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BMTC-132
DIFFERENTIAL EQUATIONS

Block

3

SECOND AND HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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BLOCK 3 SECOND AND HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

In Block 2, we discussed different methods of solving first order differential equations. But there are several real life situations in which we need to solve second and higher order differential equations. The simplest problem of the motion of a particle along a straight line is governed by a differential equation of the second order. Similarly, the motion of a pendulum, the forced vibrations of a particle, the problems of mechanical oscillations and electric circuits are all governed by differential equations of the second order. The isoperimetric problem (i.e., the problem of determining curves of a given perimeter which, under given conditions, enclose a maximum area) depends upon a differential equation of the third order. Also, the trajectories lead to differential equations of higher order depending upon the conditions imposed. In this block we will discuss the methods of solving ordinary differential equations of second, third and higher orders.

In the early years of the eighteenth century, a number of methods for solving differential equations of second and third order were found. For example, in 1712 Count Jacopo Riccati, an Italian mathematician, obtained the method of reduction of order for differential equations of the second order, viz., $f(y, y', y'') = 0$ to a first order differential equation by making use of the relation $y'' = p\dfrac{dp}{dy}$. In 1728, Leonhard Euler, Switzerland's foremost scientist and a mathematician, proposed and solved the problem of reducing a particular class of equations of second order to the equations of first order. Although John Bernoulli, a Swiss mathematician, claimed that, before the year 1700, he had studied the general equation of nth order, the systematic treatment and various methods of solving general differential equation of any order were given by Clairaut, the French mathematician, in 1734 and by Euler in 1739. Now let us see how we have presented the material in this block.

This block consists of four units. In Unit 10, which is the first unit of this block, we have classified general linear differential equations into those with constant coefficients and those with variable coefficients and further classified these equations into homogeneous and non-homogeneous equations. We have stated the conditions for the existence and uniqueness of the solution of a general non-homogeneous linear differential equation. Conditions for the solutions of linear differential equation to be linearly dependent or independent and elementary properties of these solutions are discussed. The methods of solving homogeneous linear differential equations with constant coefficients have also been discussed in this unit.

In Unit 11, we have discussed the method of undetermined coefficients for finding a particular integral of non-homogeneous linear differential equation with constant coefficients. We have identified the types of non-homogeneous terms for which the method is applicable and illustrated the method for the types identified.

In Unit 12, we have discussed the method of variations of parameters, which is due to the French mathematician Joseph Louis Lagrange (1736-1813), for finding the solutions of linear differential equations with constant as well as variable coefficients. The French mathematician D'Alembert's, method of reduction of order and the method of solving Euler's equations are also discussed in this unit.

In Unit 13, we have developed the theory for finding particular integrals with the help of operators. In particular, we have discussed the method of differential operators for finding particular integrals of differential equations with constant coefficients of the type $f(D) = X$, where $D = \frac{d}{dx}$ is a differential operator and, when $X = e^{\alpha x}$, x^m, $\sin(ax+b)$, $e^{ax}v(x)$ and $x^m v(x)$, a, b, α and m being constants. Applications of non-homogeneous differential equations with constant coefficients in the study of vibrations in mechanics and the theory of electric circuits are also discussed in this unit.

NOTATIONS AND SYMBOLS

$D^n y$ where $D = \dfrac{d}{dx}$	:	nth order derivative of y w.r.t. x
$f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2}$ $+ \cdots + a_{n-2} D^2 + a_{n-1} D + a_n$	:	nth order polynomial in D or nth order linear differential operator
$W(y_1(x), y_2(x), \ldots, y_n(x))$	:	wronskian of the n functions $y_1(x), y_2(x), \ldots, y_n(x)$
C.F.	:	complementary function
P.I.	:	particular integral
l.h.s.	:	left hand side
r.h.s.	:	right hand side
kg.	:	kilogram
m.	:	meter
sec.	:	second
cm.	:	centimeter
E	:	electromotive force
R	:	resistance
L	:	inductance
C	:	capacitance
q	:	charge
i	:	current

Greek Alphabets

σ	:	Sigma
ζ	:	Zeta
ε	:	Epsilon
τ	:	Tao
π	:	Pi
ω	:	Omega
δ	:	Delta
ξ	:	Xi

UNIT 10

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

Structure **Page No.**

10.1 INTRODUCTION

In Block 2 we defined the basic concepts related to ordinary differential equations and discussed various methods of solving first order ordinary differential equations. We also formulated and solved some of the physical problems which were governed by the first order linear ordinary differential equations. The methods developed there required mainly the skill in the techniques of integration and equations could be solved in terms of familiar elementary functions. However, this may not be the case always. As mentioned in the introduction to this block, the differential equations governing many physical or biological phenomenon may not necessarily be of the first order, they may be higher order linear or non-linear equations, may have constant or variable coefficients.

In this unit we shall classify in Sec. 10.2 the general linear nth order differential equation into two broad categories:

i) homogeneous and non-homogeneous

ii) equation with constant coefficients and variable coefficients.

Here we shall mainly confine to the methods of finding the complete solution of **linear homogeneous differential equations with constant coefficients of order two or more**. Non-homogeneous equations will be considered in the units to follow. It is usually much more difficult to solve equations if the coefficients are not constants but variables. We shall not be discussing the methods of solving such equations in this course. However, for a general linear differential equation with variable coefficients, we shall state in Sec. 10.3 the conditions under which a unique solution can be found. We shall also discuss here some properties of solutions of linear differential equations and define its general/complete solutions. The methods of finding complementary function of a linear homogeneous differential equation with constant coefficients are discussed in Sec. 10.4. At the end of the unit we have given in an appendix, a brief review of some of the concepts from matrices and determinants which you will find useful while going through the properties of the solutions of linear differential equations in Sec. 10.3.

Objectives

After reading this unit, you should be able to:

- identify linear differential equations with constant, as well as with variable, coefficients;
- identify homogeneous and non-homogeneous linear differential equations;
- describe the conditions under which a unique solution of a linear differential equation exists;
- write the complete integral of a given differential equation when its various independent integrals are known;
- classify solutions of non-homogeneous equations into complementary function and particular integral; and
- obtain the general solution of a homogenous linear differential equation with constant coefficients.

10.2 GENERAL LINEAR DIFFERENTIAL EQUATION

Let us begin by considering the following equations

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

and

$$\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} - y = x + 2$$

What is the difference between the two equations?

You know that both the equations are second order linear equations with constant coefficients. The only difference being the term $x+2$, on the right hand side of the second equation, which is a function of x plus a constant term. Such a term, called the non-homogeneous term, makes the equation non-homogeneous. Now consider an equation of the form

$$x^2 \frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} + 3y = x^2 + x + 1$$

where the coefficients are functions of x. Note that it is second order non-homogeneous linear differential equation with variable coefficients.
The most general form of linear differential equation is given by

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\ y = b(x). \tag{1}$$

The coefficients $a_0, a_1, \ldots, a_n$ and b are continuous on an interval $\alpha < x < \beta$. If, $a_0(x) \neq 0$, the differential equation is of nth order. Eqn. (1) is a **linear differential equation of nth order with variable coefficients.**

In case coefficients $a_0, a_1, \ldots, a_n$ are all constants i.e., they do not depend on x, then Eqn. (1) will be termed as a **linear differential equation of nth order with constant coefficients**.

Further, when $b(x) = 0 \ \forall \ x$, Eqn. (1) is classified as **homogeneous** linear differential equation.

But if $b(x) \neq 0$ and is a constant or a function of x, then Eqn. (1) is termed as **non-homogeneous linear** differential equation. The term $b(x)$ is the non-homogeneous term of Eqn. (1). In studying the non-homogeneous Eqn. (1) it is necessary to consider along with it the corresponding homogeneous equation obtained from it by replacing $b(x)$ by 0. Such an equation is called the **reduced equation** associated with Eqn. (1).

For example, equation

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$$

is a third order linear homogeneous differential equation with constant coefficient.

Equation $\dfrac{d^4y}{dx^4} + 4\dfrac{d^3y}{dx^3} + 3y = x^2 + 1$ is a linear non-homogeneous equation of fourth order with constant coefficients. The corresponding reduced equation is

$$\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} + 3y = 0.$$

Equation $x^3\dfrac{d^2y}{dx^2} + x^2\dfrac{dy}{dx} + xy = 2$ is a second order non-homogeneous linear differential equation with variable coefficients.

You must have **noticed** that the context in which the word homogeneous is used here is totally different from the one used in Sec. 6.3, Unit 6 of Block-2.

You may now try the following exercise.

E1) Classify the following differential equations into homogeneous and non-homogeneous equations and write the reduced equations corresponding to the non-homogeneous equations:

i) $y''' + xy'' + x^2y' + x^3y = \ln x$

ii) $y''' + 2y'' - y' = 2y$

iii) $(1-x)y'' + xy' = y + (1-x)^2$.

From your knowledge of linear algebra you might be familiar with the linear dependence and independence of a set of functions on an interval. Before we move further, let us recall these two concepts which are basic to the study of linear differential equations.

Linear Dependence and Independence of Functions

Let us consider three functions $y_1(x) = 2e^{3x}$, $y_2(x) = 5e^{3x}$ and $y_3(x) = e^{-4x}$ over any interval I. For these functions, we can always find constants c_1, c_2, c_3 not all zero such that $c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) = 0$; for instance, we can have $c_1 = -5, c_2 = 2, c_3 = 0$. Such functions y_1, y_2 and y_3 are said to be linearly dependent on I. Whereas, for functions e^x and xe^x over any interval I, $c_1 e^x + c_2 xe^x = 0$, if and only if $c_1 = 0, c_2 = 0$. Such functions e^x and xe^x are linearly independent on I.

Accordingly, we give the following two definitions.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is said to be **linearly dependent** on an interval I if there exist real numbers $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0.$$

for every x in the interval.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is said to be **linearly independent** on an interval I, if it is not linearly dependent on the interval.

In other words, a set of functions is linearly independent on an interval if the only real numbers for which

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$$

for every x in the interval, are $c_1 = c_2 = \cdots = c_n = 0$.

