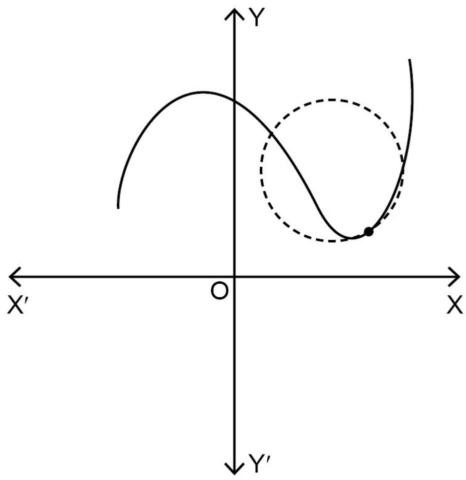

Fig. 14

We say that the curve and the circle osculate, as both the curves (the circle and the curve) have the same tangent at the point where they meet.

The **radius of curvature** of the curve at a particular point is defined as the radius of the approximating circle. The radius changes as we move along the curve. How do we find this changing radius of curvature?
We measure the curvature at $(c, f(c))$ by the ratio.

$$k(c) = \frac{f''(c)}{[1+(f'(c))^2]^{3/2}}$$

The radius of curvature at $(c, f(c))$ is denoted by $\rho(c)$ and is defined by

$\rho(c) = \frac{1}{k(c)}$, if $k(c) \neq 0$.

Let us find the radius of curvature in the following examples.

Example 11: Find the radius of curvature for the cubic $y = 2x^3 - x + 3$ at $x = 1$.

Solution: Here, $f(x) = 2x^3 - x + 3$. First, let's draw the graph of the function f as given in Fig. 15.

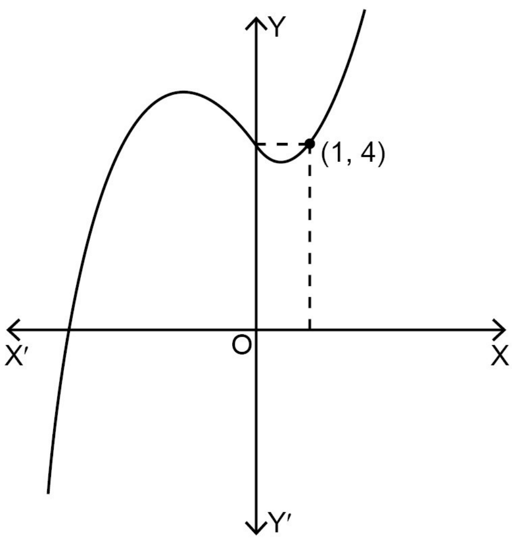

Fig. 15

To find the radius of curvature, we need derivatives. Thus, $f'(x) = 6x^2 - 1$, and $f'(1) = 5$. Also, $(f(1))^2 = 25$. Here, $f''(x) = 12x$ and $f''(1) = 12$.
Now, we are ready to substitute these values in the formula to get the radius at any point c. We get

$$\rho(c) = \frac{[1+(f'(c))]^{3/2}}{f''(c)} = \frac{(1+25)^{3/2}}{12} \approx 11.05$$

To show what we have done, let's look at the graph of the curve with the approximating circle overlaid. The circle is a good approximation for the curve at $(1, 4)$ as shown in Fig. 16.

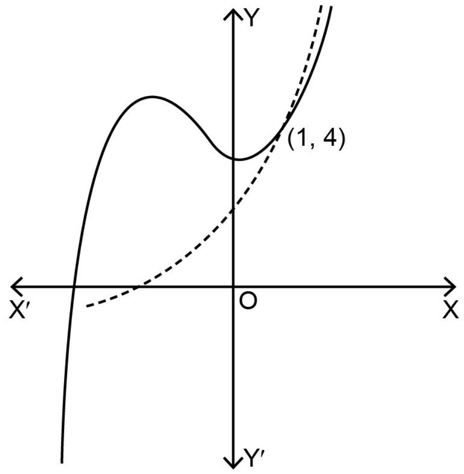

Fig. 16

Now try the following exercises.

E4) Find the radius of curvature of the function f, defined by, $f(x) = 1.5x^2 - 2.5x + 2$ at $(2, 3)$.

E5) Find the curvature at an arbitrary point of the graph of the function

i) $f(x) = x - 5,\ x \in \mathbb{R}$.

ii) $f(x) = x^2 + 9,\ x \in \mathbb{R}$

iii) $f(x) = \sin x,\ x \in \mathbb{R}$

iv) $f(x) = \sqrt{(1 - x^2)},\ -1 < x < 1$

So far, we discussed the applications of the second derivative. We go back to the first derivative now and discuss tangents and normals in the following section.

14.3 EQUATIONS OF TANGENTS AND NORMALS

In Unit 9, we have seen how a derivative can be defined precisely with the help of the slope of the tangent at any point to a curve. We have noted that the slope of a tangent to the curve $y = f(x)$ at (x_0, y_0) is given by $f'(x_0)$, whenever it exists. In fact, we had also obtained the equations of the tangents of some simple curves. Once, we know how to find the equation of a tangent, it is easy to find the equation of a normal using slope of the tangent. A normal to a curve, $y = f(x)$ at (x_0, y_0) is a line which passes through (x_0, y_0) and is perpendicular to the tangent at that point. This means that the slope of this normal will be $-\frac{1}{f'(x_0)}$, if $f'(x_0) \neq 0$. You may recall Unit 3, where we discussed that a line L_1 is perpendicular to a line L_2 iff $m_1 m_2 = -1$, where m_1 and m_1 are the slopes of L_1 and L_2, respectively.

Now, what happens when $f'(x_0) = 0$? The derivative $f'(x_0) = 0$ implies that the slope of the tangent at (x_0, y_0) is zero, that is, this tangent is parallel to the x-axis. In this case, the normal (which is perpendicular to the tangent) would be parallel to the y-axis. The equation of this normal, would then be $x = x_0$.

Now, let us look at various curves and try to obtain the equations of their tangents and normals using derivatives.

Example 12: Find the equation of the tangent and normal to the curve $y = 2\sqrt{ax}$, $a \neq 0$.

Solution: Consider the curve given in Fig. 17. Here, $\frac{dy}{dx} = \sqrt{\frac{a}{x}} = \frac{2a}{y}$. Thus, $\frac{dy}{dx}$ exists and is non-zero for all $y \neq 0$. Now, y will be zero, only if, x is zero. Thus, we can find the equations of tangent and normal to this curve at any point, except the origin $(0, 0)$. The slope of the tangent at any point (x_0, y_0) will be $2a / y_0$. The slope of the normal at (x_0, y_0) will, therefore, be $-y_0 / 2a$.

Thus, the equation of the tangent at (x_0, y_0) is $y - y_0 = \frac{2a}{y_0}(x - x_0)$

Recall Unit 3, where we learnt that the equation of a line passing through the point (x_0, y_0) and having a slope m is $y - y_0 = m(x - x_0)$.

$\Rightarrow \quad yy_0 - y_0^2 = 2ax - 2ax_0$

$\Rightarrow \quad yy_0 = 2ax + y_0^2 - 2ax_0$

$\Rightarrow \quad yy_0 = 2a(x + x_0)$ [since, $y_0^2 = 4ax_0$]

The equation of the normal at (x_0, y_0) is $y - y_0 = \frac{-y_0}{2a}(x - x_0)$, which is $2ay = 2ay_0 + x_0 y_0 - xy_0$ after simplification, as shown in Fig. 17.

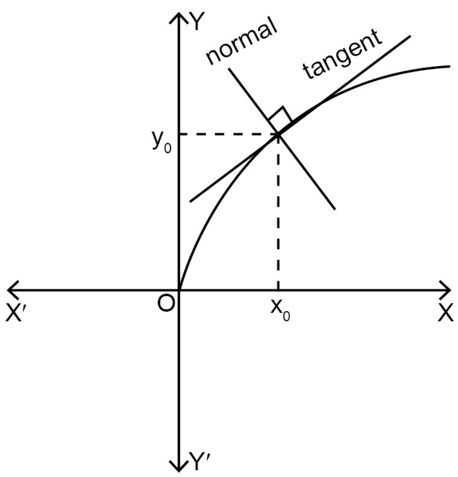

Fig. 17: Tangent and Normal of $y = 2\sqrt{ax}$.

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Example 13: The position of a moving car at time t is given by
$f(t) = t^3 - 2t^2 + 5$.

i) Suppose the car is moving around a corner and is driven over something slippery on the road (like oil, ice, water or loose gravel) and also suppose that the car starts to skid (see Fig. 18). What would be the equation of the path is which car skids at $t = 2$.

ii) If the car is moving fast around a circular track, the force that the traveller feels pushing outwards is normal to the curve of the road. The force that is making the traveller go around that corner is actually directed towards the center of the circle, that is, normal to the circle. Hence, find the equation of the normal.

iii) Draw the graph of the tangent and normal to the road at $t = 2$.

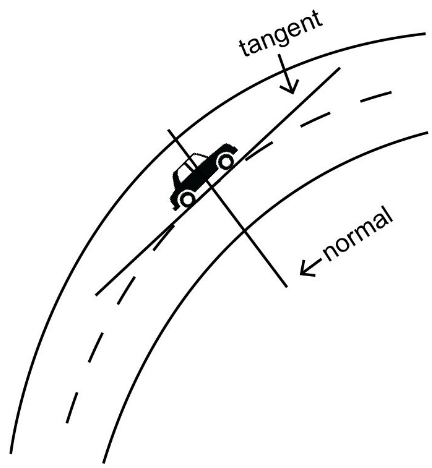

Fig. 18: A skidding car at the corner.

Solution: At $t = 2, f(2) = 5$. The car will start to skid and continue in a direction along the tangent. Here, $f'(t) = 3t^2 - 4t$.
The slope of the tangent to the road at the point where the car starts to skid is $m_1 = f'(2) = 4$.
Equation of the tangent is $y - 5 = 4(t - 2)$, that is, $4t - y - 3 = 0$.

The slope of the normal is $m_2 = -\frac{1}{m_1} = -\frac{1}{4}$.

Equation of the normal is $y - 5 = -\frac{1}{4}(t - 2)$

That is, $t + 4y - 22 = 0$.
Fig. 19 shows the graph of the tangent and normal to the curve at $t = 2$.

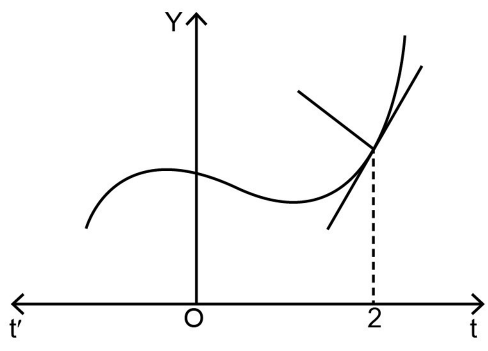

Fig. 19

Example 14: Find the points on the graph of $f(x) = -x^3 + 6x^2$ at which the tangent line is horizontal.

Solution: We use the derivative to find the slope of a tangent line, and a horizontal tangent line has slope 0. Therefore, we need to find all x for which $f'(x) = 0$. Setting $f'(x) = 0$

$$\frac{d}{dx}\left[-x^3 + 6x^2\right] = 0$$

$$-3x^2 + 12x = 0$$

$x = 0$ or $x = 4$.

To find the points, we need to find y-coordinate also, which is $f(0) = 0$ and $f(4) = 32$. Thus, the points, at which tangents are horizontal, are $(0,0)$ and $(4,32)$. These points are origin O and P, as shown in Fig. 20.

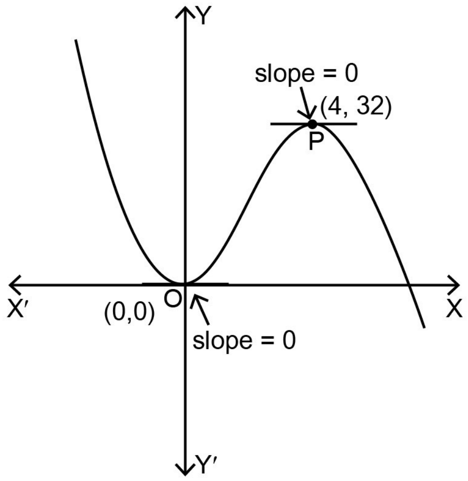

Fig. 20: Graph of f with horizontal tangents.

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Now, let us find the tangents and normal of the curves in the following examples where the equation of the curve is given in the parametric form. In Block 3, we have already discussed parametric equation.

Example 15: Find the equations of the tangent and the normal at the point $\theta = \pi/4$ to the curve given by $x = a\cos^3\theta,\ y = a\sin^3\theta$.

Solution: The rough sketch of the curve is given in Fig. 21. Let us find the derivative of y with respect to x, so that we get the slope of the tangent line.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

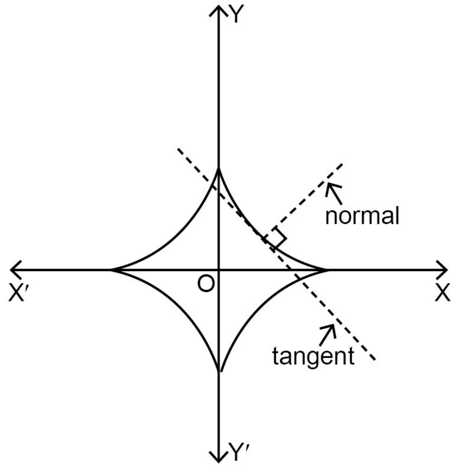

Fig. 21: Graph of parametric curve.

The slope of the tangent at $\theta = \pi/4$ is $-\tan(\pi/4) = -1$. The slope of the normal at this point, thus, comes out to be 1. Now, if $\theta = \pi/4, \cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$.

Thus, $x = \frac{a}{2\sqrt{2}}$ and $y = \frac{a}{2\sqrt{2}}$.

The equation of the tangent at $\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$ is

$$y - \frac{a}{2\sqrt{2}} = -1\left(x - \frac{a}{2\sqrt{2}}\right)$$

That is, $x + y = \frac{a}{\sqrt{2}}$ or $\sqrt{2}(x + y) = a$

The equation of the normal at $\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$ is

$$y - \frac{a}{2\sqrt{2}} = 1\left(x - \frac{a}{2\sqrt{2}}\right) \text{or } y = x.$$

***

By now, you are quite familiar with the fact that $f'(x)$ or dy/dx may not exist at some points. At such points either the tangent does not exist, or else, is parallel to the y-axis, that is vertical. To examine the existence of vertical tangents at (x_0, y_0), we examine $\left(\frac{dx}{dy}\right)_{(x_0, y_0)}$. If $\left(\frac{dx}{dy}\right)_{(x_0, y_0)} = 0$, then, we conclude that there is a vertical tangent at (x_0, y_0). In such cases, the equation of the tangent can be written as $x = x_0$.

The normal corresponding to a vertical tangent will be horizontal or parallel to the x-axis. This means, we can write its equation as $y = y_0$, as its passes through $x = x_0$.

If you consider the curve taken in Example 15, you will find that $\frac{dy}{dx}$ does not exist when $\theta = \pi/2$. Let us examine $\frac{dx}{dy}$ at this point $\frac{dx}{dy} = -\cot\theta = 0$ if $\theta = \pi/2$. This means, that the curve has a vertical tangent and, consequently, a horizontal normal at this point. Now, when $\theta = \pi/2$, $x = 0$ and $y = a$. Thus, the equation of the tangent at $(0, a)$ is $x = 0$ and that of the normal is $y = a$.

Let us now look at another example.

Example 16: A flight of a paper aeroplane, follows the trajectory $x = t - 3\sin t$ and $y = 4 - 3\cos t, (t \geq 0)$.
but crashes into a wall at time $t = 10$.

i) At what time was the aeroplane flying horizontally?

ii) At what time was it flying vertically?

Solution: i) The aeroplane was flying horizontally at those times when $dy/dt = 0$ and $dx/dt \neq 0$. From the given trajectory, we have

$$\frac{dy}{dt} = 3\sin t \text{ and } \frac{dx}{dt} = 1 - 3\cos t$$

Setting $dy/dt = 0$ yields the equation $3\sin t = 0$, or, more simply, $\sin t = 0$. This equation has four solutions in the time interval $0 \leq t \leq 10$ and are $t = 0,\ t = \pi,\ t = 2\pi,\ t = 3\pi$.

Now, $\frac{dx}{dt} = -1 - 3\cos t$

$$\left(\frac{dx}{dt}\right)_{t=0} = -2$$

$$\left(\frac{dx}{dt}\right)_{t=\pi} = 4$$

$$\left(\frac{dx}{dt}\right)_{t=2\pi} = -2$$

$$\left(\frac{dx}{dt}\right)_{t=3\pi} = 4$$

Since, $dx/dt = 1 - 3\cos t \neq 0$ for these values of t, the aeroplane was flying horizontally at times $t = 0,\ t = \pi \approx 3.14,\ t = 2\pi \approx 6.28,$ and $t = 3\pi \approx 9.42$ which is consistent with Fig. 22.

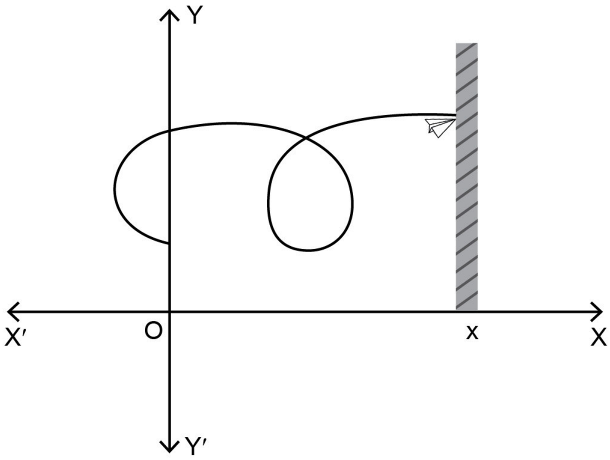

Fig. 22

ii) The aeroplane was flying vertically at those times when $dx/dt = 0$ and $dy/dt \neq 0$. Setting $dx/dt = 0$, we get $1 - 3\cos t = 0$ or $\cos t = \frac{1}{3}$

This equation has three solutions in the time interval $0 \leq t \leq 10$, these are $t = \cos^{-1}\frac{1}{3},\quad t = 2\pi - \cos^{-1}\frac{1}{3},\quad t = 2\pi + \cos^{-1}\frac{1}{3}$.

Since $dy/dt = 3\sin t$ is not zero at these points, it follows that the aeroplane was flying vertically at times

$$t = \cos^{-1}\frac{1}{3} \approx 1.23,\quad t \approx 2\pi - 1.23 \approx 5.05,\quad t \approx 2\pi + 1.23 \approx 7.51$$

***

Now, let us find the tangent line of the curve in polar form. To find a method for obtaining slopes of tangent lines to polar curves of the form $r = f(\theta)$ in which r is a differentiable function of θ. We express x and y parametrically in terms of the parameter θ by substituting $f(\theta)$ for r in the equations $x = r\cos\theta$ and $y = r\sin\theta$. This yields

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$

Now, differentiating x and y with respect to θ, we obtain

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$$

$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta = r\cos\theta + \frac{dr}{d\theta}\sin\theta$$

Thus, if $dx/d\theta$ and $dy/d\theta$ are continuous and if $dx/d\theta \neq 0$, then y is a differentiable function of x, and the derivative dy/dx with θ in place of t yields

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}}$$

Now, let us find the slope of tangent of polar curve in the following example:

Example 17: Find the slope of the tangent line to the circle $r = 4\cos\theta$ at the point where $\theta = \pi/4$.

Solution: Here, $r = 4\cos\theta$, we obtain

$$\frac{dy}{dx} = \frac{4\cos^2\theta - 4\sin^2\theta}{-8\sin\theta\cos\theta} = \frac{4\cos 2\theta}{-4\sin 2\theta} = -\cot 2\theta$$

Thus, at the point where $\theta = \pi/4$ the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -\cot\frac{\pi}{2} = 0$$

which implies that the circle has a horizontal tangent line at the point where $\theta = \pi/4$ (Fig. 23).

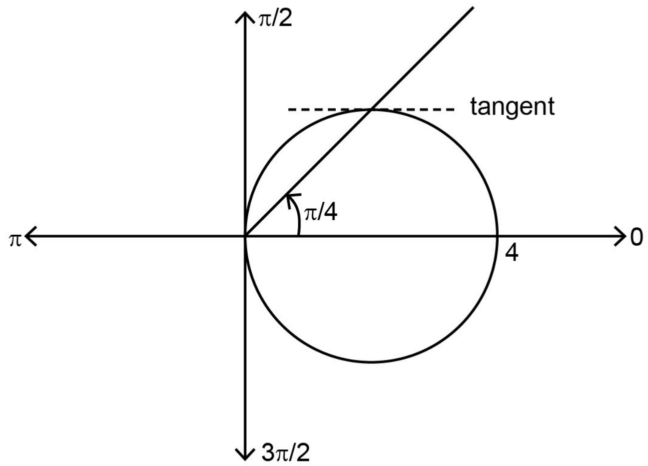

Fig. 23

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The following example illustrates the method of finding the equations of tangents and normals when the equation of the curve is given in the implicit form.

Example 18: Find the equations of the tangent and the normal to the curve defined by $x^3 + y^3 - 6xy = 0$ at a point $P(x_0, y_0)$ on it.

Solution: Fig. 24 shows the rough sketch of the curve $x^3 + y^3 - 6xy = 0$. To find the equations of tangent and normal, we first find the slope of the tangent using derivative.

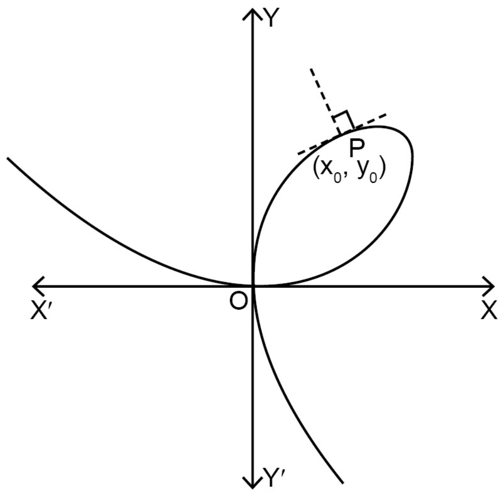

Fig. 24: Graph of $x^3 + y^3 - 6xy = 0$.

In Unit 10, we have seen how to calculate the derivative when the relation between x and y is expressed implicitly. We shall follow the same procedure again. Differentiating the given equation throughout with respect to x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} - 6y - 6x\frac{dy}{dx} = 0 \text{, which gives } \frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

Thus, the slope at the point (x_0, y_0) is $\frac{2y_0 - x_0^2}{y_0^2 - 2x_0}$.

Hence, the equation of the tangent at (x_0, y_0) is

$$y - y_0 = \frac{2y_0 - x_0^2}{y_0^2 - 2x_0}(x - x_0)$$

On simplifying and using the relation $x_0^3 + y_0^3 = 6x_0y_0$, the equation of tangent reduces to

$$(2y_0 - x_0^2)x + (2x_0 - y_0^2)y + 2x_0y_0 = 0$$

Now, the normal at (x_0, y_0) is a line passing through (x_0, y_0) and having slope $\frac{-(y_0^2 - 2x_0)}{2y_0 - x_0^2}$. Hence, the equation of the normal at (x_0, y_0) is

$$y - y_0 = -\frac{y_0^2 - 2x_0}{2y_0 - x_0^2}(x - x_0).$$

On simplifying, we get

$$(y_0^2 - 2x_0)x + (2y_0 - x_0^2)y + (x_0 - y_0)(2x_0 + x_0y_0 + 2y_0) = 0.$$

***

Example 19: Find the equations of those tangents to the curve $y = x^3$, which are parallel to the line $12x - y - 3 = 0$.

Solution: We first observe that the slope of the line $12x - y - 3 = 0$ is 12 [Recall Unit 3, and compare the equation of line with slope intercept form, $y = mx + c$]. Thus, the slope of any line parallel to this line should also be 12.

Now, the slope of the tangent to the curve $y = x^3$ at any point (x, y) is $f'(x) = 3x^2$.

If we equate $f'(x)$ to 12, we will get those points on the curve where the tangent is parallel to $12x - y - 3 = 0$.

Thus, $3x^2 = 12$, or $x^2 = 4$, that is, $x = \pm 2$.

If $x = 2, y = x^3 = 8$ and when $x = -2,\ y = x^3 = -8$.

Thus, the points are $(2,8)$ and $(-2,\ -8)$. The equations of the tangents at these points are $y - 8 = 12(x-2)$ and $y + 8 = 12(x+2)$, respectively.

***

If you have followed these examples, you should have no problem is solving the following exercises.

E6) Find the equations of the tangent and the normal to each of the following at the specified point.

i) $y = x^2 + 2x + 1$ at $(1,4)$

ii) $x = a\ \cos\ t, y = b \sin t$ at the point given by $t = \pi / 4$

iii) $x^2 + y^2 = 25$ at $(-3,\ 4)$.

E7) Find the points on the graph of $f(x) = -x^3 + 6x^2$ at which the tangent line has slope 9.

E8) Find the equations of the tangent and the normal to each of the following curves at the point 't':

i) $x = at^2,\ y = 2at$.

ii) $x = a(t + \sin t),\ y = a(1 - \cos t)$.

E9) Find the equation of the tangent to each of the following curves at the point $(x_0,\ y_0)$.

i) $x^2 + y^2 + 4x + 6y - 1 = 0$

ii) $xy = a$

E10) Prove that the line $2x + 3y = 1$ touches the curve $3y = e^{-2x}$ at a point whose x-coordinate is zero.

E11) Prove that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point $(a\sqrt{2},\ b)$ is $ax + b\sqrt{2y} = (a^2 + b^2)\sqrt{2}$

E12) Without eliminating the parameter t, find dy/dx and d^2y/dx^2 at the points $(1,1)$ and $(1,-1)$ on the semicubical parabola $x = t^2$ and $y = t^3$.

E13) A bee follows the trajectory $x = t - 2\sin t,\quad y = 2 - 2\cos t$, where $t \geq 0$. It lands on a wall at time $t = 10$.

i) At what time was the bee flying horizontally?

ii) At what time was the bee flying vertically?

E14) Find the points on the cardioid $r = 1 - \cos\theta$ at which there is a horizonatal tangent line, or a vertical tangent line.

E15) Are there any points on the following curves where the tangent is parallel to either axis? If yes, find all such points.

i) $y = x^3 - x^2 - 2x$.

ii) $y = \sin x$.

E16) Find the equation of the tangent to the curve $y = \frac{x^2}{4}$ and is parallel to $3x - y + 2 = 0$.

We have seen that the slope of the tangent at a point is its derivative at that point. We can use slopes of two curves at a point to find the angle of intersection between these two curves at that point.

In the following section, we will find the angle of intersection of two curves.

14.4 ANGLE OF INTERSECTION OF TWO CURVES

The concept of a tangent to a curve has proved very useful in describing various geometrical features of the curve. In this section, we shall look at one such features.

When two curves intersect at a point, their angle of intersection at that point can be defined with the help of their tangents there. In fact, we say that if two curves intersect at a point P, the angle of intersection of these two curves at P is an angle between the tangents to these curves at P, such that $0 \le \theta \le \pi/2$ (see Fig. 25).

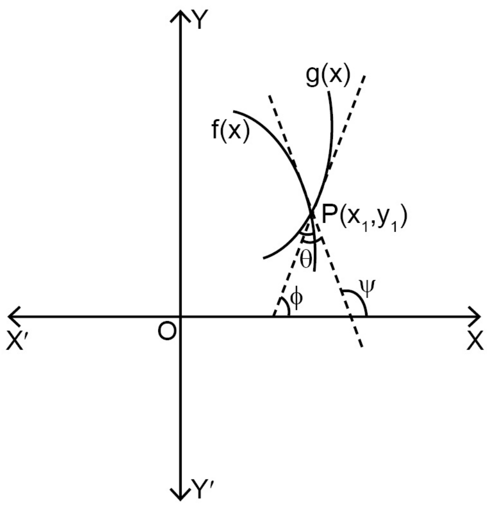

Fig. 25

We now prove a theorem which gives us the angle of intersection at a point when the equations of the two curves are known.

Theorem 1: If two curves $y = f(x)$ and $y = g(x)$ intersect at a point $P(x_1, y_1)$, then the angle θ of intersection of these curves at $P(x_1, y_1)$ is given by

$$\tan\theta = \left|\frac{f'(x_1) - g'(x_1)}{1 + f'(x_1)g'(x_1)}\right|.$$

Proof: From Fig. 25, $\tan\theta = \tan(\psi - \phi)$

$$= \frac{\tan\psi - \tan\phi}{1 + \tan\psi\tan\phi}$$

$$= \frac{f'(x_1) - g'(x_1)}{1 + f'(x_1)g'(x_1)}.$$

Fig. 25 shows that $\psi - \phi$ is an acute angle. But if the curves f and g were as in Fig. 26, then angle $\theta = \pi - (\psi - \phi)$, since, we take the acute angle as the angle of intersection.

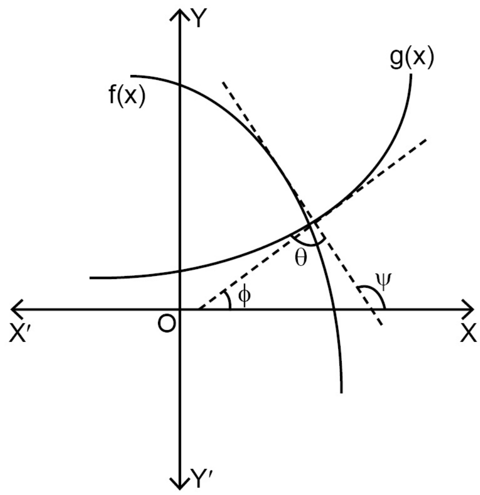

Fig. 26

In this case, $\tan\theta = \tan[\pi - (\psi - \Phi)] = -\tan(\psi - \Phi)$.

But we are not in a position to decide whether we should take $\tan\theta$ as $\tan(\psi - \Phi)$ or $-\tan(\psi - \Phi)$, unless we have drawn the curves. Since, it would be tedious to first draw the curves and then decide, therefore, we think of an alternate scheme. We observe that since θ lies between 0 and $\pi/2$, therefore, $\tan\theta$ is non-negative. Thus, we take $\tan\theta$ to $|\tan(\psi - \Phi)|$.

Hence, $\tan\theta = \left|\frac{f'(x_1) - g'(x_1)}{1 + f'(x_1)g'(x_1)}\right|$. ■

Having proved this theorem, we can easily deduce the following corollaries.

Corollary 1: Two curves $y = f(x)$ and $y = g(x)$ touch each other (x_1, y_1), that is, have common tangent at (x_1, y_1), iff $\theta = 0$, that is, iff $f'(x_1) = g'(x_1)$.

Corollary 2: Two curves cut each other at right angles, or orthogonally, at (x_1, y_1) iff $f'(x_1)g'(x_1) = -1$.

If your go through the following examples carefully, you will have no difficulty in solving the exercises later.

Example 20: Find the angle of intersection of the parabola $y^2 = 2x$ and the circle $x^2 + y^2 = 8$.

Solution: First, we find the points of intersection of these curves, if there are any. The coordinates of these points will satisfy both the equation to the parabola and the equation to the circle. So substituting $y^2 = 2x$ in $x^2 + y^2 = 8$, we get $x^2 + 2x = 8$, or $x = -4$ or 2.

It is clear from $y^2 = 2x$ that the abscissa $x(= y^2/2)$ of any common point must be non-negative. So, we reject the value $x = -4$. When $x = 2,\ y = \pm 2$. Hence, the common points are $P(2, 2)$ and $Q(2, -2)$. Since, both the curves are symmetric about the x-axis (see Fig. 27) and since, P and Q are reflections of each other w.r.t. the x-axis, then it is sufficient to find the angle at one point, say P because the angle at Q is equal to the angle at P.

Differentiating the two equations w.r.t x, we get

$$2y\frac{dy}{dx} = 2 \text{ and } 2x + 2y\frac{dy}{dx} = 0$$

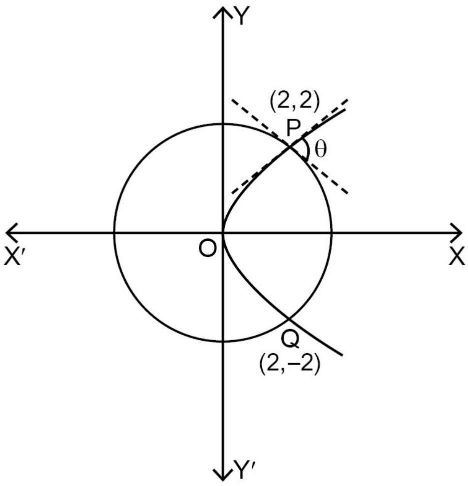

Fig. 27

Hence, the values of $f'(x)$ and $g'(x)$, that is, the slopes of the tangents to the two curves at (x, y) are $1/y$ and $-x/y$. Thus, the slopes of the tangents at $(2, 2)$ to the two curves are $1/2$ and -1 respectively. Hence, if θ is the required angle, then

$$\tan\theta = \left|\frac{1/2 - (-1)}{1 + 1/2(-1)}\right| = 3$$

Hence, $\theta = \tan^{-1} 3 \approx 71.56^\circ$.

***

Example 21: Find the angle between cubic polynomial $y = -x^3 + 6x^2 - 14x + 14$ and quadratic polynomial $y = -x^2 + 6x - 6$.

Solution: To find the point where the curves intersect, we should solve their equations simultaneously. Therefore, $-x^3 + 6x^2 - 14x + 14 = -x^2 + 6x - 6$ or $x^3 - 7x^2 + 20x - 20 = 0$. You may recall what you learnt in Unit 5, to find the root of the cubic equation. The only real root of this cubic is $x = 2$.

Then, we calculate the slopes of the tangents drown to the given cubic and the quadratic polynomial by evaluating their derivatives at $x = 2$. Thus, taking $f(x) = -x^3 + 6x^2 - 14x + 14$, we get $f'(x) = -3x^2 + 12x - 14$, then $f'(2) = -2$ and $g(x) = -x^2 + 6x - 6$, we get g', then $g'(2) = 2$.

Finally, we put the slopes of tangents into the formula to find the angle between given curves, as shown in Fig. 28.

$$\tan\theta = \left|\frac{g(x_0) - f(x_0)}{1 + g(x_0) f(x_0)}\right| = \left|\frac{-2-2}{1+2\cdot(-2)}\right| = \frac{4}{3}.$$

then, $\theta = \tan^{-1}\left(\frac{4}{3}\right)$, which is approximately $53^{\circ}\, 7'$.

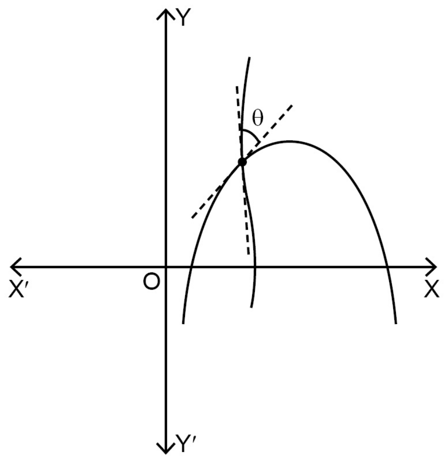

Fig. 28

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Example 22: At which point of the cubic polynomial $y = x^3 - 3x^2 + 2x - 2$ is its tangent perpendicular to the line $y = x$.

Solution: Since, the slopes of perpendicular lines are the negative reciprocals of each other, therefore, the slope of the tangent to the cubic has to be $f'(x) = -1$ which is to be perpendicular to the given line whose slope is 1.

Therefore, $f'(x) = 3x^2 - 6x + 2$, and $f'(x) = -1$, thus, $3x^2 - 6x + 2 = -1$, which gives $x = 1$, the abscissa of the tangency point. Then, we put $x = 1$ into the given cubic to calculate its ordinate, $y = x^3 - 3x + 2x - 2$, $y(1) = -2$, so the tangency point is $(1,-2)$, as shown in Fig. 29.

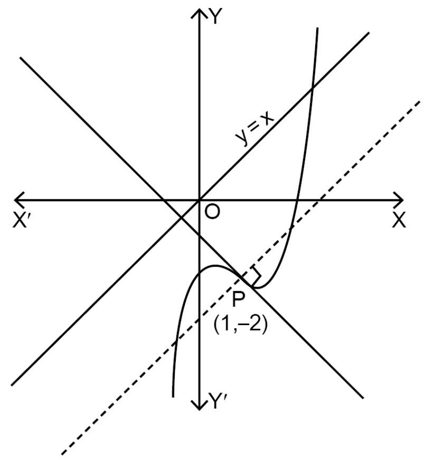

Fig. 29

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You can try these exercises now.

E17) Find the angle of intersection of the parabola $y^2 = 4x$ and $x^2 = 4y$.

E18) Show that the ellipse $x^2 + 4y^2 = 8$ and the hyperbola $x^2 - 2y = 4$ cut each other orthogonally (at right angles) at four points.

E19) Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a$ touch each other (have a common tangent) at two points.

You may recall Unit 3, where we discussed polar coordinates. Suppose we are given an initial line OX in a plane (see Fig. 30(a)). Then a point P can be located if we know

i) r, its distance from O, and

ii) θ, the angle made by OP with OX.

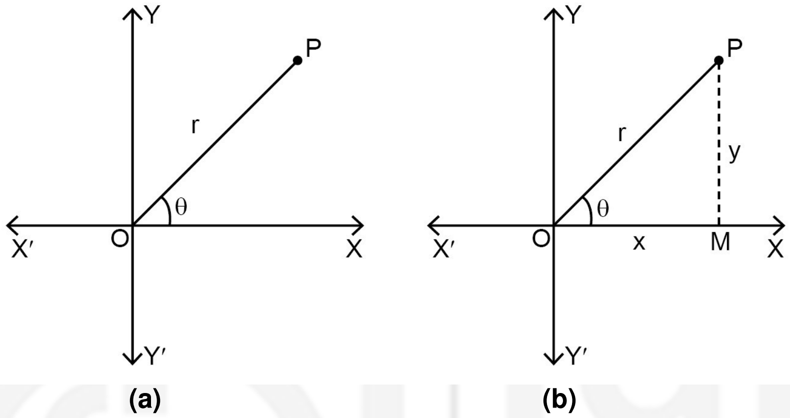


Fig. 30: Polar Coordinates.

r and θ are called the **polar coordinates** of P . From Fig. 30 (b), it is clear that if x and y are the cartesian coordinates of P, then $x = r\cos\theta$ and $y = r\sin\theta$. This also gives $r^2 = x^2 + y^2$ and $\tan\theta = y/x$. The equation of a curve is sometimes expressed in polar coordinates by an equation $r = f(\theta)$. For example, the equation of a circle with centre O and radius r is $r = a$. Now, let us turn once again to the problem of finding the angle of intersection of two curves. The model that we have been following till now, cannot be used if the equation of the curve is given in the polar form. In this case, we follow a somewhat indirect method.

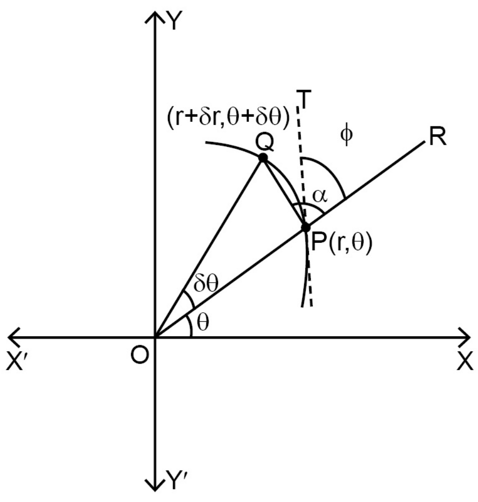


Fig. 31

Take a look at Fig. 31. It shows that a curve whose equation is given in the polar form as $r = f(\theta)$. $P(r, \theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ are two points on this curve. PT is the tangent at P and OPR is the line through the origin and the point P. We shall now try to find θ, the angle between PT and OR.

We note here, that the tangent PT is the limiting position of the secant PQ. If we denote the angle between PQ and OR by α then, we can similarly say that ϕ is the limit of α as $Q \to P$ along the curve.
Now, from ΔOPQ we have

$$\frac{OQ}{OP} = \frac{\sin \angle OPQ}{\sin \angle OQP}$$

Remember the sine rule for a ΔABC?
$$\frac{\sin A}{A} = \frac{\sin B}{B} = \frac{\sin C}{C}$$

or, $$\frac{r + \delta r}{r} = \frac{\sin(\pi - \alpha)}{\sin(\alpha - \delta\theta)}$$

or, $$1 + \frac{\delta r}{r} = \frac{\sin(\pi - \alpha)}{\sin(\alpha - \delta\theta)}$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

or, $$\frac{\delta r}{r} = \frac{\sin\alpha - \sin(\alpha - \delta\theta)}{\sin(\alpha - \delta\theta)} \quad (\text{since, } \sin(\pi - \alpha) = \sin\alpha)$$

or, $$\frac{1}{r}\frac{\delta r}{\delta\theta} = \frac{2\cos\left(\alpha - \frac{\delta\theta}{2}\right)\sin\left(\frac{\delta\theta}{2}\right)}{\sin(\alpha - \delta\theta).\delta\theta}$$

$$= \frac{2\cos\left(\alpha - \frac{\delta\theta}{2}\right)}{\sin(\alpha - \delta\theta)} \cdot \frac{\sin\left(\frac{\delta\theta}{2}\right)}{\frac{\delta\theta}{2}}$$

Recall $$\lim_{\delta\theta \to 0} \frac{\sin\left(\frac{\delta\theta}{2}\right)}{\frac{\delta\theta}{2}} = 1$$

As $Q \to P$, $\alpha \to \phi$, $\delta\theta \to 0$ and $\delta r \to 0$. Hence, as $Q \to P$, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\phi}{\sin\phi} = \cot\phi$$

So that $\tan\phi = r \cdot \frac{d\theta}{dr}$

This formula helps us to find the angle between OP and the tangent at the point P on the curve defined by the equation $r = f(\theta)$.

We shall use this result to find the angle between two curves C_1 and C_2 which intersect at P (say). If the angles between OP and the tangents to C_1 and C_2 at P are ϕ_1 and ϕ_2, respectively, then, the angle of intersection of C_1 and C_2 will be $|\phi_1 - \phi_2|$ (see Fig. 32).

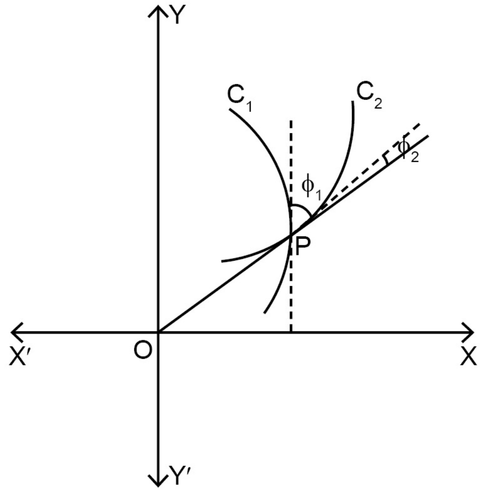


Fig. 32

This can be easily calculated as we know $\tan\phi_1$ and $\tan\phi_2$.

Thus, $\tan|\phi_1-\phi_2| = \left|\dfrac{\tan\phi_1-\tan\phi_2}{1+\tan\phi_1\tan\phi_2}\right|$

Further, if the curves intersect orthogonally, $\tan\phi_1.\tan\phi_2 = -1$. The following examples will help clarify this discussion.

Example 23: Find the angle of intersection of the curves $r = a\sin 2\theta$ and $r = a\cos 2\theta$ at the point $P(a/\sqrt{2}, \pi/8)$.

Solution: The coordinates of P satisfy both the equations $r = a\sin 2\theta$ and $r = a\cos 2\theta$.

If ϕ_1 is the angle between OP and the tangent to $r = a\sin 2\theta$, then

$$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{a\sin 2\theta}{dr/d\theta} = \frac{a\sin 2\theta}{2a\cos 2\theta} = \frac{1}{2}\tan 2\theta = \frac{1}{2}$$

Similarly, if ϕ_2 is the angle between OP and the tangent to $r = a\cos 2\theta$, then

$$\tan\phi_2 = r\frac{d\theta}{dr} = -\frac{1}{2}\cot 2\theta = -\frac{1}{2}$$

Thus, $\tan|\phi_1-\phi_2| = \left|\dfrac{\tan\phi_1-\tan\phi_2}{1+\tan\phi_1\tan\phi_2}\right| = \left|\dfrac{1/2+1/2}{1-1/4}\right| = \dfrac{4}{3}$

Thus, $\phi_1-\phi_2 = \tan^{-1}(4/3) \approx 53.13^\circ$, which is the required angle.

***

Now try to do a few exercises on your own.

E20) Find the angle between the line joining a point $P(r, \theta)$ on the curve to the origin and the tangent for each of the following curves

i) $r^2 = a^2\cos 2\theta$ ii) $1/r = 1 + e\cos\theta$

iii) $r^m = a^m\cos m\theta$ iv) $r^m = a^m(\cos m\theta - \sin m\theta)$

E21) Check whether the following two curves intersect orthogonally.

i) $r = ae^\theta$ and $re^\theta = b$

ii) $r = a(1+\sin\theta)$ and $r = a(1-\sin\theta)$.

In the following section, we shall discuss the special points which show behaviour different from ordinary points. Such points are called singular points.

14.5 SINGULAR POINTS

An equation of the type $y = f(x)$ determines a unique value of y for a given value of x. This means, every straight line parallel to the y-axis meets the curve $y = f(x)$ in a unique point. However, the equation of a curve is often given as $f(x, y) = 0$. If $f(x, y)$ is not a linear expression in y, then it may not be possible to write $f(x, y) = 0$ in the form $y = F(x)$ uniquely. For example, if $f(x, y) = y^2 - x^3$, then $f(x, y) = 0$ gives $y^2 = x^3$.

This gives us two relations $y = +x^{3/2}$ and $y = -x^{3/2}$ of type $y = F(x)$.

The curve has 2 branches, as you can see from Fig 33.

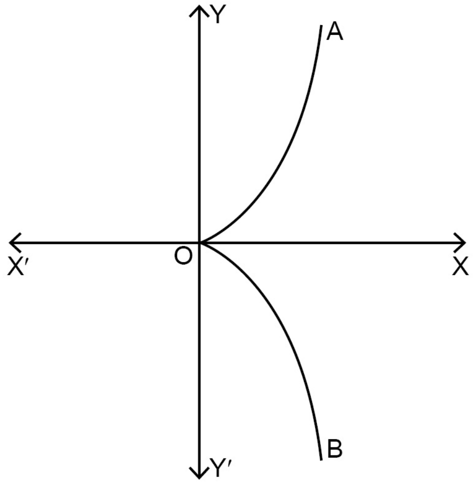


Fig 33: Graph of $y^2 = x^3$.

The origin is common to the two branches of the curve. We can say that two branches of the curve $y^2 = x^3$ pass through points A and B and meet at the origin. We have a generic name, '**singular point'**, for points like O. Also, $y^2 = x^3$ gives $\frac{dy}{dx} = \frac{3x^2}{2y}$, which is indeterminate at $(0,0)$. We call such points also **singular points**. We cannot make a general statement about the behaviour of curves at singular points, we analyse these case by case. A precise definition is as follows.

Definition: If k branches of a curve pass through a point P on the curve $f(x, y) = 0$ and $k > 1$, then P is said to be a **singular point** or a **multiple point** of order k.

Fig. 34 shows the graph of the function with multiple points.

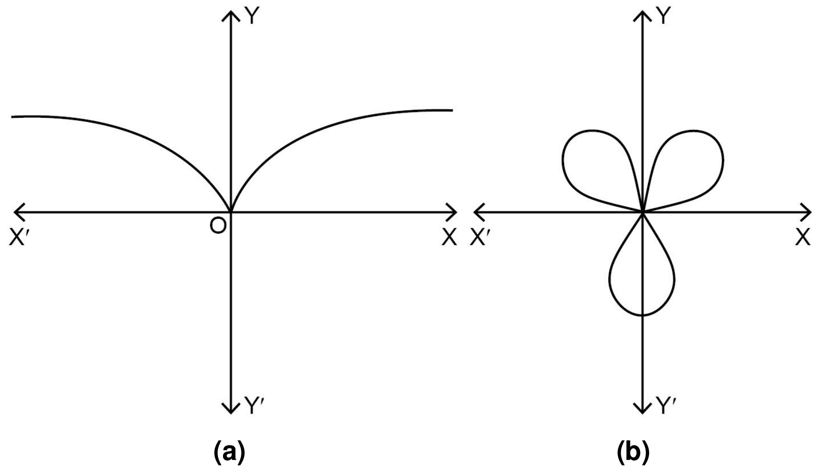


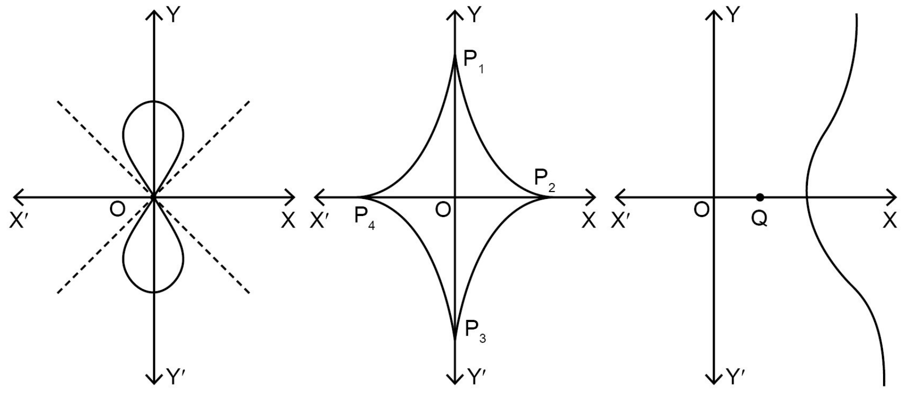
Fig. 34: Double and Multiple point.

Multiple point of order two is known as **double point**. Thus, the point O in Fig. 34 (a) is a double point. Obviously, a curve will have more than one tangent at a double point (one corresponding to each branch). Depending upon whether tangents at double points are distinct, coincident or imaginary, we shall give special names to such points such as node, cusp and conjugate as given in the following definition:

Definition: A double point is known as

i) a **node** if the two tangents at that point are real and distinct,
ii) a **cusp** if the two tangents are real and coincident,
iii) a **conjugate** (or **isolated**) point if the two tangents are imaginary.

In Fig 35, we show example of each. In Fig. 35 (a), for the curve $f(x, y) = 0$, the origin is a node. In Fig. 35 (b), for the curve $g(x, y) = 0$, the points P_1, P_2, P_3 and P_4 are cusps, while the point Q on the curve $h(x, y) = 0$ is a conjugate point in Fig. 35 (c). Thus, a conjugate point is an isolated point.



(a) Node **(b) Cusp** **(c) Isolated Point**

Fig. 35

Example 24: Determine whether the graph of the given functions has a vertical tangent or a cusp at the origin.

i) $y^3 = x^2(x+3)^3$

ii) $y^3 = x(x+1)^3$

Solution: i) We have $y^3 = x^2(x+3)^3$ or $y = x^{2/3}(x+3)$

$$y = x^{5/3} + 3x^{2/3}$$

$$y' = \frac{5}{3}x^{2/3} + 2x^{-1/3}$$

$$= \frac{1}{3}(5x^{2/3} + 6x^{-1/3})$$

$$= \frac{1}{3}x^{-1/3}(5x+6)$$

when $x \to 0^-, y' \to -\infty$, and as $x \to 0^+, y' \to +\infty$. In this case, where y' approaches $+\infty$ from one side of a point and $-\infty$ from other side of the point, the curve is said to have a cusp at that point. Hence, there is a cusp at the origin as shown in Fig. 36.

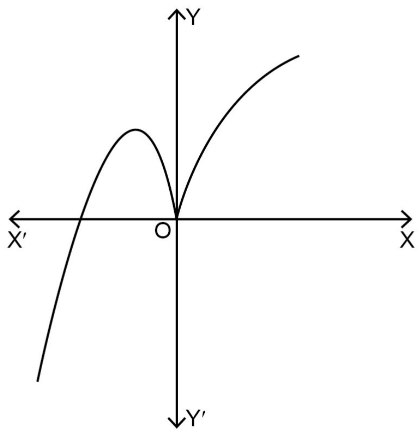


Fig. 36

ii) $y = x^{1/3}(x+1)$

$$y' = \frac{4}{3}x^{1/3} + \frac{1}{3}x^{-2/3}$$

$$= \frac{1}{3}x^{-2/3}(4x+1)$$

As $x \to 0^-, y' \to \infty$ and $x \to 0^+, y \to \infty$. Since $y' \to \infty$ as $x \to 0$ from both the sides. This means that a vertical tangent occurs at origin as shown in Fig. 37.

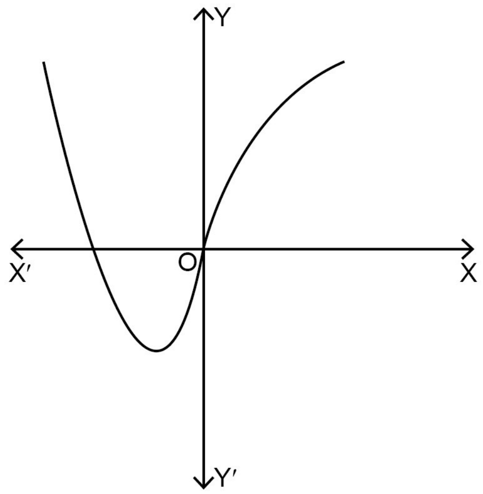


Fig. 37

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Now, try the following exercises.

E22) Check whether the following curves has a vertical tangent or a cusp at the origin.

i) $y = 5x^{7/5} - x^{3/5}$

ii) $y^5 = x^2(2x+1)^5$

Now, let us summarize what we have studied in this unit.

14.6 SUMMARY

In this unit, we have covered the following points.

1. If $f''(x) > 0$ on some interval then f is convex on it. If $f''(x) < 0$, then f is concave on it.

2. If $f''(x_0) = 0$ or does not exist and f'' changes sign in passing through x_0, then f has a point of inflection at $x = x_0$. This means that the tangent at $(x_0, f(x_0))$ crosses the graph of f at this point.

3. The radius of curvature at point c on the function f is $\rho(c) = \frac{(1+(f'(c)^2)^{3/2}}{|f''(c)|}$ and curvature $k(c) = \frac{1}{\rho(c)}$.

4. The equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is $y - y_0 = f'(x_0)(x - x_0)$.

5. The curve has a vertical tangent at (x_0, y_0) if $\frac{dx}{dy} = 0$ at this point.

6. The angle θ of intersection of two curves $y = f(x)$ and $y = g(x)$ is the acute angle between the tangents at that point to the curves. It is given by the relation $\tan\theta = \left|\frac{f'(x) - g'(x)}{1 + f'(x)g'(x)}\right|$.

7. $y = f(x)$ and $y = g(x)$ cut each other orthogonally at (x_0, y_0) if $f'(x_0)g'(x_0) = -1$.

8. The angle ϕ between the tangent and the radius vector of the curve $r = f(\theta)$ at the point θ is given by $\tan\phi = r\frac{d\theta}{dr}$.

9. If k branches of a curve pass through a point P on the curve $f(x, y) = 0$ and $k > 1$, then P is said to be a singular point or a multiple point of order k. Singular points of order two are known as double points. A double point is known as a node, a cusp or a conjugate (isolated) point according as the two tangents at that point are real and distinct, real but coincident, or imaginary.

14.7 SOLUTIONS/ANSWERS

E1) From condition (i), it is clear that f is increasing on $]-\infty, 2[$ and decreasing on $]2, \infty[$. From (ii), it can be said that f is concave upward on $]-\infty, -3[$ and $]3, \infty[$ and f is concave downward on $]-3, 3[$.

$\lim_{x\to-\infty} f(x) = -4$ says that f approaches -4 as x approaches $-\infty$ and

$\lim_{x\to\infty} f(x) = 1$ says that f approaches 1 as x approaches ∞.

The graph sketch of f will look like this (Fig. 38).

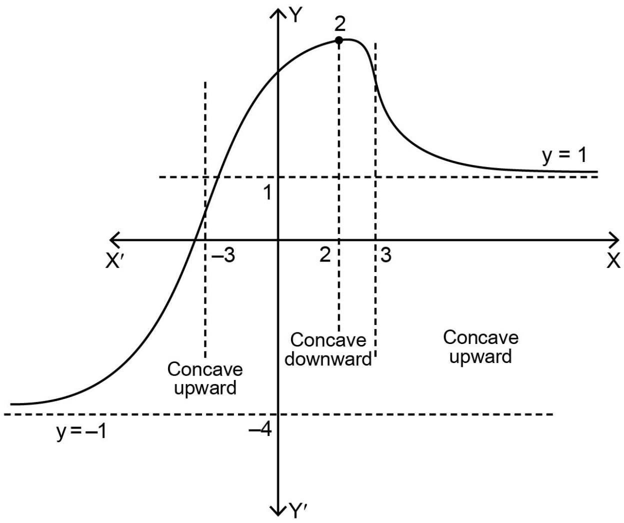

Fig. 38

E2) $f(x) = x^4 - 4x^3$, we get

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

Thus, $f''(x) > 0$ when $x > 2$ and $x < 0$. Also, $f''(x) < 0$ when $0 < x < 2$. Hence, the curve is concave upward on $]-\infty, 0[$ and $]2, \infty[$ and the curve is concave downward on $]0, 2[$. The possible graph is shown in Fig. 39.

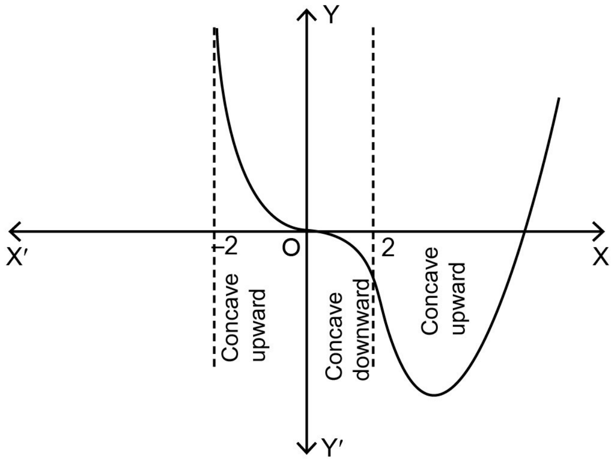

Fig. 39

E3) i) Concave upward in $]0, \infty[$; concave downward in $]-\infty, 0[$; point of inflection $(0, 0)$

ii) Concave upward in $]-\infty, 0[$; concave downward in $]0, \infty[$; point of inflection: $(0, 0)$

iii) Concave upward in $]-\infty, -1[\cup]2, \infty[$; concave downward in $]-1, 2[$; Points of inflection are $(-1, -8), (2, -47)$

iv) Concave upward if $x > 3$; concave downward if $x < 3$; no point of inflection

v) Concave downward for all $x > 0$; no point of inflection

vi) Concave upward in $]\pi/2, 3\pi/2[$; concave downward in $]0, \pi/2[\cup]3\pi/2, 2\pi[$; point of inflection are $(\pi/2, 0)$ and $(3\pi/2, 0)$.

E4) We see that this parabola passes through each of the 3 points A, B and C as shown in Fig. 40.

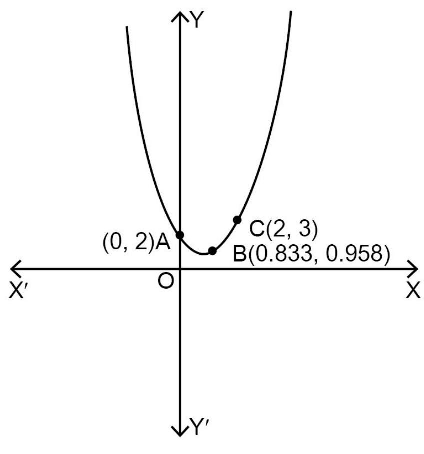

Fig. 40

Using the given function, we get $\frac{dy}{dx} = 3x - 2.5$. At $x = 2$, $\frac{dy}{dx} = 3.5$

Now, $\frac{d^2y}{dx^2}$ at $x = 2$ is 3.

The radius of curvature at $(2,3)$ is $\frac{[1+(3.5)^2]^{3/2}}{3} \approx 16.08$. The corresponding circle is shown in Fig. 41.

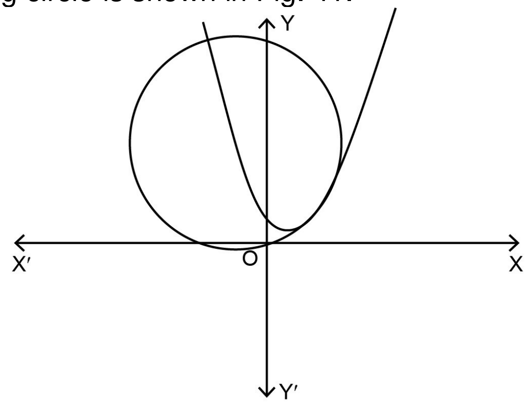

Fig. 41

E5) i) 0

ii) $\frac{2}{(1+4x^2)^{3/2}}$

iii) $\frac{-\sin x}{(1+\cos^2 x)^{3/2}}$

iv) -1

E6) i) $\left(\frac{dy}{dx}\right)_{x=1} = 4$.

Equation of the tangent at $(1, 4)$ is $(y-4) = 4(x-1)$, which is $4x - y = 0$

Slope of the normal at $(1, 4) = -1/4$

Equation of the normal at $(1, 4)$ is $(y-4) = (-1/4)\ (x-1)$, which is $x + 4y = 17$.

ii) Slope of the tangent at $t = \frac{\pi}{4}$ is $-b/a$

Slope of the normal at $t = \frac{\pi}{4}$ is a/b

At $t = \pi/4,\ x = \frac{a}{\sqrt{2}},\ y = \frac{b}{\sqrt{2}}$.

Equation of the tangent: $\left(y - \frac{b}{\sqrt{2}}\right) = \frac{-b}{a}\left(x - \frac{a}{\sqrt{2}}\right)$

Equation of the normal: $y - \frac{b}{\sqrt{2}} = \frac{a}{b}\left(x - \frac{a}{\sqrt{2}}\right)$.

iii) Slope of the tangent $= 3/4$

Slope of the normal $= -4/3$

Equation of the tangent: $y-4=(3/4)\ (x+3)$

Equation of the normal: $y-4=(-4/3)\ (x+3)$.

E7) $f'(x)=-3x^2+12x$

$f'(x)=9 \Rightarrow -3x^2+12x=9$

$-3x^2+12x-9=0$

$x^2-4x+3=0$

$x=1$ or $x=3$.

$f(1)=5$ and $f(3)=27$

Thus, the points at which slope of the tangent is 9 are $(1,5)$ and $(3,27)$.

E8) i) Equation of the tangent: $ty=x+at^2$

Equation of the normal: $y+tx=at(2+t^2)$

ii) Equation of the tangent: $(1+\cos t)y=\sin t(x-at)$ which is

$\sin(t/2)x-\cos(t/2)y=at\sin(t/2)$

Equation of the normal:

$\sin(t/2)y+\cos(t/2)x=2a\sin(t/2)+at\cos(t/2)$

E9) i) $y-y_0=-\left(\dfrac{x_0+2}{y_0+3}\right)(x-x_0)$.

ii) $y-y_0=(-y_0/x_0)\ (x-x_0)$.

E10) $3y=e^{-2x} \Rightarrow \left.\dfrac{dy}{dx}\right|_{x=0}=-\dfrac{2}{3}$. When $x-$coordinate is zero, $(0, 1/3)$ is a point on this curve. The tangent at $(0, 1/3)$ is given by $y-\dfrac{1}{3}=-\dfrac{2}{3}x$ or, $2x+3y=1$.

E11) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \Rightarrow \left.\dfrac{dy}{dx}\right|_{(a\sqrt{2},\, b)}=\dfrac{b\sqrt{2}}{a}$

$\Rightarrow$ Slope of the normal $=-\dfrac{a}{b\sqrt{2}}$

$\Rightarrow$ Equation of the normal is $y-b=\dfrac{-a}{b\sqrt{2}}\ (x-a\sqrt{2})$

E12) Since, $\dfrac{dx}{dt}=2t$ and $\dfrac{dy}{dt}=3t^2$, therefore

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3t^2}{2t}=\frac{3}{2}t \qquad (t\neq 0)$$

Also, $\dfrac{d^2y}{dx^2}=\dfrac{dy'}{dx}=\dfrac{dy'/dt}{dx/dt}=\dfrac{\frac{d}{dt}(3/2t)}{dx/dt}=\dfrac{3/2}{2t}=\dfrac{3}{4t}$

Since the point $(1,1)$ on the curve corresponds to $t=1$ in the parametric equations, therefore we get

$\left.\dfrac{dy}{dx}\right|_{t=1}=\dfrac{3}{2}$ and $\left.\dfrac{d^2y}{dx^2}\right|_{t=1}=\dfrac{3}{4}$

Similarly, the point $(1,-1)$ corresponds to $t=-1$ in the parametric equations, we get

$$\left.\frac{dy}{dx}\right|_{t=-1} = -\frac{3}{2} \text{ and } \left.\frac{d^2y}{dx^2}\right|_{t=-1} = -\frac{3}{4}$$

The graph shown in Fig. 42 also verifies the values we obtained for the first and second derivatives. Since at $(1,1)$ on the upper branch of the graph, the tangent line has positive slope and the curve is concave up, and at $(1,-1)$ on the lower branch, the tangent line has negative slope and the curve is concave down.

Finally, we observe observe that we were able to find dy/dx and d^2y/dx^2 for both $t=1$ and $t=-1$, even though the points $(1,1)$ and $(1,-1)$ lie on different branches.

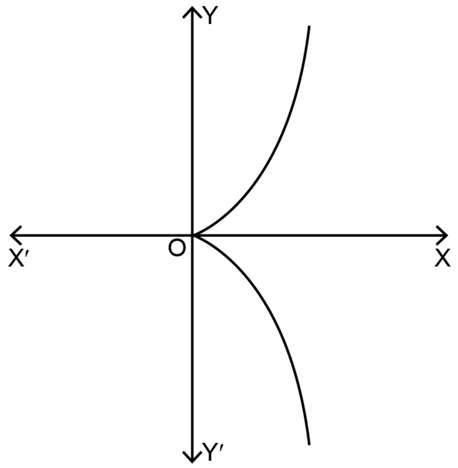

Fig. 42

E13) $\frac{dx}{dt} = 1-2\cos t$ and $\frac{dy}{dt} = 2\sin t$, therefore, $\frac{dy}{dx} = \frac{2\sin t}{1-2\cos t}$.

E14) The curve will have a horizontal tangent line when $dy/d\theta = 0$ and $dx/d\theta \neq 0$. Similarly, the curve has a vertical tangent line when $dy/d\theta \neq 0$ and $dx/d\theta = 0$, and a singular point when $dy/d\theta = 0$ and $dx/d\theta = 0$. We can find these derivatives. However, an alternative approach is to go back to basic principles and express the cardioid parametrically by substituting $r = 1-\cos\theta$ in the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$. We get $x = (1-\cos\theta)\cos\theta$, $y = (1-\cos\theta)\sin\theta$ where $0 \le \theta \le 2\pi$.

Differentiating these equations with respect to θ and then simplifying, we get $\frac{dx}{d\theta} = \sin\theta(2\cos\theta - 1)$ and $\frac{dy}{d\theta} = (1-\cos\theta)\ (1+2\cos\theta)$.

Thus, $dx/d\theta = 0$ if $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$, and $dy/d\theta = 0$ if $\cos\theta = -\frac{1}{2}$.

Hence, the solutions of $dx/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are $\frac{dx}{d\theta} = 0 : \theta = 0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3},\ 2\pi$ and the solutions of $dy/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are $\frac{dy}{d\theta} = 0 : \quad \theta = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ 2\pi$.

Thus, the curve has horizontal tangent lines at $\theta = 2\pi/3$ and $\theta = 4\pi/3$, vertical tangent lines at $\theta = \pi/3,\ \pi,$ and $5\pi/3$.

E15) i) Tangents are parallel to the x-axis at $x = \frac{(1 \pm \sqrt{7})}{3}$.

ii) Tangents are parallel to the x-axis at all points where $x = n\pi + \frac{\pi}{2}$ for some integer n. There is no tangent parallel to the y-axis.

E16) $\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$

Since, $\frac{x}{2} = 3 \Rightarrow x = 6$ when $x = 6, y = 9$.

The equation of the tangent is $y - 9 = 3(x - 6)$

$3x - 4 = 9$.

E17) $y^2 = 4x \Rightarrow x = y^2/4 \Rightarrow x^2 = y^4/16 = 4y$ at the point of intersection.

$\Rightarrow y^4 - 64y = 0$

$\Rightarrow y(y^3 - 64) = 0$

$\Rightarrow y(y-4)\ (y^2 + 4y + 16) = 0$

$\Rightarrow y = 0,\ 4$ (other roots are complex)

$\Rightarrow x = 0$ or 4.

Slope of the tangent to $y^2 = 4x$ at $(4, 4) = 1/2$

Slope of the tangent to $x^2 = 4y$ at $(4, 4) = 2$

$\Rightarrow$ angle of intersection $= \tan^{-1}(3/4)$

The tangent at $(0, 0)$ to $y^2 = 4x$ is vertical, and the tangent at $(0, 0)$ to $x^2 = 4y$ is horizontal.

Hence the angle of intersection of the curves at $(0, 0)$ is $\pi/2$.

E18) The four points are $(4/\sqrt{3}, \pm\sqrt{2/3}), (-4/\sqrt{3}, \pm\sqrt{2/3})$.

$\frac{dy}{dx}$ for $x^2 + 4y^2 = 8$ is $-x/4y$

$\therefore \left.\frac{dy}{dx}\right|_{x=\frac{4}{\sqrt{3}},\ y=\sqrt{\frac{2}{3}}} = \frac{-1}{\sqrt{2}}$

$\frac{dy}{dx}$ for $x^2 - 2y^2 = 4$ is $x/2y$

$\left.\frac{dy}{dx}\right|_{x=4/\sqrt{3},\ y=\sqrt{\frac{2}{3}}} = \sqrt{2}$

$\therefore$ They cut orthogonally.

E19) The points are (a, a) and $(-a, -a)$

E20) i) $2r = -2a^2 \sin 2\theta \frac{d\theta}{dr}$

$\Rightarrow \frac{d\theta}{dr} = \frac{-r}{a^2 \sin 2\theta}$

$\Rightarrow$ angle $= \tan^{-1}\left(r\frac{d\theta}{d\pi}\right) = \tan^{-1}(-\cot 2\theta)$

$$= \tan^{-1}\left[\tan\frac{(2n+1)\pi}{2} + 2\theta\right]$$

$$= (2n+1).\frac{\pi}{2} + 2\theta$$

ii) $\tan^{-1}\left(\frac{1+e\cos\theta}{e\sin\theta}\right)$

iii) $(2n+1)\frac{\pi}{2} + m\theta, n \in \mathbb{Z}$

iv) $m\theta - \frac{\pi}{4}$.

E21) i) $r = ac^{\theta} \Rightarrow 1 = ae^{\theta}.\frac{d\theta}{dr} \Rightarrow \frac{d\theta}{dr} = \frac{1}{ae^{\theta}}$

$\Rightarrow \tan\phi_1 = r\frac{d\theta}{dr} = \frac{r}{ae^{\theta}} = 1$

$re^{\theta} = b \Rightarrow r = be^{-\theta} \Rightarrow 1 = -be^{-\theta}\frac{d\theta}{dr}$

$\Rightarrow \frac{d\theta}{dr} = \frac{-1}{be^{-\theta}} \Rightarrow \tan\phi_2 = r\frac{d\theta}{dr} = \frac{-r}{be^{-\theta}} = -1$

$\Rightarrow \tan\phi_1 \tan\phi_2 = -1 \Rightarrow$ the curves cut orthogonally.

ii) The curves cut orthogonally.

E22) i) $y' = 7x^{\frac{2}{5}} - \frac{3}{5}x^{\frac{-2}{5}}$

$= x^{\frac{-2}{5}}\left(7x^{\frac{4}{5}} - \frac{3}{5}\right)$

$y' = 0 \Rightarrow 7x^{\frac{4}{5}} - \frac{3}{5} = 0$

As $x \to 0^-, y' \to \infty$ and as $x \to 0^+, y' \to \infty$, therefore, the curve has vertical tangent at origin.

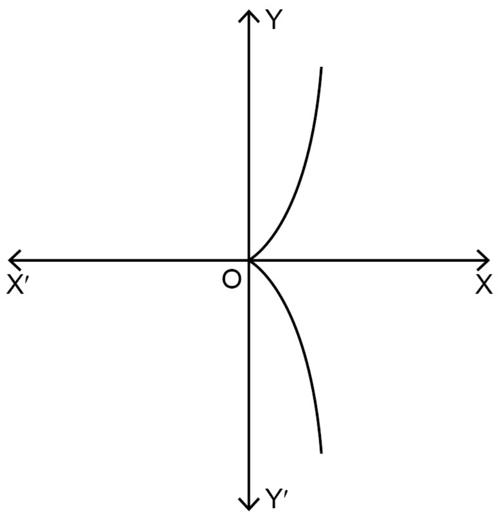

Fig. 43

ii) $y = 2x^{\frac{7}{5}} + x^{\frac{2}{5}}$

$$y' = \frac{14}{5}x^{\frac{2}{5}} + \frac{2}{5}x^{\frac{-3}{5}}$$

$$= \frac{2}{5}x^{\frac{-3}{5}}(7x+1)$$

As $x \to 0^-, y' \to -\infty$, and as $x \to 0^+, y' \to +\infty$, therefore, origin is a cusp.

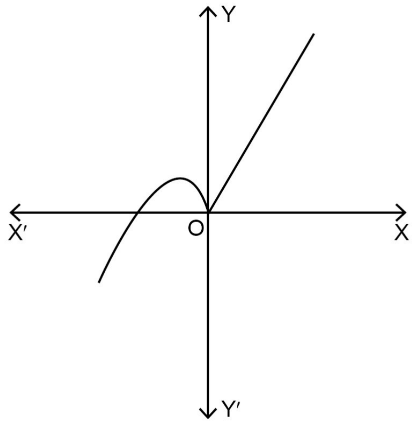

Fig. 44

UNIT 15

ASYMPTOTES

Structure

15.1 INTRODUCTION

In the previous two units, we discussed the applications of first and second derivatives to visualise the graph of the function. In this unit, we shall discuss lines that approach a given curve as close as possible. Such lines are called **asymptotes**. You will see in Sec. 15.2 and Sec. 15.3 that there are three kinds of asymptotes: horizontal, vertical and slant/oblique asymptotes. You will see how all these will prove very useful when you learn curve tracing in the next unit.