Let us understand these definitions in the case of two functions y_1 and y_2. If the functions are linearly dependent on an interval, then there exists real numbers c_1 and c_2, not both zero, such that, for every x in the interval

$$c_1 y_1(x) + c_2 y_2(x) = 0.$$

If $c_1 \neq 0$, it follows that

$$y_1(x) = -\frac{c_2}{c_1} y_2(x).$$

That is, **if two functions are linearly dependent, then one is a constant multiple of the other**. Conversely, if $y_1(x) = c_2 y_2(x)$ for some constant c_2, then

$$(-1) y_1(x) + c_2 y_2(x) = 0$$

for every x on some interval. Hence if the functions are linearly dependent, at least one of the constants (namely, $c_1 = -1$) is not zero. We thus conclude that **two functions are linearly independent on an interval I when neither is a constant multiple of the other** on the interval.
Functions $y_1(x) = \sin 2x$ and $y_2(x) = \sin x \cos x$ are linearly dependent on the interval $]-\infty, \infty[$ since $c_1 \sin 2x + c_2 \sin x \cos x = 0$ is satisfied for every real x with $c_1 = \frac{1}{2}$ and $c_2 = -1$. Also we can see that $y_1(x) = 2 y_2(x)$.

Let us consider the following examples.

Example 1: Determine whether the functions $\cos^2 x, \sin^2 x, \sec^2 x$ and $\tan^2 x$ are linearly independent or linearly dependent on the interval $]-\pi/2, \pi/2[$.

Solution: Let us consider $c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$. Then we can choose $c_1 = 1, c_2 = 1, c_3 = -1$ and $c_4 = 1$ for which

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$$

is satisfied for $-\pi/2 < x < \pi/2$. Thus the given functions are linearly dependent.

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Example 2: Show that the functions e^x and e^{2x} are linearly independent on any interval.

Solution: Let us assume that

$$c_1 e^x + c_2 e^{2x} = 0$$

for all x in an interval. Then we must show that $c_1 = c_2 = 0$.

Let x_0 and x_1 be two **distinct** points in the interval. We then obtain a homogeneous system of equations

$$c_1 e^{x_0} + c_2 e^{2x_0} = 0$$
$$c_1 e^{x_1} + c_2 e^{2x_1} = 0$$

The determinant of the coefficients is

$\begin{vmatrix} e^{x_0} & e^{2x_0} \\ e^{x_1} & e^{2x_1} \end{vmatrix} = e^{x_0} e^{2x_1} - e^{x_1} e^{2x_0} = e^{x_0} e^{x_1}(e^{x_1} - e^{x_0}) \neq 0$ in the interval since $x_1 \neq x_0$.

You know from your knowledge of linear algebra if the determinant of coefficients $\neq 0$, the only solution of the homogeneous system in the interval is the trivial solution i.e., we must have $c_1 = c_2 = 0$ (ref. Appendix at the end of the unit). Hence e^x and e^{2x} are linearly independent on any interval.

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In the consideration of linear dependence or linear independence, the interval on which the functions are defined is important. We now illustrate it through an example.

Example 3: Show that the functions $y_1(x) = x$ and $y_2(x) = |x|$ are

i) linearly independent on the interval $]-\infty, \infty[$.

ii) linearly dependent on the interval $]0, \infty[$.

Solution: (i) It is clear from Fig. 1 that in the interval $]-\infty, \infty[$ neither function is a constant multiple of the other.

Thus in order to have $c_1 y_1(x) + c_2 y_2(x) = 0$ that is, $c_1 x + c_2 |x| = 0$ for every real x, we must have $c_1 = 0$ and $c_2 = 0$.

ii) For $y_1(x) = x$ and $y_2(x) = |x|$ in the interval $]0, \infty[$

$$c_1 x + c_2 |x| = 0$$

$\Rightarrow$ $c_1 x + c_2 x = 0$ since $|x| = x$ for $0 < x < \infty$

$\Rightarrow$ $(c_1 + c_2)x = 0$

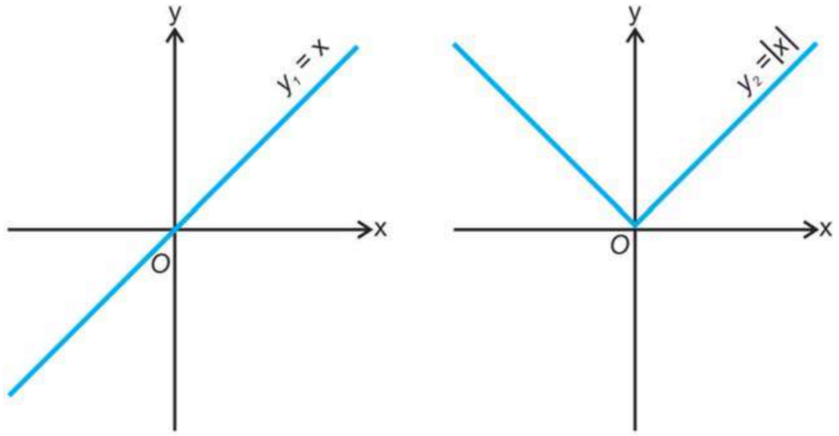

Fig. 1

which is satisfied for any non-zero choice of c_1 and c_2 for which $c_1 = -c_2$. Thus $y_1(x)$ and $y_2(x)$ are linearly dependent on the interval $]0, \infty[$.

***

You may now try the following exercise.

E2) Check whether the set $\{y_1, y_2\}$ of functions over the given interval is linearly dependent or independent.

i) $y_1 = e^x$ and $y_2 = e^{-x}$ over $-\infty < x < \infty$

ii) $y_1 = 2\cos 3x$ and $y_2 = 3\sin\left(3x + \frac{\pi}{2}\right)$ over $-\infty < x < \infty$.

iii) $y_1 = e^x$ and $y_2 = xe^x$ over $-\infty < x < \infty$.

The procedure given above for examining the linear dependence or independence of a set of functions appears to be a bit involved. We now give a theorem which provides a **sufficient condition** for examining the linear independence of a set of n functions on an interval.

Theorem 1: Suppose $y_1, y_2, \ldots, y_n$ are n functions on an interval $\mathbf{I}$ possessing derivatives at least upto $(n-1)$th order. If the determinant

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ . & . & \cdots & . \\ . & . & \cdots & . \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is non-zero for even one point in the interval I, then the functions $y_1, \ldots, y_n$ are linearly independent on I.

Proof: For simplicity, we consider the case when $n = 2$ and prove the theorem by contradiction. Let us assume that $W(y_1, y_2)(x_0) \neq 0$ for some x_0 in the interval I and that y_1 and y_2 are linearly dependent on the interval.

Now since y_1 and y_2 are linearly dependent, there exist constants c_1 and c_2, not both zero, for which

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

for every x in I. Evaluating the expression above and its derivative at x_0, we get the system of linear equations

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= 0 \end{aligned} \qquad (2)$$

The linear dependence of y_1 and y_2 implies that the system (2) possesses a non-trivial solution for c_1 and c_2. Hence

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$$

for every x_0 in I. But this is a contradiction to our assumption that $W(y_1, y_2)(x_0) \neq 0$. Hence we conclude that y_1 and y_2 are linearly independent.

We have proved the result for $n = 2$. The result for n functions $y_1, y_2, \ldots, y_n$ can be poved on the similar lines but we shall not go into those details here.

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As a consequence of this theorem we have the following result, which is the contrapositive of Theorem 1.

Corollarly 1: If the functions $y_1, y_2, \ldots, y_n$ possess derivatives at least upto $(n-1)\text{th}$ order and are linearly dependent on an interval I, then

$$W(y_1, y_2, \ldots, y_n)(x) = 0$$

for every x in I.

The determinant $W(y_1, y_2, \ldots, y_n)$ in the above theorem is called the **Wronskian** of the functions which is named after a Polish mathematician Josef Maria Hoene Wronski (1778-1853).

The functions $y_1(x) = \sin^2 x$ and $y_2(x) = 1 - \cos 2x$, for instance, are linearly dependent on $]-\infty, \infty[$, since $c_1 \sin^2 x + c_2(1 - \cos 2x) = 0$ is satisfied for all x if we choose $c_1 = -2$ and $c_2 = 1$. By above result $W(\sin^2 x, 1 - \cos 2x) = 0$ for every real number, since

$$\begin{aligned} \begin{vmatrix} \sin^2 x & 1 - \cos 2x \\ 2\sin x \cos x & 2\sin 2x \end{vmatrix} &= 2\sin^2 x \sin 2x - 2\sin x \cos x + 2\sin x \cos x \cos 2x \\ &= \sin 2x\,[2\sin^2 x - 1 + \cos 2x] \\ &= \sin 2x\,[2\sin^2 x - 1 + \cos^2 x - \sin^2 x] \\ &= \sin 2x\,[\sin^2 x + \cos^2 x - 1] \\ &= 0 \end{aligned}$$

Let us apply Theorem 1 to the two functions $f(x) = e^x$ and $g(x) = e^{2x}$ considered in Example 2. For any point x_0 we have the Wronskian of the two functions as

$$W(f, g)(x_0) = \begin{vmatrix} e^{x_0} & e^{2x_0} \\ e^{x_0} & 2e^{2x_0} \end{vmatrix} = e^{3x_0} \neq 0.$$

Thus by Theorem 1, functions e^x and e^{2x} are linearly independent on every interval.

In Example 3 we saw that $y_1(x) = x$ and $y_2(x) = |x|$ are linearly independent on $]-\infty, \infty[$. However, in this case, we cannot compute the Wronskian as y_2 is not differentiable at $x = 0$.

Remember that in Theorem 1 the condition of the non-vanishing of the Wronskian at a point in the interval provides only a **sufficient condition**. That is, if $W(y_1, y_2, \ldots, y_n)(x) = 0$ for every x in an interval, it does not necessarily mean that the functions $y_1, y_2, \ldots, y_n$ are linarly dependent on the interval. There may be functions linearly independent on some interval and yet have a vanishing Wronskian at a point in the interval. Can you think of an example of such a function? In the next exercise we are asking you to do that.

E3) Construct an example to show that a set of functions could be linearly independent on some interval and yet have a vanishing Wronskian at a point in the interval.

E4) In the following problems determine whether the given functions are linearly independent or dependent on the interval indicated alongside.

i) $f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = 4x - 3x^2;\ ]-\infty, \infty[$.

ii) $f_1(x) = 5,\ f_2(x) = \cos^2 x,\ f_3(x) = \sin^2 x;\ ]-\infty, \infty[$.

iii) $f_1(x) = x^{1/2},\ f_2(x) = x^2;\ ]0, \infty[$.

iv) $f_1(x) = \sin x,\ f_2(x) = \operatorname{cosec} x;\ ]0, \pi[$.