Now we shall list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

Objectives

After going through this unit, you should be able to:

- find the asymptotes parallel to axes;
- define oblique asymptotes and obtain their equations.

We shall now study a feature of curves which will prove very useful in tracing curves as you will see in the next unit. This involves taking limits as $x \to \pm\infty$ or $y \to \pm\infty$. In the following section, we will discuss asymptotes parallel to axes.

15.2 ASYMPTOTES PARALLEL TO AXES

Consider a rectangular hyperbola $xy = c, c > 0,$ shown in Fig. 1. The equation $xy = c$ implies $y = c/x$ and this implies that as $x \to \infty$ or $-\infty$, $y \to 0$. Now

$|y|$ is the distance of a point $P(x, y)$ on the hyperbola from the x-axis. So, we can say, that as $x \to \infty$ or $-\infty$, the distance of a point, $P(x, y)$ on the hyperbola from the x-axis approaches zero. In other words, this means that the $x-$axis is a line which seems to merge with the hyperbola. Such lines are called **asymptotes**. We give the definition of asymptote below.

Definition: A straight line is said to be an asymptote to a curve, if as a point P moves to infinity along the curve, the perpendicular distance of P from the straight line tends to zero.

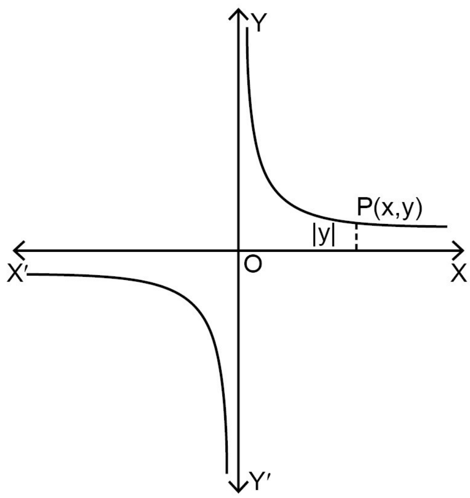

Fig. 1: Graph of $xy = c$ **.**

Writing $xy = c$ as $x = c/y$, and repeating the arguments exactly as above, we can see that the y-axis is also an asymptote of the hyperbola. Let us discuss following example.

Example 1: Prove that the x-axis is an asymptote of the curve $y = \frac{10}{1+x}$.

Solution: From the equation of the curve, we see that $y \to 0$ as $x \to \infty$ or $-\infty$. Again, this means that the distance of the point $P(x, y)$ on the curve from the x-axis tends to zero as $x \to \infty$ or $-\infty$. This proves that the x-axis is an asymptote of the curve. Fig. 2 also shows this.

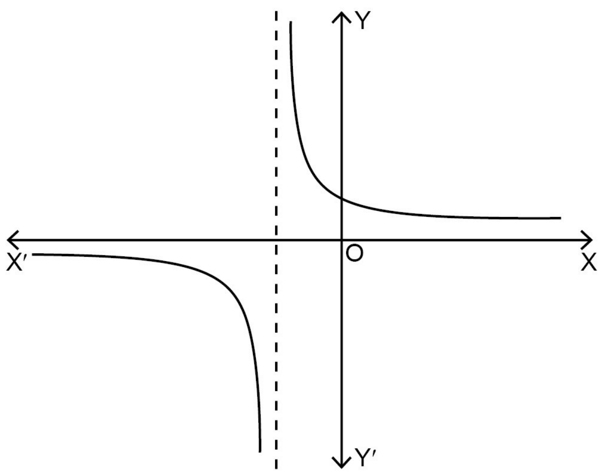

Fig. 2: Graph of $y = \frac{10}{1+x}$.

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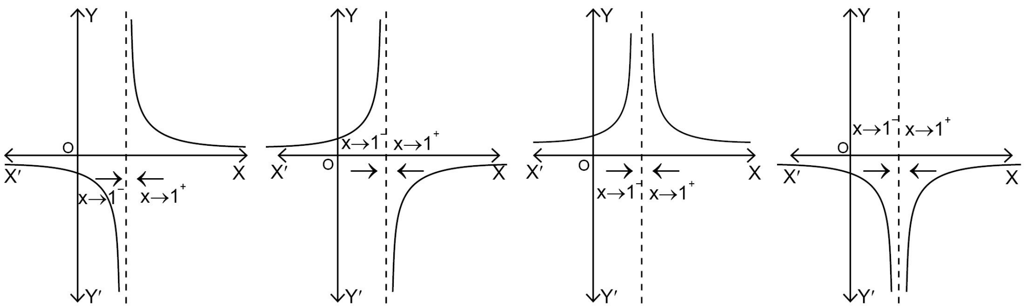
The asymptotes which are parallel to axes are either vertical or horizontal. We shall begin our discussion with vertical asymptotes. For this, let us look at some situations in the graphs of the functions given in Fig. 3.



(a) $f(x)=\dfrac{1}{x-1}$ **(b)** $f(x)=\dfrac{-1}{(x-1)}$ **(c)** $f(x)=\dfrac{1}{(x-1)^2}$ **(d)** $f(x)=\dfrac{-1}{(x-1)^2}$

Fig. 3

In Fig. 3 (a), the function f increases indefinitely as x approaches 1 from the right and decreases indefinitely as x approaches 1 from the left.

Therefore, $\lim\limits_{x\to1^+}\dfrac{1}{x-1}=+\infty$ and $\lim\limits_{x\to1^-}\dfrac{1}{x-1}=-\infty$.

Similarly, in Fig. 3 (b), the function f decreases indefinitely as x approaches 1 from the right and increases indefinitely as x approaches 1 from the left.

Therefore, $\lim\limits_{x\to1^+}\dfrac{-1}{x-1}=-\infty$ and $\lim\limits_{x\to1^-}\dfrac{-1}{x-1}=+\infty$

Similarly, in Fig. 3 (c), the function f increases indefintely as x approaches 1 from both the left and right.

Therefore, $\lim\limits_{x\to1}\dfrac{1}{(x-1)^2}=\lim\limits_{x\to1^+}\dfrac{1}{(x-1)^2}=\lim\limits_{x\to1^-}\dfrac{1}{(x-1)^2}=+\infty$.

Also in Fig. 3 (d), the function f decreases indefinitely as x approaches 1 from both the left and right.

Therefore, $\lim\limits_{x\to1}\dfrac{-1}{(x-1)^2}=\lim\limits_{x\to1^+}\dfrac{-1}{(x-1)^2}=\lim\limits_{x\to1^-}\dfrac{-1}{(x-1)^2}=-\infty$

We can say that if $f(x)\to+\infty$ as $x\to1^+$ or as $x\to1^-$, then the graph of f rises without bound and comes closer to the vertical line $x=1$ on any side. If $f(x)\to-\infty$ as $x\to1^+$ or as $x\to1^-$, then the graph of f falls without bound and comes closer to the vertical line $x=1$ on any side of $x=1$. In all these cases, the distance between any point $P(x, y)$ on the curve and the straight line $x=1$ tends to zero. Thus, we call the line $x=1$ vertical asymptote or asymptote parallel to $y-$axis. This leads to the following definition.

Definition: A line $x=a$ is called a **vertical asymptote** to the curve of the function f if $f(x)\to+\infty$ or $f(x)\to-\infty$ as x approaches a from either side.
In vertical asymptotes, we have used limits to describe the behaviour of $f(x)$ as x approaches a.

Let us find vertical asymptotes in the following examples:

Example 2: Determine the vertical asymptotes of the function f given by
$f(x)=\dfrac{x^2-4}{x^2+x-12}$.

Solution: We can rewrite $f(x)$ as $f(x) = \frac{(x+2)(x-2)}{(x+4)(x-3)}$.

You may note that as x gets closer to 3 from the left, the value of the function gets smaller and smaller negatively, approaching $-\infty$. And as x gets closer to 3 from the right, the value of the function f gets larger and larger positively approaching $+\infty$. Thus, $\lim_{x\to 3^-} f(x) = -\infty$ and $\lim_{x\to 3^+} f(x) = \infty$.

For this function, the line $x = 3$ is a vertical asymptote. Similarly, $x = -4$ is another vertical asymptote as shown in Fig. 4.

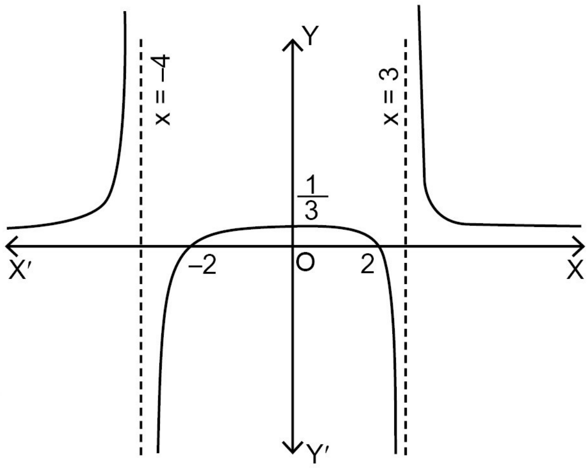


Fig. 4: Vertical asymptotes.

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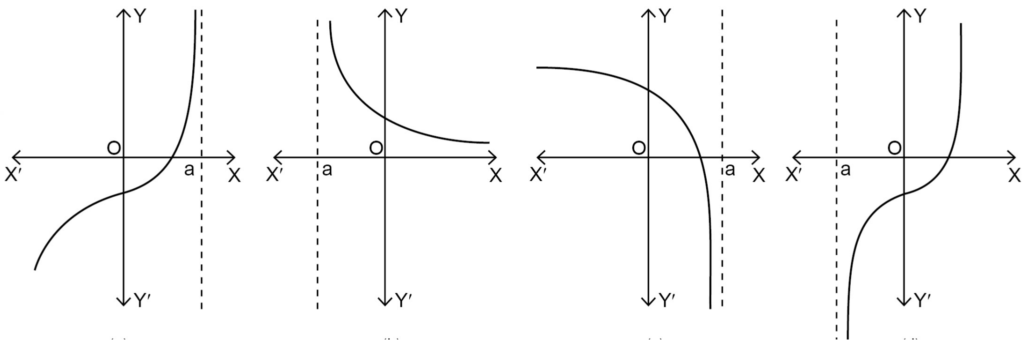
In case of rational functions, it may not always be true that a factor in denominator is an asymptote. For example, $f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)}$ does not have a vertical asymptote at $x = 1$, even though $x = 1$ makes the denominator 0. This is because when we simplify $\frac{(x^2-1)}{(x-1)}$, it has $(x-1)$ as a common factor of the numerator and the denominator. Few possible ways in which a vertical asymptote can occur are given in Fig. 5.



(a) $\lim_{x\to a^-} f(x) = +\infty$ **(b)** $\lim_{x\to a^+} f(x) = +\infty$ **(c)** $\lim_{x\to a^-} f(x) = -\infty$ **(d)** $\lim_{x\to a^+} f(x) = -\infty$

Fig. 5: Vertical Asymptotes.

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Example 3: Find the vertical asymptotes of $y = \cot x$.

Solution: The contagent function is a periodic function with period π.

However, from the identity $y = \cot x = \frac{\cos x}{\sin x}$, you can see that the cotangent function has vertical asymptotes when $\sin x$ is zero, which occurs at $x = n\pi$, where n is an integer. The graph of the cotangent function is shown in Fig. 6.

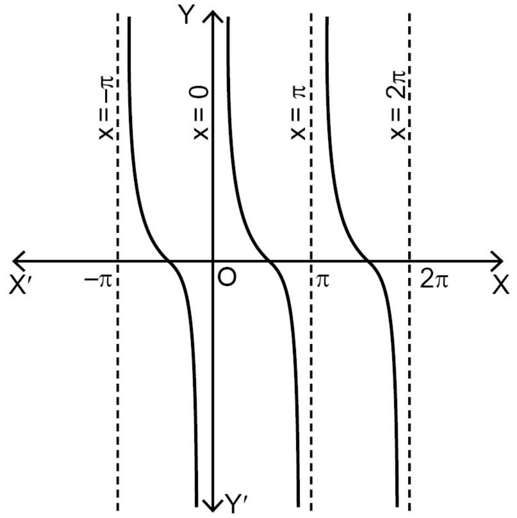

Fig. 6: Graph of $\cot x$ **.**

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Example 4: Find the vertical asymptotes of $y = \sec x$.

Solution: The secant function is a periodic function with period 2π. This function has vertical asymptotes when $\cos x$ is 0, which occurs at $x = \frac{(2n+1)\pi}{2}$, where n is an integer. The graph of secant function is shown in Fig. 7.

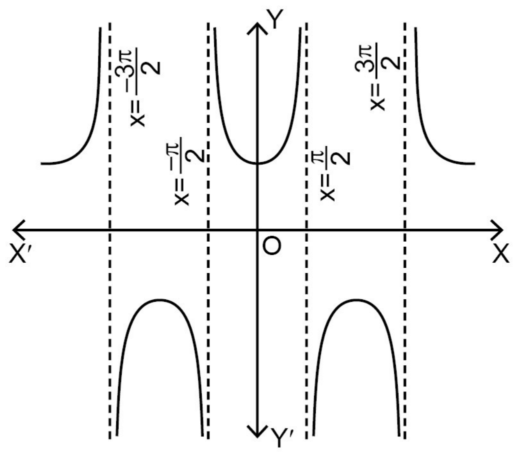

Fig. 7: Graph of $\sec x$ **.**

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Example 5: Find the vertical asymptotes of $y = \ln(x^2 - 3x - 4)$.

Solution: $y = \ln(x^2 - 3x - 4)$ has vertical asymptotes, when $x^2 - 3x - 4 = 0$. Thus, $x = 4$ and $x = -1$ are the vertical asymptotes of $y = \ln(x^2 - 3x - 4)$. The graph of $y = \ln(x^2 - 3x - 4)$ is shown in Fig. 8.

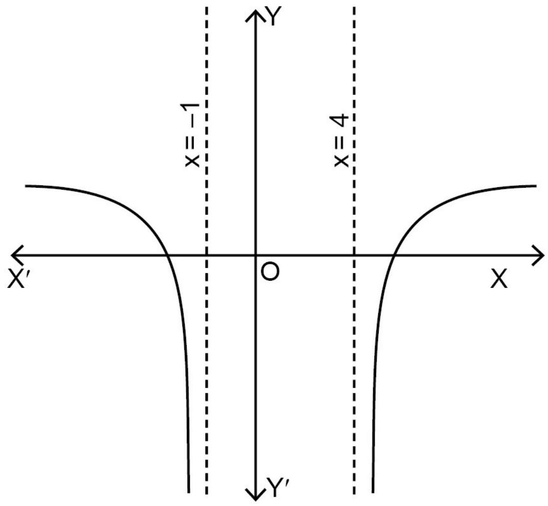

Fig. 8: Graph of $y = \ln(x^2 - 3x - 4)$**.**

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Now, try the following exercises:

E1) Determine the vertical asymptotes of the following

i) $f(x) = \frac{1}{x(x^2-1)}$

ii) $f(x) = \frac{7}{x^2+49}$

iii) $f(x) = \frac{2x}{x^2-16}$

iv) $f(x) = \frac{x}{x+2}$

v) $f(x) = 2\cot\frac{x}{3}$

vi) $f(x) = \ln\left(\frac{1}{x}\right)$

E2) Is $x = 0$ is an asymptote of $f(x) = \frac{x^2-2x}{x^3-x}$? Justify your answer.

Similarly, we define horizontal asymptotes. For this, going back to Fig. 1, we can see that as x increases without bound, the value of $f(x) = \frac{c}{x}$ is positive, but gets closer and closer to 0, and as x decreases without bound, the value of $f(x) = \frac{c}{x}$ is negative, and gets closer and closer to 0. We write these limits as $\lim\limits_{x\to+\infty} \frac{c}{x} = 0$ and $\lim\limits_{x\to-\infty} \frac{c}{x} = 0, c > 0$.

However, sometimes we will not be concerned with the behaviour of $f(x)$ near a specific x –value, but rather with how $f(x)$ behaves as x increases without bound or decreases without bound. This is sometimes called the **end behaviour** of the function because it describes how the function behaves for the values of x that are farther from the origin.

In general, we can say that if $f(x) \to L$ as $x \to \infty$, then the graph of $y = f(x)$ gets closer and closer to the line $y = L$ as shown in Fig. 9 (a). We can also say that if $f(x) \to L$ as $x \to -\infty$, then the graph of $y = f(x)$ gets closer and closer to the line $y = L$ as shown in Fig. 9 (b). In either case, we call the line $y = L$ a **horizontal asymptote** or **asymptote parallel to** $x-$**axis**. This leads to the following definition.

Definition: A line $y = L$ is called a **horizontal asymptote** to the curve of the function f if $f(x) \to L$ as $x \to +\infty$ or as $x \to -\infty$.

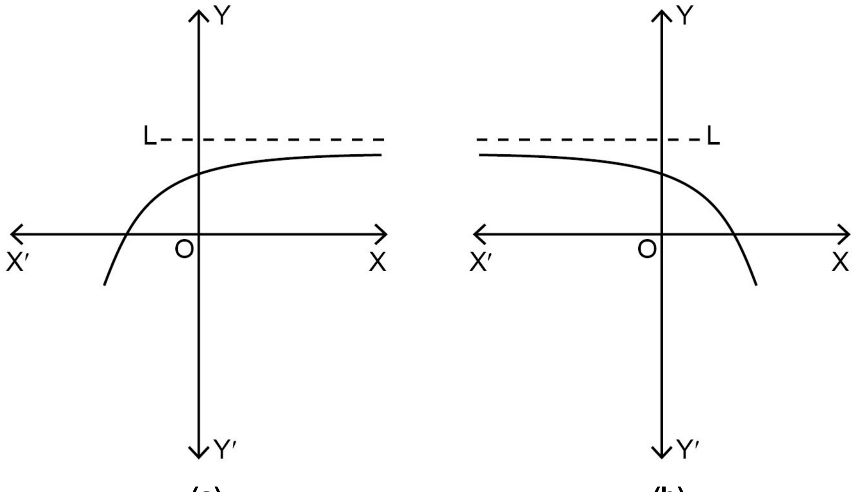

Fig. 9

Let us find the horizontal asymptotes in the following examples.

Example 6: Determine the horizontal asymptotes of the function f given by $f(x) = \frac{2x+5}{x}$.

Solution: To find the horizontal asymptotes, we find

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x+5}{x}$$
$$= \lim_{x\to\infty}\left(2 + \frac{5}{x}\right)$$
$$= 2$$

Thus, the horizontal asymptote is the line $y = 2$.

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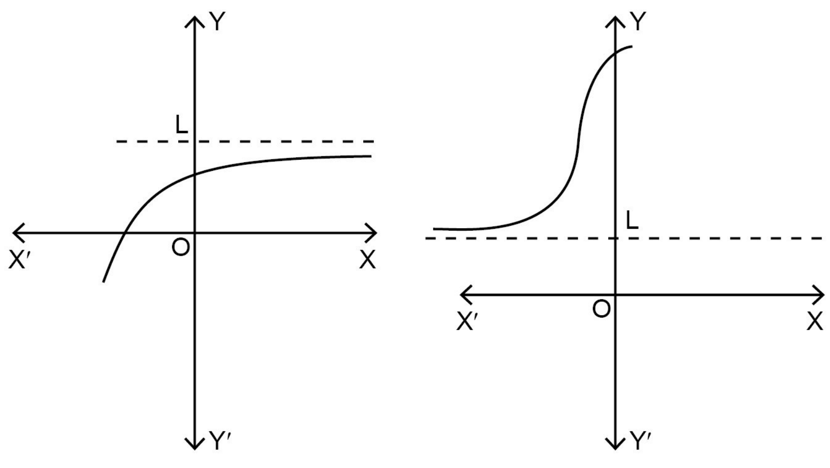
In Fig. 10 (a) and Fig. 10 (b), we see two ways in which a horizontal asymptote can occur.



(a) $\lim_{x\to\infty} f(x) = L$ **(b)** $\lim_{x\to-\infty} f(x) = L$

Fig. 10

Example 7: Find the horizontal asymptotes of $y = \frac{5+2^x}{1-2^x}$.

Solution: To find the horizontal asymptote, let us find the limits at infinity.

$$\lim_{x\to+\infty} \frac{5+2^x}{1-2^x} = \lim_{x\to+\infty} \frac{\frac{5}{2^x}+1}{\frac{1}{2^x}-1}$$

$$= \frac{0+1}{0-1} = -1$$

and

$$\lim_{x\to-\infty} \frac{5+2^x}{1-2^x} = \frac{5+0}{1-0} = 5$$

Hence, its horizontal asymptotes are $y = -1$ and $y = 5$, which are shown in Fig. 11.

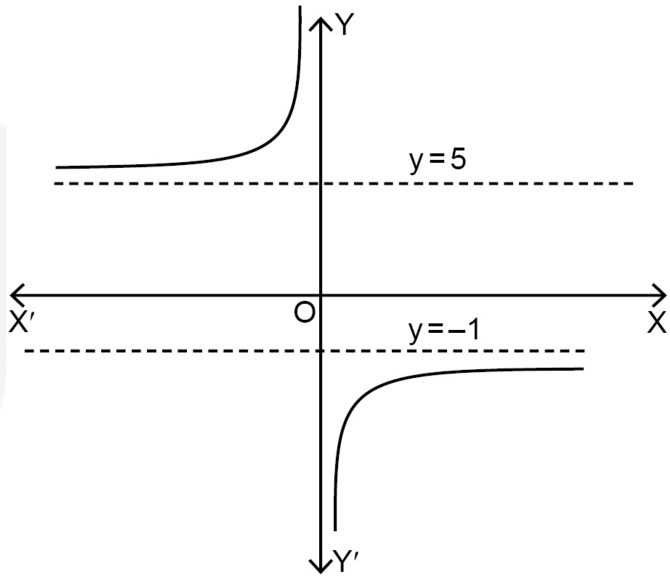


Fig. 11: Graph of $y = \frac{5+2^x}{1-2^x}$**.**

***

Example 8: Find the horizontal asymptote(s) of $y = x\,e^x$.

Solution: Let us find limits at infinity.

$$\lim_{x\to\infty} x\,e^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to-\infty} \frac{1}{-e^{-x}} \text{ (Applying L'Hôptial's rule)}$$

$$= \frac{1}{-\infty} = 0.$$

Thus, $y = 0$ is the horizontal asymptote of $y = x\,e^x$, which is shown in Fig. 12.

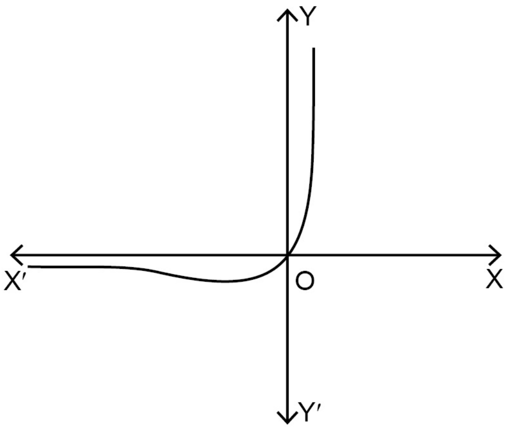


Fig. 12: Graph of $y = x e^{x}$.

***

Now, try the following exercises.

E3) Determine the horizontal asymptote of each of the following.

i) $f(x) = \dfrac{3x+4}{x^3 - x^2 + 5}$

ii) $f(x) = \dfrac{2x^4 - 3x^3}{x^3 + x}$

iii) $f(x) = \dfrac{-1}{x^2 + 2}$

iv) $f(x) = \dfrac{1}{x}$

v) $f(x) = (1.001)^x$

vi) $f(x) = e^{-3x} \cos x$

vii) $f(x) = x + \dfrac{1}{x}$

viii) $f(x) = \dfrac{\sin 2x}{x}$

E4) Determine a rational function f, which has a horizontal asymptote at $y = 0$, and vertical asymptotes at $x = -2$ and $x = 3$, and $f(1) = 1$.

Now, we will see the procedure to compute asymptotes parallel to axes of a polynomial function. Here we shall derive tests to decide whether a given curve has asymptotes parallel to the x and y axes. For this, we shall consider a curve given by $f(x, y) = 0$, where $f(x, y)$ is a polynomial in x and y.

Theorem 1: A straight line $y = c$ is an asymptote of a curve $f(x, y) = 0$ iff $y - c$ is a factor of the co-efficient of the highest power of x in $f(x, y)$.

This theorem can also be interpreted as follows.

Asymptotes parallel to the x-axis are obtained by equating to zero the real linear factors of the co-efficient of the highest power of x in the equation of the curve.

We can also state another theorem, similar to Theorem 1, giving a test to decide whether a given curve has an asymptote parallel to the y-axis or not.

Theorem 2: A straight line $x = c$ is an asymptote parallel to the y-axis iff $x - c$ is a factor of the co-efficient of the highest power of y in $f(x, y)$.

Now, let us see some examples to find asymptotes parallel to axes.

Example 9: Find the asymptotes parallel to either axis for the curve $y = x + \frac{1}{x}$.

Solution: Writing the given equation in the form $f(x, y) = 0$, we get $x^2 - xy + 1 = 0$. You can see the graph of this curve in Fig. 13. In $f(x, y) = 0$, the highest power of x is 2 and the co-efficient of x^2 is 1. It has no factors of the form $y - c$. Hence, there are no asymptotes parallel to the x-axis. The highest power of y in $f(x, y) = 0$ is 1 and the co-efficient of y when equated to zero gives $x = 0$. Hence, there is one asymptote parallel to the y-axis and moreover, it is the y-axis itself.

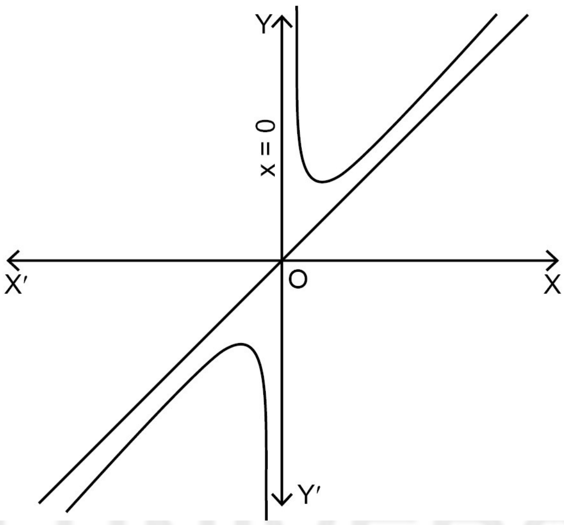


Fig. 13

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Example 10: Find the asymptotes parallel to axes of the curve $y^2(x^2 - a^2) = x$.

Solution: The given equation can be re-written as $y^2(x^2 - a^2) - x = 0$.

Asymptotes Parallel to $x-$axis: Equating the coefficient of x^2 (highest power of x) to zero, we get $y^2 = 0$. Which gives $y = 0$ as an asymptote.

Asymptotes Parallel to $y-$axis: Equating the coefficient of y^2 (highest power of y) equal to zero, we get $x^2 - a^2 = 0 \Rightarrow x = \pm a$ i.e., $x = a$, $x = -a$ are the asymptotes parallel to $y-$axis.

The required asymptotes are $x = \pm a, y = 0$.

***

Example 11: Find the asymptotes parallel to the co-ordinate axes of the curve $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$.

Solution: The given equation can be re-written as $a^2y^2 - b^2x^2 - x^2y^2 = 0$.

Asymptotes Parallel to $x-$axis: Equating the coefficient of x^2 (highest power of x) to zero, we get $y^2+b^2=0 \Rightarrow y=\pm ib$, which gives two imaginary asymptotes.

Asymptotes Parallel to $y-$axis: Equating the coefficient of y^2 (highest power of y) to zero, we get $x^2-a^2=0 \Rightarrow x=\pm a$. Thus, $x=+a,\ x=-a$ are the asymptotes parallel to $y-$axis.

***

Example 12: Find the asymptotes of the curve f given by $y=\dfrac{2x+1}{x}$.

Solution: We find the horizontal or vertical asymptotes by limits. Here, $y=2+\dfrac{1}{x}$ and $y\to 2$ as $x\to\infty$ or $-\infty$. Therefore, $y=2$ is an asymptote parallel to $x-$axis. Also, as $x\to 0, y\to\infty$, therefore, $x=0$ is the vertical asymptote as shown in Fig. 14.

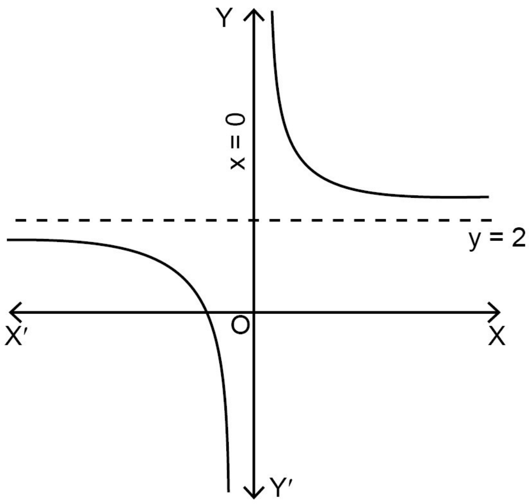

Fig. 14

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See if you can do this exercise on your own.

E5) For each of the following curves, find asymptotes parallel to either axis, if there are any.

i) $x^2y=2+y$ ii) $xy^2=16x^2+20y^2$

iii) $(x+y)^2=x^2+4$. iv) $x^2y^2=9(x^2+y^2)$.

v) $y=\dfrac{1}{x^2+1}$. vi) $y=\dfrac{3-10x}{x^2+10}$.

So far, we were finding the asymptotes which were parallel to axes. In the following section, we will find asymptotes which are not parallel to axes. These are called **oblique or slant asymptotes**.

15.3 SLANT/OBLIQUE ASYMPTOTES

You may be wondering whether an asymptote must always be parallel to a coordinate axis. No, there are many curves having asymptotes which are neither vertical nor horizontal. For example, consider $f(x) = \frac{x^2 - 1}{x - 2}$. The graph of f is shown in Fig. 15.

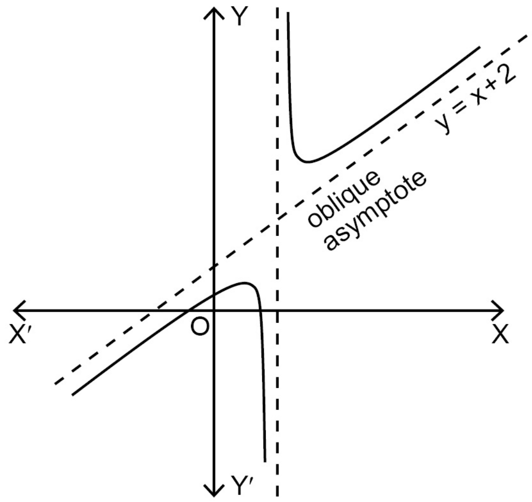

Fig. 15: Graph of $\frac{x^2 - 1}{x - 2}$.

We can write $f(x) = (x + 2) + \frac{3}{(x - 2)}$. We see that as $x \to \infty, \frac{3}{x - 2} \to 0$, therefore, when x gets very large, y gets closer and closer to $x + 2$. Thus, the line $y = x + 2$ is called the **slant asymptote** or **oblique asymptote**.

Definition: A line $y = mx + c\,(m \neq 0)$ is an **oblique asymptote or slant asymptote** to the graph of the function f if $\lim_{x \to +\infty} [f(x) - (mx + c)] = 0$ or $\lim_{x \to -\infty} [f(x) - (mx + c)] = 0$.

For example, if we say that the lines $y = x - 1$ and $y = -\frac{x}{2} + 1$ are asymptotes to any curve, that means

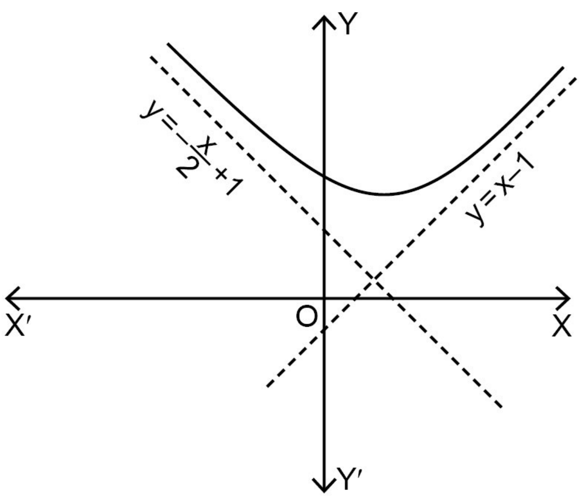

Fig. 16: Slant asymptotes.

that, as $x \to \infty$, the graph of f approaches the line $y = x - 1$, so $y = x - 1$ is an oblique asymptote to the graph of f at ∞. Similarly, as $x \to -\infty$, the graph of f approaches the line $y = -\frac{x}{2} + 1$, so $y = -\frac{x}{2} + 1$ is an oblique asymptote to the graph of f at $-\infty$, as shown in Fig. 16.

Going back to the definition of the oblique asymptotes, we can say that in the first case, the line $y = mx + c$ is an oblique asymptote of $f(x)$ when x tends to ∞, and in the second case the line $y = mx + c$ is an oblique asymptote of $f(x)$ when x tends to $-\infty$. The oblique asymptote, for the function $f(x)$ will be given by the equation $y = mx + c$. The value of m is computed first and is given by the following limit:

Suppose $y = mx + c$ is a slant asymptote to f at $\pm\infty$, then $\lim_{x \to \pm\infty} [f(x) - (mx + c)] = 0$.

On dividing this equation both the sides by x, we get

$$\lim_{x \to \pm\infty} \left[\frac{f(x)}{x} - \frac{mx + c}{x} \right] = 0$$

$$\lim_{x \to \pm\infty} \left[\frac{f(x)}{x} - m - \frac{c}{x} \right] = 0$$

$$\lim_{x \to \pm\infty} \left[\frac{f(x)}{x} - m \right] = 0 \qquad \left[\because \lim_{x \to \pm\infty} \frac{c}{x} = 0 \right]$$

Thus, $m = \lim_{x \to \pm\infty} \frac{f(x)}{x}$.

We can solve m separately for two cases as $x \to \infty$ and as $x \to -\infty$. If this limit does not exist or is equal to zero, then, there is no oblique asymptote in that direction.

Having m, then the value of c can be computed by $c = \lim_{x \to \pm\infty} [f(x) - mx]$. If this limit does not exist, then there is no oblique asymptote in that direction, even if a limit defining m exists.

Let us find slant asymptotes in the following examples:

Example 13: Find the slant asymptotes of $y = \frac{x^3}{x^2 - 1}$.

Solution: We shall find m and c.

$$m = \lim_{x \to \pm\infty} \frac{y}{x} = \lim_{x \to \pm\infty} \frac{1}{x} \left(\frac{x^3}{x^2 - 1} \right)$$

$$= \lim_{x \to \pm\infty} \frac{1}{1 - \frac{1}{x^2}} = 1$$

$$c = \lim_{x \to \pm\infty} (y - mx) = \lim_{x \to \pm\infty} \left(\frac{x^3}{x^2 - 1} - (1)x \right)$$

$$= \lim_{x \to \pm\infty} \frac{x}{x^2 - 1} = 0$$

Therefore, the slant asymptote is $y = x$.

***

Example 14: Find the slant asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solution: $y^2 = b^2\left(\frac{x^2}{a^2} - 1\right) = \frac{b^2}{a^2}(x^2 - a^2)$

$$m = \lim_{x \to \pm\infty} \frac{y}{x} = \lim_{x \to \pm\infty} \frac{1}{x}\left[\pm \frac{b}{a}\sqrt{x^2 - a^2}\right]$$

$$= \pm \frac{b}{a} \lim_{x \to \pm\infty} \sqrt{1 - \frac{a^2}{x^2}}$$

$$= \pm \frac{b}{a}(1) = \pm \frac{b}{a}.$$

$$c = \lim_{x \to \pm\infty}\left(y - \left(\pm \frac{b}{a}x\right)\right) = \pm \frac{b}{a} \lim_{x \to \pm\infty}\left[\sqrt{x^2 - a^2} - x\right]$$

$$= \pm \frac{b}{a} \lim_{x \to \pm\infty} \frac{-a^2}{\sqrt{x^2 - a^2} + x} = 0.$$

Thus, the slant asymptotes are $y = \pm \frac{b}{a}x$. Fig. 17 shows these asymptotes.

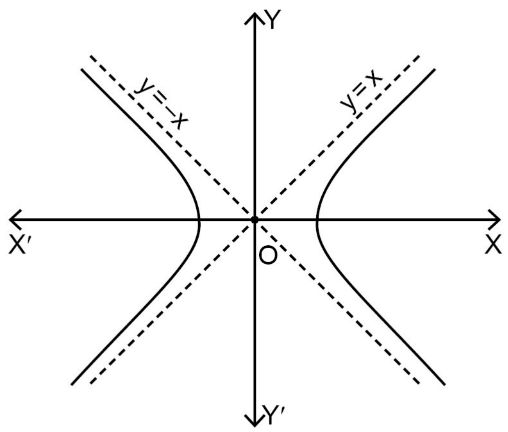

Fig. 17: Slant asymptotes in a Hyperbola.

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Example 15: Find the slant asymptote to the curve $y = \sqrt{x^2 + 9x}$.

Solution: Slope $m = \lim_{x \to \pm\infty} \frac{\sqrt{x^2 + 9x}}{x}$

$$= \lim_{x \to \pm\infty} \frac{\sqrt{(x^2)\left(1 + \frac{9}{x}\right)}}{x}$$

$$= \lim_{x \to \pm\infty} \frac{|x|\sqrt{1 + \frac{9}{x}}}{x}$$

$$= \lim_{x\to+\infty} \frac{x\sqrt{1+\frac{9}{x}}}{x} \text{ and } \lim_{x\to-\infty} \frac{-x\sqrt{1+\frac{9}{x}}}{x}$$

$$= \lim_{x\to+\infty} \sqrt{1+\frac{9}{x}} \text{ and } \lim_{x\to-\infty} -\sqrt{1+\frac{9}{x}}$$

$$= 1 \text{ and } -1$$

Now, let us find c for both the values of m. Where $m = 1$, we get:

$$c = \lim_{x\to\pm\infty} \left[\sqrt{x^2+9x} - x\right]$$

$$= \lim_{x\to\pm\infty} \frac{x^2+9x-x^2}{\sqrt{x^2+9x}+x}$$

$$= \lim_{x\to\pm\infty} \frac{9x}{\sqrt{x^2+9x}+x}$$

$$= \lim_{x\to\pm\infty} \frac{9x}{x\left(\sqrt{1+\frac{9}{x}}+1\right)}$$

$$= \lim_{x\to\pm\infty} \frac{9}{\sqrt{1+\frac{9}{x}}+1}$$

$$= \frac{9}{2}$$

Similarly, when $m = -1$, we get $c = \lim_{x\to\pm\infty} \left[\sqrt{x^2+9x}+x\right] = -\frac{9}{2}$

Hence, the slant asymptote to f are $y = x + \frac{9}{2}$ and $y = -x - \frac{9}{2}$.

***

Example 16: Show that $f(x) = x + \sqrt{x}$ does not have a slant asymptote at ∞.

Solution: We shall do a proof by contradiction. Suppose f has a slant asymptote $y = mx + c$. Then we must have

$$m = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \left(\frac{x+\sqrt{x}}{x}\right) = \lim_{x\to\pm\infty} \left(1+\frac{1}{\sqrt{x}}\right) = 1 \text{ or does not exist.}$$

so, $y = x + c$.

And then, we get

$$c = \lim_{x\to\pm\infty} (f(x) - x) = \lim_{x\to\pm\infty} \left(x + \sqrt{x} - x\right) = \lim_{x\to\pm\infty} \sqrt{x} = \infty \text{ or does not exist.}$$

Which is a contradiction (since c must be finite).

Hence, f cannot have a slant asymptote at ∞.

***

Let us find the slant asymptote to a curve, where the equation of the curve is of the form $f(x, y) = 0$.

Example 17: Find the oblique asymptotes for curve $x^3 - y^3 = 3xy$.

Solution: Suppose that the given curve has an oblique asymptote $y = mx + c$. The equation of the curve can be written as

$$x^3 - y^3 - 3xy = 0.$$

Dividing throughout by x^3 we get

$$1 - \frac{y^3}{x^3} - \frac{3y}{x}.\frac{1}{x} = 0$$

Thus, $\lim_{x\to\pm\infty}\left[1 - \frac{y^3}{x^3} - \frac{3y}{x}.\frac{1}{x}\right] = 0$

$$\Rightarrow \quad 1 - \lim_{x\to\pm\infty}\left(\frac{y^3}{x^3}\right) - 3\lim_{x\to\infty}\left(\frac{y}{x}\right)\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow \quad 1 - \lim_{x\to\pm\infty}\left(\frac{y^3}{x^3}\right) = 0, \left[\text{Since, } \lim_{x\to\infty}\frac{1}{x} = 0\right]$$

$$\Rightarrow \quad 1 - \left[\lim_{x\to\pm\infty}\left(\frac{y}{x}\right)\right]^3 = 0$$

$\Rightarrow \quad m^3 = 1 \Rightarrow m = 1$, as the other roots of $m^3 - 1 = 0$ are complex numbers.

Rewriting the equation of the curve as $(x - y)(x^2 + xy + y^2) = 3xy$, we have

$$c = \lim_{x\to\pm\infty}(y - mx) = \lim_{x\to\pm\infty}(y - x) = \lim_{x\to\pm\infty}\left[\frac{-3xy}{x^2 + xy + y^2}\right]$$

$$= \lim_{x\to\pm\infty}\left[\frac{-3}{\frac{x^2}{xy} + \frac{xy}{xy} + \frac{y^2}{xy}}\right]$$

$$= \frac{-3}{1+1+1}, \text{ [since, } \lim_{x\to\pm\infty}\frac{x}{y} = \lim_{x\to\pm\infty}\left(\frac{y}{x}\right)^{-1} = \left(\frac{1}{m}\right) = 1]$$

$$= -1$$

Hence, the required asymptote is $y = x - 1$.

***

If a rational function $\frac{P(x)}{Q(x)}$ is such that the degree of the numerator exceeds the degree of the denominator by **one**, then the graph of $\frac{P(x)}{Q(x)}$ may have an oblique asymptote.

To find the asymptote we write $\frac{P(x)}{Q(x)} = (ax + b) + \frac{R(x)}{Q(x)}$, where, degree of $R(x) <$ degree of $Q(x)$.

Now, $\lim_{x\to\infty}\left[\frac{P(x)}{Q(x)} - (ax + b)\right] = 0$ [since $\lim_{x\to\infty}\frac{R(x)}{Q(x)} = 0$]

and $\lim_{x\to-\infty}\left[\frac{P(x)}{Q(x)} - (ax + b)\right] = 0$ [since $\lim_{x\to\infty}\frac{R(x)}{Q(x)} = 0$]

We can say that the graph of $\frac{P(x)}{Q(x)}$ approaches the line $y = ax + b$ as $x \to \infty$ or $x \to -\infty$. This line $y = ax + b$ is oblique asymptote.

Now, in the next example we shall find the oblique asymptote of a rational function.

Example 18: Find the oblique asymptote of $y = \frac{2x^3 - 3x + 4}{x^2}$.

Solution: We can write $y = 2x + \frac{-3x + 4}{x^2}$

Here $\lim_{x\to\infty} \frac{-3x+4}{x^2} = 0$ and $\lim_{x\to-\infty} \frac{-3x+4}{x^2} = 0$

Thus, the line $y = 2x$ is an oblique asymptote.

***

Try to solve these exercises now.

E6) Find oblique asymptotes to each of the following curves.

i) $x^3 + y^3 = 3ax^2$ ii) $x^4 - y^4 + xy = 0$

iii) $y = \frac{2x^3 + x^2 + 11x + 5}{x^2 + 5}$ iv) $y = 2x - x^2 + 2$.

E7) The cost-function to produce x units of a product is given by $C(x) = 3x^2 + 80$. Find the oblique asymptote for the average cost and interpret its significance.

Now, let us summarize what we have studied in this unit.

15.4 SUMMARY

In this unit, we have covered the following points.

1. A straight line is said to be an asymptote to an infinite branch of a curve, if, as a point P on the curve moves to infinity along the curve, the perpendicular distance of P from the straight line tends to zero.

2. Asymptotes parallel to the coordinate axes are obtained by equating to zero the real linear factors in the co-efficients of the highest power of x and the highest power of y in the equation of the curve.

3. A line $y = mx + c\,(m \neq 0)$ is an oblique asymptote or slant asymptote to the graph of the function f if $\lim_{x\to+\infty}[f(x) - (mx + c)] = 0$ or $\lim_{x\to-\infty}[f(x) - (mx + c)] = 0$. The values of m and c are $m = \lim_{x\to\pm\infty} \frac{f(x)}{x}$ and $c = \lim_{x\to\pm\infty}[f(x) - mx]$.

15.5 SOLUTIONS/ANSWERS

E1) i) The vertical asymptotes are $x = 0, x = 1$ and $x = -1$

ii) No vertical asymptotes.

iii) The vertical asymptotes are $x = 4$ and $x = -4$.

iv) The vertical asymptote is $x = -2$.

v) The vertical asymptotes are $x = 3n\pi$, where $n \in \mathbb{Z}$.

vi) The vertical asymptote is $x = 0$.

E2) The line $x = 0$ is not a vertical asymptote because $\lim_{x\to 0} f(x) = 2$.

E3) i) $$f(x) = \frac{\frac{3}{x^2} + \frac{4}{x^3}}{1 - \frac{1}{x} + \frac{5}{x^3}}$$

Since, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$, therefore, the line $y = 0$, is a horizontal asymptote.

ii) No horizontal asymptotes.

iii) The horizontal asymptote is $y = 0$.

iv) The horizontal asymptote is $y = 0$

v) $\lim_{x\to\infty} (1.001)^x = \infty$ and $\lim_{x\to-\infty} (1.001)^{-x} = 0$. Thus, $y = 0$ is horizontal asymptote.

vi) $\lim_{x\to\infty} e^{-3x} \cos(x) = 0$. Thus, $y = 0$ is horizontal asymptote.

vii) No horizontal asymptote.

viii) The horizontal asymptote is $y = 0$

E4) One of the possible function is $f(x) = \dfrac{x-7}{(x+2)(x-3)}$.

E5) i) $x^2 y = 2 + y \Leftrightarrow x^2 y - y - 2 = 0$

Highest power of x is 2, and the coefficient of x^2 is y. Hence $y = 0$ is an asymptote.

Highest power of y is 1, and the coefficient of y is $x^2 - 1 = (x-1)(x+1)$.

Hence, $x = -1$ and $x = 1$ are two asymptotes parallel to $y-$axis.

ii) No asymptote parallel to the x-axis.
$x = 20$ is an asymptote parallel to $y-$axis.

iii) No asymptote parallel to the y-axis.
$y = 0$ is an asymptote.

iv) $y = \pm 3$ are asymptotes parallel to $x-$axis.

$x = \pm 3$ are asymptotes parallel to $y-$axis.

v) $y = 0$ is an asymptote.

vi) $y = 0$ is an asymptote.

E6) i) $x^3 + y^3 = 3ax^2$

$\Rightarrow \quad 1 + (y/x)^3 = 3a/x$

$\Rightarrow \quad 1 + \lim\limits_{x\to\infty}\left(\dfrac{y}{x}\right)^3 = \lim\limits_{x\to\infty}\dfrac{3a}{x}$

$\Rightarrow \quad 1 + m^3 = 0 \Rightarrow m^3 = -1 \Rightarrow m = -1$.

$$c = \lim_{x\to\infty}(y - mx) = \lim_{x\to\infty}(y + x)$$

$$= \lim_{x\to\infty}\frac{3ax^2}{x^2 - xy + y^2} = \lim_{x\to\infty}\frac{3a}{1 - y/x + (y/x)^2}$$

$$= \frac{3a}{1+1+1} = a$$

Hence, the equation of the asymptote is $y + x = a$

ii) $m = 1, c = 0$, Equation: $y = x$

$m = -1, c = 0$, Equation: $y + x = 0$.

iii) $y = 2x + 1$.

iv) $\lim\limits_{x\to\infty}[f(x) - (mx + 1)]$ is not zero as 2^x is unbounded. Therefore, considering $\lim\limits_{x\to-\infty}[f(x) - (mx + c)]$

$$= \lim_{x\to-\infty}[2x - 2^x + 2 - (mx + c)]$$

$$= \lim_{x\to-\infty}[(2 - m)x - 2^x + (2 - c)]$$

This limit is 0, only if $m = 2$ and $c = 2$. Thus, the equation of the oblique asymptote is $y = 2x + 2$.

E7) The average-cost function $A(x) = \dfrac{C(x)}{x}$

$$A(x) = \frac{3x^2 + 80}{x} = 3x + \frac{80}{x}$$

Here, $\lim\limits_{x\to\infty}\dfrac{80}{x} = 0$ and $\lim\limits_{x\to-\infty}\dfrac{80}{x} = 0$.

Thus, $y = 3x$ is an oblique asymptote, as shown in Fig. 18.

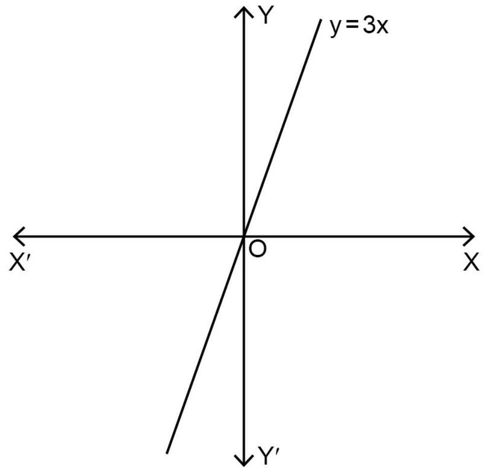


Fig.18

UNIT 16

CURVE TRACING

Structure **Page No.**

16.1 INTRODUCTION

A picture is worth a thousand words. A curve which is the visual image of a function gives us a lot of information. Of course, we can also obtain this information by analysing the equation which defines the functional relation. But studying the associated curve is often easier and quicker. In addition to this, a curve which represents a relation between two quantities also helps us to easily find the value of one quantity corresponding to a specific value of the other. In Sec. 16.2, we shall try to understand what is meant by the picture or the graph of a function like $y = f(x)$ and the curve with more than one branches at any point, expressed in the form $f(x, y) = 0$ and how to sketch them. In Sec. 16.3 and Sec. 16.4, we shall discuss the tracing of a curve in parametric and polar form, respectively. We shall be using many results from the earlier units here. With this unit we come to the end of Block 4, in which we have studied various geometrical features of functional relations with the help of differential calculus.

Now we shall list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

Objectives

After studying this unit, you should be able to:

- list the properties which can be used for tracing a curve;
- trace some curves whose equations are given in cartesian, parametric or polar form.

16.2 TRACING A CURVE: CARTESIAN EQUATION

You may recall from Unit 2, that by the graph of a function $f : D \to \mathbb{R}$, we mean the set of points $\{(x, f(x)) : x \in D\}$. Graphing a function means showing the points of the corresponding set in a plane. Thus, essentially curve tracing means plotting the points which satisfy a given relation. However, there are some difficulties involved in this. Let's see what these are and how to overcome them.

It is often not possible to plot all the points on a curve. The standard technique is to plot some suitable points and to get a general idea of the shape of the curve by considering tangents, asymptotes, singular points, extreme points, inflection points, concavity, monotonicity, periodicity etc. Then, we draw a free hand curve as nearly satisfying the various properties as is possible.

The curve or graph that we draw has a limitation. If the range of values of either (or both) variables is not finite, then it is not possible to draw the complete graph. In such cases, the graph is not only approximate, but is also incomplete. For example, consider the simplest curve, a straight line. Suppose we want to draw the graph of $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 1$. We know that this is a line parallel to the x-axis. But it is not possible to draw a complete graph as this line extends infinitely on both sides. We indicate this by arrows at both ends as in Fig 1.

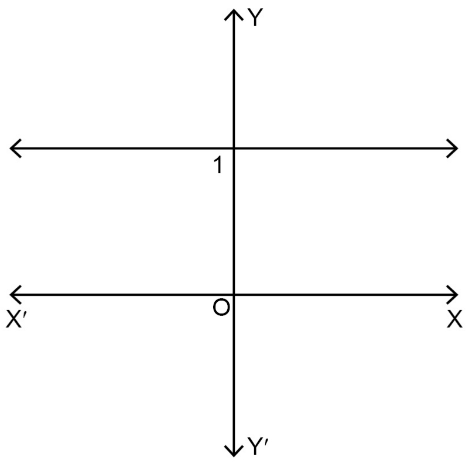


Fig. 1

Now, we shall take up the problem of graphing a function by hand, when the equation is given in the cartesian form.

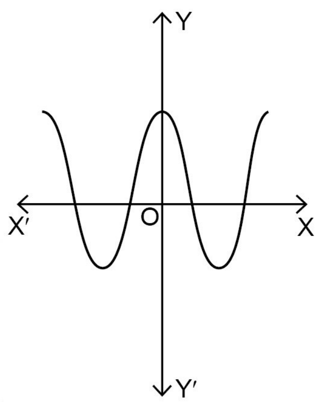
Let us list some properties which, when taken, will simplify our job of tracing this curve. We have discussed all these properties. Now, we shall summarize these one by one.

i) **Simplify:** If possible, simplify the function $y = f(x)$, you wish to sketch. For example, if f is defined by $f(x) = \dfrac{x^2 + x - 2}{(x-1)}, x \neq 1$, you must write it as $f(x) = x + 2, x \neq 1$ before beginning the procedure listed here.

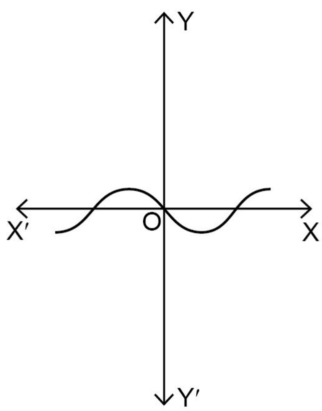
ii) **Domain and Range:** In case of $y = f(x)$, we find the domain and range and mark the regions accordingly.

iii) **Periodicity:** Recall Unit 6, wherein we discussed periodic function. **Periodicity** is the tendency of a function to repeat itself in a regular pattern at a fixed interval. For example, all trigonometric functions have periodicity. If $f(x+p)=f(x)$ for all x in D, where p is a positive constant, then, f is called a periodic function and smallest p is called the **period of the function**. While tracing a curve, if we know that the function is periodic and the period is p, we can keep on translating to sketch the entire curve [Recall from Unit 3 for translation].

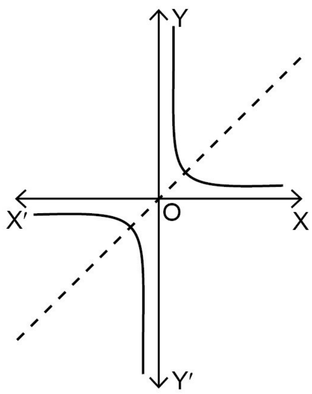
iv) Symmetry: The next step is to find out if the curve is symmetrical about any line, or about the origin. A curve is symmetrical about a line if, when we fold the curve on the line, the two positions of the curve exactly coincide. A curve is symmetrical about the origin if we get the same curve after rotating it through 180^{o}. We have already discussed symmetry of curves in Unit 6. Fig. 2, shows you some examples of symmetric curves.



(a) Symmetric about the x-axis.



(b) Symmetric about the origin.



(c) Symmetric about the line $y=x$.

Fig. 2

Here, we give you some hints which will help you to determine the symmetry of a curve.

a) **Symmetry about y-axis:** The graph of a function $y=f(x)$ is said to be symmetric about y-axis, if f is an even function, that is, the equation of the curve is unchanged when x is replaced by $-x$. For example, $y=\cos x$, $y=x^2$, $y=|x|$, etc. This means that our work is cut in half. If we know what the curve looks like for $x \geq 0$, then we only need to reflect about the $y-$axis to obtain the complete curve.

b) **Symmetry about origin:** Recall Unit 6, wherein we learnt that odd functions are symmetric about origin. If $f(x)=-f(-x)$, then the curve is symmetrical about the origin. In such cases, it is enough to draw the part of the graph above the x-axis and rotate it through 180° to get the complete graph. Some such functions are: $y = x^3$, $y = \sin x, y = x$, etc.

c) **Symmetry about the line $y = x$:** If the equation of the curve does not change when we interchange x and y, then the curve is symmetric about the line $y = x$.

v) Points of intersection with axes: The next step is to determine the points where the curve intersects the axes. If we put $y = 0$ in $y = f(x)$, and solve the resulting equation for x, we get the points of intersection with the x-axis. Similarly, putting $x = 0$ and solving the resulting equation for y, we can find the points of intersection with the y-axis.

For example, in the curve $y = 3x^2 - x^3$, if $y = 0$, we get $x = 0, 3$ and if $x = 0$, we get $y = 0$. Thus, the curve intersects axes at $(0,0)$ and $(3,0)$. You can omit this step, if the equation is difficult to solve.

vi) Points of discontinuity: Try to locate the points where the function is discontinuous.

vii) Intervals of increasing and decreasing functions: For this, calculate $\frac{dy}{dx}$. This will help you in locating the portions where the curve is rising $\left(\frac{dy}{dx} > 0\right)$ or falling $\left(\frac{dy}{dx} < 0\right)$. You may recall Unit 13.

viii) Concavity and point(s) of inflection: Recall from Unit 14, and calculate second derivative of w.r.t. x. From $\frac{d^2y}{dx^2}$, you can find concavity. The curve is concave upward where $\frac{d^2y}{dx^2} > 0$ and concave downward where $\frac{d^2y}{dx^2} < 0$. Inflection point occurs where the direction of concavity changes. These will give you a good idea about the shape of the curve.

ix) Relative extrema: Recall from Unit 13, we use the second-derivative test to find the relative maxima or minima. We substitute the first-order critical numbers x_0 in the following test:

- If $\left(\frac{d^2y}{dx^2}\right)_{x=x_0} > 0$, then relative minimum at x_0.
- If $\left(\frac{d^2y}{dx^2}\right)_{x=x_0} < 0$, then relative maximum at x_0.
- If $\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = 0$, then the test fails.

We can also use first derivative test.

x) Tangents and normals: Compute the equations of the vertical tangents and corresponding normals. You may recall Unit 14.

xi) Asymptotes: The next step is to find the asymptote(s), if there are any. We can find asymptotes parallel to axes and oblique as discussed in Unit 15. They indicate the trend of the branches of the curve extending to infinity.

xii) Singular point: Another important step is to determine the singular points. The shape of the curve at these points is, generally, more complex, as more than one branch of the curve passes through them. [Recall from Unit 14].

xiii) Plot points: Plot points where f has a relative maxima, minima or point(s) of inflection, x-intercepts, y-intercepts, etc.

xiv) Sketch the curve: Now, try to draw tangents to the curve at some of these plotted points. Now join the plotted points by a smooth curve (except at points of discontinuity). The tangents will guide you in this, as they give you the direction of the curve. Sketch the asymptotes by dash lines. Finally, draw the curve using the information in items i) to xiii).

We shall now illustrate this procedure through a number of examples. You will notice that it may not be necessary to take all the steps mentioned above, in each case. We begin by tracing some functions which were introduced in Unit 2 and Unit 6.

Example 1: Sketch the graph of the function $y = |x|$.

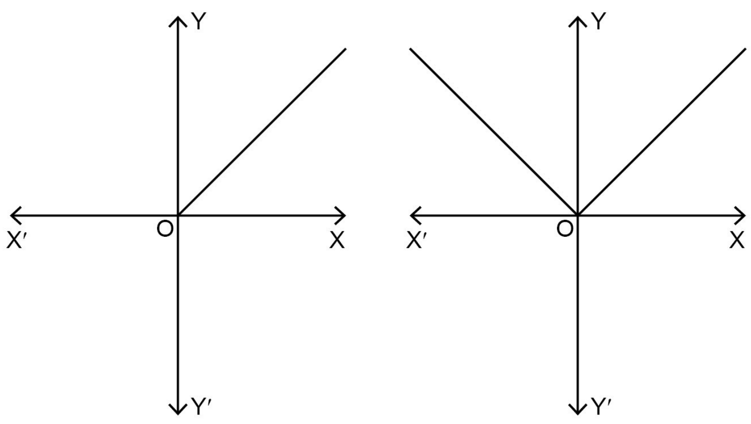
Solution: Let us begin using steps of curve tracing. We can rewrite y as

$$y = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}.$$

i) Domain and Range: The domain of this function is $\mathbb{R}$ and the range is non-negative reals. Therefore, y can take only positive values. Thus, the graph lies above the x-axis.

ii) Symmetry: Since, $|x| = |-x|$, therefore, the function $y = |x|$ is symmetric about the y-axis.

iii) Points of Intersection with axes: If $x = 0$, then, $y = 0$, therefore, the curve meets the axes only at the origin. On the right of the y-axis, $x > 0$, and, so, $|x| = x$. Thus, the graph reduces to that of $y = x$ and you know that this a straight line equally inclined to the axes (Fig. 3(a) below).Taking its reflection in the y-axis, we get the complete graph as shown in Fig 3(b).



(a) Graph on the right of the y-axis. **(b) Complete graph.**

Fig 3

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Example 2: Sketch the greatest integer function $y = [x]$.

Solution: Let us see which properties of curve tracing will be used to trace greatest integer function.

i) **Domain and Range:** The domain of the function is $\mathbb{R}$ and the range is set of all integers. The curve lies in the first and third quadrant, because either $x \geq 0$ and $y \geq 0$ or $x \leq 0$ and $y \leq 0$.

ii) **Symmetry:** If we replace x by $-x$, we get different value of y. Therefore, $[x]$ is not symmetrical about y-axis. Also, $y = [x]$ is not an odd function, thus, not symmetrical about origin.

iii) **Points of intersection with axes:** When $x = 0, y = 0,$ thus, the curve passes through the origin. Also, when $y = 0, 0 \leq x < 1$ therefore, the graph lies on x-axis.

iv) **Points of Discontinuity:** $y = [x]$ is discontinuous at every integer point. Hence, there is a break in the graph at every integer point n. In every interval $[n,\ n+1[$ its value is constant, and is equal to n.

v) **Relative extrema:** No maximum or minimum.

vi) **Asymptotes:** There are no asymptotes.

vii) **Concavity:** The graph is neither concave upward nor concave downward.
Hence, the graph is as shown in Fig 4. Note that a hollow circle around a point indicates that the point is not included in the graph.

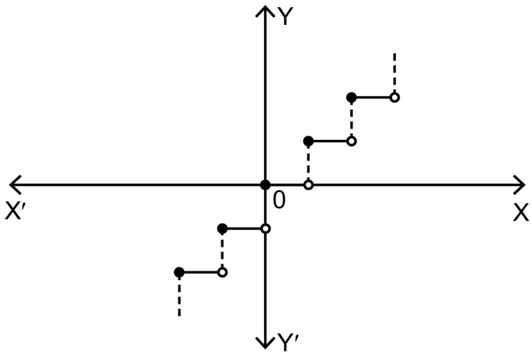

Fig 4: Graph of $y = [x]$.

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Example 3: Sketch the graph of $y = x^3$.

Solution: Let us check for the properties for the curve $y = x^3$.

i) **Domain and Range:** The domain and range of the function is $\mathbb{R}$. When $x > 0, y > 0$ and when $x < 0, y < 0$. Thus, there is no portion of the graph in the second and fourth quadrants.

ii) **Symmetry:** The function is an odd function. This means that the curve is symmetric about the origin. Thus, it is sufficient to draw the graph above the x-axis and join it to the portion obtained by rotating it through 180°.

iii) **Points of intersection with axes:** If $x = 0,$ then, $y = 0$. Therefore, the curve meets the axes only at the origin.

iv) **Tangents at origin:** We have $\frac{dy}{dx} = 3x^2$, which is 0 at the origin. Thus, the tangent at origin is the x-axis.

v) Monotonicity: We find, $\frac{dy}{dx} = 3x^2$, which is always non-negative. This means that as x increases, so does y. Thus, the graph keeps on rising.

vi) Relative extrema: Here, $\frac{dy}{dx} = 0$ at $(0, 0)$ and $\frac{d^2y}{dx^2} = 6x$ is 0 at $(0, 0)$. Since the second derivative test is not helpful to find extrema, let us look at the sign of $\frac{dy}{dx}$ on each side.

$$\frac{dy}{dx} = 3x^2 = \begin{cases} > 0, & \text{for } x > 0 \\ > 0, & \text{for } x < 0 \end{cases}$$

Since, the sign of $\frac{dy}{dx}$ does not change, therefore, there are no extreme points.

vii) Concavity and Point of Inflection: Here, $\frac{d^2y}{dx^2} = 0$ at the origin. Also, $\frac{d^2y}{dx^2} < 0$ when $x < 0$ and $\frac{d^2y}{dx^2} > 0$ when $x > 0$. Therefore, the curve is concave upward when $x > 0$ and concave downward when $x < 0$. Since the concavity is changing at origin, therefore, the point of inflection is $(0, 0)$.

viii) Asymptotes: The graph has no asymptotes parallel to the axes. Further $\lim_{x \to \infty} \frac{y}{x} = \lim_{x \to \infty} x^2$ and this does not exist. This means that the curve does not have any oblique asymptote.

ix) Singular Points: The curve has no singular points.
The graph is shown in Fig 5.

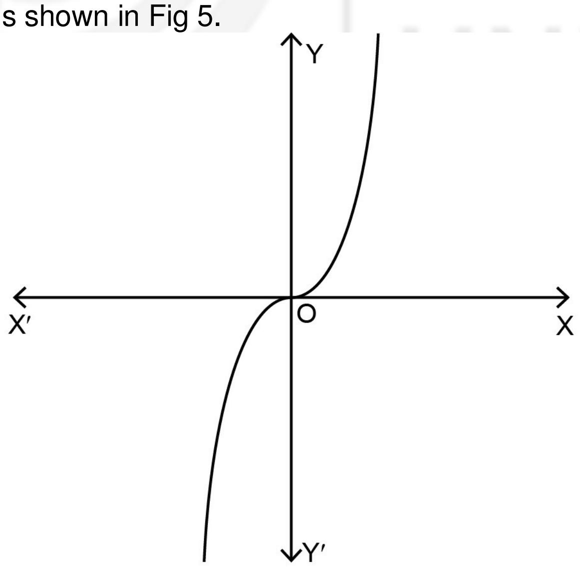

Fig 5: Graph of $y = x^3$.

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Example 4: Sketch the graph of $y = \frac{1}{x^2}$.

Solution: Let us list the properties to trace the curve.

i) **Domain and Range:** The domain of the function is $\mathbb{R}-\{0\}$ and the range of the function is non-negative reals. The y-coordinates of any point on the curve cannot be negative. So, the curve must be above the x-axis.

ii) **Symmetric:** Here, $f(x)=f(-x)$, thus the curve is symmetric about the y-axis. Hence, we shall draw the graph to the right of the y-axis first.

iii) **Points of intersection on axes:** The curve does not intersect at the axes at all.

iv) **Monotonicity:** We have $\frac{dy}{dx}=-\frac{2}{x^3}$ and $\frac{d^2y}{dx^2}=\frac{6}{x^4}$. Since $\frac{dy}{dx}<0$ for all $x>0$, therefore, the function is decreasing in $]0,\infty[$, that is, the graph keeps on falling as x increases. Also, since $\frac{dy}{dx}>0$ for all $x<0$, therefore, the function is increasing in $]-\infty,0[$.

v) **Discontinuity:** The graph of y is continuous in the domain of the function.

vi) **Relative extrema:** Further, since, $\frac{dy}{dx}$ is non-zero for all x in the domain, thus, there is no extreme point.

vii) **Concavity and point of inflection:** Since, $\frac{d^2y}{dx^2}$ is positive in the domain, therefore, the function is concave upward everywhere in the domain. Since, the concavity does not change, therefore, there is no point of inflection.

viii) **Asymptotes:** Since $\lim_{x\to 0}\frac{1}{x^2}=\infty$, therefore, $x=0$ is the vertical asymptote. Also, $\lim_{x\to\infty} f(x)=0$, thus, $y=0$ is the horizontal asymptote.

The curve is shown in Fig 6.

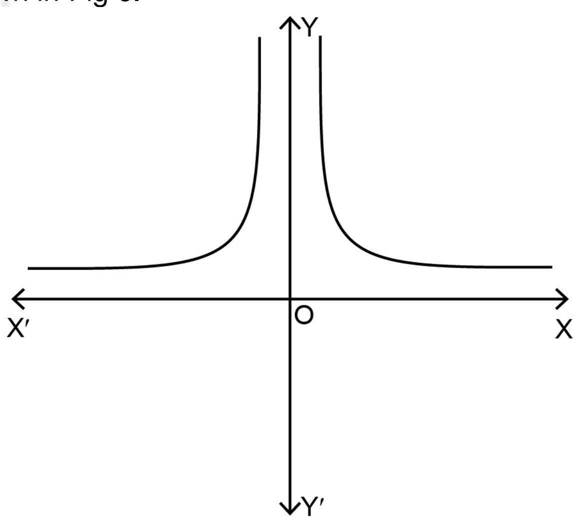

Fig 6: Graph of $y=1/x^2$.

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Example 5: Sketch the graph of $y = \frac{1}{x}$.

Solution:

i) **Doman and Range:** The domain of the function is $\mathbb{R} - \{0\}$ and the range is $\mathbb{R}$. Here, we can see that either x and y both will be positive or both will be negative. This means that the curve lies in the first and the third quadrants.

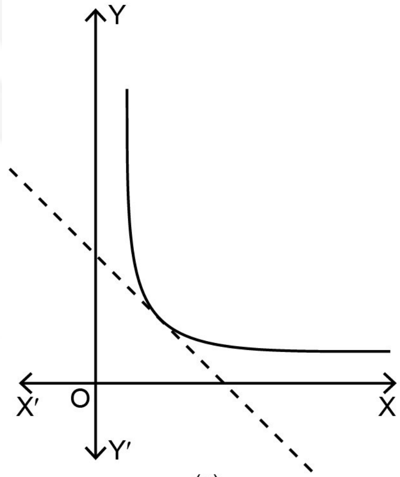
ii) **Symmetric:** Here, $f(x) = \frac{1}{x}$, and f is not an even function, therefore, it is not symmetric about y-axis. Further, it is symmetric about the origin and hence, it is sufficient to trace it in the first quadrant and rotate this through 180° to get the portion of the curve in the third quadrant.

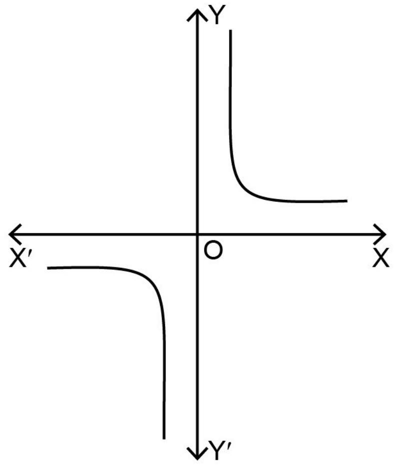
iii) **Interval of increasing or decreasing:** Here, $\frac{dy}{dx} = \frac{-1}{x^2}$, which means that $y < 0$ for all values of x in the domain. Hence, as x increases, y decreases.

iv) **Asymptotes:** Since, $\lim_{x \to \infty} f(x) = 0$, therefore, $y = 0$ is the horizontal asymptote. Also, $\lim_{x \to 0} f(x) = \infty$, therefore, $x = 0$ is the vertical asymptote.

v) **Relative extrema:** We have $\frac{dy}{dx} = \frac{-1}{x^2} \neq 0$ for any x in the domain. That is, there are no extrema.

Considering all these points we can trace the curve in the first quadrant (see Fig 7(a)). Fig 7(b) gives the complete curve.





(a) Graph of $xy = 1$ in the first Quadrant. **(b) Complete graph.**

Fig. 7

***

The curve traced in Example 5 is a **hyperbola**. If we cut a double cone by a plane as in Fig 8(a), we get a **parabola**. It is a section of a cone. For this reason, it is also called a **conic section**. Fig. 8(b) and Fig. 8(c) show a circle and an ellipse respectively. The curve in Fig 8(d) is called a **hyperbola** and that in Fig 8(e) is the pair of straight lines.

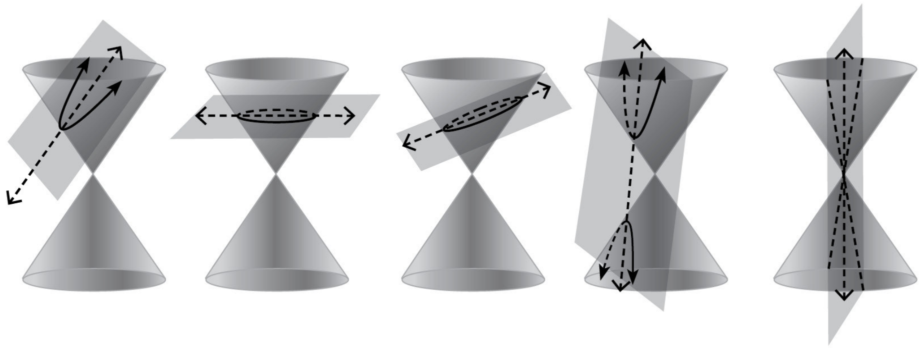


Fig 8: Conic Sections [(a) Parabola, (b) Circle, (c) Ellipse (d) Hyperbola (e) Pair of straight lines.]

The earliest mention of these curves is found in the works of a Greek Mathematician Menaechmas (fourth century B.C.). Later Apollonius (third century B.C.) studied them extensively and gave them their current names. In the seventeenth century, Rene Descartes discovered that the conic sections can be characterised as curves which are governed by a second degree equation in two variables. Blaise Pascal (1623-1662) presented them as projections of a circle. (Why don't you try this experiment? Throw the light of a torch on a wall at different angles and watch the different conic sections on the wall). Galileo (1564-1642) showed that the path of a projectile thrown obliquely (Fig 9) is a parabola.

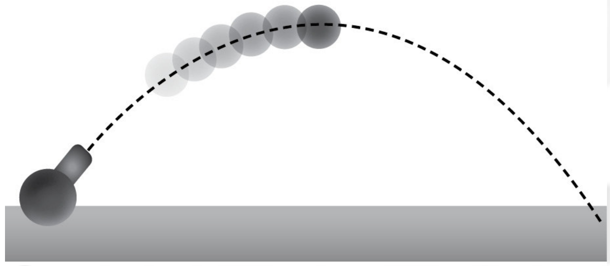


Fig 9: Projectile path

Paraboloid curves are also used in arches and suspension bridges (Fig 10). Paraboloid surfaces are used in telescopes, search lights, solar heaters and radar receivers.