E5) i) Observe that for the functions f_1 and f_2 defined by $f_1(x) = 2$ and $f_2(x) = e^x$

$$1.f_1(0) - 2.f_2(0) = 0.$$

Does this imply that f_1 and f_2 are linearly dependent on any interval containing $x = 0$? Give reasons for your answer.

ii) The Wronskian of two functions at a point x is $W(x) = x^2 \sin x$ on $]-\infty, \infty[$. Are the functions linearly independent or linearly dependent? Why?

In the next section we shall study the conditions under which the solution of linear differential Eqn. (1) exists and is unique. We shall also disucss here the elementary properties of the solutions of the equation.

10.3 SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

You may recall that in Unit 6 of Block-2 we introduced you to the first order initial value problem (IVP). In an IVP we look for the solution of a given differential equation which satisfies certain conditions at a single value of the independent variable. For the first order IVP $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$ where

x_0 is a number in some interval I and y_0 is an arbitrary real number, we look for the solution of the equation on an interval I whose graph passes through (x_0, y_0).

Depending on the context, I could represent $]a, b[,]0, \infty[$, $]-\infty, \infty[$ and so on.

For a linear second order equation, solution to the initial value problem

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)\ y=b(x)$$

$$y(x_0)=y_0,\ y'(x_0)=y_1$$

can be interpreted geometrically as a function defined on I whose graph passes through the point (x_0, y_0) such that the slope of the curve at the point is the number y_1.

Similarly, for an n^{th} order IVP we are required to find the solution of Eqn. (1) on some interval I satisfying, at some point $x_0 \in \mathrm{I}$, the conditions,

$$y(x_0)=y_0,\ y'(x_0)=y_1, \ldots,\ y^{(n-1)}\ (x_0)=y_{n-1} \qquad (3)$$

where $y_0, y_1, \ldots, y_{n-1}$ are given real numbers. The values given in Eqn. (3) are the **initial conditions.**

E6) Interpret geometrically the function defining the solution of an IVP, $y''-y=0,\ y(0)=2,\ y'(0)=-1$ on the interval $]-\infty, \infty[$.

You may **note** here that an equation of the form (1) may not always have a solution. Moreover, even if its solution exists it may not be unique.

We shall now study the conditions under which the solution of Eqn. (1), if it exists, shall be unique.

10.3.1 Conditions for the Existence of a Unique Solution

Let us consider a theorem which gives the conditions whose fulfilment guarantee the existence and uniqueness of the solution of Eqn. (1).

Theorem 2: Consider the n^{th} order IVP

$$a_0(x)y^{(n)}+a_1(x)y^{(n-1)}+\cdots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=b(x)$$

$$y(x_0)=y_0,\ y'(x_0)=y_1, \ldots,\ y^{(n-1)}(x_0)=y_{n-1}$$

where x_0 is any point in an interval I and $y_0, y_1, \ldots, y_{n-1}$ are given real numbers. If $a_0, a_1, a_2, \ldots, a_n$ and b are continuous on the interval I with $a_0(x) \neq 0$ for every x in the interval then the solution $y(x)$ of the IVP **exists** on the interval and is **unique**.

You may **observe** here that the theorem says three things.

1. The initial value problem **has** a solution; that is, a solution **exists.**
2. The initial value problem has **only one** solution; that is, the solution is **unique.**

3. The solution is at least n-times differentiable function **in the interval** I where the coefficients are continuous.

We shall not be proving this theorem here as it is beyond the scope of the present course but illustrate it with the help of a few examples. However, if the functions $a_0, a_1, \ldots, a_n$ are constants, we shall give the solutions of the corresponding equations in Sec. 10.4 when $b(x)=0$ and in Units 11, 12 and 13 when $b(x)\neq 0$.

Let us consider the following examples.

Example 4: Show that $y(x)=3e^{2x}+e^{-2x}-3x$ is a unique solution of the initial value problem

$$y''-4y=12x$$

$$y(0)=4,\ y'(0)=1.$$

on any interval containing zero.

Solution: We have $y(x)=3e^{2x}+e^{-2x}-3x$, therefore,

$$y'=6e^{2x}-2e^{-2x}-3 \text{ and } y''=12e^{2x}+4e^{-2x}$$

Now $y''-4y=12e^{2x}+4e^{-2x}-4(3e^{2x}+e^{-2x}-3x)$

$$=12e^{2x}+4e^{-2x}-12e^{2x}-4e^{-2x}+12x$$

$$=12x$$

Therefore, $y(x)=3e^{2x}+e^{-2x}-3x$ satisfy the given differential equation.

Also, $y(0)=3e^{2.0}+e^{-2.0}-3.0=4$

$$y'(0)=6e^{2.0}-2e^{-2.0}-3=1$$

which shows that $y(x)$ satisfy the given initial conditions also.

Thus, $y(x)=3e^{2x}+e^{-2x}-3x$ is a solution of the given initial value problem. Moreover, the given differential equation is linear and all its coefficients are constant functions and hence continuous on any interval containing $x=0$. Also, $a_0(x)=1\neq 0$ on any interval containing $x=0$. We conclude from Theorem 2 that the given function y is the unique solution of the given initial value problem.

***

Remember that both the requirements in Theorem 2, that is, $a_i, i=0,1,\ldots,n$ and b be continuous and $a_0(x)\neq 0$, for every x in some interval say I, are important. Specifically, if $a_0(x)=0$ for some x in the interval, then the solution of a linear initial value problem may not be unique or it may not even exist.

We now illustrate this point through an example.

Example 5: Obtain the values of c for which the function

$$y(x)=cx^2+x+3$$

is a solution of the initial value problem

$$x^2 y'' - 2xy' + 2y = 6,$$

$$y(0) = 3, \ y'(0) = 1$$

on an interval $]-\infty, \infty[$. Does the problem have a unique solution?

Solution: Since $y' = 2cx + 1$ and $y'' = 2c$, it follows that

$$\begin{aligned} x^2 y'' - 2xy' + 2y &= x^2(2c) - 2x(2cx+1) + 2(cx^2 + x + 3) \\ &= 2cx^2 - 4cx^2 - 2x + 2cx^2 + 2x + 6 \\ &= 6 \end{aligned}$$

Also, $y(0) = c.(0)^2 + 0 + 3 = 3$
and $y'(0) = 2c.0 + 1 = 1$

Thus, $y = cx^2 + x + 3$ **for all real values of** $\mathbf{c}$ is a solution of the given problem in the given interval which implies that the problem has infinite number of solutions or we can say that the problem does not have a unique solution. You may **observe** that the given equation is linear and all its coefficients, being polynomials, are continuous everywhere but the coefficient of y'' i.e., $a_0(x) = x^2$ becomes zero at $x = 0$.

***

You may now try the following exercises.

E7) Is the function $y = \frac{1}{4}\sin 4x$ a unique solution to the initial value problem

$$y'' + 16y = 0$$

$$y(0) = 0, \ y'(0) = 1\,?$$

E8) Find the largest interval in which the solution of the following initial value problem exists:

$$(x^2 - 3x)y'' + xy' - (x+3)y = 0$$

$$y(1) = 2, \ y'(1) = 1.$$

After discussing the conditions for the existence of a unique solution of general n^{th} order non-homogeneous linear differential Eqn. (1), we shall now discuss some properties of the solutions of both the non-homogeneous Eqn. (1) and its corresponding homogeneous equation.

10.3.2 Properties of the Solutions of Linear Differential Equations

We shall first discuss the properties of the solutions of the homogeneous equation. Consider the general n^{th} order **homogeneous** linear differential equation given by

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\ y = 0 \qquad (4)$$

We shall be giving these properties in the form of elementary theorems, which hold true for n^{th} order Eqn. (4). However, for simplicity, we shall be proving these properties for the case $n = 2$ that is, for the second order homogeneous linear equation.

We start by considering the following example.

Example 6: Show that if $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the equation $x^3 y''' - 2xy' + 4y = 0$ on the interval $]\,0, \infty\,[$, then for arbitrary constants c_1 and c_2, $y = c_1 x^2 + c_2 x^2 \ln x$ is also a solution of the equation on the given interval.

Solution: If $y = c_1 x^2 + c_2 x^2 \ln x$ is a solution of the given equation then it must satisfy the equation. Let us check if it is true.

We have: $y' = 2c_1 x + 2c_2 x \ln x + c_2 x^2 \dfrac{1}{x} = 2c_1 x + 2c_2 x \ln x + c_2 x$

$$y'' = 2c_1 + 2c_2 \ln x + 2c_2 x \frac{1}{x} + c_2 = 2c_1 + 2c_2 \ln x + 3c_2$$

$$y''' = \frac{2c_2}{x}$$

Substituting the above values of y, y', y''' in the given equation, we get

$$\begin{aligned} x^3 y''' - 2xy' + 4y &\\ &= x^3 \left(\frac{2c_2}{x} \right) - 2x(2c_1 x + 2c_2 x \ln x + c_2 x) + 4c_1 x^2 + 4c_2 x^2 \ln x \\ &= 2c_2 x^2 - 4c_1 x^2 - 4c_2 x^2 \ln x - 2c_2 x^2 + 4c_1 x^2 + 4c_2 x^2 \ln x \\ &= 0 \end{aligned}$$

Thus, $y = c_1 x^2 + c_2 x^2 \ln x$ satisfies the given equation and hence is a solution of the equation on the given interval.

***

The result shown in Example 6 holds true in general, for Eqn. (4) and is stated as follows:

Theorem 3: If $y_1, y_2, \ldots, y_n$ are the solutions of the differential Eqn. (4) on an interval I, then the linear combination $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is also a solution of Eqn. (4) on I, where $c_i, i = 1, 2, \ldots, n$ are arbitrary constants.

Proof: Let us prove the theorem when $n = 2$ i.e., for the second order equation

$$a_0(x) y'' + a_1(x) y' + a_2(x) y = 0 \tag{5}$$

having y_1 and y_2 as its two solutions.

Since y_1 and y_2 be the two solutions of Eqn. (5), we have

$$a_0(x) y_1'' + a_1(x) y_1' + a_2(x) y_1 = 0 \tag{6}$$

and

$$a_0(x) y_2'' + a_1(x) y_2' + a_2(x) y_2 = 0. \tag{7}$$

We know that

$$(c_1y_1 + c_2y_2)' = (c_1y_1)' + (c_2y_2)'$$
$$= c_1y_1' + c_2y_2'$$

Similarly, $(c_1y_1 + c_2y_2)'' = c_1y_1'' + c_2y_2''$.