Fig 10

In the seventeenth century, Johannes Kepler discovered that planets move in elliptical orbits around the sun. Halley's comet is also known to move along a very elongated ellipse. A comet or meteorite coming into the solar system

from a great distance moves in a hyperbolic path. Hyperbolas are also used in sound ranging and navigation systems.
Let's look at the next example now.

Example 6: Sketch the graph of $y = x^3 + x^2$.

Solution:

i) **Domain and Range:** The domain and range of the function are $\mathbb{R}$.

ii) **Symmetry:** The function is neither even nor odd, thus, not symmetric about y-axis and origin.

iii) **Points of intersection:** If $x = 0$, then $y = 0$, and if $y = 0, x = 0, -1$. Thus, the curve meets the axes at $(0, 0)$ and $(-1, 0)$.

iv) **Tangents:** We have $\frac{dy}{dx} = 3x^2 + 2x$. The x-axis is the tangent at the origin as $\frac{dy}{dx} = 0$, at $x = 0$. Since, $\frac{dy}{dx} = 1$ when $x = -1$, therefore, the tangent at $(-1, 0)$ makes an angle of 45° with the x-axis (Fig 11(a)).

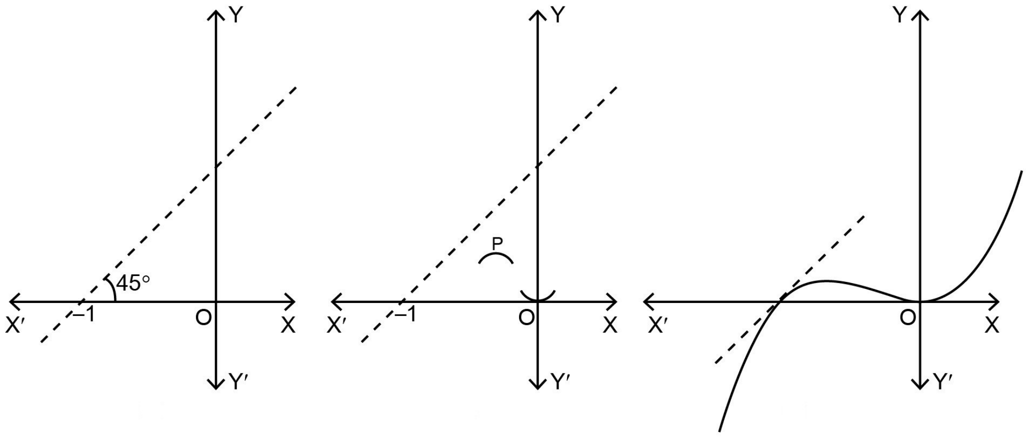
v) **Relative extrema:** Further, $\frac{dy}{dx} = 0$ gives $x = 0$ and $x = \frac{-2}{3}$. Now, $\frac{d^2y}{dx^2} = 6x + 2$. Since, $\frac{d^2y}{dx^2} > 0$ at $(0,0)$, therefore, the point $(0,0)$ has a relative minimum. The point $\left(\frac{-2}{3}, \frac{4}{27}\right)$ has a relative maximum as $\frac{d^2y}{dx^2} < 0$ at $x = \frac{-2}{3}$. Thus, in Fig 11(b), O is a valley and P is a peak.

vi) **Point of inflection:** Here, $\frac{d^2y}{dx^2} = 0$ at $x = -\frac{1}{3}$ and changes sign from negative to positive as x passes through $-1/3$. Hence, $\left(\frac{-1}{3}, \frac{2}{27}\right)$ is a point of inflection. Since, $\frac{d^2y}{dx^2} < 0$ on $\left]-\infty, -\frac{1}{3}\right[$, therefore, the curve is concave downward. Also, since $\frac{d^2y}{dx^2} > 0$ on $\left]-\frac{1}{3}, \infty\right[$, therefore, the curve is concave upward in this interval.

vii) **Interval of increasing or decreasing function:** If $-\frac{2}{3} < x < 0$, then $\frac{dy}{dx} < 0$. Thus, the graph rises in $]-\infty, -2/3[$ and $]0, \infty[$, but falls in $]-2/3, 0[$.

viii) **Asymptotes:** As x tends to infinity, so does y. As $x \to -\infty$, so does y. There is neither horizontal nor vertical asymptote.For oblique asymptote, $\lim_{x\to\infty} [(x^3 + x^2) - (mx + c)]$ does not exist, therefore, no oblique asymptote.

Hence, the graph is as shown in Fig 11(c).



(a) (b) (c)

Fig. 11

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Example 7: Sketch the curve $y = \frac{3x^2}{x^2-1}$.

Solution:

i) **Domain and Range:** The domain is $\mathbb{R} - \{-1, 1\}$.

ii) **Symmetry:** Since the powers of x are even, therefore, the curve is symmetric about the y-axis.

iii) **Point of intersection with the axes:** The curve passes through origin.

iv) **Asymptotes:** Since, $\lim_{x\to\pm\infty} \frac{3x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{3}{1-\frac{1}{x^2}} = 3$

Therefore, the line $y = 3$ is the horizontal asymptote.

Also, $\lim_{x\to 1^+} \frac{3x^2}{x^2-1} = \infty$, $\lim_{x\to 1^-} \frac{3x^2}{x^2-1} = -\infty$, $\lim_{x\to -1^+} \frac{3x^2}{x^2-1} = \infty$ and $\lim_{x\to -1^-} \frac{3x^2}{x^2-1} = -\infty$

Therefore, the lines $x = 1$ and $x = -1$ are vertical asymptotes. We can draw these asymptotes as shown in Fig.12 (a).

v) **Monotonicity:** Here, $y' = \frac{5x(x^2-1) - 3x^2 \cdot 2x}{(x^2-1)^2} = \frac{-6x}{(x^2-1)^2}$. Since, $y' > 0$ when $x < 0$ and $y' < 0$ when $x > 0$, therefore, f is increasing on $]-\infty, -1[$ and $]-1, 0[$ and decreasing on $]0, 1[$ and $]1, \infty[$.

vi) **Relative Extrema:** When $y' = 0, x = 0$. Since, y' changes from positive to negative at 0, therefore, there is a local maximum by the first derivative test.

vii) **Concavity:** We have $f''(x) = \frac{-6(x^2-1)^2 + 6x \cdot 2(x^2-1)2x}{(x^2-1)^4} = \frac{6(1+3x^2)}{(x^2-1)^3}$

Since, $6(1+3x^2) > 0$ for all x, we have $y'' > 0 \Leftrightarrow x^2 - 1 > 0 \Leftrightarrow |x| > 1$ and $y'' < 0 \Leftrightarrow |x| < 1$. Thus, the curve is concave upward on the intervals

$]-\infty,-1[$ and $]1,\infty[$ and concave downward on $]-1,1[$. Since, 1 and -1 are not in the domain of f, therefore, there is no point of inflection.
Using the information in i) to vii), we sketch the curve in Fig. 12(b).

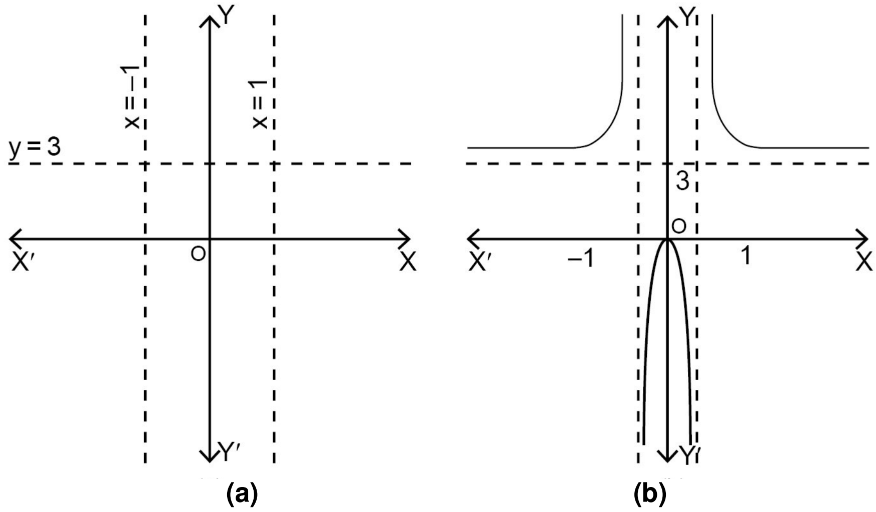

Fig. 12

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In the next example, we will trace a curve with exponential function.

Example 8: Trace the curve $y = xe^x$.

Solution:

i) **Domain and Range:** The domain is $\mathbb{R}$.

ii) **Points of intersection with the axes:** The curve passes through origin.

iii) **Symmetry:** There is no symmetry.

iv) **Asymptotes:** Because both x and e^x become large as $x \to \infty$, we have $\lim\limits_{x\to\infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and so, we have an indeterminate product that requires the use of L'Hôpital's Rule:

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} (-e^x) = 0$$

Thus, the $x-$axis is a horizontal asymptote.

v) **Monotonicity:** We have $y' = xe^x + e^x = (x+1)e^x$. Since, e^x is always positive, therefore, $y' > 0$ when $x+1>0$, and $y' < 0$ when $x+1<0$. So, y is increasing on $]-1,\infty[$ and decreasing on $]-\infty,-1[$.

vi) **Relative Extrema:** Since $\left(\frac{dy}{dx}\right)_{x=-1} = 0$ and $\frac{dy}{dx}$ changes from negative to positive at $x=-1$, therefore, $(-1,-e^{-1})$ a local minimum.

vii) **Concavity:** We have $y'' = (x+1)e^x + e^x = (x+2)e^x$. Since, $y''>0$ if $x > -2$ and $y'' < 0$ if $x < -2$, therefore, the curve is concave upward on $]-2,\infty[$ and concave downward on $]-\infty,-2[$. The inflection point is $]-2,-2e^{-2}[$.

We use this information to trace the curve in Fig. 13.

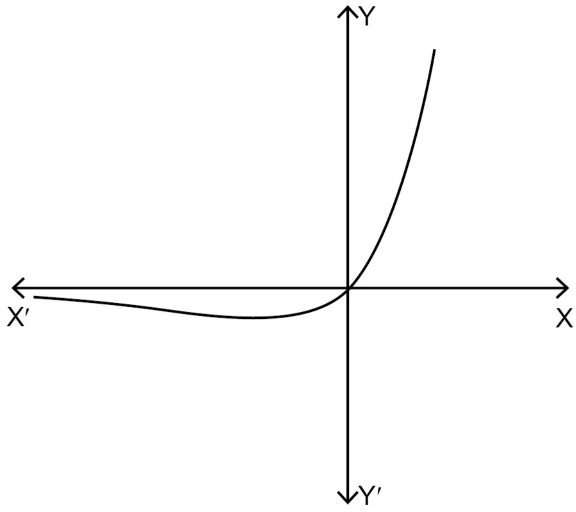

Fig. 13

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In the following example, we will trace a curve with trigonometric functions.

Example 9: Trace the curve $y = \frac{\cos x}{2+\sin x}$.

Solution:

i) **Domain and Range:** The domain is $\mathbb{R}$.

ii) **Points of intersection with the axes:** The curve passes through $\left(0, \frac{1}{2}\right)$ and $\left(\frac{(2n+1)\pi}{2}, 0\right)$ where n is an interger.

iii) **Symmetry and Periodicity:** The curve is not symmetric about any of the axes. We have, $f(x+2\pi) = f(x)$ for all x and so f is periodic and has period 2π. Thus, we need to consider only $0 \le x \le 2\pi$ and then extend the curve by translation.

iv) **Asymptotes:** There is no asymptote.

v) **Monotonicity:** We have $\frac{dy}{dx} = \frac{(2+\sin x)(-\sin x) - \cos x(\cos x)}{(2+\sin x)^2}$.

$$= -\frac{2\sin x + 1}{(2+\sin x)^2}$$

Thus, $\frac{dy}{dx} > 0$ when $2\sin x + 1 < 0 \Leftrightarrow \sin x < -\frac{1}{2} \Leftrightarrow \frac{7\pi}{6} < x < \frac{11\pi}{6}$. So, f is increasing on $\left]\frac{7\pi}{6}, \frac{11\pi}{6}\right[$ and decreasing on $\left]0, \frac{7\pi}{6}\right[$ and $\left]\frac{11\pi}{6}, 2\pi\right[$.

vi) **Relative Extrema:** From the first derivative test, we see that the local minimum value is $\frac{-1}{\sqrt{3}}$ and the local maximum value is $\frac{1}{\sqrt{3}}$.

vii) **Concavity:** If we differentiate $f(x)$ again and simplify, we get

$$\frac{d^2y}{dx^2} = -\frac{2\cos x(1-\sin x)}{(2+\sin x)^3}$$

Since, $(2+\sin x)^3 > 0$ and $1-\sin x \geq 0$ for all x, also we know that $\frac{d^2y}{dx^2} > 0$ when $\cos x < 0$, that is, $\frac{\pi}{2} < x < \frac{3\pi}{2}$, therefore, the curve is concave upward on $\left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$ and concave downward on $\left]0, \frac{\pi}{2}\right[$ and $\left]\frac{3\pi}{2}, 2\pi\right[$. The inflection points are $\left]\frac{\pi}{2}, 0\right[$ and $\left]\frac{3\pi}{2}, 0\right[$.

We draw the graph of the function only to $0 \leq x \leq 2\pi$ is shown in Fig. 14 (a). Then, we extend it, using periodicity, to the complete graph in Fig. 14 (b).

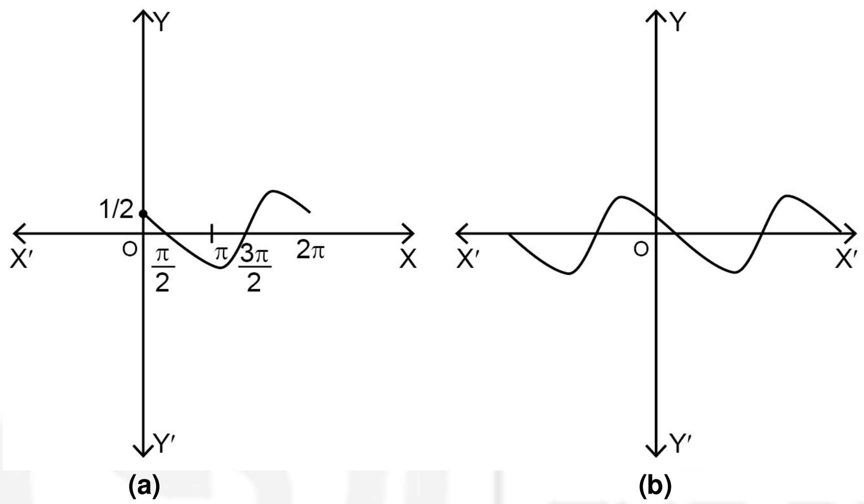

Fig. 14

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So far, all our curves were graphs of functions. We shall now trace some curves which are not the graphs of functions, but have more than one branch. These curves are of the form $f(x, y) = 0$.

Example 10: Trace the **semi cubical parabola** $y^2 = x^3$.

Solution:

i) **Regions where the curve lies:** We note that x^3 is always non-negative for points on the curve. This means that x is always non-negative and no portion of the curve lies on the left of the y-axis.

ii) **Symmetry:** There is symmetry about the x-axis (even powers of y).

iii) **Point of intersection with axes:** The curve meets the axes only at the origin.

iv) **Double point:** Here, $y = \pm x^{3/2}$. The derivative $\frac{dy}{dx} = \pm\frac{3}{2}x^{1/2}$. Here, $y' \to 0$ as $x \to 0^+$ and does not exist as $x \to 0^-$. There are two real and equal tangents at origin, therefore, origin is a cusp.

v) **Increasing and decreasing behaviour:** In the first quadrant y increases with x and $y \to \infty$ as $x \to \infty$.

vi) **Asymptotes:** There are no asymptotes.

We first draw the curve in the first quadrant as shown in Fig. 15 (a), and then take its reflection in the x-axis and we get the complete graph as shown in Fig. 15 (b).

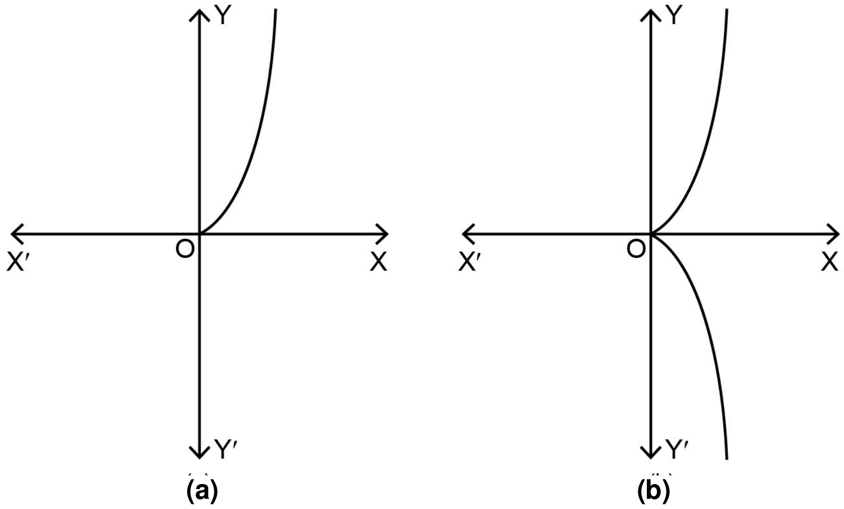

Fig 15

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Example 11: Trace the curve $y^2 = (x-2)\ (x-3)\ (x-4)$.

Solution:

i) **Region where curve lies:** We can see that $(x-2)(x-3)(x-4)$ is non-negative. If $x < 2$, we get a negative value for y^2 which is impossible. So, no portion of the curve lies to the left of the line $x = 2$. For the same reason, no portion of the curve lies between the lines $x = 3$ and $x = 4$. Therefore, the curve lies between the lines $x = 2$ and $x = 3$ and right to the line $x = 4$.

ii) **Symmetry:** Since, y occurs with even powers alone, therefore, the curve is symmetrical about the x-axis. Thus, we draw the curve above $x-$axis, and then get a reflection below the x-axis to complete the graph.

iii) **Point of intersection with axes:** The curve meets the axes at points $(2, 0), (3, 0)$ and $(4, 0)$.

iv) **Tangents and normals:** Here, $\frac{dy}{dx} = \frac{1}{2y}[(x-2)(x-3)+(x-2)(x-4)$ $+(x-3)(x-4)]$. Thus, the curve has vertical tangent at $(2,0), (3,0)$ and $(4,0)$. Combining these facts, the shape of the curve near $A(2,0), B(3,0),\ C(4,0)$ must be as shown in Fig 16 (a).

v) **Interval of increasing or decreasing:** Let us take $y > 0$ (i.e., consider point of the curve above the x-axis). Then,
$\frac{dy}{dx} = \frac{3x^2-18x+26}{2\sqrt{(x-2)\ (x-3)(x-4)}}$. This is zero at $x = 3 \pm 1/\sqrt{3}$. If $\alpha = 3+1/\sqrt{3}$ and $\beta = 3-1/\sqrt{3}$ then α lies between 3 and 4, and can therefore be ignored. Also, $3x^2 - 18x + 26 = 3(x-\beta)\ (x-\alpha)$ and $2 < \beta < 3 < \alpha < 4$. For $x \in\ ]2, 3[,\ x-\alpha$ remains negative. Hence, for

$2 < x < \beta, \frac{dy}{dx} > 0$ since, $(x-\alpha)$ and $(x-\beta)$ are both negative.

Similarly, for $\beta < x < 3, \frac{dy}{dx} < 0$. Hence, the graph rises in $]2, \beta[$ and falls in $]\beta, 3[$. Thus, the shape of the curve is oval above the x-axis, and by symmetry about the x-axis, we can complete the graph between $x = 2$ and $x = 3$ as in Fig 16. (b).

vi) **Concavity:** Now let us consider the portion of the graph to the right of $x = 4$. Shifting the origin to $(4, 0)$, the equation of the curve becomes $y^2 = x(x+1)(x+2) = x^2 + 3x^2 + 2x$.
As x increases, so does y. As $x \to \infty$, so does y (considering points above the x-axis). When x is very small, x^3 and $3x^2$ are negligible as compared to $2x$, so that near the (new) origin, the curve is approximately of the shape of $y^2 = 2x$. The large values of $x, 3x^2$ and $2x$ are negligible as compared to x^3, so that the curve shapes like $y^2 = x^3$ for large x. Thus, at some point the curve changes its convexity.
This conclusion could also be drawn by showing the existence of a point of inflection.

vii) **Asymptotes:** There are no asymptotes.

viii) **Multiple point:** There is no multiple point.

Considering the reflection along the x-axis, we have the complete graph as shown in Fig 16(c).

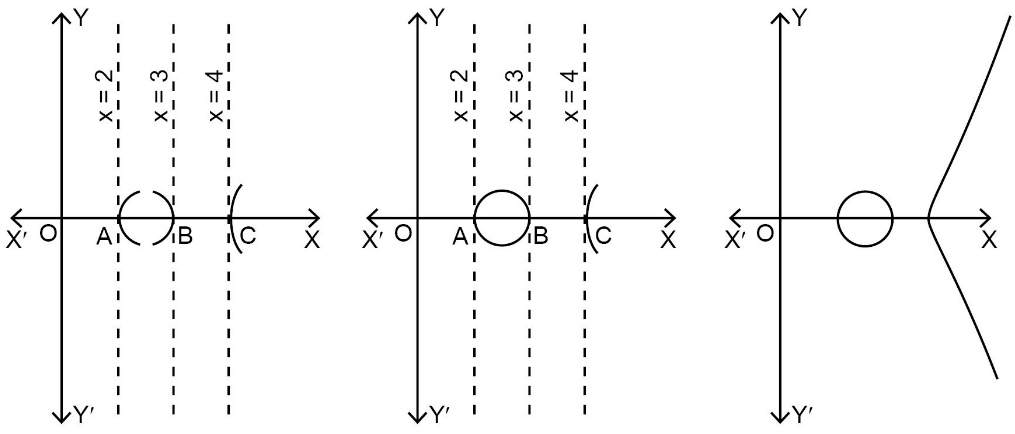

Fig. 16

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Example 12: Trace the curve $(x^2 - 1)(y^2 - 4) = 4$.

Solution:

i) **Region, where the curve lies:** Here, $y^2 = \frac{4}{x^2 - 1} + 4$, therefore,

$x \notin]-1,1[$. Similarly, $x^2 = \frac{4}{y^2 - 4} + 1$, therefore, $y \notin]-2,2[$.

ii) **Symmetry:** There is symmetry about both axes. We can therefore, sketch the graph in the first quadrant only and then take its reflection in the y-axis to get the graph above the x-axis. The reflection of this graph in the x-axis will give the complete graph.

iii) **Point of intersection with axes:** Notice that the origin is a point on the curve. The curve does not meet the axes at any other points.

iv) **Tangent at origin:** The curve has tangents at origin and these are given by $4x^2 + y^2 = 0$. These being imaginary, the origin is an isolated point on the graph.

v) **Asymptotes:** Equating to zero the coefficients of the highest powers of x and y, we get $y = \pm 2$ and $x = \pm 1$ as asymptotes of the curve. Thus, the portion of the curve in the first quadrant approaches the lines $x = 1$ and $y = 2$ in the region far away from the origin. As $x \to \infty,\ y \to 2$ and as $y \to \infty,\ x \to 1$.

vi) **Increasing and decreasing:** In the first quadrant, as x increases, so does $x^2 - 1$, and since $x^2 - 1 = \frac{4}{(y^2 - 4)}$, therefore, y decreases as x increases.

vii) **Relative extrema:** There are no extreme points.

There are no singular points or points of inflection. Hence, the graph is as shown in Fig 17.

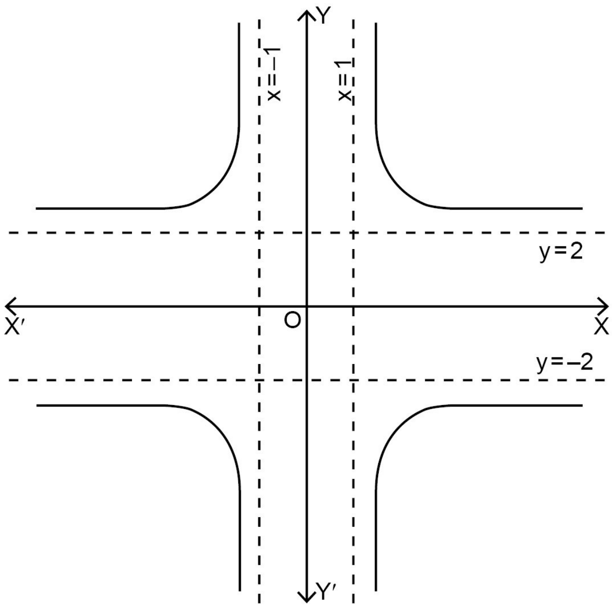

Fig 17

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Example 13: Trace the curve $y^2 = (x-1)\,(x-2)^2$.

Solution:

i) **Symmetry:** Since, the power of y is even, therefore, there is symmetry about the x-axis.

ii) **Region:** No portion of the curve lies to the left of $x = 1$, as y^2 cannot be negative.

iii) **Points of intersection with axes:** Points of intersection with the axes are $(1, 0)$ and $(2, 0)$.

iv) Tangent: The tangent at $(1, 0)$ is vertical. Shifting the origin to $(2, 0)$, the curve transforms into $y^2 = x^2(x+1)$. The tangents at the new origin are given by $y^2 = x^2$. This means that the point $(2,0)$ is a node, and the tangents at $(2,0)$ are equally inclined to the axes.

Let us try to build up the graph above the x-axis between $x = 1$ and $x = 2$. Differentiating the equation of the curve with respect to x, we get

$$2yy' = (x-2)^2 + 2(x-1)\,(x-2) = (x-2)(3x-4)$$

or, $$y' = \frac{(x-2)\,(3x-4)}{2y}$$

when $1 < x < 2,\ (x-2) < 0$. If y is positive, then $y' > 0$ provided $3x - 4 < 0$. Thus, $y' > 0$ when $x \in \left]1, \frac{4}{3}\right[$ and $y' < 0$ when $x \in \left]\frac{4}{3}, 2\right[$. The tangent is parallel to the x-axis when $3x - 4 = 0$, that is, when $x = 4/3$ (see Fig 18 (a)). Hence, for $1 < x < 2$, the curve shapes as in Fig 18 (b).

v) Intervals for increasing and decreasing: As $x \to \infty,\ y \to \infty$, in the first quadrant. Note that when $(2, 0)$ is taken as the origin, the equation of the curve reduces to

$$y^2 = x^2(x+1) = x^3 + x^2.$$

This shows that when $x > 0$ and $y > 0$, the curve lies above the line $y = x$ (on which $y^2 = x^2$). Hence, the final sketch (Fig 18 (c)) shows the complete graph.

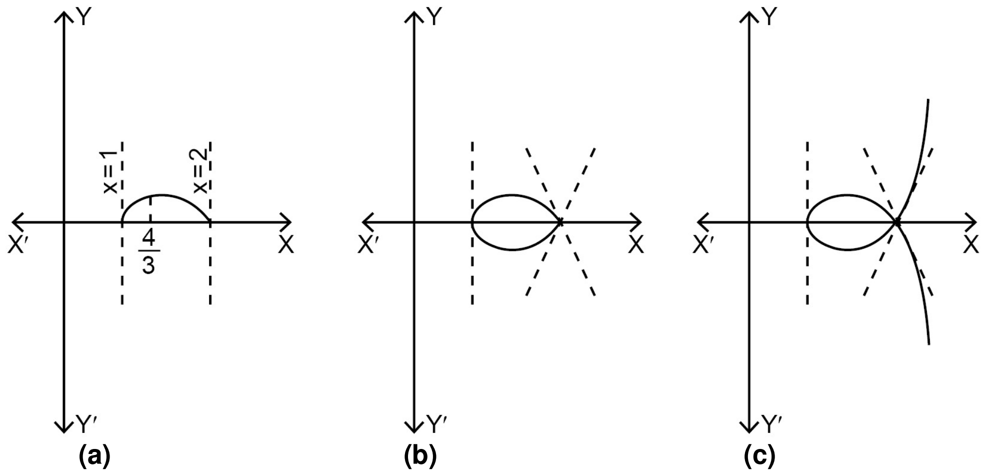

Fig. 18

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If you have gone through Examples 1-13 carefully, you should be able to do the following exercise.

E1) Trace the following curves by stating all the properties you use to trace:

i)	$y = x^2$	ii)	$y^2 = (x-2)^3$
iii)	$y(1+x^2) = x$	iv)	$y^2 = x^2(1-x^2)$
v)	$y = x\,e^{-1/x}$	vi)	$y = \sin^3 x$
vii)	$y = x(\ln x)$	viii)	$y = x - 5x^{1/3}$

ix) $y = \ln(\sin x)$ x) $y = x\sqrt{5-x}$

xi) $y = \dfrac{x^2}{x-1}$ xii) $y = \dfrac{x^3+4}{x^2}$

E2) Find the oblique asymptotes of the curve $y = x - \tan^{-1} x$ and hence, trace the curve using this fact.

E3) In the theory of relativity, the mass of a particle is $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$,

where m_0 is the mass of the particle, m is the mass when the particle moves, with speed v relative to the observer, and c is the speed of light. Trace the curve for m as a function of v.

In the following section, we will trace the curves which are in parametric form.

16.3 TRACING A CURVE: PARAMETRIC EQUATION

Sometimes a functional relationship may be defined with the help of a parameter. In such cases, we are given a pair of equations which relate x and y with the parameter. For example, imagine a particle that moves along a curve and the x and y coordinates are defined in terms of time t as shown in Fig. 19.

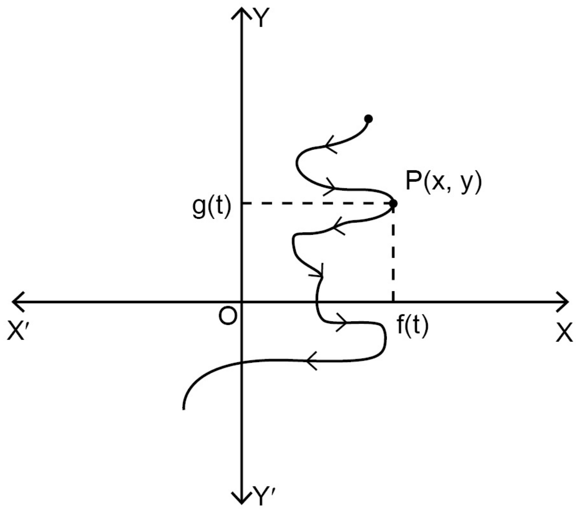


Fig. 19

In this case, we write $x = f(t)$ and $y = g(t)$, where t is the third variable called **parameter**. The equations $x = f(t)$ and $y = g(t)$ are known as parametric equations. The parameter t does not necessarily represent time. You may recall what you learnt in Appendix I of Block 3.

Now, we shall see how we can trace a curve whose equation is in the parametric form.

We shall illustrate the process with an example.

Example 14: Trace the cycloid $x = a(t + \sin t),\ y = a(1 - \cos t)$ as t varies from $-\pi$ to π.

Solution: Here $\frac{dx}{dt} = a(1+\cos t)$, $\frac{dy}{dt} = a \sin t$, so that $\frac{dy}{dx} = \tan(t/2)$. Since, $\frac{dy}{dx} > 0$ for all $t \in]-\pi, \pi[, x$ increases with t from $-a\pi$ (at $t = -\pi$) to 0 (at $t = 0$) to $a\pi$ (at $t = \pi$).

Also, $\frac{dy}{dx}$ is negative when $t \in]-\pi, 0[$ and positive when $t \in]0, \pi[$. Hence, y decreases from $2a$ to 0 in $[-\pi, 0]$ and increases from 0 to $2a$ in $[0, \pi]$. Let us tabulate this data in Table 2.

Table 2

$t \in [-\pi, 0]$	$t \in [0, \pi]$
i) x increases from $-a$ to 0 ii) y decreases from $2a$ to 0 iii) Hence, the curve falls	i) x increases from 0 to a ii) y increases from 0 to $2a$ iii) Hence, the curve rises

Also, at the terminal points $-\pi, 0$ and π of the intervals $[-\pi, 0]$ and $[0, \pi]$, we summarize this in Table 3.

Table 3

t	(x, y)	$\frac{dy}{dx}$	$\frac{dx}{dy}$	Tangent
$-\pi$	$(-a\pi, 2a)$	not defined	0	vertical
0	$(0, 0)$	0	not defined	horizontal
π	$(a\pi, 2a)$	not defined	0	vertical

On the basis of the data tabulated in Table 3, the graph is drawn in Fig 20.

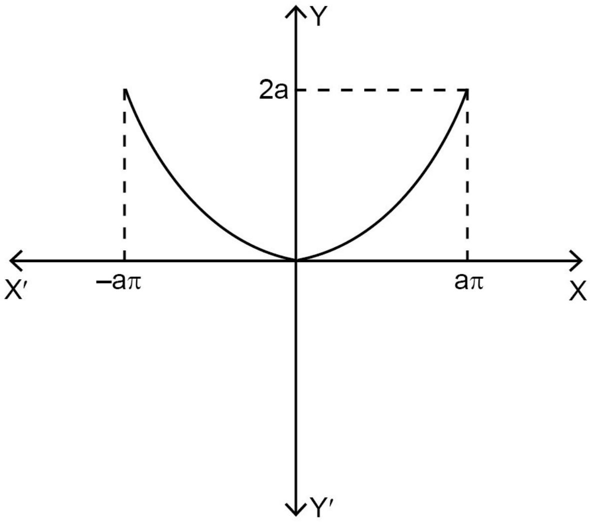


Fig. 20

If t is increased by 2π, x is increased by $2\pi a$ and y does not change. Thus, the complete graph can be obtained in intervals $\ldots[-5\pi, -3\pi], [-3\pi, -\pi], [\pi, 3\pi], [3\pi, 5\pi]\ldots$ by translation through a proper distance.

***

The cycloid is known as the Helen of geometry because it was the cause of many disputes among mathematicians. It has many interesting properties.

We shall describe just one of them here. Consider this question: What shape should be given to a trough connecting two points A and B, so that a ball rolls from A to B in the shortest possible time?

Now, we know that the shortest distance between A and B would be along the line AB (Fig. 21). But since we are interested in the shortest time rather than distance, we must also consider the fact that the ball will roll quicker, if the trough is steeper at A. The Swiss mathematician Jakob and Johann Bernoulli proved by exact calculations that the trough should be made in the form of an arc of a cycloid. Because of this, a cycloid is also called the curve of the quickest descent.

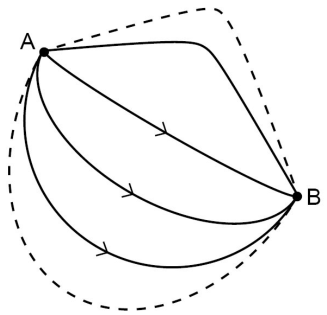


Fig. 21

The cycloid is used in clocks and in teeth for gear wheels. It can be obtained as the locus of a fixed point on a circle as the circle rolls along a straight line.

Now, let us trace another curve in parametric form.

Example 15: The position of a particle at time t is given by the parametric equations $x = t^2 - 2t$ and $y = t + 1$. Sketch and identify the path along which the particle moves.

Solution: We have $\frac{dx}{dt} = 2t - 2$ and $\frac{dy}{dt} = 1$, so that $\frac{dy}{dx} = \frac{1}{2t-2}$. Here, $\frac{dy}{dx} > 0$, when $t > 1$ and $\frac{dy}{dx} < 0$, when $t < 1$. That means y is increasing when $t > 1$ and y is decreasing when $t < 1$. At $t = 1$, the tangent of the curve is vertical. In Fig. 22 we plot the curve.

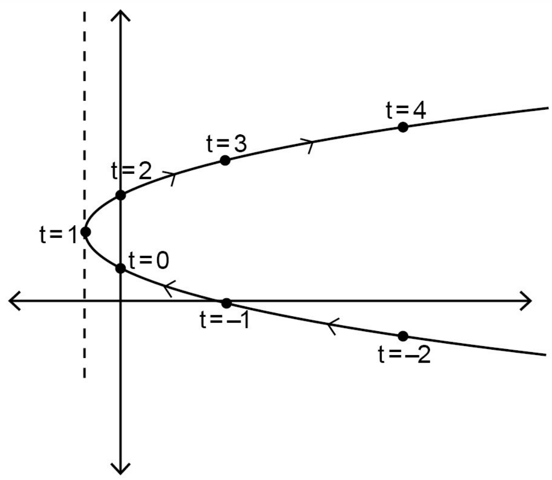


Fig. 22

We can also mark the points given in the Table 4.

Table 4

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

A particle whose position is given by the parametric equations moves along the curve in the direction of the arrows as t increases. Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as t increases.

It appears from Fig. 22 that the curve traced out by the particle may be a parabola. We can confirm this by eliminating the parameter t as follows: We obtain $t = y - 1$ from the second equation and substitute into the first equation. This gives $x = t^2 - 2t = (y-1) = y^2 - 4y + 3$ and so the curve represented by the given parametric equations is the parabola $x = y^2 - 4y + 3$.

***

Example 16: What curve is represented by the following parametric equations?

$x = \cos t$, $y = \sin t$, where $0 \le t \le 2\pi$

Solution: If we plot points, it appears that the curve is a circle. We can confirm this impression by eliminating t. Observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

Thus, the point (x, y) moves on the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians) shown in Fig. 23. As t increases from 0 to 2π, the point $P(\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point $(1, 0)$.

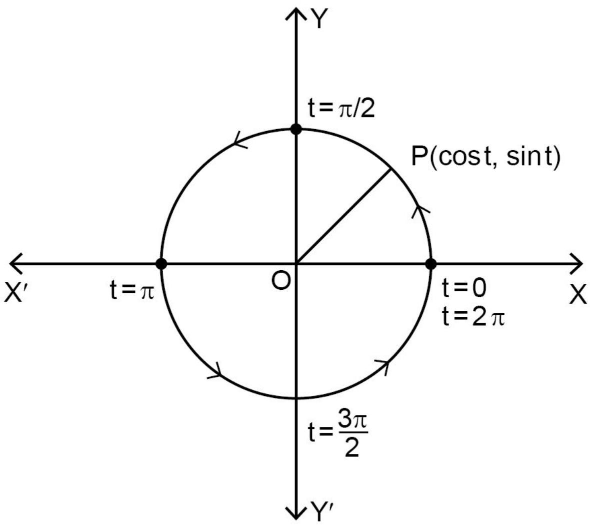


Fig. 23

***

See if you can do this exercise now.

E4) Trace the following curves:

i) $x = a(t + \sin t),\ y = a(1 + \cos t),\ -\pi \le t \le \pi, a > 0$.

ii) $x = a \sin 2t(1 + \cos 2t),\ y = a \cos 2t(1 - \cos 2t),\ 0 \le t \le \pi, a > 0$.

iii) $x = at^2,\ y = 2at,\ 0 \le t \le 1$.

iv) $x = \sin 2t, y = \cos 2t, 0 \le t \le 2\pi$.

v) $x = \sin t,\ y = \sin^2 t$.

So far, we have discussed tracing of curves in cartesian and parametric forms. In the following section, we will discuss tracing of a curve in polar form.

16.4 TRACING A CURVE: POLAR EQUATION

In this section, we shall consider the problem of tracing those curves, whose equations are given in the polar form. You may recall Unit 3 for polar coordinates. In such a coordinate system, we can associate each point P in the plane with a pair of polar coordinates (r, θ), where r is the number of units between P and the pole and θ is an angle from the polar axis to the ray OP as shown in Fig. 24. If r is negative, then the point is located on the opposite side of the origin. Thus, r is a position on a rotated axis.

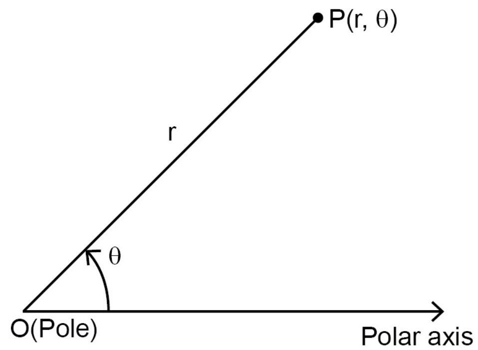


Fig. 24

The following considerations can be useful in this connection.

i) Symmetry: If the equation remains unchanged when θ is replaced by $-\theta$, then the curve is symmetric with respect to the initial line.
If the equation does not change when r is replaced by $-r$, then the curve is symmetric about the pole (or the origin).
Finally, if the equation does not change when θ is replaced by $\pi - \theta$, then the curve is symmetric with respect to the line $\theta = \pi/2$.

ii) Region: Find the limits within which r must lie for the permissible values of θ. If $r < a (r > a)$ for some $a > 0$, then the curve lies entirely within (outside) the circle $r = a$. If r^2 is negative for some values of θ, then the curve has no portion in the corresponding region.

iii) Angle between the line joining a point of the curve to the origin and the tangent: At suitable points, this angle can be determined easily. It helps in knowing the shape of the curve at these points. You may recall from Unit 14 that the angle ϕ is given by the relation $\tan\phi = r\frac{d\theta}{dr}$.

We shall illustrate the procedure through some examples of graphing equations of the form $r = f(\theta)$ in polar coordinates, where θ is assumed to be measured in radians. Study them carefully, so that you can trace some curves on your own later.

Example 17: Trace the cardioid $r = a(1+\cos\theta)$.

Solution: We can make the following observations.

i) **Symmetry:** Since, $\cos\theta = \cos(-\theta)$, therefore, the curve is symmetric with respect to the initial line. That means we need to trace the curve only above the initial line, rest half curve would be the reflection along the initial line.

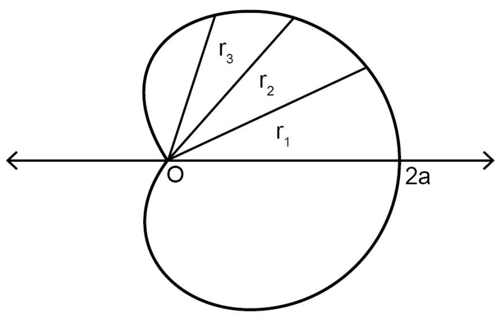
ii) **Region:** Since, $-1 \le \cos\theta \le 1$, therefore, the curve lies inside the circle $r = 2a$.

iii) **Tangents:** $\frac{dr}{d\theta} = -a\sin\theta$. Hence, $\frac{dr}{d\theta} < 0$, when $0 < \theta < \pi$. Thus, r decreases as θ increases in the interval $]0, \pi/2[$. Similarly, r increases with θ in $]\pi/2, \theta[$. Some corresponding values of r and θ are given in Table 5.

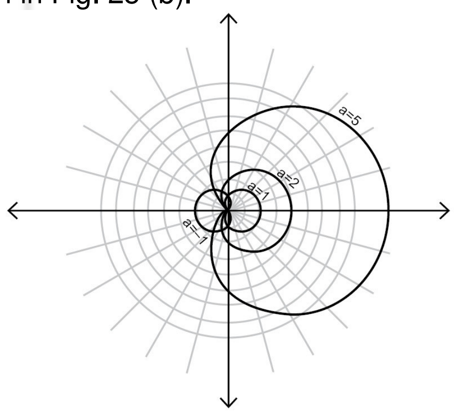
Table 5

θ	0	$\pi/2$	π
r	$2a$	a	0

Combining the above facts, we can easily draw the graph above the initial line. By reflecting this portion in the initial line we can completely draw the curve as shown in Fig 25 (a). Notice the decreasing radii $2a, r_1, r_2, r_3$ etc. If we allow a to vary and keep is positive, then the size of cardioid varies. If a is negative, then the cardioid changes its direction. These cardioids are shown in Fig. 25 (b).



(a)



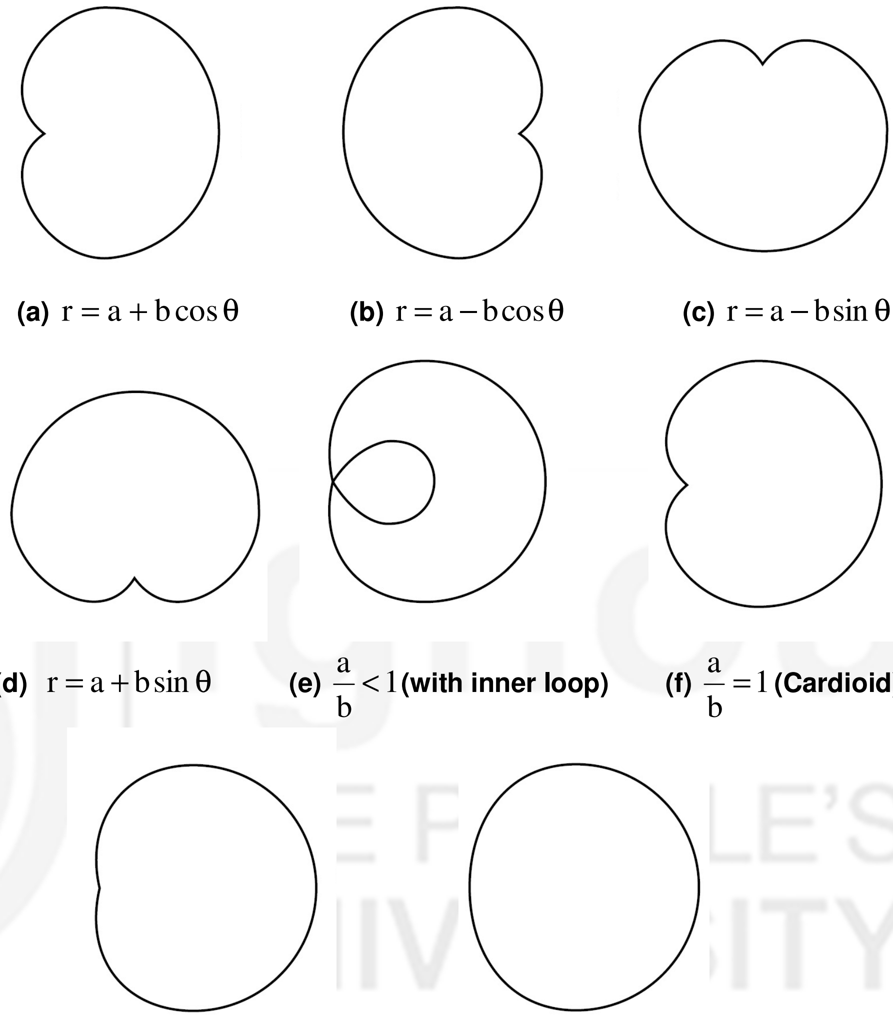
(b)

Fig. 25: (a) Curve $r = a(1+\cos\theta)$; (b) Curve $r = a(1+\cos\theta)$ for $a = 1, 2, 5, -1$.

This curve is called a cardioid since it resembles a heart.

***

You may note that the equations with any of the four forms $r = a \pm b\sin\theta$ and $r = a \pm b\cos\theta$ in which $a > 0$ and $b > 0$ represent polar curves called **limacons** (from the Latin word "limax" for a snail-like creature that is commonly called a slug) as shown in Fig. 26 (a) to Fig. 26 (d). There are four possible shapes for a limacon in each of the four cases that are determined by the ratio a/b (Fig 26(e) to Fig. 26 (h)). If $a = b$ (the case $a/b = 1$), then the limacon is called a cardioid because of its heart-shaped appearance, as noted in Example 17.



(a) $r = a + b\cos\theta$ **(b)** $r = a - b\cos\theta$ **(c)** $r = a - b\sin\theta$

(d) $r = a + b\sin\theta$ **(e)** $\frac{a}{b} < 1$ **(with inner loop)** **(f)** $\frac{a}{b} = 1$ **(Cardioid)**

(g) $1 < \frac{a}{b} < 2$ **(Dimpled)** **(h)** $\frac{a}{b} > 2$ **(Convex)**

Fig 26

Example 18: Trace the equiangular spiral $r = ae^{\theta\cot\alpha}$.

Solution: We proceed as follows.

i) **Region:** When $\theta = 0,\ r = a$.

ii) **Symmetry:** There is no symmetry.

iii) **Tangents:** $\frac{dr}{d\theta} = r\cot\alpha$, which is positive, assuming $\cot\alpha > 0$. Hence as θ increases so does r. $r\frac{d\theta}{dr} = \tan\alpha$. Thus, at every point, the angle between the line joining a point on the curve to the origin and the tangent is the same, namely α. Hence the name.

Combining these facts, we get the shape of the curve as shown in Fig 27.

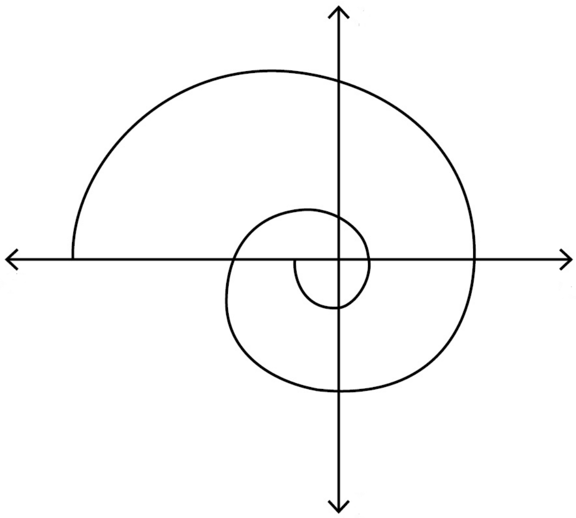


Fig. 27: Curve $r = a e^{\theta \cot \alpha}$**.**

The equiangular (or logarithmic) spiral $r = ae^{\theta \cot \alpha}$ is known as the **curve of pursuit**. Suppose four dogs start from the four corners of a square, each pursues the dog in front with the same uniform velocity (always following the dog in front), then each will describe an equiangular spiral. Several shells and fossils have forms which are quite close to equiangular spirals (Fig 28). Seeds in the sunflower or blades of pine cones are also arranged in this form.

Fig. 28: Spiral.

The first discussion of this spiral occur in letters written by Descartes to Mersenne in 1638. The name logarithmic spiral is due to Jacques Bernoulli. He was so fascinated by it that he willed that an equiangular spiral be carved on his tomb with the words 'Though changed, I rise unchanged' inscribed below it.

The spiral $r = a\theta$ is known as the **Archimedean spiral**. Its study was, however, initiated by Conon. Archimedes used this spiral to square the circle, that is, to find a square of area equal to that of a given circle. This spiral is widely used as a cam to produce uniform linear motion. It is also used as casings of centrifugal pumps to allow air which increases uniformly in volume with each degree of rotation of the fan blades to be conducted to the outlet without creating back-pressure.

The spiral $r\theta = a$, due to Varignon, is known as the **reciprocal or hyperbolic** (recall that $xy = a$ is a hyperbola) **spiral**. It is the path of a particle under a central force which varies as the cube of the distance.

***

Now let's consider the next example.

Example 19: Trace the curve $r = a\sin 3\theta,\ a > 0$.

Solution:

i) **Symmetry:** You may note that there is symmetry about the line $\theta = \pi/2$ as the equation is unchanged if θ is replaced by $\pi - \theta$.

ii) **Region:** The curve lies inside the circle $r = a$, because $\sin 3\theta \le 1$. The origin lies on the curve and this is the only point where the initial line meets the curve.

iii) **Tangents:** $r = 0 \Rightarrow \theta = n\pi/3$, where n is any integer. Hence the origin is a multiple point, the lines $\theta = 0,\ \frac{\pi}{3},\ \frac{2\pi}{3},\ \pi,\ \frac{4\pi}{3},\ \frac{5\pi}{3},\ 2\pi$ etc. being tangents at the pole.

iv) **Monotonicity:** $\frac{dr}{d\theta} = 3\cos 3\theta$. Hence r increases in the intervals $\left]0, \frac{\pi}{6}\right[, \left]\frac{\pi}{2}, \frac{\pi}{6}\right[$, and $\left]\frac{7\pi}{6}, \frac{3\pi}{2}\right[$, and decreases in the intervals $\left]\frac{\pi}{6}, \frac{\pi}{2}\right[, \left]\frac{5\pi}{6}, \frac{7\pi}{6}\right[$ and $\left]\frac{3\pi}{2}, \frac{5\pi}{3}\right[$. Notice that r is negative when $\theta \in \left]\frac{\pi}{3}, \frac{2\pi}{3}\right[$ or $\theta \in \left]\pi, \frac{4\pi}{3}\right[$ or $\theta \in \left]\frac{5\pi}{3}, 2\pi\right[$. Hence, the curve consists of three loops as shown in Fig 29. The function is periodic and the curve retraces itself as θ increases from 2π on.

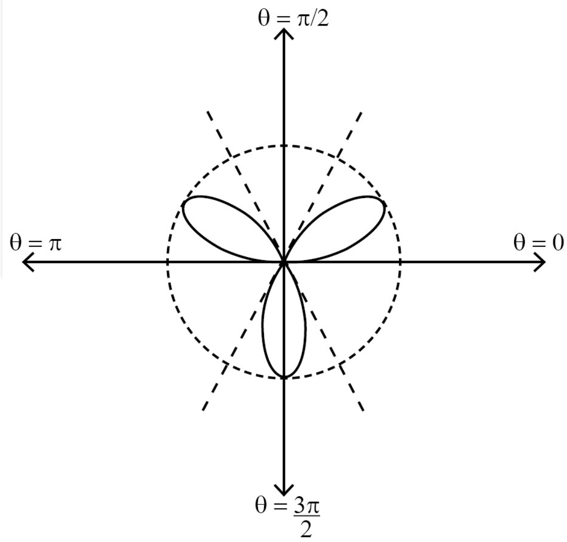
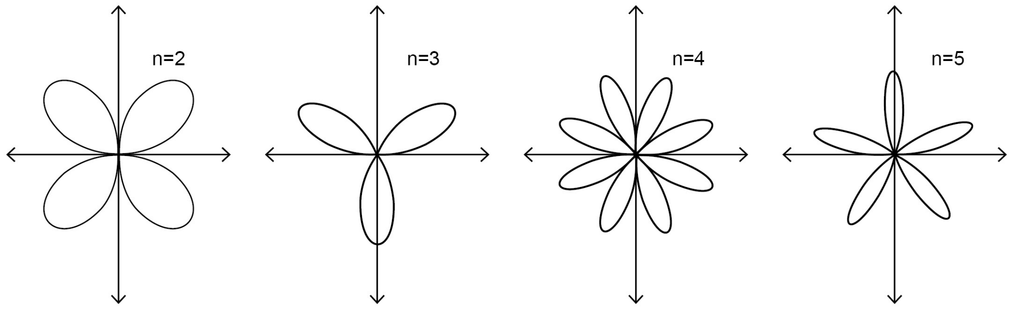


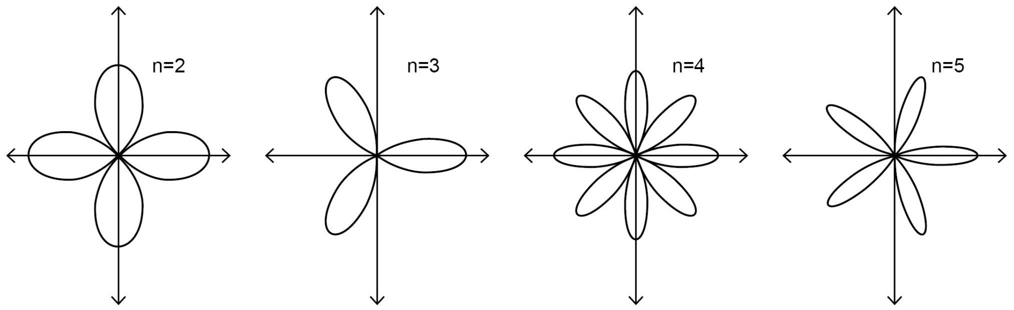
Fig. 29: Curve $r = a\sin 3\theta$.

***

In polar coordinates, equations of the form $r = a\sin n\theta$ and $r = a\cos n\theta$, in which $a > 0$ and n is a positive interger represent families of flower-shaped curves called roses (Fig. 30). The rose consists of n equally spaced petals of radius a if n is odd and $2n$ equally spaced petals of radius a if n is even. It can be shown that a rose with an even number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < 2\pi$ and a rose with an odd number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < \pi$. A three-petal rose of radius a was graphed in Example 19.



(a) $r = a \sin n\theta$.



(b) $r = a \cos n\theta$.

Fig. 30: Rose Curves

Now try to trace a few curves on your own.

E5) Trace the following curves in polar coordinates.

i) $r = 1$

ii) $\theta = \frac{\pi}{4}$

iii) $r = \theta \quad (\theta \geq 0)$

E6) Trace the following curves by stating all the properties you used:

i) $r = a(1 - \cos\theta),\ a > 0$.

ii) $r = 2 + 4\cos\theta$.

iii) $r = a\cos 3\theta,\ a > 0$.

iv) $r = a\sin 2\theta,\ a > 0$

Now, let us summarize what we have studied in this unit.

16.5 SUMMARY

In this unit, we have covered the following points.

1. Tracing a curve $y = f(x)$ or $f(x, y) = 0$ means plotting the points which satisfy this relation.

2. Criteria for symmetry and monotonicity, equations of tangents, asymptotes and points of inflection are used in curve tracing.

3. Curve tracing is illustrated by some examples when the equation of the curve is given in

   i) Cartesian form

   ii) Parametric form

   iii) Polar form.

16.7 SOLUTIONS/ANSWERS

Dotted lines represents tangents or asymptotes throughout.

E1) i) **Domain:** $\mathbb{R}$ and the curve lies in first and second quadrant.
Point of intersection with axes: $(0,0)$
Symmetry: About $y-$axis
Asymptotes: None
Monotonicity: Increasing on $]0,\infty[$ and decreasing on $]-\infty,0[$
Relative extrema: Minimum at 0 and $f(0)=0$.
The corresponding sketch of the curve is given in Fig. 31.

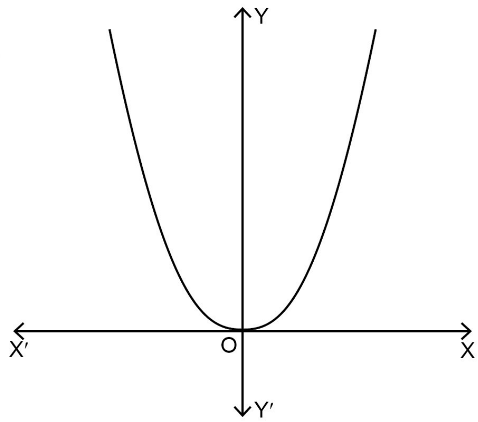


Fig. 31

ii) **Region of existence:** $[2,\infty[$ and the curve lies in first and fourth quadrant.
Point of intersection with axes: $(2,0)$
Symmetry: About $x-$axis
Asympotes: None
Double Point: If we shift origin at $(2,0)$, then, $(2,0)$ is double point and is cusp.
The corresponding curve is traced in Fig. 32.

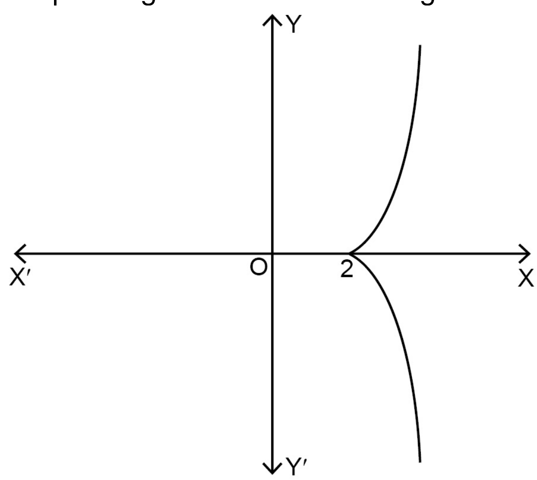


Fig. 32

iii) **Domain:** $\mathbb{R}$ and the curve is in first and third quadrant as either x, y are both positive or both negative.

Symmetry: About origin.

Asymptote: x-axis is an asymptote.

Monotonicity: Function rises in $]-1, 1[$ and falls elsewhere.

Tangents: $y = x$ is the tangent at the origin

Concavity: $(0, 0), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right), \left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$ are points of inflection.

The graph is shown in Fig. 33.

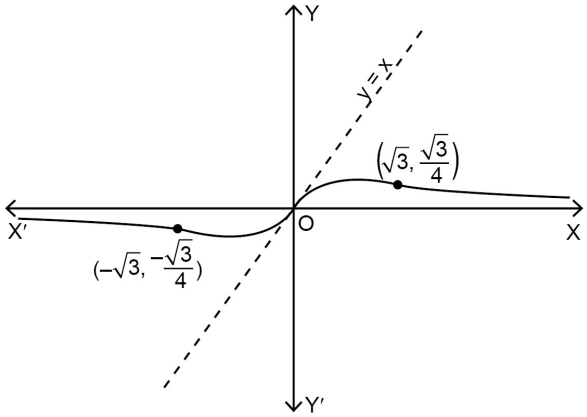


Fig. 33

iv) **Region of existence:** The curve $\frac{y^2}{x^2} = 1 - x^2$ shows that the entire curve lies within the lines $x = \pm 1$.

Point of intersection with axes: $(0,0), (1,0)$ and $(-1,0)$

Symmetry: About $x-$axis, $y-$axis and origin.

Tangents: Tangents at the origin are $y = \pm x$. Tangents at $x = \pm 1$ are vertical.

Relative extrema: Maxima at $\left(\pm 1/\sqrt{2}, 1/4\right)$, and minima at $\left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{4}\right)$

Multiple point: $y = \pm x\sqrt{1-x^2}$, y is defined if $1 - x^2 \geq 0$, or $-1 \leq x \leq 1$. If we equate lowest degree term to 0, we get $y^2 = x^2$, which gives $y = \pm x$. Therefore, the curve has two tangents at origin, namely, $y = x$ and $y = -x$, and the origin is a node. The curve is sketched in Fig. 34.

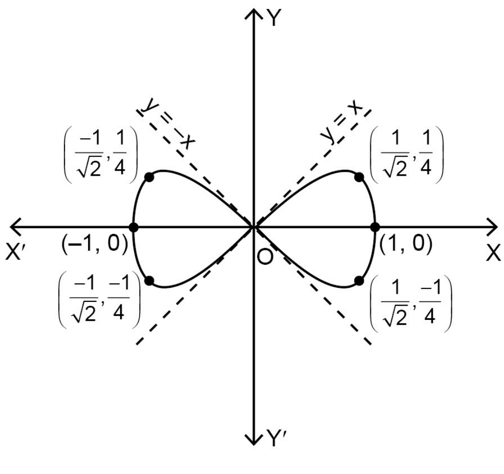


Fig. 34

v) **Domain and Range:** $]-\infty,0[\,\cup\,]0,\infty[$

Symmetry: None

Point of intersection with axes: None

Concavity: Concave up on $]0,\infty[$ and concave down on $]-\infty,0[$

Relative extrema: Maxima on -1 and the relative maximum value is $f(-1)=-e$.

Monotonicity: Increasing on $]-\infty,-1[$ and on $]0,\infty[$.
decreasing on $]-1,0[$.

Point of continuity: Origin is the point of discontinuity

The corresponding curve is given in Fig. 35.

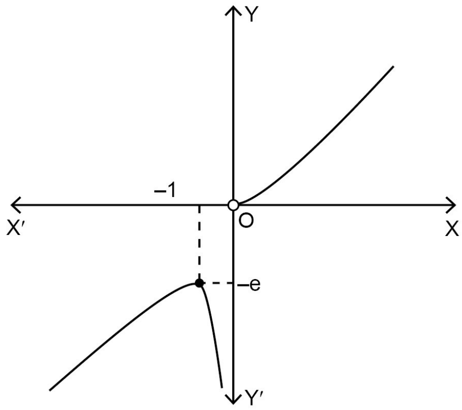


Fig. 35

vi) **Domain:** $\mathbb{R}$

Symmetry: About the origin

Periodicity: Period 2π

Point of intersection: Origin $(0,0),(n\pi,0)$ (n is integer)

Monotonicity: Increasing on $]0,\pi/2[$ and $\left]\frac{3\pi}{2},2\pi\right[$,decreasing on $\left]\frac{\pi}{2},\frac{3\pi}{2}\right[$

Relative extrema: Maximum at $\pi/2, f(\pi/2)=1$ and minima at $\frac{3\pi}{2}, f\left(\frac{3\pi}{2}\right)=-1$.

Concavity: Concave upward on $]0,a[,]\pi-a,\pi[$ and concave downward on $]a,\pi-a[$ where $a=\sin^{-1}\sqrt{2/3}$.

Point of inflection: $x=0,\pi,\pi-a$.

Curve is traced in Fig. 36.

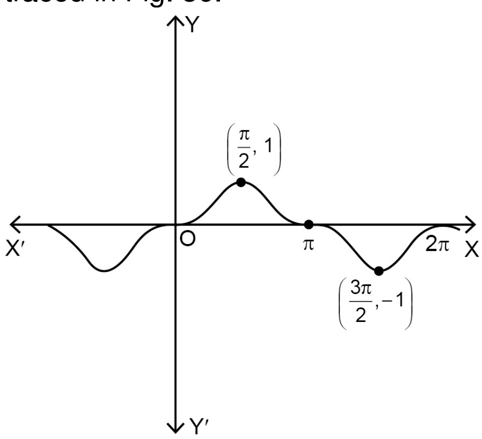


Fig. 36

vii) **Domain:** $]0, \infty[$

Point of intersection with axes: $(1, 0)$

Symmetry: None

Asymptote: None

Monotonicity: Increasing on $\left]\frac{1}{e}, \infty\right[$ and decreasing on $\left]0, \frac{1}{e}\right[$.

Relative extrema: Minima at $x = \frac{1}{e}, f\left(\frac{1}{e}\right) = -\frac{1}{e}$

Concavity: Concave upward on $]0, \infty[$

The curve is drawn in Fig. 37.

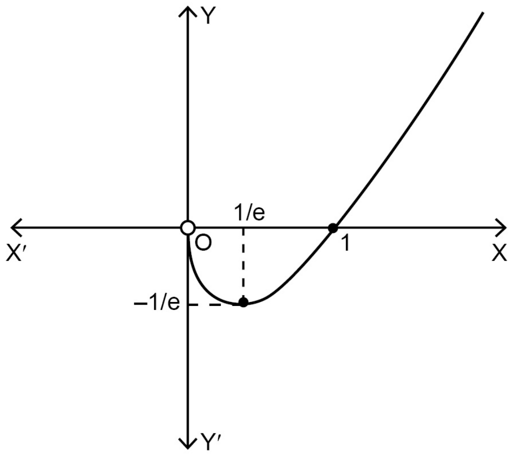


Fig. 37

viii) **Domain:** $\mathbb{R}$

Point of intersection with axes: $(0,0), (\pm 3\sqrt{3}, 0)$

Symmetry: About the origin

Asymptote: None

Monotonicity: Increasing on $]-\infty, -1[,]1, \infty[$ and decreasing on $]-1, 1[$

Relative extrema: Maximum at -1 and $f(-1) = 2$, Minimum at 1 and $f(1) = -2$

Concavity: Concave upward on $]0, \infty[$ and concave downward on $]-\infty, 0[$

Point of inflection: $(0, 0)$ is point of inflection.

The corresponding curve is traced in Fig. 38.

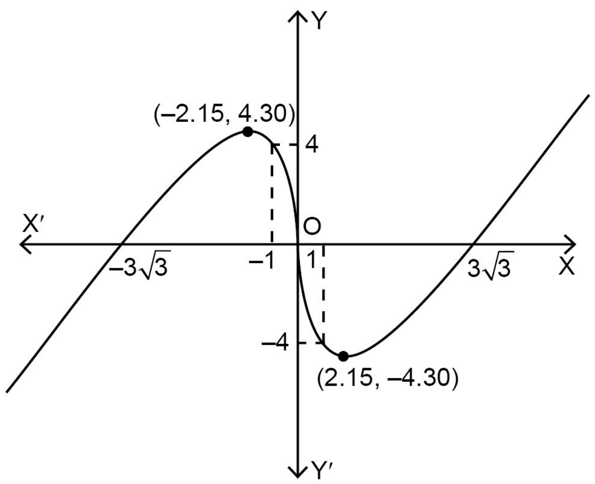


Fig. 38

ix) **Domain:** $x \in]2n\pi, (2n+1)\pi[$,where n is an interger. The value of y is always negative. Therefore, the curve lies in third and fourth quadrant.

Point of intersection with axes: $\left(\frac{\pi}{2} + 2n\pi, 0\right)$.

Symmetry: None

Periodicity: Period 2π

Asymptotes: Vertical asymptotes at $x = n\pi$

Monotonicity: Increasing on $\left]2n\pi, \frac{\pi}{2} + 2n\pi\right[$ and decreasing on $\left]\frac{\pi}{2} + 2n\pi, (2n+1)\pi\right[$.

Relative extrema: Maximum at $\frac{\pi}{2} + 2n\pi$ and $f\left(\frac{\pi}{2} + 2n\pi\right) = 0$.

The corresponding curve is traced in Fig. 39.

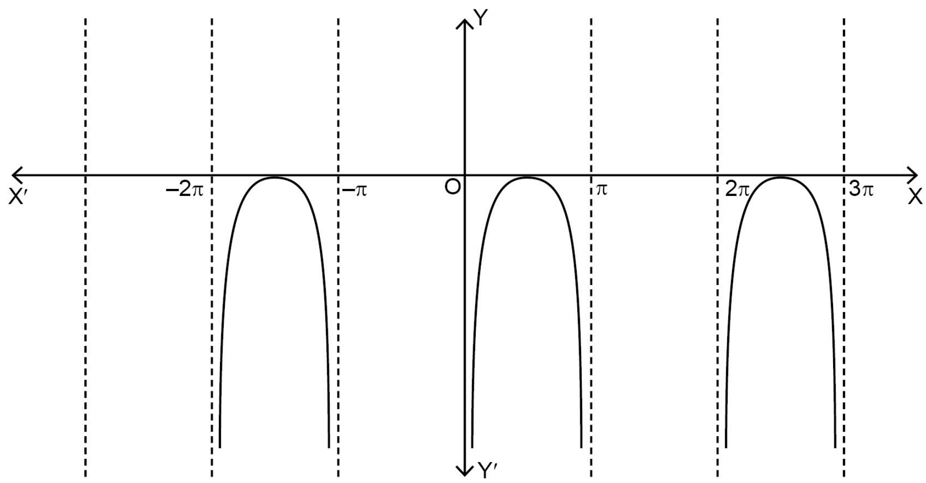


Fig. 39

x) **Domain:** $]-\infty, 5]$

Point of intersection with axes: $(0,0), (5,0)$

Symmetry: None

Asymptote: None

Monotonicity: Increasing on $\left]-\infty, \frac{10}{3}\right[$ and decreasing on $\left]\frac{10}{3}, 5\right[$.

Relative extrema: Maximum at $\frac{10}{3}$ and $f\left(\frac{10}{3}\right) = \frac{10}{9}\sqrt{5}$.

Concavity: Concave downward on $]-\infty, 5[$

The corresponding curve is traced in Fig. 40.

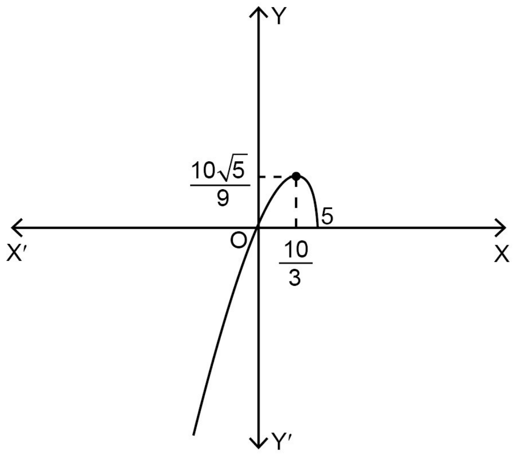


Fig. 40

xi) **Domain:** $]-\infty,1[\ \cup\ ]1,\infty[$

Point of intersection with axes: $(0,0)$

Symmetry: None

Asymptotes: $x=1, y=x+1$

Monotonicity: Increasing on $]-\infty,0[$ and $]2,\infty[$

Decresing on $]0,1[$ and $]1,2[$

Relative extrema: Maximum at $x=0$ and $f(0)=0$ and minimum at $x=2$ and $f(2)=4$.

Concavity: Concave upward on $]1,\infty[$ and concave downward on $]-\infty,1[$.

The corresponding curve is traced in Fig. 41.

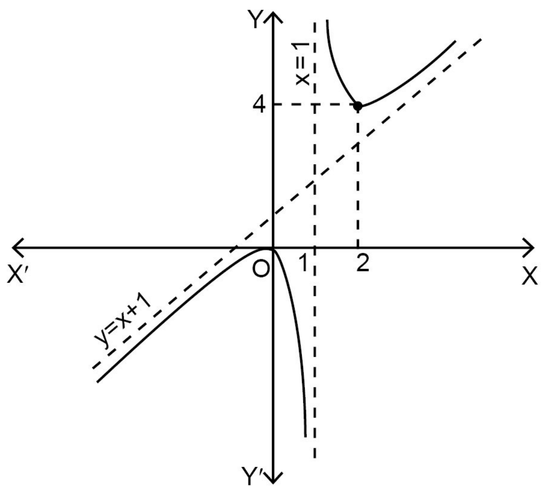


Fig. 41

xii) **Domain:** $]-\infty,0[\ \cup\ ]0,\infty[$

Point of intersection with axes: $(-(4)^{1/3},0)$

Symmetry: None

Asymptotes: $x=0, y=x$

Monotonicity: Increasing on $]-\infty,0[$ and $]2,\infty[$ and decreasing on $]0,2[$.

Relative extrema: Minimum at $x=2$ and $f(2)=3$

Concavity: Concave upward on $]-\infty,0[$ and $]0,\infty[$.

The corresponding curve is traced in Fig. 42.