If we define $y = c_1y_1 + c_2y_2$, then

$$a_0(x)y'' + a_1(x)y' + a_2(x)y$$
$$= a_0(x)\,[c_1y_1'' + c_2y_2''] + a_1(x)[c_1y_1' + c_2y_2'] + a_2(x)[c_1y_1 + c_2y_2]$$
$$= c_1[a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1] + c_2[a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2]$$
$$= c_1.0 + c_2.0 \qquad \text{[using Eqns. (6) and (7)]}$$
$$= 0$$

That is, y also satisfies Eqn. (5).

Hence if y_1 and y_2 are the solutions of Eqn. (5) then $y = c_1y_1 + c_2y_2$ is also a solution of Eqn. (5)

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Theorem 3 is known as the **superposition principle**. A special case of Theorem 3 occurs if either c_1 or c_2 is zero. We give the result in the following corollary.

Corollary 2: A constant multiple $y = c_1y_1(x)$ of a solution $y_1(x)$ of Eqn. (4) on an interval I is also its solution on I for various values of the constant c_1.

The proof of Corollary 2 is very simple. We are leaving it for you to do it yourself (see E9).) As an illustration, consider the function $y = x^2$. It is a solution of the homogeneous linear equation

$$x^2y'' - 3xy' + 4y = 0 \text{ on }]0, \infty[$$

Hence $y = cx^2$ should also be a solution of the given equation for various values of c. You can check that $y = 3x^2$, $y = ex^2$, $y = 0, \ldots$ are all solutions of the equation on the given interval.

Let us consider another example to illustrate Theorem 3.

Example 7: Show that for the differential equation $y''' - 6y'' + 11y' - 6y = 0$, Theorem 3 holds true for the functions $y_1 = e^x$, $y_2 = e^{2x}$ and $y_3 = e^{3x}$ on the interval $]-\infty, \infty[$.

Solution: You can easily check that functions e^x, e^{2x} and e^{3x} all satisfy the given homogeneous equation on $]-\infty, \infty[$. By Theorem 3, the function $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$ should also satisfy the given equation for arbitrary constants c_1, c_2 and c_3.

Now $y' = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}$

$$y'' = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}$$
$$y''' = c_1e^x + 8c_2e^{2x} + 27c_3e^{3x}$$

Substituting the above values of y, y', y'' and y''' in the given equation, we get

$$
\begin{aligned}
&y''' - 6y'' + 11y' - 6y \\
&= c_1 e^x + 8c_2 e^{2x} + 27c_3 e^{3x} - 6(c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}) \\
&+ 11(c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x}) - 6(c_1 e^x + c_2 e^{2x} + c_3 e^{3x}) \\
&= 0
\end{aligned}
$$

Thus $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$ is also a solution of the given equation on $]-\infty, \infty[$. Theorem 3 holds true for the functions in the given problem.

***

Have you noticed that **a homogeneous linear differential equation always possesses the trivial solution $y = 0$?** If not, you can check it for Examples 6 and 7 above.

Theorem 3 and corollary 2 represent properties that non-linear differential equations, in general, do not possess. This will become more clear to you after you have done the following exercises.

E9) Prove Corollary 1 for Eqn. (5).

E10) Verify that $y = 1/x$ is a solution of the non-linear differential equation $y'' = 2y^3$ on the interval $]\,0, \infty\,[$, but the constant multiple $y = c/x$ is not a solution of the equation except when $c = 0$, and $c = \pm 1$.

E11) Verify that the functions $y_1 = 1$ and $y_2 = \ln x$ are solutions of the non-linear differential equation $y'' + (y')^2 = 0$ on the interval $]\,0, \infty\,[$. Further, check whether $c_1 y_1 + c_2 y_2$ is a solution of the equation, for arbitrary constants c_1 and c_2 or not.

As we mentioned earlier there are functions which are linearly independent on an interval and yet have a vanishing Wronskian there (ref. E3)). We shall now give you a theorem, which is a stronger version of Theorem 1 and which asserts that this would not be the case if the functions involved, are the solutions of some homogeneous linear equations.

You may observe that in Example 7 functions $y_1 = e^x,\ y_2 = e^{2x},\ y_3 = e^{3x}$ are the solutions of linear homogeneous equation $y''' - 6y'' + 11y' - 6y = 0$ and are linearly independent on $]-\infty, \infty[$. For any non-zero c_1, c_2 and c_3, $c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0$ for $-\infty < x < \infty$ only when $c_1 = 0, c_2 = 0$ and $c_3 = 0$. Also, $W(y_1, y_2, y_3) \neq 0$, since we have

$$
W(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}
$$

$$
\begin{aligned}
&= e^x(18e^{5x} - 12e^{5x}) \\
&- e^{2x}(9e^{4x} - 3e^{4x}) \\
&+ e^{3x}(4e^{3x} - 2e^{3x}) \\
&= 6e^{6x} - 6e^{6x} + 2e^{6x} = 2e^{6x} \neq 0 \text{ for } -\infty < x < \infty .
\end{aligned}
$$

The result above holds true in general. We shall now show that if the functions involved are the solutions of linear homogeneous equation then the non-vanishing of the Wronskain of a set of such solutions/functions on an interval I is both **necessary** and **sufficient** for linear independence of the functions on I.

Let us consider the following theorem.

Theorem 4: Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear n^{th}-order differential Eqn. (4) on an interval I. Then the set of solutions is linearly independent on I **if and only if**

$$W(y_1, y_2, \ldots, y_n) \neq 0$$

for every x in the interval.

Proof: We prove Theorem 4 for the case $n = 2$ by considering Eqn. (5).
The first part of the proof, that is, if $W(y_1, y_2) \neq 0$ for every x in I, then y_1 and y_2 are linearly independent, immediately follows from Theorem 1. Now we need to show that if the two solutions y_1 and y_2 of Eqn. (5) are linearly independent on I, then $W(y_1, y_2) \neq 0$ for every x in I. We will prove this by contradiction.
Let y_1 and y_2 be linearly independent and there is some fixed x_0 in I for which $W(y_1, y_2)\,(x_0) = 0$.
Since $W(y_1, y_2)\,(x_0) = 0$, the system of equations

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= 0 \end{aligned} \tag{8}$$

for c_1 and c_2 has a non-trivial solution (ref. Appnedix given at the end of the unit). Using these values of c_1 and c_2, let us define $y(x) = c_1 y_1(x) + c_2 y_2(x)$. Then y is also a solution of Eqn. (5) and by Eqn. (8) y also satisfies the initial conditions

$$y(x_0) = 0, \; y'(x_0) = 0 \tag{9}$$

But we also know that the identically zero function $f(x) = 0 \; \forall x$ in I satisfies both the differential equation and the initial conditions (9). Thus by the uniqueness part of Theorem 2, f is the only solution, i.e., $y = f$.
In other words, for constants c_1 and c_2, not both zero,

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

for every x in I. This contradicts the assumption that y_1 and y_2 are linearly independent on the interval. This proves the second part of the theorem.

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From the above discussion we conclude that when y_1 and y_2 are the two solutions of Eqn. (5) on an interval I, then either the Wronskian of y_1 and y_2 is identically zero or is never zero on the interval.

The solutions y_1 and y_2 of Eqn.(5), with a non-zero Wronskian, are said to form a fundamental set of solutions of Eqn. (5). More generally, we give the following definition.

Definition: A set of n linearly independent solutions $y_1, y_2, \ldots, y_n$ of the homogeneous, n^{th}-order, linear differential Eqn. (4) on an interval I is said to be a **fundamental set of solutions** of the differential equation on I.

In Example 7, functions y_1, y_2 and y_3 form a fundamental set of solutions of the given equation on the interval $]-\infty, \infty[$ since $W(y_1, y_2, y_3) = 2e^{6x} \neq 0$ for $-\infty < x < \infty$. Similarly, functions x^2 and $x^2 \ln x$ in Example 6 form a fundamental set of solutions of the given equation as $W(x^2, x\ln x) = x^3 \neq 0$ in $]0, \infty[$.

As yet another example, consider the equation

$$y'' - 4y = 0, \quad -\infty < x < \infty \tag{10}$$

You can easily check that both $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$ are the solutions of Eqn. (10) on $]-\infty, \infty[$ and thus their linear combination

$$y = c_1 e^{2x} + c_2 e^{-2x} \tag{11}$$

for arbitrary c_1 and c_2 is also a solution of Eqn. (10) on the interval.
Also since $W(y_1, y_2)(x) = -4 \neq 0$, y_1 and y_2 for every value of x form a fundamental set of solutions of Eqn. (10). Further, it can be checked that the function $y = 4\cosh 2x + 3e^{2x}$ is another solution of Eqn. (10) in the interval.
We have $y' = 8\sinh 2x + 6e^{2x}$, $y'' = 16\cosh 2x + 12e^{2x}$ and thus
$y'' - 4y = 16\cosh 2x + 12e^{2x} - 4(4\cosh 2x + 3e^{2x}) = 0$. It is interesting to note here that the solution $y = 4\cosh 2x + 3e^{2x}$ of Eqn. (10) can be determined from the solution (11) by considering suitable values of the constants c_1 and c_2.
For $c_1 = 5$ and $c_2 = 2$, we can write Eqn. (11) as

$$\begin{aligned} y &= 5e^{2x} + 2e^{-2x} \\ &= 2e^{2x} + 2e^{-2x} + 3e^{2x} \\ &= 4\left(\frac{e^{2x} + e^{-2x}}{2}\right) + 3e^{2x} \\ &= 4\cosh 2x + 3e^{2x}. \end{aligned}$$

This leads us to a property which relates any solution of Eqn. (4) to its fundamental set of solutions.

Theorem 5: Let $\{y_1, y_2, \ldots, y_n\}$ be a fundamental set of solutions of the n^{th}-order, homogeneous, linear differential Eqn. (4) on an interval I. Then, for any solution $Y(x)$ of Eqn. (4) constants $c_1, c_2, \ldots, c_n$ can be determined so that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x).$$

Proof: As usual, let us prove it for $n = 2$. Consider Eqn. (5) i.e.,

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

corresponding to the case $n = 2$.
Let $\{y_1, y_2\}$ form the fundamental set of solutions of the equation on the interval I.

Let $Y(x)$ be any solution of the equation on the interval I. Being a fundamental set of solutions y_1 and y_2 are linearly independent and by

Theorem 4, $W(y_1, y_2)(x) \neq 0$ for every x in I. Let x_0 be one such point in this interval for which $W(y_1, y_2)(x_0) \neq 0$ and that the values of $Y(x_0)$ and $Y'(x_0)$ are given by

$$Y(x_0) = k_1,\ Y'(x_0) = k_2.$$

Now consider the system of non-homogeneous equations

$$c_1 y_1(x_0) + c_2 y_2(x_0) = k_1$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = k_2,$$

Constants c_1 and c_2 can be uniquely determined provided the determinant of the coefficient matrix is non zero (ref. Appendix at the end of the unit), that is,

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

which is true as this determinant is the Wronskian $W(y_1, y_2)(x_0)$, which is assumed to be non-zero. Thus if we define the function $F(x)$ by

$$F(x) = c_1 y_1(x) + c_2 y_2(x)$$

then we observe:

i) $F(x)$ satisfies the differential equation since it is the superposition of two known solutions y_1 and y_2.

ii) $F(x)$ satisfies the initial conditions

$$F(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) = k_1$$
$$F'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0) = k_2.$$

iii) $Y(x)$, as per our assumptions, also satisfy the same linear equation and the same initial conditions.

Since the solution of linear initial value problem is unique (ref. Theorem 2), we have $Y(x) = F(x)$, or

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

which completes the proof.

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We can thus conclude from Theorem 5 that any solution of Eqn. (5) is obtained from a linear combination of functions in the fundamental set of solutions of the equation. Solution $y = c_1 y_1 + c_2 y_2$ of second order Eqn. (5), involving two arbitrary constants c_1 and c_2, is called its general solution or complete integral. In order to find the general solution, and therefore all solutions, of the second order equation of the form (5), we need only to find two solutions of the given equation whose Wronskian is non-zero.

We now give the following definition.

Definition: Let $y_1, y_2, \ldots, y_n$ form a fundamental set of solutions of the homogeneous linear n^{th}-order differential Eqn. (4) on an interval I. Then

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_i,\ i = 1, 2, \ldots, n$ are arbitrary constants is defined to be the **general solution** or the **complete integral** of the equation on I.

For instance, the second order equation $y'' - y' - 12y = 0$ possesses two solutions

$y_1 = e^{4x}$ and $y_2 = e^{-3x}$ on $]-\infty, \infty[$

Since $W(e^{4x}, e^{-3x}) = \begin{vmatrix} e^{4x} & e^{-3x} \\ 4e^{4x} & -3e^{-3x} \end{vmatrix} = (-3e^{x} - 4e^{x}) = -7e^{x} \neq 0$

for every value of x, y_1 and y_2 form a fundamental set of solutions on $]-\infty, \infty[$. The general solution of the differential equation on the interval is

$$y = c_1\, e^{4x} + c_2\, e^{-3x}.$$

Let us consider some more examples.

Example 8: Show that $y_1(x) = \sin x,\ y_2(x) = \cos x,\ y_3(x) = x,\ y_4(x) = 1$ form a fundamental set of solutions of the equation

$$y^{IV} + y'' = 0 \text{ for } -\infty < x < \infty.$$

Also write the general solution of the equation.

Solution: Consider $y_1(x) = \sin x$, we have

$$y_1'(x) = \cos x,\ y_1''(x) = -\sin x,\ y_1'''(x) = -\cos x,\ y^{IV}(x) = \sin x.$$

Substituting these values in the given equation, we get

$$y^{IV} + y'' = \sin x - \sin x = 0$$

thus $y_1(x)$ is a solution of the given equation.

Similarly, you can check that $y_2(x),\ y_3(x)$ and $y_4(x)$ also satisfy the given equation and hence are the solutions of it.

To check the linear independence of the solutions $y_1(x),\ y_2(x),\ y_3(x)$ and $y_4(x)$ consider

$$W(\sin x, \cos x, x, 1) = \begin{vmatrix} \sin x & \cos x & x & 1 \\ \cos x & -\sin x & 1 & 0 \\ -\sin x & -\cos x & 0 & 0 \\ -\cos x & \sin x & 0 & 0 \end{vmatrix}$$

$$= \sin x.0 - \cos x.0 + x.0 - 1(-\sin^2 x - \cos^2 x)$$

$$= 1 \neq 0\ \forall x \in]-\infty, \infty[$$

Thus y_1, y_2, y_3 and y_4 form a fundamental set of solutions on $]-\infty, \infty[$. The general solution of the differential equation on the interval is

$$y = c_1 \sin x + c_2 \cos x + c_3 x + c_4.$$

***

Example 9: Show that $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ form a fundamental set of solutions, of the equation

$$y'' - 9y = 0 \text{ for } -\infty < x < \infty.$$

Further verify Theorem 5 for the solution $y = 4\sinh 3x - 5e^{-3x}$ of the given equation.

Solution: We have $y_1' = 3e^{3x},\ y_1'' = 9e^{3x}$

$$\therefore\ y_1'' - 9y_1 = 9e^{3x} - 9e^{3x} = 0.$$

Thus y_1 is a solution of the given equation. Similarly,

$$y_2'' - 9y_2 = 9e^{-3x} - 9e^{-3x} = 0.$$

Further, $W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$.

Therefore, for every value of x, y_1 and y_2 form a fundamental set of solutions on $]\infty, \infty[$. The general solution of the given equation on the interval is

$y = c_1 e^{3x} + c_2 e^{-3x}$, c_1 and c_2 are constants.

Now consider $y = 4\sinh 3x - 5e^{-3x}$. We have

$y'' - 9y = 36\sinh 3x - 45e^{-3x} - 9(4\sinh 3x - 5e^{-3x}) = 0$

Thus y also satisfy the given equation. Choosing $c_1 = 2$ and $c_2 = -7$ in the general solution $y = c_1 e^{3x} + c_2 e^{-3x}$, we obtain

$$\begin{aligned} y &= 2e^{3x} - 7e^{-3x} \\ &= 2e^{3x} - 2e^{-3x} - 5e^{-3x} \\ &= 4\left(\frac{e^{3x} - e^{-3x}}{2}\right) - 5e^{-3x} = 4\sinh 3x - 5e^{-3x} \end{aligned}$$

Thus, Theorem 5 holds true for the solution $y = 4\sinh 3x - 5e^{-3x}$ of the equation.

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You may now try the following exercises.

E12) Show that $y_1(x) = x^{1/2}$ and $y_2(x) = x^{-1}$ form a fundamental set of solutions of the equation

$$2x^2 y'' + 3xy' - y = 0, \ x > 0.$$

Also write the general solution of the equation.

E13) Show that the functions $y_1(x) = e^x$, $y_2(x) = e^{2x}$ and $y_3(x) = e^{3x}$ form a fundamental set of solutions of the equation

$$y''' - 6y'' + 11y' - 6y = 0, \ -\infty < x < \infty.$$

Also write the general solution of the equation.

E14) Show that $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on any interval. Find a particular solution for which $y(0) = 2$ and $y'(0) = 3$.

So far we have discussed the properties pertaining to the solutions of homogeneous linear equations. We now turn our attention to the **non-homogeneous linear equations of the form (1)**.

Consider a non-homogeneous equation

$$y'' - 9y = 3x^2 \tag{12}$$

The function $y_p(x) = -\frac{x^2}{3} - \frac{2}{27}$ satisfies the non-homogeneous Eqn. (12) since we have $y_p' = -\frac{2x}{3}$, $y_p'' = -\frac{2}{3}$ and

$$y_p'' - 9y_p = -\frac{2}{3} - 9\left(-\frac{x^2}{3} - \frac{2}{27}\right) = -\frac{2}{3} + 3x^2 + \frac{2}{3} = 3x^2.$$

Thus y_p is a solution of non-homogeneous Eqn. (12).

Also from Example 9, we know that the general solution of the associated homogeneous equation $y''-9y=0$ is $y=c_1e^{3x}+c_2e^{-3x}$ for c_1 and c_2 being arbitrary constants.

You can now check that $y=c_1e^{3x}+c_2e^{-3x}-\dfrac{x^2}{3}-\dfrac{2}{7}$ which is the sum of the solution of the non-homogeneous equation and the solution of its corresponding homogeneous equation is also a solution of Eqn. (12). We have

$$y''-9y=9c_1e^{3x}+9c_2e^{-3x}-\frac{2}{3}-9\left(c_1e^{3x}+c_2e^{-3x}-\frac{x^2}{3}-\frac{2}{27}\right)$$

$$=3x^2.$$

Thus $y=c_1e^{3x}+c_2e^{-3x}+y_p(x)$ satisfy Eqn. (12).

This example leads us to the following theorem which defines the general solution of n^{th}-order non-homogeneous linear Eqn. (1).

Theorem 6: If $y_p(x)$ is any solution of the non-homogeneous linear differential Eqn. (1) on an interval I and if $y_1, y_2, \ldots, y_n$ be the linearly independent solutions of the corresponding homogeneous linear differential Eqn. (4) on the interval, then

$$y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)+y_p(x)$$

is also a solution of the non-homogeneous Eqn. (1) on the interval I for any constants $c_1, c_2, \ldots, c_n$.

The proof of the theorem is simple and we are leaving it for you to do it yourself.

E15) Prove Theorem 6 for the second order equation i.e., corresponding to case $n=2$.

From the above discussion and referring to Example 9, we can now say that

$$y=4\sinh 3x-5e^{-3x}-\frac{x^2}{3}-\frac{2}{27}$$

is another solution of Eqn. (12). Following the procedure shown in Example 9, this solution can be obtained from the solution

$$y=c_1e^{3x}+c_2e^{-3x}+y_p(x) \text{ where, } y_p(x)=\frac{-x^2}{3}-\frac{2}{27},$$

by suitably choosing the constants c_1 and c_2. In this case $c_1=2$ and $c_2=-7$ serves our purpose.

We can thus consider the following analogue of Theorem 5 for non-homogeneous differential equations.