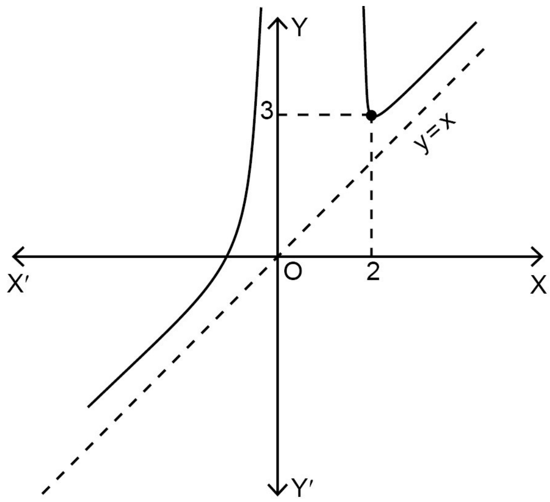


Fig. 42

E2) $y = x - \tan^{-1} x$ oblique asymptotes are $y = x \pm \frac{\pi}{2}$, which are shown in Fig. 43.

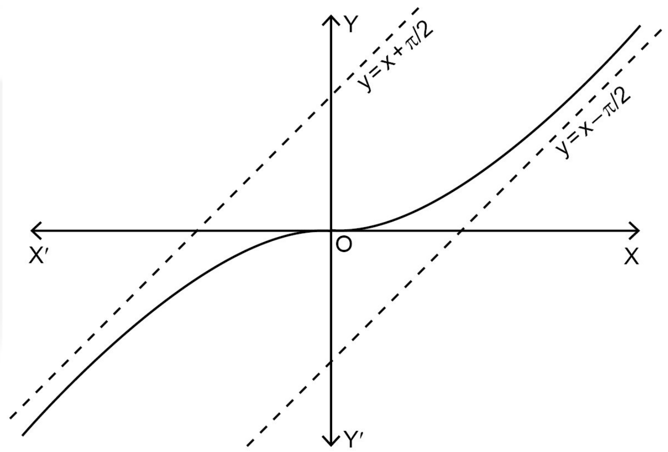


Fig. 43

E3) **Domain:** $[0, c[$

Point of intersection with axes: $(0, m_0)$.

Symmetry: None

Asymptotes: $v = c$

Monotonicity: Increasing on $[0, c[$

The corresponding curve is traced in Fig. 44.

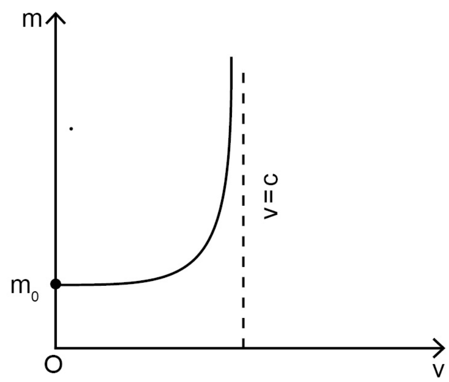


Fig. 44

E4) i)

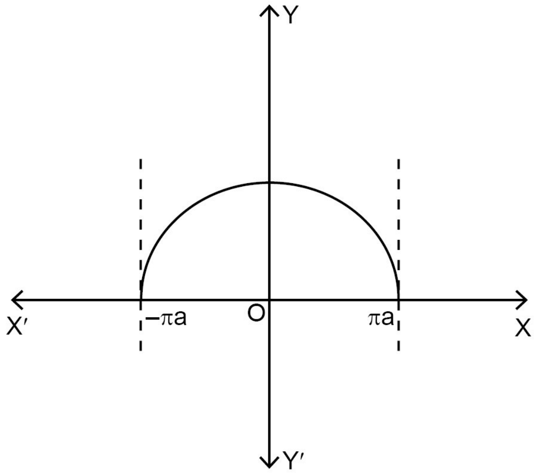


Fig. 45

ii)

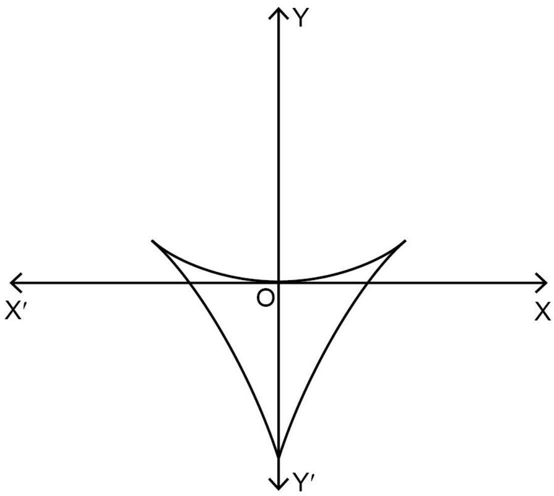


Fig. 46

iii)

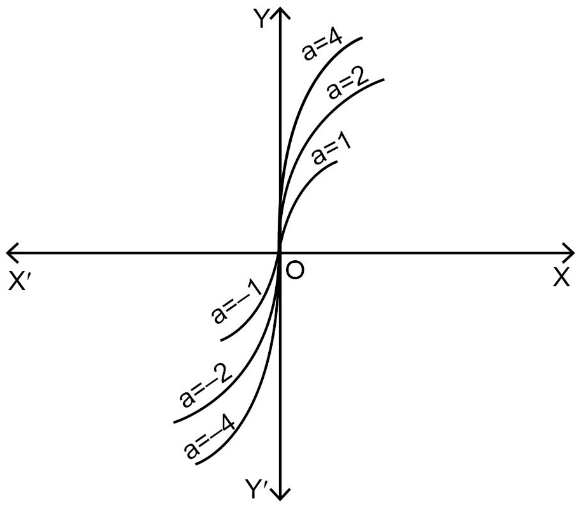


Fig. 47

iv) Again we have $x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1$
So the parametric equations again represent the unit circle $x^2 + y^2 = 1$. But as t increases from 0 to 2π, the point

$(x, y) = (\sin 2t, \cos 2t)$ starts at $(0,1)$ and moves twice around the circle in the anti clockwise direction as indicated in Fig. 48

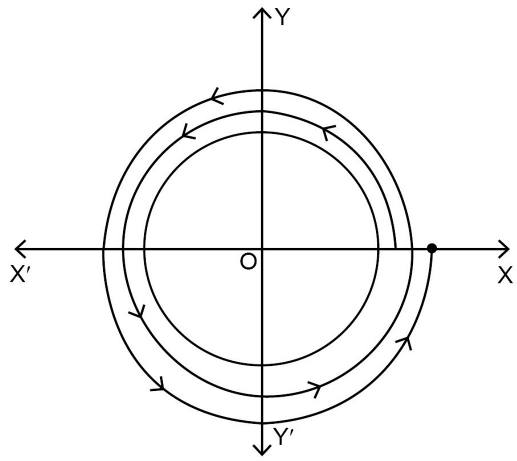


Fig. 48

v) You may observe that $y = (\sin t)^2 = x^2$ and so the point (x, y) moves on the parabola $y = x^2$. But note also that, since $-1 \le \sin t \le 1$, we have $-1 \le x \le 1$, so the parametric equations represent only the part of the parabola for which $-1 \le x \le 1$. Since $\sin t$ is periodic, the point $(x, y) = (\sin t, \sin^2 t)$ moves back and forth infinitely often along the parabola from $(-1,1)$ to $(1,1)$. (See Fig 49).

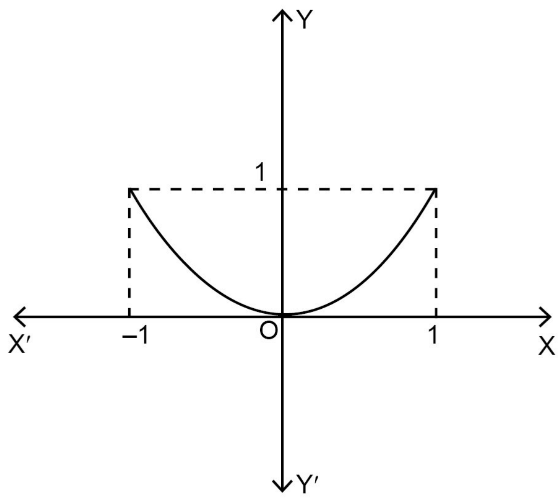
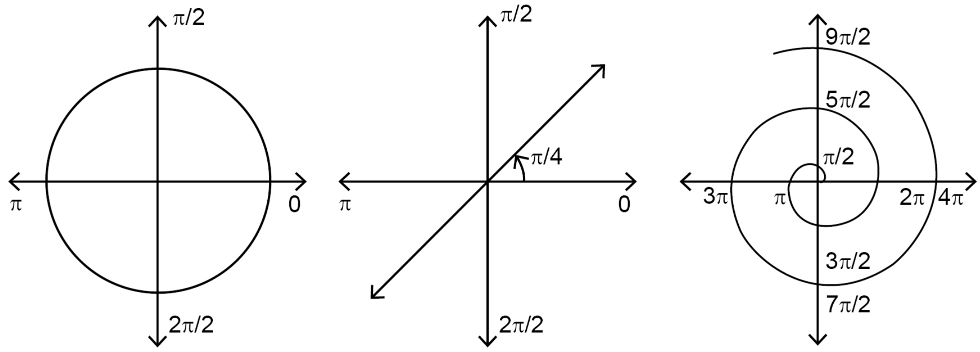


Fig. 49

E5) i) For all values of θ, the point $(1, \theta)$ is 1 unit away from the pole. Thus, the graph is the circle of radius 1 centered at the pole (Fig. 50 (a))

ii) For all values of r, the point $(r, \pi/4)$ lies on a line that makes an angle of $\pi/4$ with the polar axis (Fig. 50(b)). Positive values of r correspond to points on the line in the first quadrant and negative values of r to points on the line in the third quadrant. Thus, in the absence of any restriction on r, the graph is the entire line. Observe, however, that had we imposed the restriction $r \ge 0$, the graph would have been just the ray in the first quadrant.

iii) Observe that as θ increases, so does r; thus, the graph is a curve that spirals out from the pole as θ increases. A reasonably accurate sketch of the spiral can be obtained by plotting the intersections with the $x-$and $y-\text{axes}$ for values of θ that are multiples of $\pi/2$, keeping in mind that the value of r is always equal to the value of θ (Fig. 50(c)).



(a) $r=1$ **(b)** $\theta=\pi/4$ **(c)** $r=\theta$

Fig. 50

E6) i)

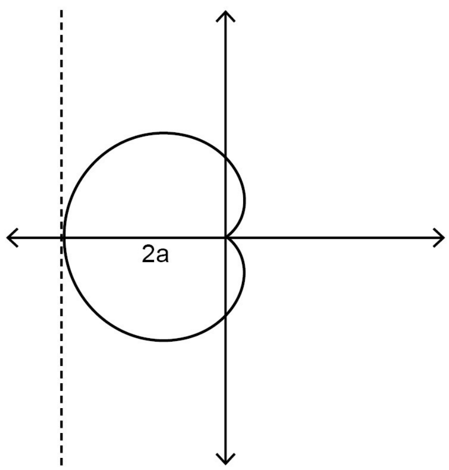


Fig. 51

ii)

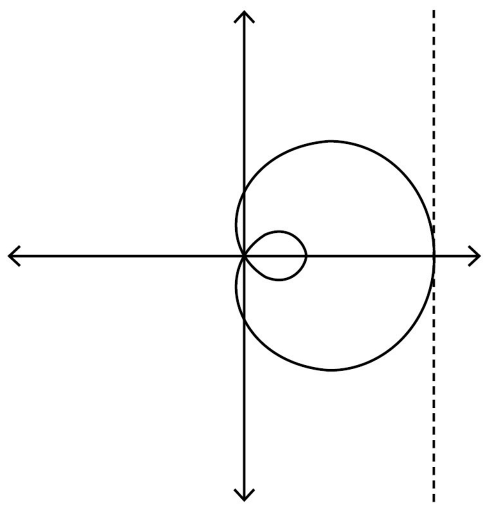


Fig. 52

iii)

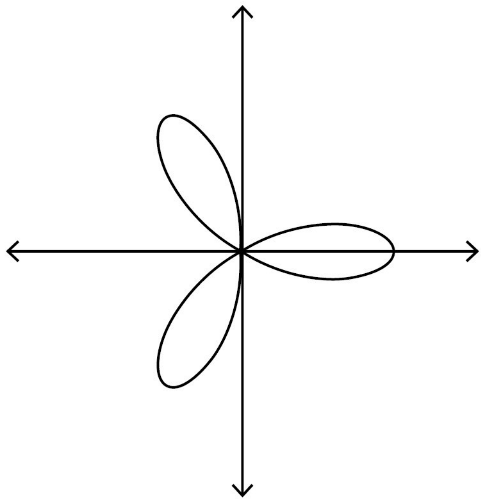


Fig. 53

iv)

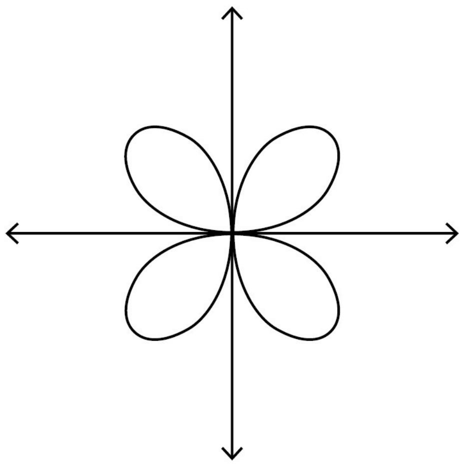


Fig. 54

MISCELLANEOUS EXAMPLES AND EXERCISES

The examples and exercises given below cover the concepts and processes you have studied in this block. Doing them will give you a better understanding of the concepts concerned, as well as practice in solving such problems.

Example 1: Find the intervals on which the following functions are increasing and the intervals on which they are decreasing.

i) $f(x) = x^2 - 6x + 5$ ii) $f(x) = x^3$

Solution: i) The graph of f in Fig. 1 suggests that f is decreasing for $x \le 3$ and increasing for $x \ge 3$. To confirm this, we differentiate f to obtain $f'(x) = 2x - 6 = 2(x-3)$.

It follows that $f'(x) < 0$ if $x < 3$, and $f'(x) > 0$ if $x > 3$.

Since, f is continuous at $x = 3$, using the first derivative test, we can say that f is decreasing on $]-\infty, 3[$ and f is increasing on $]3, +\infty[$.

We can also conclude these from the graph of f in Fig. 1.

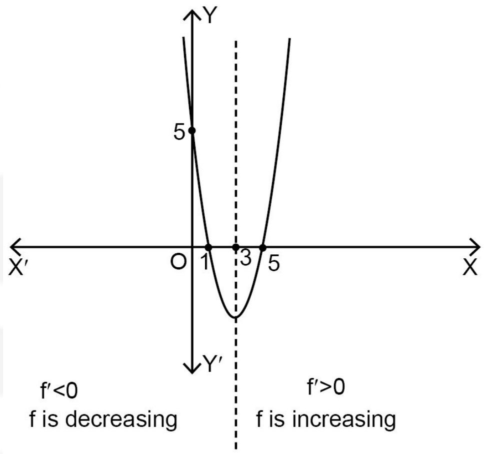


Fig. 1: Graph of $x^2 - 6x + 5$.

ii) The graph of f in Fig. 2 suggests that f is increasing above the x-axis. To confirm this, we differentiate f and obtain $f'(x) = 3x^2$. Thus, $f'(x) > 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$.

Since, f is continuous at $x = 0$, therefore, using first derivative test f is increasing on $]-\infty, 0[$ and $]0, +\infty[$.

Hence, f is increasing over the entire interval $]-\infty, +\infty[$, we also conclude the same from the graph in Fig. 2.

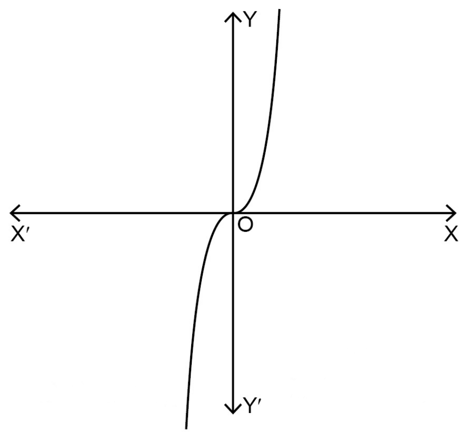


Fig. 2: Graph of x^3.

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Example 2: Use the graph of the function of defined as $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 5$ given in Fig. 3 to mark the intervals on which f is increasing or decreasing. Also, verify it using derivatives.

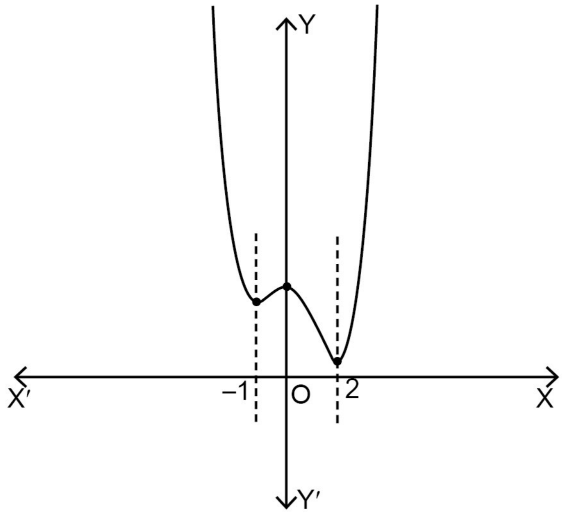

Fig. 3: Graph of $\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 5$.

Solution: The graph suggests that f is decreasing if $x \le -1$, increasing if $-1 \le x \le 0$, decreasing if $0 \le x \le 2$, and increasing if $x \ge 2$.

On differentiating f we obtain

$$f'(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x+1)(x-2)$$

The sign of f' is given in Table 1, which confirms the conclusion derives from the graph.

Table 1

Interval	**Sign of $(x)(x+1)(x-2)$**	**Sign of $f'(x)$**	**Conclusion**
$x < -1$	$(-)(-)(-)$	$-$	f is decreasing on $]-\infty, -1[$
$-1 < x < 0$	$(-)(+)(-)$	$+$	f is increasing on $]-1, 0[$
$0 < x < 2$	$(+)(+)(-)$	$-$	f is decreasing on $]0, 2[$
$x > 2$	$(+)(+)(+)$	$+$	f is increasing on $]2, +\infty[$

***

Example 3: Find the intervals on which the following functions are concave upward and concave downward.

i) $f(x) = x^2 - 6x + 5$ ii) $f(x) = x^3$ iii) $f(x) = \frac{1}{3}x^3 - x^2 + 2$

Solution: i) Calculating the first two derivatives we obtain $f'(x) = 2x$ and $f''(x) = 2$. Since, $f''(x) > 0$ for all x, the function f is concave upward on $]-\infty, +\infty[$. You can verify this from the graph given in Fig. 1.

ii) Calculating the first two derivatives, we obtain $f'(x) = 3x^2$ and $f''(x) = 6x$. Since, $f''(x) < 0$ if $x < 0$ and $f''(x) > 0$ if $x > 0$, therefore, the function f is concave downward on $]-\infty, 0[$ and concave upward on $]0, +\infty[$ as shown in Fig. 2.

iii) Calculating the first two derivatives, we obtain $f'(x) = x^2 - 2x$ and $f''(x) = 2x - 2 = 2(x-1)$. Since, $f''(x) > 0$ if $x > 1$ and $f''(x) < 0$ if $x < 1$, we conclude that f is concave upward on $]1, +\infty[$ and f is concave downward on $]-\infty, 1[$. Fig. 4 shows the graph and verifies this.

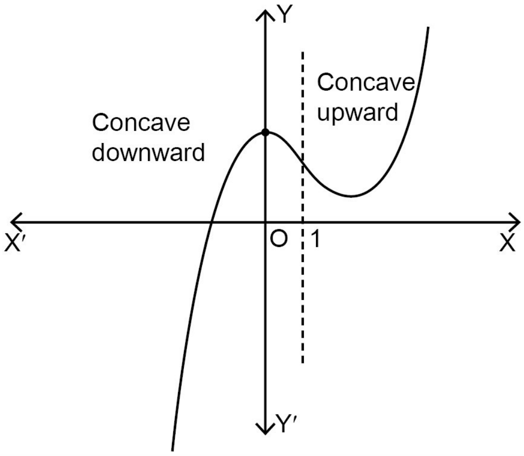

Fig. 4: Graph of $\frac{1}{3}x^3 - x^2 + 2$.

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Example 4: Consider the graph of the function f defined by $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 5$ as shown in Fig. 3. Find the inflection points from the graph, and check your answer by finding the inflection points using derivatives.

Solution: From the graph shown in Fig. 3, it is clear that the graph changes from concave upward to concave downward between -1 and 0, say roughly at $x = -0.50$, and the graph changes from concave downward to concave upward somewhere between 1 and 2, say roughly at $x = 1.25$. To check this result with the exact inflection points, we obtain the second derivative of f. We get $f'(x) = x^3 - x^2 - 2x$ and $f''(x) = 3x^2 - 2x - 2 = [3x - (1+\sqrt{7})][3x - (1-\sqrt{7})]$.

We check sign of the second derivative at different intervals as given in Table 2. Thus, from the sign of f'' in Table 2, we can say that f has inflection points at the values $x = \frac{1-\sqrt{7}}{3} \approx -0.55$ and $x = \frac{1+\sqrt{7}}{3} \approx 1.22$. The graph of f'' is shown in Fig. 5 verifies this.

Table 2

Interval	Sign of f''	Result
$x < \frac{1-\sqrt{7}}{3}$	$+$	f is concave upward
$\frac{1-\sqrt{7}}{3} < x < \frac{1+\sqrt{7}}{3}$	$-$	f is concave downward
$x > \frac{1+\sqrt{7}}{3}$	$+$	f is concave upward

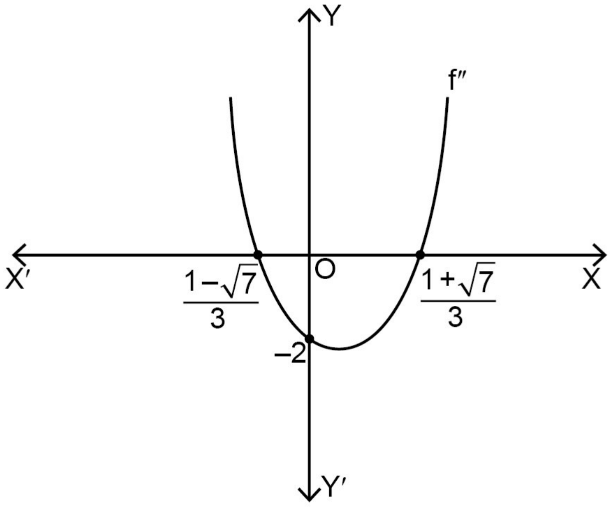

Fig. 5: Graph of f''.

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Example 5: Find the inflection points of $f(x) = \sin x$ on the interval $[0, 2\pi]$, and verify your results with the graph of the function.

Solution: Calculating the first two derivatives of f, we obtain $f'(x) = \cos x$ and $f''(x) = -\sin x$.

Thus, $f''(x) < 0$ if $0 < x < \pi$, and $f''(x) > 0$ if $\pi < x < 2\pi$, which implies that the graph is concave downward on $0 < x < \pi$ and concave upward on $\pi < x < 2\pi$. Thus, there is an inflection point at $x = \pi \approx 3.14$ as shown in Fig. 6.

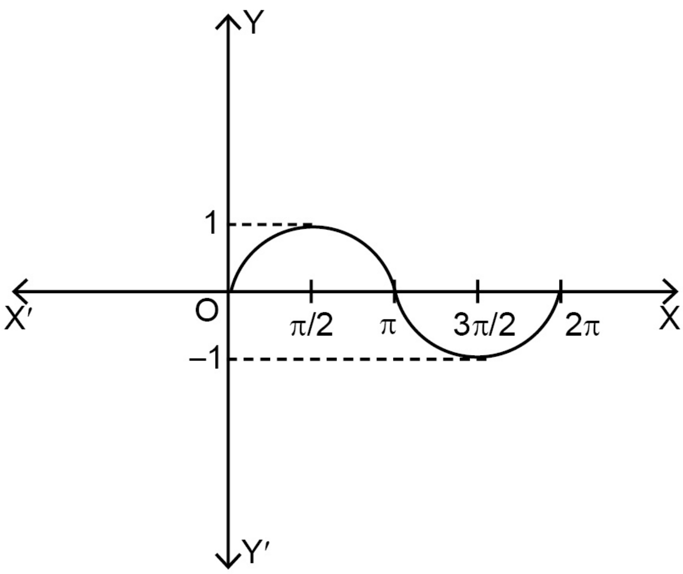

Fig. 6

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Example 6: Use the graph of $y = f'(x)$ given in the Fig. 7 to fill the boxes with $<, =, \text{or} >$. Give reasons for your answer.

i) $f(0)$ ☐ $f(1)$ ii) $f(1)$ ☐ $f(2)$ iii) $f'(0)$ ☐ 0

iv) $f'(1)$ ☐ 0 v) $f''(0)$ ☐ 0 vi) $f''(2)$ ☐ 0

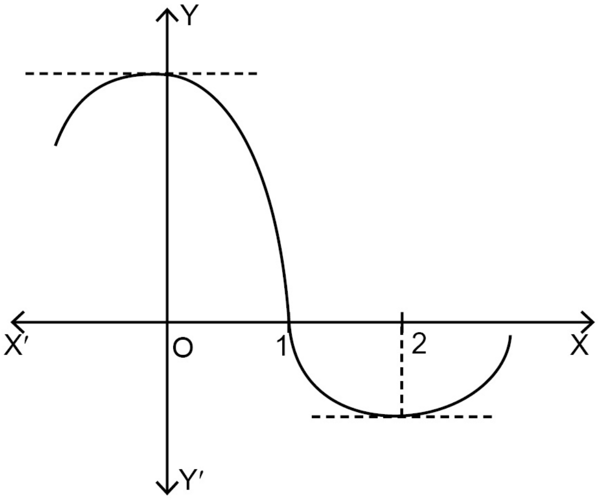

Fig. 7

Solution: i) Since, $f' > 0$ on $[0,1]$, therefore, f is increasing on $[0,1]$ and $f(0) < f(1)$.

ii) Since $f' < 0$ on $[1,2]$, therefore, f is decreasing on $[1,2]$ and $f(1) > f(2)$.

iii) $f'(0) > 0$

iv) $f'(1) = 0$, the graph of f' intersects the x-axis at $x = 1$.

v) $f''(0) = 0$

vi) $f''(2) = 0$.

***

Example 7: Show that $\sqrt[3]{1+x} < 1 + \frac{1}{3}x$ if $x > 0$.

Solution: Let $f(x) = 1 + \frac{x}{3} - \sqrt[3]{1+x}$.

Then, $f'(x) = \frac{1}{3} - \frac{1}{3}(1+x)^{\frac{-2}{3}} = \frac{1}{3} - \frac{1}{3(1+x)^{2/3}} = \frac{1}{3}\left[1 - \frac{1}{(1+x)^{2/3}}\right]$.

Here, $f'(x) > 0$, when $x > 0$, therefore, f is an increasing function on $]0,\infty[$.
Hence, $f(0) < f(x) \; \forall \, x \in]0,\infty[$.

Which gives $0 < 1 + \frac{x}{3} - \sqrt[3]{1+x}$ $[\because f(0) = 0]$

Thus, $\sqrt[3]{1+x} < 1 + \frac{x}{3}$

***

Example 8: Find the relative maxima and minima of $f(x) = x^4 - 2x^2$. Mark these on the graph of f.

Solution: We have $f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$, and $f''(x) = 12x^2 - 4$.
On solving $f'(x) = 0$, we get critical points, which are $x = 0, x = 1$ and $x = -1$.
Using second derivative test, we get
$f''(0) = -4 < 0$, thus maxima at $x = 0$.
$f''(1) = 8 > 0$, thus minima at $x = 1$.
$f''(-1) = 8 > 0$, thus minima at $x = -1$.

So, there is a relative maximum at $x = 0$ and relative minima at $x = 1$ and at $x = -1$ as shown in Fig. 8.

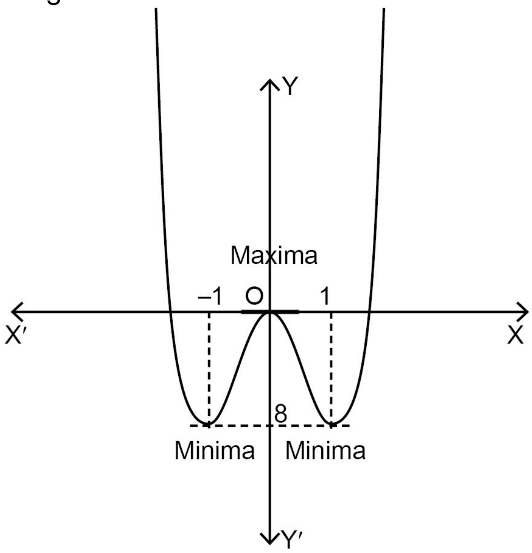

Fig. 8

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Example 9: Trace the curve $y = \frac{x^3 - x^2 - 8}{x - 1}$ by showing all the properties you use to trace.

Solution:

i) **Symmetry:** There is no symmetry about x-axis, y-axis or about origin.

ii) **Point of intersection with axes:** Setting $y = 0$, gives the equation $x^3 - x^2 - 8 = 0$. The LHS of this equation changes its sign in the interval $[2,3]$, therefore, the graph of y intersects x-axis between 2 and 3. Also, the curve passes through the point $(0,8)$.

iii) **Asymptotes:** Here $\lim_{x \to 1} \frac{x^3 - x^2 - 8}{x - 1}$ tends to ∞, therefore $x = 1$ is a vertical asymptote. There are no horizontal asymptotes.

iv) **Relative maxima or minima:** We get $\frac{dy}{dx} = \frac{d}{dx}\left[x^2 - \frac{8}{x-1}\right]$

$$= 2x + \frac{8}{(x-1)^2}$$

and $\frac{d^2y}{dx^2} = 2 - \frac{16}{(x-1)^3}$

Here, $y' = 0$ when $2x = -\frac{8}{(x-1)^2}$ or when $2(x^3 - 2x^2 + x + 4)$

$= 2(x+1)(x^2 - 3x + 4) = 0$. The only real solution to this equation is $x = -1$. Therefore, there is a relative minimum at $x = -1$, and the minimum value of y is 5.

v) **Increasing or decreasing function:** Here, $y' < 0$, when $x < -1$, therefore, f is decreasing, and $y' > 0$, when $-1 < x < \infty$, thus, f is increasing on $]-1, \infty[$ except at $x = 1$.

vi) Concavity: Here, $y'' = 0$, when $2 = \frac{16}{(x-1)^3}$ or when $(x-1)^3 = 8$. Then, $x - 1 = 2$, so, $x = 3$. Thus, there is an inflection point at $x = 3$. The coordinates of the inflection point are $(3, 5)$.

Combining all the properties, we discussed from (i) to (vi), we can trace the curve $y = f(x)$. Fig. 9 shows the curve.

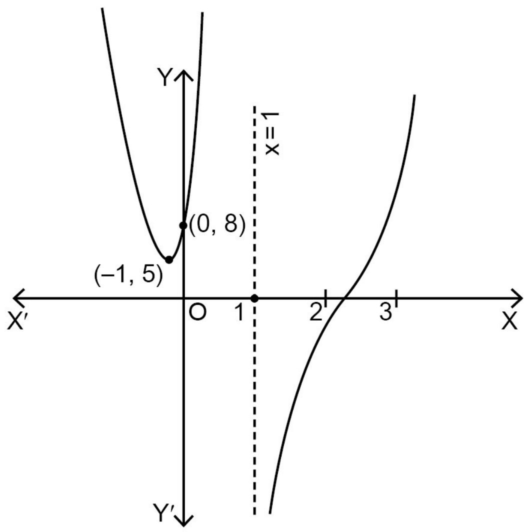

Fig. 9

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Example 10: The position function of a moving particle is given by $s(t) = 2t^3 - 21t^2 + 60t + 3$. Find the velocity and acceleration of the particle and also, determine the interval in which velocity and acceleration are increasing or decreasing.

Solution: The velocity and acceleration at time t are

$$v(t) = s'(t) = \frac{ds}{dt} = 6t^2 - 42t + 60 = 6(t-2)(t-5) \text{ and}$$

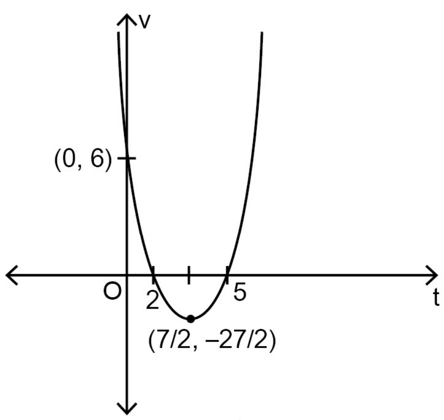
$$a(t) = v'(t) = \frac{d^2s}{dt^2} = 12t - 42 = 12\left(t - \frac{7}{2}\right).$$

At each instant we can determine the direction of the motion from the sign of $v(t)$ and whether the particle is speeding up or slowing down from the signs of $v(t)$ and $a(t)$ together (Fig. 10 (a) and (b)).

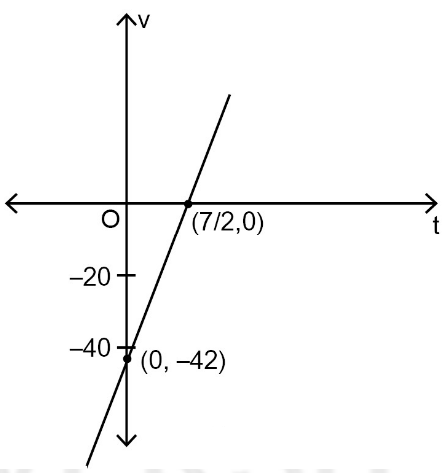
Table 3

Time	**Velocity** $v(t)$	**Acceleration** $a(t)$	**Interpretation**
$0 < t \le 2$	$v(0) = 60\,m/s$ $v(2) = 0\,m/s$	$a(0) = -42\,m/s^2$ $a(2) = -18\,m/s^2$	Since, the acceleration is negative, the speed of the particle is decreasing. The particle continuous moving with decreasing speed.

Time	Velocity $v(t)$	Acceleration $a(t)$	Interpretation
$2 \le t \le \frac{7}{2}$	$v(2) = 0 m/s$ $v\left(\frac{7}{2}\right) = -\frac{27}{2} m/s$	$a(2) = -18 m/s^2$ $a\left(\frac{7}{2}\right) = 0 m/s^2$	The particle begins to slow down.
$\frac{7}{2} \le t \le 5$	$v\left(\frac{7}{2}\right) = \frac{-27}{2} m/s$ $v(5) = 0 m/s$	$a\left(\frac{7}{2}\right) = 0 m/s^2$ $a(5) = 18 m/s^2$	The particle continues moving until time $t = 5$, when it stops at $t = 5$, $s(5) = 28m$, it reverses direction again, and begins to speed up with acceleration $a(5) = 18$. The particle then continues moving right thereafter with increasing speed.



(a) Graph of $v(t)$



(b) Graph of $a(t)$

Fig. 10

The motion of the particle is described schematically by the curved line in Fig. 11.

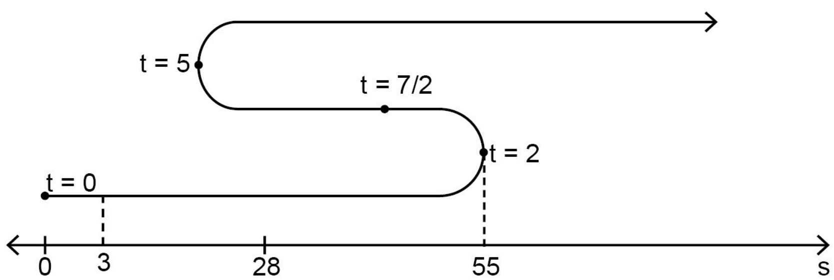


Fig. 11

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Example 11: Find all absolute extrema of the function $f(x) = x^3 - 3x^2 + 4$ on the interval $]-1, 2]$.

Solution: Here, $f(x) = x^3 - 3x^2 + 4$. On differentiating, we get

$f'(x) = 3x^2 - 6x$ and $f''(x) = 6x - 6$.

$f'(x) = 0$ gives $3x^2 - 6x = 0$, which implies $x = 0, 2$.

Since, $f''(0) = -6$, therefore, f has a relative maxima at $x = 0$. The maximum value is $f(0) = 4$.

Since, $f''(2) = 6 > 0$, therefore, f has a relative minima at $x = 2$. The minimum value is $f(2) = 0$.

Since, f has only one relative maximum value and one relative minimum value on $]-1, 2[$, therefore, the relative extrema would be absolute extrema. Thus, f has an absolute maximum at $x = 0$, and the absolute minimum at $x = 2$. You can see the graph of f in Fig. 12.

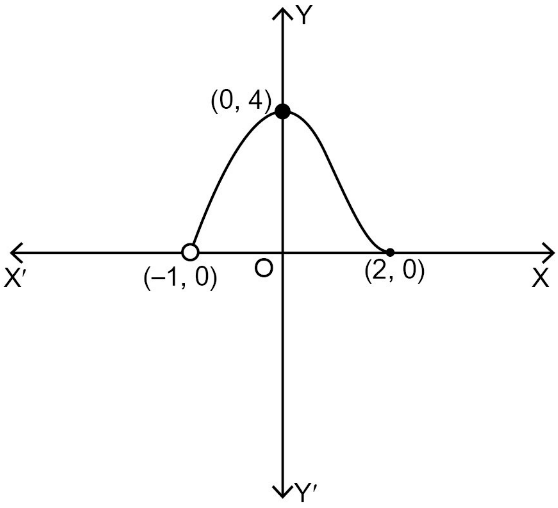

Fig. 12

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Example 12: Find the radius and height of the right circular cylinder of the largest volume that can be inscribed in a right circular cone with radius 12 cm and height 20 cm.