Theorem 7: Let y_p be a given solution of the non-homogeneous linear n^{th}-order differential Eqn. (1) on an interval I and let $y_1, y_2, \ldots, y_n$ form a fundamental set of solutions of the associated homogeneous Eqn. (4) on the interval. Then for any solution $Y(x)$ of Eqn. (1) on I, constants $c_1, c_2, \ldots, c_n$ can be determined so that

$$Y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)+y_p(x).$$

Proof: We prove the case when $n = 2$. Suppose Y and y_p are the solutions of the second order non-homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = b(x) \tag{13}$$

and y_1, y_2 form a fundamental set of solutions of the homogeneous equation corresponding to Eqn. (13). Let $u(x)$ be a function such that $u(x) = Y(x) - y_p(x)$. Then

$$\begin{aligned} &a_2(x)u'' + a_1(x)u' + a_0(x)u \\ &= a_2(x)[Y'' - y_p''] + a_1(x)[Y' - y_p'] + a_0(x)[Y - y_p] \\ &= a_2(x)Y'' + a_1(x)Y' + a_0(x)Y - [a_2(x)y_p'' + a_1(x)y_p' + a_0(x)y_p] \\ &= b(x) - b(x) = 0 \end{aligned}$$

Therefore $u(x)$ is a solution of the homogeneous equation corresponding to Eqn. (13). By Theorem 5, we can write

$u(x) = c_1y_1(x) + c_2y_2(x)$ for constants c_1 and c_2.

$\therefore\ Y(x) - y_p(x) = c_1y_1(x) + c_2y_2(x)$

or, $Y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$

which completes the proof.

– ■ –

With this background, we can now have the following definition of the general solution of non-homogeneous differential Eqn. (1).

Definition: Let $y_p(x)$ be a given solution of the n^{th}-order non-homogeneous linear differential Eqn. (1) on an interval I and let

$$y_c = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

be the general solution of its associated homogeneous Eqn. (4). Then the **general solution** of the non-homogeneous Eqn. (1) on the interval is defined to be

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x) \\ &= \boldsymbol{y_c(x) + y_p(x)}. \end{aligned}$$

In the above definition the linear combination

$$y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

involving n arbitrary constants is the general solution of Eqn. (4) and is called **complementary function** of Eqn. (1). The solution $y_p(x)$ of Eqn. (1) which is free of arbitrary parameters is called a **particular integral/solution** of Eqn. (1). Thus, the **complete solution/general solution** of the non-homogeneous Eqn. (1) is given by

$\boldsymbol{y} =$ **complementary function + any particular integral**

For example, $y_p = x^3 - x$ is a particular solution of the differential equation

$$x^2y'' + 2xy' - 8y = 4x^3 + 6x$$

since $y_p' = 3x^2 - 1,\ y_p'' = 6x$ and

$$\begin{aligned} x^2y_p'' + 2xy_p' - 8y_p &= x^2(6x) + 2x(3x^2 - 1) - 8(x^3 - x) \\ &= 4x^3 + 6x \end{aligned}$$

Similarly, you can check that the function

$$y_p = \frac{-11}{12} - \frac{1}{2}x$$

is a particular solution of the non-homogeneous differential equation

$$y''' - 6y'' + 11y' - 6y = 3x, \ -\infty < x < \infty.$$

Further, from E13) you know that the general solution of the homogeneous equation corresponding to the above equation in the interval $]-\infty, \infty[$ is

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

Hence the general solution of the given non-homogeneous equation in the interval $]-\infty, \infty[$ is

$$y = y_c + y_p$$
$$= c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x.$$

After having gone through the above discussion a natural question which may come to your mind is – How to find the complementary function and a particular integral of Eqn. (1)?

In the next section we give you the methods of finding the complementary function $y_c(x)$ of the linear equation with constant coefficients and discuss various methods of finding a particular solution in the units to follow.

10.4 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Euler (1707-1783)

Since the complementary function refer to the solution of the homogeneous equation, we begin the discussion by considering the second order linear equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \ -\infty < x < \infty, \tag{14}$$

where the coefficients a, b and c are real constants.

The method of solving Eqn. (14) was given in the year 1739 by Leonhard Euler (1707-1783) who was born in Basel, Switzerland and was one of the most distinguished mathematicians of the eighteenth century.

Before taking up Eqn. (14), let us look at a simple example. Consider the equation

$$y'' - y = 0 \tag{15}$$

which is of the form (14) with $a = 1, b = 0$ and $c = -1$. In words, Eqn. (15) says that we look for a function y with the property that the second derivative of the function is the same as the function itself i.e., $y'' = y$. From your knowledge of calculus you can think of at least one well known function with this property, namely, $y_1(x) = e^x$, the exponential function. Other function coming to your mind would be $y_2(x) = e^{-x}$. Further, you can say that the multiples of these two solutions are also the solutions. For example, the functions $4e^x$ and $7e^{-x}$ also satisfy Eqn. (15). In the same way, functions $c_1 y_1(x) = c_1 e^x$ and $c_2 y_2(x) = c_2 e^{-x}$ satisfy the differential Eqn. (15) for all values of the constants c_1 and c_2. Further, their linear combination i.e., $y = c_1 y_1(x) + c_2 y_2(x) = c_1 e^x + c_2 e^{-x}$ is also the solution of Eqn. (15).

Let us now come back to our general second order Eqn. (14) having arbitrary (real) constant coefficients. Based on our experience with Eqn. (15) we assume that $y = e^{mx}$ is a solution of Eqn. (14) for some number m. On putting the values of y and its derivatives in Eqn. (14), we get

$$(am^2 + bm + c)e^{mx} = 0 \tag{16}$$

Since $e^{mx} \neq 0$ for real values of x, Eqn. (16) is satisfied if

$$am^2 + bm + c = 0 \tag{17}$$

Eqn. (17) is called the **auxiliary equation** or **characteristic equation** corresponding to differential Eqn. (14).

Let m_1 and m_2 be the two roots of the auxiliary Eqn. (17). Then the following three possibilities arise:

I) Roots of the auxiliary equation may be **real** and **distinct**,

II) Roots of the auxiliary equation may be **real**, but **some of the roots may be repeated**.

III) The auxiliary equation may have **complex roots**.

We now proceed to find the solution of Eqn. (17) for these three possibilities one by one.

Case I: Auxiliary equation has real and distinct roots:

Let the roots m_1 and m_2 of the auxiliary Eqn. (14) be real and distinct.

Since m_1 is a root of Eqn. (17), $y_1 = e^{m_1 x}$ is a solution of Eqn. (14) on the interval $]-\infty, \infty[$. Similarly, $y_2 = e^{m_2 x}$ is a solution of Eqn. (14). Also, $e^{m_1 x}$ and $e^{m_2 x}$ are linearly independent on the interval, since

$$W(e^{m_1 x}, e^{m_2 x}) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= (m_2 - m_1)\, e^{(m_1 + m_2)x} \neq 0 \text{ for } m_1 \neq m_2$$

Thus the solutions y_1 and y_2 form a fundamental solution set of Eqn. (14) on the interval $]-\infty, \infty[$ and its general solution can then be expressed as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x},$$

where c_1 and c_2 are arbitrary constants.

For an n^{th} order equation having n distinct and real roots $m_1, m_2, \ldots, m_n$ the general solution is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} \tag{18}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

We now illustrate this case for differential equations of various orders.

Example 10: Solve $2\dfrac{d^2 y}{dx^2} + 5\dfrac{dy}{dx} - 12y = 0$

Solution: The auxiliary equation for the given equation is

$$2m^2 + 5m - 12 = 0$$

$\Rightarrow \quad (2m-3)\ (m+4)=0$

$\Rightarrow \quad m=3/2, -4$

The roots are real and distinct.
Hence the complete solution of the given differential equation is
$y = c_1 e^{(3/2)x} + c_2 e^{-4x}$, where c_1 and c_2 are arbitrary constants.

***

Let us look at another example.

Example 11: Consider the equation $\frac{d^2y}{dx^2} - a^2 y = 0$, where a is a non-zero constant. Show that $y = A\cosh ax + B\sinh ax$ is the complete solution of the given equation.

Solution: The auxiliary equation corresponding to the given differential equation is

$$m^2 - a^2 = 0$$

$\Rightarrow \quad (m-a)\ (m+a)=0$

$\Rightarrow \quad m=a, -a$.

Roots being real and distinct, the general solution of the given equation is

$$y = c_1 e^{ax} + c_2 e^{-ax}$$

From the definition of hyperbolic functions, we know that

$$\cosh ax = \frac{1}{2}\ (e^{ax} + e^{-ax}) \qquad (19)$$

and $\quad \sinh ax = \frac{1}{2}\ (e^{ax} - e^{-ax}) \qquad (20)$

Adding relations (19) and (20), we get

$$e^{ax} = \cosh ax + \sinh ax$$

Subtracting relation (20) from (19), we get

$$e^{-ax} = \cosh ax - \sinh ax\ .$$

Using these expressions for e^{ax} and e^{-ax}, the general solution of given differential equation can be written as

$$y = c_1(\cosh ax + \sinh ax) + c_2(\cosh ax - \sinh ax)$$

$\Rightarrow \quad y = (c_1 + c_2)\ \cosh ax + (c_1 - c_2)\ \sinh ax$

$\Rightarrow \quad y = A\cosh ax + B\sinh ax$,

where $A = c_1 + c_2$ and $B = c_1 - c_2$ are two arbitrary constants.

***

Let us now consider the initial value problems.

Example 12: Solve the equation

$$\frac{d^2x}{dt^2} - 4x = 0$$

with the conditions that when $t=0,\ x=0$ and $\frac{dx}{dt} = 3$.

Solution: The auxiliary equation corresponding to the given equation is

$$m^2 - 4 = 0$$

$\Rightarrow \quad m = 2, -2$

Hence the general solution of the differential equation is

$$x = c_1 e^{2t} + c_2 e^{-2t}.$$

We now apply the given conditions at $t = 0$.

We have

$$\frac{dx}{dt} = 2c_1 e^{2t} - 2c_2 e^{-2t}$$

Condition that $x = 0$ when $t = 0$ gives

$$0 = c_1 + c_2,$$

and the condition that $\frac{dx}{dt} = 3$ when $t = 0$ gives

$$3 = 2c_1 - 2c_2$$

From the two equations for c_1 and c_2, we conclude that

$c_1 = \frac{3}{4}$ and $c_2 = -\frac{3}{4}$. Therefore,

$$x = \frac{3}{4}(e^{2t} - e^{-2t})$$

which can also be put in the form $x = \frac{3}{2}\sinh 2t$.

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Example 13: Solve the equation

$$\frac{d^3y}{dx^3} + 13\frac{d^2y}{dx^2} + 36\frac{dy}{dx} = 0$$

under the conditions that at $x = 0,\ y = 0,\ \frac{dy}{dx} = 1$ and $\frac{d^2y}{dx^2} = -7$.

Solution: The auxiliary equation corresponding to the given equation is

$$m^3 + 13m^2 + 36m = 0$$

$\Rightarrow \quad m(m+4)\,(m+9) = 0$

$\Rightarrow \quad m = 0, -4, -9.$

Hence the general solution of the differential equation is

$$y = c_1 + c_2 e^{-4x} + c_3 e^{-9x}$$

Now we find the values of c_1, c_2 and c_3 by applying conditions at $x = 0$.

We have

$$\frac{dy}{dx} = -4c_2 e^{-4x} - 9c_3 e^{-9x} \text{ and}$$

$$\frac{d^2y}{dx^2} = 16c_2 e^{-4x} + 81c_3 e^{-9x}$$

Condition that $y = 0$ when $x = 0$ gives

$$0 = c_1 + c_2 + c_3 \qquad (21)$$

$\frac{dy}{dx} = 1$ at $x = 0$ gives

$$1 = -4c_2 - 9c_3 \qquad (22)$$

$\frac{d^2y}{dx^2} = -7$ at $x = 0$ gives

$$-7 = 16c_2 + 81c_3 \qquad (23)$$

Solving Eqns. (21), (22) and (23) for c_1, c_2 and c_3, we get

$$c_1 = \frac{1}{6},\ c_2 = -\frac{1}{10} \text{ and } c_3 = -\frac{1}{15}.$$

Thus the general solution of the given equation is

$$y = \frac{1}{6} - \frac{1}{10}e^{-4x} - \frac{1}{15}e^{-9x}.$$

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Let us consider a fourth order equation.

Example 14: Solve the equation

$$\frac{d^4y}{dx^4} - 10\frac{d^3y}{dx^3} + 35\frac{d^2y}{dx^2} - 50\frac{dy}{dx} + 24y = 0.$$

Solution: The auxiliary equation corresponding to the given equation is

$$m^4 - 10m^3 + 35m^2 - 50m + 24 = 0$$

$$\Rightarrow \quad (m-1)\,(m^3 - 9m^2 + 26m - 24) = 0$$

$$\Rightarrow \quad (m-1)\,(m-2)\,(m^2 - 7m + 12) = 0$$

$$\Rightarrow \quad (m-1)\,(m-2)\,(m-3)\,(m-4) = 0$$

$$\Rightarrow \quad m = 1, 2, 3, 4$$

Hence the general solution of the given equation is

$$y = c_1e^{x} + c_2e^{2x} + c_3e^{3x} + c_1e^{4x}.$$

***

You may now try the following exercises.

E16) Solve the following differential equations:

i) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0.$

ii) $9\frac{d^2y}{dx^2} + 18\frac{dy}{dx} - 16y = 0$

iii) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$

E17) Solve the following differential equations under the conditions given alongside:

i) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$; when $x = 0,\ y = 4$ and $y' = 0$

ii) $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 0$ when $x = 0,\ y = 0,\ y' = 0$ and $y'' = 2$.

We now take up the case when the roots of the auxiliary Eqn. (17) are real but repeated.

Case II: Auxiliary Equation has real and repeated roots:

Let the roots m_1, m_2 of the auxiliary Eqn. (17) be equal, i.e, $m_1 = m_2$. This case occurs when the discriminant $b^2 - 4ac$ of the quadratic Eqn. (17) is zero and the two equal roots are $m_1 = m_2 = -b/2a$. Both these roots yield the same solution i.e., $y_1(x) = e^{-bx/2a} = y_2(x)$.

The solution of Eqn. (17) in this case becomes $y(x) = (c_1 + c_2)e^{-bx/2a}$. Since the constants $(c_1 + c_2)$ can be replaced by a single constant, this solution practically involves only one arbitrary constant. But we know that the general or complete solution of the 2[nd] order linear differential equation must contain two arbitrary constants. Hence the above solution having one arbitrary constant is not the general solution. We thus need to find a second solution of Eqn. (14).

Let us take up an example to understand the method of finding a second solution in the case of repeated roots.

Example 15: Solve the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

Solution: The auxiliary equation of the given equation is

$$m^2 + 4m + 4 = 0$$

$$\Rightarrow \quad m = -2, -2$$

One solution of the given equation is $y_1(x) = e^{-2x}$. To find its general solution we need a second linearly independent solution that is not a multiple of y_1. The method of finding this second solution was given by D'Alembert (1717-1783), a French physicist and mathematician, in the eighteenth century. You already know from Corollary 2 that if $y_1(x)$ is a solution of Eqn. (14) then so is $cy_1(x)$ for any constant c. The basic idea is to generalise this observation by replacing c by a function $v(x)$ and then try to determine $v(x)$ so that the product $v(x)y_1(x)$ is a solution of Eqn. (14).

Let us substitute $y = v(x)y_1(x) = v(x)e^{-2x}$, in the given equation.

We know that $y' = v'(x)e^{-2x} - 2v(x)e^{-2x}$ and

$y'' = v''(x)e^{-2x} - 4v'(x)e^{-2x} + 4v(x)e^{-2x}$.

Substituting for y, y' and y'' in the given equation, we obtain

$$[v''(x) - 4v'(x) + 4v(x) + 4v'(x) - 8v(x) + 4v(x)]e^{-2x} = 0$$

which simplifies to

$$v''(x) = 0.$$

Therefore

$$v'(x) = c_1$$

or, $\quad v(x) = c_1 x + c_2$.

Thus, we get

$$y(x) = v(x)y_1(x) = c_1 xe^{-2x} + c_2 e^{-2x} \tag{24}$$

where c_1 and c_2 are arbitrary constants. The second term on the right hand side of Eqn. (24) corresponds to the original solution $y_1(x) = e^{-2x}$, but the first term arises from a second solution, namely, $y_2(x) = xe^{-2x}$. You can easily check that these two solutions are linearly independent as

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix} = e^{-4x} - 2xe^{-4x} + 2xe^{-4x} = e^{-4x} \neq 0.$$

Therefore $y_1(x) = e^{-2x}$ and $y_2(x) = xe^{-2x}$ form a fundamental set of solutions of the given equation and its general solution is given by Eqn. (24).

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The procedure used in Example 15 when extended to Eqn. (14) yield two solutions

$$y_1(x) = e^{-bx/2a} = e^{m_1 x} \text{ and } y_2(x) = xe^{-bx/2a} = xe^{m_1 x}$$

Since $W(y_1, y_2)(x) = e^{-bx/a} \neq 0$, the solutions y_1 and y_2 are a fundamental set of solutions of Eqn. (14). Thus in the case of equal roots the general solution of Eqn. (14) is given by

$$y = (c_1 + c_2 x)e^{-bx/2a}$$
$$= (c_1 + c_2 x)e^{m_1 x}$$

c_1, c_2 being constants.

In words, we say that in the case of repeated roots of Eqn. (14), there is one exponential solution corresponding to the repeated root, while a second solution is obtained by multiplying the exponential solution by x. The method given above for finding the second solution is known as the method of **reduction of order**. This method is applicable to linear homogeneous equations with variable coefficients as well. We shall be discussing the method later in Unit 12.

For the n^{th} order equation having n roots $m_1, m_2, \ldots, m_n$, if a root m_1 is repeated r times and the remaining $(n-r)$ roots are distinct then the solution corresponding to this r times repeated root will be of the form

$$Y = e^{m_1 x}(A_1 + A_2 x + A_3 x^2 + \cdots + A_r x^{r-1}) \tag{25}$$

and the general solution will then be of the form

$$y = e^{m_1 x}(A_1 + A_2 x + A_3 x^2 + \cdots + A_r x^{r-1}) + A_{r+1}e^{m_{r+1}x} + \cdots + A_n e^{m_n x} \tag{26}$$

where $A_1, A_2, \ldots, A_n$ are arbitrary constants.

We now illustrate the above discussion with the help of following examples.

Example 16: Solve $\dfrac{d^4 y}{dx^4} - \dfrac{d^3 y}{dx^3} - 9\dfrac{d^2 y}{dx^2} - 11\dfrac{dy}{dx} - 4\ y = 0$.

Solution: The auxiliary equation of the given differential equation is

$$m^4 - m^3 - 9m^2 - 11m - 4 = 0$$

$$\Rightarrow \quad (m+1)(m^3 - 2m^2 - 7m - 4) = 0$$

$$\Rightarrow \quad (m+1)\ (m+1)\ (m^2 - 3m - 4) = 0$$

$$\Rightarrow \quad m = -1, -1, -1, 4$$

Here the root -1 is repeated three times and the root 4 is distinct. Using Eqn. (25), the solution corresponding to three times repeated root -1 can be written as

$$Y = (A + Bx + Cx^2)\ e^{-x}$$

and the general solution from Eqn. (26) will then be

$$y = (A + Bx + Cx^2)\ e^{-x} + De^{4x}$$

where A, B, C and D are arbitrary constants.

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Example 17: Find the complete solution of the differential equation

$$\frac{d^4y}{dx^4} - 8\frac{d^2y}{dx^2} + 16y = 0$$

Solution: In this case the auxiliary equation is

$$m^4 - 8m^2 + 16 = 0$$

$$\Rightarrow \quad (m-2)\ (m^3 + 2m^2 - 4m - 8) = 0$$

$$\Rightarrow \quad (m-2)\ (m-2)\ (m^2 + 4m + 4) = 0$$

$$\Rightarrow \quad m = 2,\ 2,\ -2,\ -2$$

Here the roots 2 and -2 are both repeated. The solution corresponding to the repeated root 2 can be written as

$$Y_1 = (A + Bx)\, e^{2x}$$

and that corresponding to the repeated root -2 can be written as

$$Y_2 = (C + Dx)\ e^{-2x}.$$

The two together give the complete solution of the given differential equation as

$$y = (A + Bx)\ e^{2x} + (C + Dx)\ e^{-2x}.$$

where A, B, C and D are arbitrary constants.

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And now some exercises for you.