Solution: Let r be the radius (in cm) of the cylinder, h be the height (in cm) of the cylinder and V be the volume (in cubic cm) of the cylinder as shown in Fig. 13 (a).

The volume of the inscribed cylinder is $V = \pi r^2 h$.

Since, the volume has two variables, we can eliminate one of the variables using relationship between r and h. For this, we use similar triangles (Fig. 13 (b)),

We obtain $\frac{BC}{CD} = \frac{BO}{OA}$

$\frac{20 - h}{r} = \frac{20}{12}$ or $h = 20 - \frac{5}{3}r$

Putting h in terms of r in the formula of V, we get

$$V = \pi r^2\left(20 - \frac{5}{3}r\right) = 20\pi r^2 - \frac{5}{3}\pi r^3$$

which expresses V in terms of r alone. Because r represents a radius, it cannot be negative, and because the radius of the inscribed cylinder cannot exceed the radius of the cone, the variable r must satisfy $0 \le r \le 12$.

On differentiating V with respect to r, we get $\frac{dV}{dr} = 40\pi r - 5\pi r^2 = 5\pi r(8-r)$ and

$\frac{d^2V}{dr^2} = 40\pi - 10\pi r$.

Setting, $dV/dr = 0$ gives $5\pi r(8-r) = 0$, so, $r = 0$ and $r = 8$ are critical numbers. Since, these lie on the interval $[0,12]$, the maximum must occur at one of the values $r = 0, \quad r = 8, \quad r = 12$.

$(V)_{\text{at } r=0} = 0$

$(V)_{\text{at } r=8} = \frac{1280\pi}{3}$

$(V)_{\text{at } r=12} = 0$

Here, the maximum volume $V = \frac{1280\pi}{3} \text{cm}^3$ occurs when the inscribed cylinder has radius 8 cm. When $r = 8\text{cm}, h = \frac{20}{3}\text{cm}$. Thus, the inscribed cylinder of the largest volume has radius 8 cm and height $\frac{20}{3}\text{cm}$.

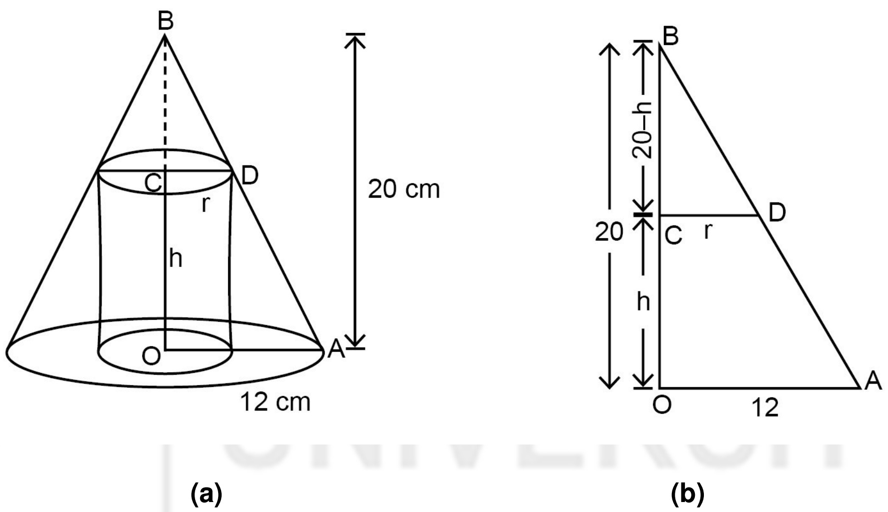

Fig. 13

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Example 13: A pharmaceutical firm sells liquid form of penincillin in units at a price of ₹100 per unit. The total production cost (in ₹) for x units is $C(x) = 100{,}000 + 20x + 0.004x^2$ and the production capacity of the firm is at most 20,000 units in a specified time. Find the number of sells a liquid form of penicillin at units of penicillin, manufactured and sold in that time to maximise the profit?

Solution: The total revenue for selling x units is $R(x) = 100x,$ the profit $P(x)$ on x units will be $P(x) = R(x) - C(x)$

$= 100x - (100{,}000 + 20x + 0.004x^2) = 80x - 100{,}000 - 0.004x^2$

On differentiating, $P(x)$ with respect to x, we get, $\frac{dP}{dx} = 80 - 0.008x$.

Setting, $\frac{dP}{dx} = 0$ gives $80 - 0.008x = 0,$ which gives $x = 10{,}000$.

Since, the capacity is at most 20,000 units, the critical number lies in the interval $[0, 20{,}000]$. Hence, the maximum profit must occur at one of the values $x = 0, x = 10{,}000,$ or $x = 20{,}000$

Now, the value of $P(x)$ at each critical number is

$P(0) = -100{,}000$

$P(10{,}000) = 300000$

$P(20{,}000) = -100{,}000$

Thus, the firm must manufacture 10,000 units to maximise the profit.

***

Example 14: Trace the curve $y = e^{-x^2/2}$ by stating all the properties you use to trace.

Solution: i) Symmetry: Since, the power of x is even, therefore, the curve is symmetrical about the y-axis.

ii) **Points of intersection with axes:** Setting $y = 0$, we get $e^{-x^2/2} = 0$, which has no solution, because all powers of e have positive values. Thus, there are no x-intercepts. Now, setting $x = 0$ gives $y = 1$. Therefore, the curve passes through the point $(0,1)$.

iii) **Asymptotes:** There are no vertical asymptotes, since $e^{-x^2/2}$ is defined and continuous on $]-\infty, +\infty[$.

Also, $\lim\limits_{x\to-\infty} e^{-x^2/2} = \lim\limits_{x\to+\infty} e^{-x^2/2} = 0$

Thus, the curve $y = e^{-x^2/2}$ has horizontal asymptote, which is $y = 0$.

iv) **Increasing and decreasing function:** On differentiating, we get

$$\frac{dy}{dx} = e^{-x^2/2}\frac{d}{dx}\left(-\frac{x^2}{2}\right) = -xe^{-x^2/2}$$

Here, $y' > 0,$ when $x < 0,$ thus, y is increasing on $]-\infty, 0[$, and $y' < 0,$ when $x > 0,$ thus, y is decreasing on $]0, \infty[$.

v) **Relative extrema:** Since, $e^{-x^2/2} > 0$ for all $x,$ the sign of $dy/dx = -xe^{-x^2/2}$ is the same as the sign of $-x$.

Therefore, y has a relative minimum $e^0 = 1$ at $x = 0$.

vi) **Concavity:** Here, $$\frac{d^2y}{dx^2} = -x\frac{d}{dx}\left[e^{-x^2/2}\right] + e^{-x^2/2}\frac{d}{dx}[-x]$$
$$= x^2e^{-x^2/2} - e^{-x^2/2} = (x^2 - 1)e^{-x^2/2}$$

Since, $e^{-x^2/2} > 0$ for all $x,$ the sign of $d^2y/dx^2 = (x^2 - 1)e^{-x^2/2}$ is the same as the sign of $(x^2 - 1)$, and the sign of $(x^2 - 1)$ would change at $x = 1$ and at $x = -1$. Thus, the inflection points occur at $x = -1$ and at $x = 1$. These inflection points are $(-1, e^{-1/2}) \approx (-1, 0.607)$ and $(1, e^{-1/2}) \approx (1, 0.607)$.

We combine all these points, and trace the curve as shown in Fig. 14.

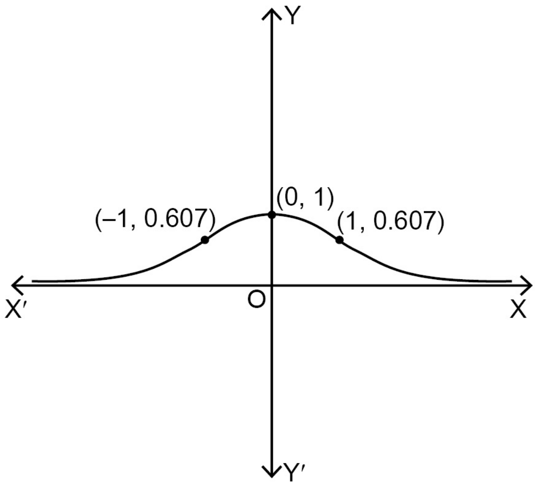

Fig. 14: Curve $y = e^{-x^2/2}$.

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Example 15: Find the type of the indeterminate forms in the following limits. Also, find the limit.

i) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$ ii) $\lim\limits_{x\to \pi/2} \dfrac{1-\sin x}{\cos x}$

iii) $\lim\limits_{x\to \infty} \dfrac{e^x-1}{x^3}$ iv) $\lim\limits_{x\to +\infty} \dfrac{x^{-4/3}}{\sin(1/x)}$

Solution: i) The numerator and denominator are 0 as $x \to 3$. Therefore, the limit is an indeterminate form of type $0/0$. Applying L'Hôpital's rule, we get

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{\frac{d}{dx}[x^2-9]}{\frac{d}{dx}[x-3]} = \lim_{x\to 3} \frac{2x}{1} = 6$$

Alternatively, you may find this limit by factoring

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6.$$

ii) The numerator and denominator are ∞ as $x \to \frac{\pi}{2}$, therefore, the limit is an indeterminate form of type $0/0$. Applying L'Hôpital's rule, we get

$$\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x} = \lim_{x\to \frac{\pi}{2}} \frac{\frac{d}{dx}[1-\sin x]}{\frac{d}{dx}[x^3]} = \lim_{x\to \frac{\pi}{2}} \frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0$$

iii) The numerator and denominator are 0 as $x \to \infty$, therefore, the limit is an indeterminate form of type $\frac{\infty}{\infty}$. Applying L'Hôpital's rule repeatedly, we get

$$\lim_{x\to \infty} \frac{e^x-1}{x^3} = \lim_{x\to \infty} \frac{\frac{d}{dx}[e^x-1]}{\frac{d}{dx}[x^3]} = \lim_{x\to \infty} \frac{e^x}{3x^2} = \lim_{x\to \infty} \frac{e^x}{6x} = \lim_{x\to \infty} \frac{e^x}{6} = \infty.$$

iv) The numerator and denominator are 0 as $x \to \infty$, so the limit is an indeterminate form of type $0/0$. Applying L'Hôpital's rule, we get

$$\lim_{x\to+\infty}\frac{x^{-4/3}}{\sin(1/x)}=\lim_{x\to+\infty}\frac{-\frac{4}{3}x^{-7/3}}{(-1/x^2)\cos(1/x)}=\lim_{x\to+\infty}\frac{\frac{4}{3}x^{-1/3}}{\cos(1/x)}=\frac{0}{1}=0.$$

***

Example 16: Show that $\lim_{x\to 0}(1+x)^{1/x}=e$.

Solution: Let $y=(1+x)^{1/x}$ and taking the natural logarithm of both the sides.

$$\ln y=\ln(1+x)^{1/x}=\frac{1}{x}\ln(1+x)=\frac{\ln(1+x)}{x}$$

Thus, $\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(1+x)}{x}$, which is an indeterminate form of type $0/0$.

Using L'Hôpital's rule, we get $\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{1/(1+x)}{1}=1$

Since, we have $\ln y\to 1$ as $x\to 0$, the continuity of the exponential function implies that $e^{\ln y}\to e^1$ as $x\to 0$, and this implies that $y\to e$ as $x\to 0$. Thus, $\lim_{x\to 0}(1+x)^{1/x}=e$.

***

Now you may try the following exercises.

E1) Find the inflection points, if any, for the curve $y=x^4$.

E2) Use the graph of $y=f(x)$ given in Fig. 15 to find the following:

i) The intervals on which f is increasing.

ii) The intervals on which f is decreasing.

iii) The intervals on which f is concave upward.

iv) The intervals on which f is concave downward.

v) The values of x at which f has an inflection point.

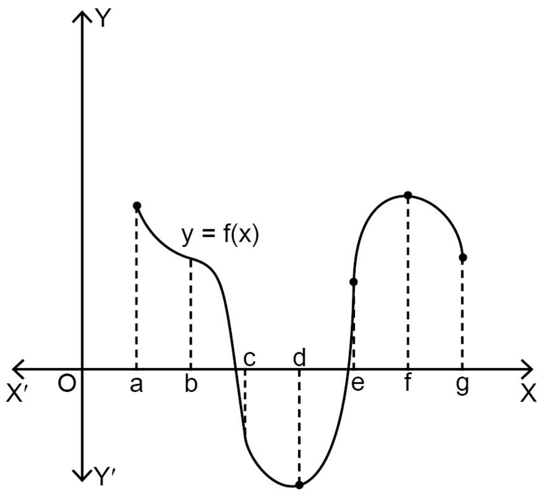

Fig. 15

E3) Show that $x<\tan x$ if $0<x<\pi/2$.

E4) Find the intervals on which f is increasing, the intervals on which f is decreasing, the open intervals on which f is concave upward, the open intervals on which f is concave downward, and the x-coordinates of all inflection points for the functions defined as follows:

i) $f(x) = (x+2)^3$ iii) $f(x) = \sqrt[3]{x+2}$

ii) $f(x) = \dfrac{x^2}{x^2+2}$ iv) $f(x) = x^{1/3}(x+4)$

E5) Prove that a general cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ $(a \neq 0)$ has exactly one inflection point.

E6) Find the relative extrema of the functions defined as follows using both the first and second derivative tests.

i) $f(x) = 1 - 4x - x^2$

ii) $f(x) = 2x^3 - 9x^2 + 12x$

iii) $f(x) = \sin^2 x,\ 0 < x < 2\pi$

iv) $f(x) = |x^2 - 4|$

E7) Trace the following curves by stating all the properties you use to trace.

i) $y = (x-4)^{2/3}$

ii) $y = 6x^{1/3} + 3x^{4/3}$.

E8) Let $f(x) = x^2 + px + q$. Find the values of p and q such that $f(1) = 3$ is an extreme value of f on $[0, 2]$. Is this value a maximum or minimum?

E9) Find the absolute maximum and minimum values of f, if any, on the stated interval.

i) $f(x) = (x^2 - 1)$ on $]-\infty, +\infty[$

ii) $f(x) = x^{2/3}(20 - x)$ on $[-1, 20]$

iii) $f(x) = 2\sec x - \tan x$ on $[0, \pi/4]$

iv) $f(x) = \sin(\cos x)$ on $[0, 2\pi]$

E10) Suppose that the equations of motion of a paper aeroplane during the first 12 seconds of flight are $x = t - 2\sin t, y = 2 - 2\cos t$, $0 \le t \le 12$. What are the highest and lowest points in the trajectory, and when is the aeroplane at those points?

E11) A closed cylindrical can is to be made to hold 1 litre (1000cm^3) of liquid. What should be the height and radius of the can to minimize the amount of material needed to manufacture the can?

E12) Find a point on the curve $y = x^2$ that is closest to the point $(18, 0)$.

E13) A firm determines that x units of its product can be sold daily at p rupees per unit, where $x = 1000 - p$. The cost of producing x units per day is $C(x) = 3000 + 20x$.

i) Find the revenue function $R(x)$.

ii) Find the profit function $P(x)$.

iii) Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximise the profit.

iv) Find the maximum profit.

v) What price per unit must be charged to obtain the maximum profit?

E14) Find the point on the curve $y = (1 + x^2)^{-1}$, at which the tangent line has the greatest slope?

E15) Trace the curve $y = \frac{(\ln x)}{x}$ by stating all the properties you use to trace it.

E16) Trace the curve $y = \frac{L}{1 + Ae^{-kt}}$, where y is the population at time $t\,(t \geq 0)$ and A, k and L are positive constants.

E17) Suppose that a hollow tube rotates with a constant angular velocity of ω rad/s about a horizontal axis at one end of the tube, as shown in the Fig. 16. Assume that an object is free to slide without friction in the tube while the tube is rotating. Let r be the distance from the object to the pivot point at time $t \geq 0$, and assume that the object is at rest and $r = 0$ when $t = 0$. If the tube is horizontal at time $t = 0$ and rotating, then $r = \frac{g}{2\omega^2}[\sinh(\omega t) - \sin(\omega t)]$ during the period that the object is in the tube. Assume that t is in seconds and r is in meters, and use $g = 9.8\,m/s^2$ and $\omega = 2\,rad/s$.

i) Trace the curve $r = f(t)$ for $0 \leq t \leq 1$.

ii) If the length of the tube is $1\,m$, then find the limit taken by the object to reach the end of the tube?

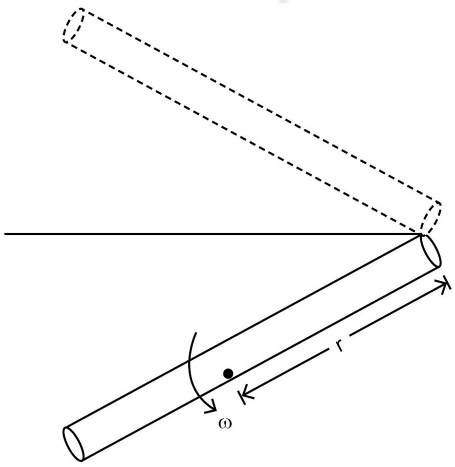

Fig. 16

E18) Trace the following curves given in polar coorindates:

i) $\theta = -\frac{3\pi}{4}$ iii) $r^2 = \sin 2\theta$

ii) $r-2=2\cos\theta$ iv) $r=4\theta$

E19) Find the slope of the tangent line to the following polar curves for the given value of θ.

i) $r=2\cos\theta;\ \theta=\pi/3$

ii) $r=\frac{1}{\theta};\ \theta=2$

E20) Show that the curve with parametric equations $x=t^3-4t$, $y=t^2$ intersects itself at the point $(0,4)$, and find equations for the two tangent lines to the curve at the point of intersection.

SOLUTIONS/ANSWERS

E1) Calculating the first two derivatives of f we obtain $f'(x)=4x^3, f''(x)=12x^2$. Here, $f''(x)>0$ for $x<0$ and for $x>0$, which implies that f is concave upward for $x<0$ and for $x>0$. In fact, f is concave upward on $]-\infty,+\infty[$. Thus, there are no inflection points, and in particular, there is no inflection point at $x=0$, even though $f''(0)=0$. (See Fig. 17)

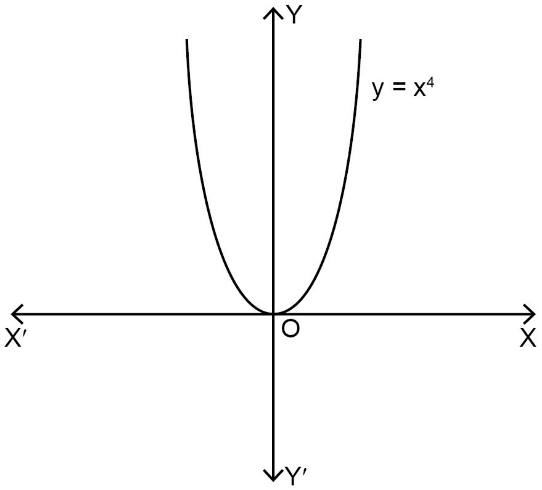

Fig. 17

E2) i) The function f is increasing on $[d,f]$.

ii) The function f is decreasing on $[a,d]$ and $[f,g]$.

iii) The function f is concave upward on the intervals $]a,b[$ and $]c,e[$.

iv) The function f is concave downward on $]b,c[$ and $]e,g[$.

v) The points of inflection are at $x=b, x=c$ and $x=d$.

E3) Let $f(x)=\tan x-x$

$f'(x)=\sec^2 x-1$

$f'(x)\geq 0$, as $\sec^2 x\geq 1$ on $0<x<\frac{\pi}{2}$.

Therefore, f is increasing on $0<x<\frac{\pi}{2}$.

Hence, $f(0)<f(x)$

Here, $f(0)=0$. Thus, $0<\tan x-x$ which gives, $x<\tan x$.

E4) i) Increasing on $]-\infty,+\infty[$.

Not decreasing anywhere on $\mathbb{R}$.

Concave upward on $]-2,+\infty[$.

Concave downward on $]-\infty,-2[$.

Point of inflection is at $x=-2$.

ii) Increasing on $[0,+\infty[$.

Decreasing on $]-\infty,0]$

Concave upward on $\left]-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right[$

Concave downward on $\left]-\infty,-\sqrt{\frac{2}{3}}\right[$ and on $\left]\sqrt{\frac{2}{3}},+\infty\right[$.

Points of inflection are $\left(+\sqrt{\frac{2}{3}},\frac{1}{3}\right)$ and $\left(-\sqrt{\frac{2}{3}},\frac{1}{3}\right)$

iii) Increasing on $]-\infty,+\infty[$.

Not Decreasing on $\mathbb{R}$.

Concave upward on $]-\infty,-2[$.

Concave downward on $]-2,+\infty[$.

Point of inflection at $x=-2$.

iv) Increasing on $[-1,+\infty[$

Decreasing on $]-\infty,-1]$

Concave upward on $]-\infty,0[$ and $]2,+\infty[$.

Concave downward on $]0,2[$.

Points of inflection are $(0,0)$ and $(2,6(2)^{1/3})$.

E5) $f(x)=ax^3+bx^2+cx+d$

$f'(x)=3ax^2+2bx+c$

$f''(x)=6ax+2b$

$f''(x)=0 \Rightarrow x=-\frac{2b}{6a}$.

Thus, f has exactly one inflection point.

E6) i) Relative maximum at $x=-2$ and $f(-2)=5$.

ii) Relative maximum of 5 at $x=1$ and relative minimum of 4 at $x=2$.

iii) Relative minimum of 0 at $x=\pi$ and relative minima of 1 at $x=\frac{\pi}{2},\frac{3\pi}{2}$.

iv) Relative maximum of 4 at $x=0$ and relative minima of 0 at $x=2$ and -2.

E7) i) The properties to trace the curve are as follows:

i) **Symmetry:** There are no symmetries about the coordinate axes or the origin. However, the graph of $y=(x-4)^{2/3}$ is symmetric about the line $x=4$, since, it is a translation (four units to the right) of the graph of $y = x^{2/3}$, which is symmetric about the y-axis.

ii) **Point of intersection with axes:** $(4, 0)$ and $(0, 2.52)$

iii) **Asymptotes:** None, since $f(x) = (x-4)^{2/3}$ is continuous everywhere and also, $\lim_{x \to +\infty} (x-4)^{2/3} = +\infty$ and $\lim_{x \to -\infty} (x-4)^{2/3} = +\infty$.

iv) **Relative extrema:** The derivatives are

$$\frac{dy}{dx} = f'(x) = \frac{2}{3}(x-4)^{-1/3} = \frac{2}{3(x-4)^{1/3}}$$

$$\frac{d^2y}{dx^2} = f''(x) = -\frac{2}{9}(x-4)^{-4/3} = -\frac{2}{9(x-4)^{4/3}}$$

There is a critical number at $x = 4$, since f is not differentiable there; and by the first derivative test there is a relative minimum at that critical number, since $f'(x) < 0$ if $x < 4$ and $f'(x) > 0$ if $x > 4$.

v) **Concavity:** Since $f''(x) < 0$ if $x \neq 4$, the graph is concave down for $x < 4$ and for $x > 4$.

vi) **Vertical tangent lines:** Since $f(x) = (x-4)^{2/3}$ is continuous at $x = 4$ and

$$\lim_{x \to 4^+} f'(x) = \lim_{x \to 4^+} \frac{2}{3(x-4)^{1/3}} = +\infty$$

$$\lim_{x \to 4^-} f'(x) = \lim_{x \to 4^-} \frac{2}{3(x-4)^{1/3}} = -\infty$$, therefore is a vertical tangent line and cusp at $x = 4$.

Combining all the properties, we can trace the curve as shown in Fig. 18.

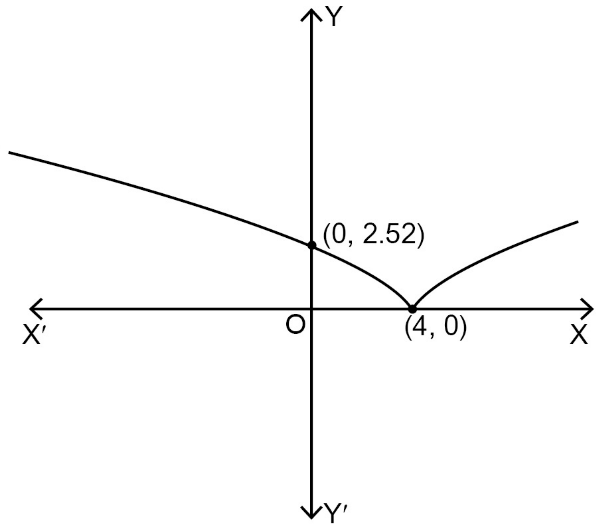

Fig. 18

ii) The properties used to trace the given curve are as follows:

i) **Symmetry:** There are no symmetry about the coordinate axes or the origin.

ii) **Points of intersection with axes:** $(0,0)$ and $(-2,0)$.

iii) **Asymptotes:** None, since $f(x) = 6x^{1/3} + 3x^{4/3}$ is continuous everywhere, also, since

$$\lim_{x\to+\infty}(6x^{1/3} + 3x^{4/3}) = \lim_{x\to+\infty} 3x^{1/3}(2+x) = +\infty$$

$$\lim_{x\to-\infty}(6x^{1/3} + 3x^{4/3}) = \lim_{x\to-\infty} 3x^{1/3}(2+x) = +\infty$$

iv) **Relative extrema:** The derivatives are

$$\frac{dy}{dx} = f'(x) = 2x^{-2/3} + 4x^{1/3} = 2x^{-2/3}(1+2x) = \frac{2(2x+1)}{x^{2/3}}$$

and

$$\frac{d^2y}{dx^2} = f''(x) = -\frac{4}{3}x^{-5/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-5/3}(-1+x) = \frac{4(x-1)}{3x^{5/3}}.$$

The critical number is $x = -\frac{1}{2}$. The sign of dy/dx changes at $x = \frac{-1}{2}$ from negative to positive, therefore,from the first derivative test, there is a relative minimum at $x = -\frac{1}{2}$.

v) **Increasing and decreasing:** The function f is decreasing when $x < -0.5$ and increasing when $x > -0.5$.

vi) **Tangents:** There is a point of vertical tangency at $x = 0$, since

$$\lim_{x\to0^+} f'(x) = \lim_{x\to0^+} \frac{2(2x+1)}{x^{2/3}} = +\infty$$

$$\lim_{x\to0^-} f'(x) = \lim_{x\to0^-} \frac{2(2x+1)}{x^{2/3}} = +\infty$$

vii) **Concavity:** Here $\frac{d^2y}{dx^2} > 0,$ when $x < 0,$ therefore, the curve is concave upward and $\frac{d^2y}{dx^2} < 0,$ when $0 < x < 1,$ therefore, the curve is concave downward, and the curve is again concave upward for $x > 1$. There are inflection points at $(0,0)$ and $(1,9)$.

Combining all the properties, we trace the curve as shown in Fig. 19.

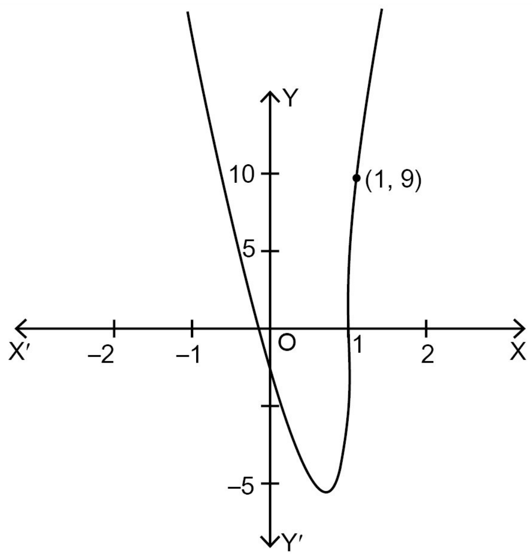

Fig. 19

E8) $f(1) = 3 = 1 + p + q \Rightarrow p + q = 2$

$f'(x) = 2x + p = 0 \Rightarrow x = -p/2$

Since, $x = 1$ is an extreme value, therefore, $p = -2$, which gives $q = 4$. Now, $f''(x) = 2$ and $f''(1) = 2 > 0$, therefore, this extreme value is minimum value.

E9) i) Minimum value -1 at $x = 0$
No maximum.

ii) Maximum value 48 at $x = 8$

Minimum value 0 at $x = 0, 20$.

iii) Maximum value 2 at $x = 0$

Minimum value $\sqrt{3}$ at $x = \frac{\pi}{6}$.

iv) Maximum value 0.841 at $x = 0, 2\pi$.
Minimum value -0.841 at $x = \pi$.

E10) Maximum $y = 4$ at $t = \pi, 3\pi$.
Minimum $y = 0$ at $t = 0, 2\pi$.

E11) The surface area of the can $S = 2\pi r^2 + 2\pi rh$, where r and h are the radius and height of the can, respectively. Also, $\pi r^2 h = 1000\,cm^3$.

Thus, $S = 2\pi r^2 + \frac{2000}{r}$, which gives $\frac{dS}{dr} = 4\pi r - \frac{2000}{r^2}$

$\frac{dS}{dr} = 0 \Rightarrow r = \frac{10}{(2\pi)^{1/3}}$ cm.

Now, $\frac{d^2S}{dr^2} = 4\pi + \frac{4000}{r^3}$ and $\left(\frac{d^2S}{dr^2}\right)_{\text{at } r = \frac{10}{(2\pi)^{1/3}}} > 0$, therefore minimum.

Hence, the height $h = \frac{20}{(2\pi)^{1/3}}$ cm.

E12) Let the point be (x, y). The distance between (x, y) and $(18, 0)$ is $D = \sqrt{(x-18)^2 + (y-0)^2}$. Since, the point (x, y) lies on $y = x^2$, therefore, $D = \sqrt{(x-18)^2 + x^4}$.

Suppose, $L = D^2$

$$L = (x-18)^2 + x^4$$

$$\frac{dL}{dx} = 2(x-18) + 4x^3$$

The critical number is $x = 2$.

$$\frac{d^2L}{dx^2} = 2 + 12x^2$$

$$\left(\frac{d^2L}{dx^2}\right)_{\text{at } x=2} = 50 > 0.$$

Thus, the distance is minimum, when $x = 2$. Thus, the point is $(2, 4)$.

E13) i) The revenue function $R(x) = xp = x(1000 - x)$

ii) The profit function $P(x) = R(x) - C(x) = 980x - x^2 - 3000$

iii) It is given that $x \le 500$. Here, $P'(x) = 980 - 2x$

$P'(x) = 0$ gives $x = 490$.

$P''(x) = -2 < 0$, thus, maxima.

Therefore, the company must produce and sell 490 units to maximise the profit.

iv) $P(490) =$ ₹237100

v) $p = 1000 - 490 =$ ₹510

E14) The required point is $\left(-\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$.

E15) i) **Symmetry:** None

ii) **Points of intersection with axes:** $(1, 0)$

iii) **Asymptotes:** Since, $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ and $\lim_{x \to 0^+} \ln x = -\infty$ it follows that values of $y = \frac{\ln x}{x} = \frac{1}{x}(\ln x)$ will decrease without bound as $x \to 0^+$, so, $\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$ and the graph has a vertical asymptote $x = 0$.

You may note that $\frac{(\ln x)}{x} > 0$ for $x > 1$. The limit $\lim_{x \to +\infty} \frac{\ln x}{x} = 0$. Thus, $(\ln x)/x$ is asymptotic to $y = 0$ as $x \to +\infty$.

iv) **Increasing and decreasing function:** The derivatives are

$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$ and

$\frac{d^2y}{dx^2} = \frac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \frac{2x \ln x - 3x}{x^4} = \frac{2 \ln x - 3}{x^3}$.

Since, $x^2 > 0$ for all $x > 0$, the sign of $\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$ is the same as the sign of $1 - \ln x$. But $\ln x$ is an increasing function with $\ln e = 1$, so, $1 - \ln x$ is positive for $x < e$ and negative for $x > e$.

v) **Relative extrema:** There is a relative maximum $\frac{(\ln e)}{e} = 1/e \approx 0.37$ at $x = e$.

vi) **Concavity:** Since, $x^3 > 0$ for all $x > 0$, the sign of $\frac{d^2y}{dx^2} = \frac{2 \ln x - 3}{x^3}$ is the same as the sign of $2 \ln x - 3$. Now, $2 \ln x - 3 = 0$ when $\ln x = \frac{3}{2}$, or $x = e^{3/2}$. Again, since, $\ln x$ is an increasing function, $2 \ln x - 3$ is negative for $x < e^{3/2}$ and positive for $x > e^{3/2}$.

Thus, an inflection point occurs at $\left(e^{3/2}, \frac{3}{2}e^{-3/2}\right) \approx (4.48, 0.33)$.

Fig. 20 shows the curve.

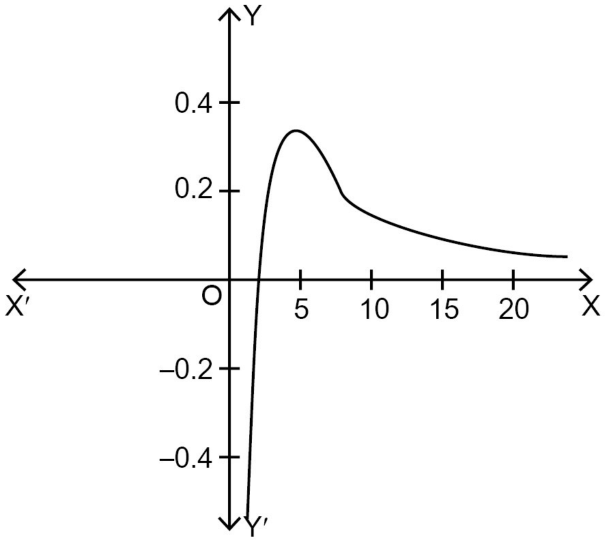

Fig. 20

E16) You may like to trace the curve by assuming values of L, A and k.

E17) You may like to trace the curve yourself.

E18) Fig. 21(a), 21 (b), 21 (c) and 21 (d) shows the graphs of the polar curves of (i), (ii), (iii) and (iv) respectively.

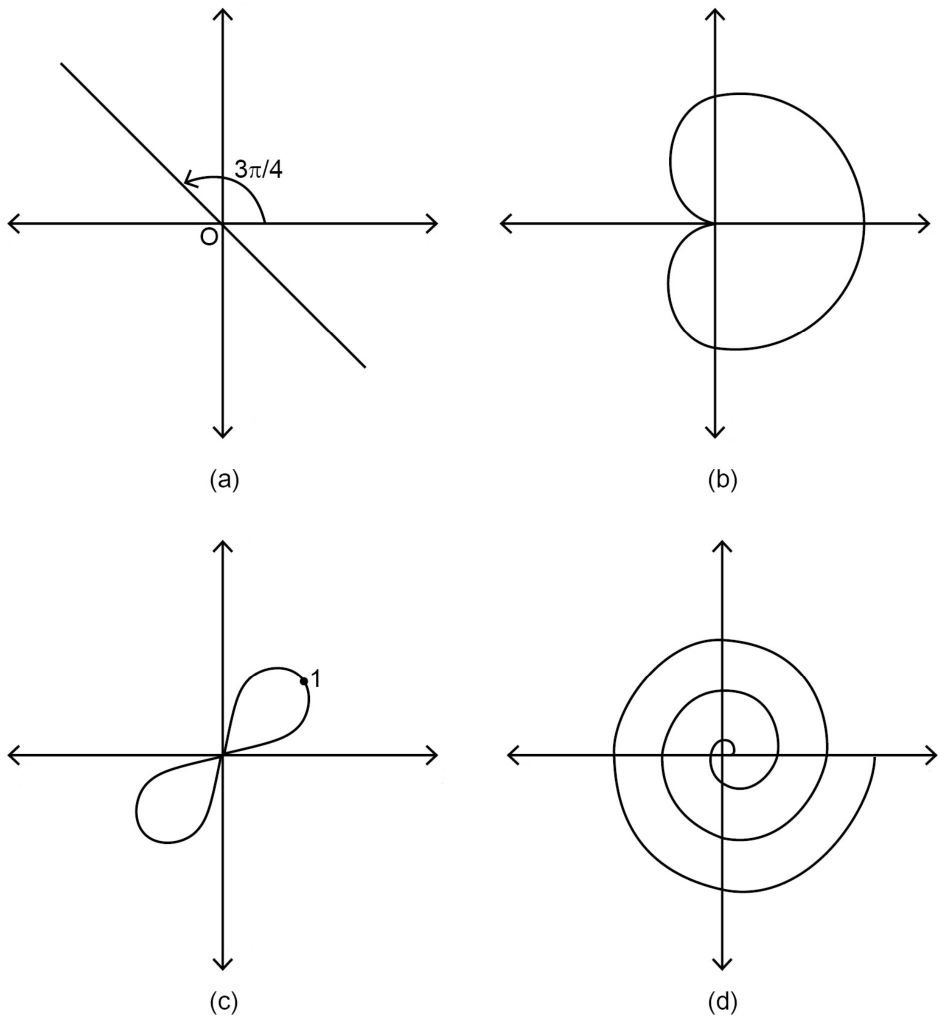

Fig. 21

E19) i) $\frac{1}{\sqrt{3}}$

ii) $\frac{\tan 2 - 2}{2\tan 2 + 1}$

E20) $y = \pm\frac{x}{2} + 4$.