E18) Find the complete solution of the following equations:

i) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0.$

ii) $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

iii) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$

iv) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

E19) Find the solution of the following equations subject to the conditions mentioned alongside:

i) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 4y = 0$; when $x = 0,\ y = 1$ and $y' = -1$

ii) $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} - 2y = 0$; when $x = 0,\ y = 0,\ y' = 9$ and $y'' = 0$.

iii) $\frac{d^4y}{dx^4} + 3\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 0$; when
$x = 0,\ y = 0,\ y' = 4,\ y'' = -6,\ y''' = 14$.

Now we shall discuss the case when the auxiliary Eqn. (17) may have the complex roots.

Case III: Auxiliary Equation has complex roots:

Let the roots m_1, m_2 of Eqn. (17) are complex. We know from the theory of equations, that if all the coefficients of a polynomial equation are real, then its complex roots occur in conjugate pairs. In Eqn. (17), the coefficients are assumed to be real constants and hence the complex roots must occur in conjugate pairs.

Let the complex roots of Eqn. (17) be $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where α and β are real and $i^2 = -1$. Formally, there is no difference between this case and Case I, and hence the corresponding terms of solution are

$$y = c_1 e^{(\alpha+i\beta)\,x} + c_2 e^{(\alpha-i\beta)\,x}$$

$$= e^{\alpha x}\,[c_1 e^{i\beta x} + c_2\, e^{-i\beta x}] \tag{27}$$

However, in practice we would prefer to work with real functions instead of complex exponentials. To achieve this, we make use of the **Euler's formula,** namely,

$$e^{i\theta} = \cos\theta + i\sin\theta \text{ and } e^{-i\theta} = \cos\theta - i\sin\theta,$$

where θ is any real number. Using these results, the right hand side of expression (27), becomes

$$e^{\alpha x}\,[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)]$$

$$= e^{\alpha x}[(c_1 + c_2)\cos\beta x + (c_1 - c_2)\; i\sin\beta x]$$

Since c_1 and c_2 are arbitrary constants, we may write

$$A = c_1 + c_2 \text{ and } B = i\,(c_1 - c_2),$$

so that A and B are again arbitrary constants though not both real. Expression (27) now takes the form

$$e^{\alpha x}[A\cos\beta x + B\sin\beta x] \tag{28}$$

Further, if the complex root is repeated, then the complex conjugate root will also be repeated and the corresponding terms in the solution can be written, using the form (25), as

$$e^{x(\alpha+i\beta)}(c_1 + c_2 x) + e^{x(\alpha-i\beta)}(c_3 + c_4 x)$$

Proceeding as above and writing

$$A = c_1 + c_3,\; B = i(c_1 - c_3),\; C = c_2 + c_4,\; D = i(c_2 - c_4)$$

the solution can be expressed in the form

$$y = e^{\alpha x}[(A + Cx)\cos\beta x + (B + Dx)\sin\beta x] \tag{29}$$

In the case of multiple repetition of the complex roots, the results are obtained analogous to those in the case of multiple repetition of the real roots.

We now illustrate the case of the complex roots with the help of following examples.

Example 18: For the differential equation

$\frac{d^4y}{dx^4} - m^4 y = 0$, show that its solution can be expressed in the form

$y = c_1 \cos mx + c_2 \sin mx + c_3 \cosh mx + c_4 \sinh mx$.

Solution: In this case since m is used as a constant in the given differential equation, we can use some other letter λ, say, in the auxiliary equation.

So, the auxiliary equation corresponding to the given equation is

$$(\lambda^4 - m^4) = 0$$

$$\Rightarrow \quad (\lambda^2 - m^2)\,(\lambda^2 + m^2) = 0$$

$$\Rightarrow \quad \lambda = m, -m, \pm im$$

So the roots are a mix of real and imaginary roots. The solution corresponding to the real roots $+m$ and $-m$ can be obtained as we have done in Example 11 and write it as

$$c_3 \cosh mx + c_4 \sinh mx \tag{30}$$

where c_3 and c_4 are arbitrary constants.
Solution corresponding to the imaginary roots $+im$ and $-im$ will be

$$Ae^{imx} + Be^{-imx}$$

which can be written as

$$A(\cos mx + i \sin mx) + B(\cos mx - i \sin mx)$$

$$= (A+B)\cos mx + i\ (A-B)\sin mx$$

$$= c_1 \cos mx + c_2 \sin mx \tag{31}$$

where $c_1 = (A+B)$ and $c_2 = i(A-B)$ are constants.
Hence combining Eqns. (30) and (31), the general solution of the given differential equation is

$$y = c_1 \cos mx + c_2 \sin mx + c_3 \cosh mx + c_4 \sinh mx$$

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Let us look at another example.

Example 19: Solve $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$.

Solution: The auxiliary equation in this case is

$$m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$$

$$\Rightarrow \quad (m^2 - 2m + 2)^2 = 0$$

$$\Rightarrow \quad [m - (1+i)]^2\ [m - (1-i)]^2 = 0$$

$$\Rightarrow \quad m = 1+i, 1+i, 1-i, 1-i.$$

Roots are complex and repeated in this case.
Hence the general solution can be written as

$$y = (c_1 + xc_2)\ e^{(1+i)x} + (c_3 + xc_4)\ e^{(1-i)x}$$

$$= e^x[(c_1 + xc_2)\ e^{ix} + (c_3 + xc_4)\ e^{-ix}]$$

$$= e^x[(c_1 + xc_2)\ (\cos x + i\sin x) + (c_3 + xc_4)(\cos x - i\sin x)]$$

$$= e^x\{[(c_1 + c_3) + x(c_2 + c_4)]\cos x + i[(c_1 - c_3) + x(c_2 - c_4)]\sin x\}$$

$$= e^x[(A + Bx)\cos x + (C + Dx)\sin x]$$

where $A = (c_1 + c_3)$, $B = (c_2 + c_4)$, $C = i(c_1 - c_3)$ and $D = i(c_2 - c_4)$ are all constants.

***

We now take up an example to obtain the differential equation whose solutions are known.

Example 20: Determine a linear differential equation with constant coefficients having the functions $4e^{6x}$, $3e^{-3x}$ as its solutions.

Solution: We know that if $y_1 = 4e^{6x}$ and $y_2 = 3e^{-3x}$ are two solutions of a linear differential equation then their linear combination

$$y = c_1 y_1 + c_2 y_2 = 4c_1 e^{6x} + 3c_2 e^{-3x} \tag{32}$$

where c_1, c_2 are arbitrary constants, is also a solution of the equation. We shall now eliminate c_1 and c_2 from Eqn. (32) to obtain the required equation. We have

$$y' = 24c_1 e^{6x} - 9c_2 e^{-3x} \tag{33}$$

$$y'' = 144c_1 e^{6x} + 27c_2 e^{-3x} \tag{34}$$

From Eqns. (32) and (34) we get

$$y'' = 9(c_1 e^{6x} + y). \tag{35}$$

From Eqns. (33) and (34) we get

$$c_1 e^{6x} = \frac{y'' + 3y'}{18} \tag{36}$$

From Eqns. (35) and (36) we thus obtain

$$y'' - 3y' - 18y = 0.$$

as the required differential equation.

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You may now try the following exercises.

E20) Find the general solution of the following equations:

i) $\dfrac{d^2y}{dx^2} - 2\alpha\dfrac{dy}{dx} + (\alpha^2 + \beta^2)\ y = 0.$

ii) $\dfrac{d^4y}{dx^4} + a^4 y = 0$

iii) $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 25y = 0.$

E21) Find the corresponding differential equations if the roots of an auxiliary equations are

i) $m_1 = 4,\ m_2 = m_3 = -5$

ii) $m_1 = \dfrac{-1}{2},\ m_2 = 3+i,\ m_3 = 3-i$

E22) Determine a linear differential equation with constant coefficients having the given solutions

i) $10\cos 4x,\ -5\sin 4x$

ii) $3,\ 2x,\ -e^{7x}$

We now end this unit by giving a summary of what we have covered in it.

10.5 SUMMARY

In this unit, we have covered the following:

1. The general linear differential equation with dependent variable y and independent variable x is termed as

   i) an equation with variable coefficients if not all the coefficients of y and its derivatives are constant functions of x.

   ii) an equation with constant coefficients if the coefficients of y and its derivatives are all constant functions.

   iii) homogeneous equation if the term free from y and any of its derivatives is absent.

   iv) non-homogeneous equation if there is a non-zero term free from y and any of its derivatives.

2. The solution $y(x)$ of an IVP
   $$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = b(x)$$
   $$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1},$$
   where x_0 is any point in an interval I and $y_0, y_1, \ldots, y_{n-1}$ are given real numbers, exists and is unique on the interval I, if $a_0, a_1, \ldots, a_n$ and b are continuous on I with $a_0(x) \neq 0$ for every x in I.

3. A set of functions $y_1, y_2, \ldots, y_n$ defined on an interval I is linearly dependent on I if for constant $c_1, c_2, \ldots, c_n$ not all zero, we have for every x in I,
   $$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0.$$

4. A set of functions $y_1, y_2, \ldots, y_n$ defined on an interval I is linearly independent on I if it is not linearly dependent on I.

5. If $y = y_1$ is a solution of homogeneous linear differential equation on I, so is $y = cy_1$ on I, for all real numbers c.

6. If $y = y_1,\ y_2,\ \ldots,\ y_m$ are solutions of linear homogenous differential equation on I, so is $y = c_1y_1 + c_2y_2 + \cdots + c_my_m$ on I, where $c_1, c_2, \ldots, c_m$ are arbitrary constants.

7. The n solutions $y_1, y_2, \ldots, y_n$ of an n^{th} order linear homogeneous equation on an interval I are linearly independent on I if and only if, the Wronskian of the n functions i.e., $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

8. A set of n linearly independent solutions of the homogeneous n^{th} order linear differential equation on an interval I constitute a fundamental set of solutions of the equation on I.

9. If $y_1, y_2, \ldots, y_n$ form a fundamental set of solutions of an n^{th} order homogeneous linear differential equation on an interval I, then

   $$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

   (where $c_1, c_2, \ldots, c_n$ being arbitrary constants)
   is defined to be the **general solution** or the **complete integral** of the given equation on I.

10. For a non-homogeneous equation
    i) the complete integral of the corresponding homogeneous equation is called its complementary function.
    ii) a particular solution of the non-homogeneous equation involving no arbitrary constant is called its particular integral.
    iii) complementary function and particular integral together constitute its general solution.

11. Solution y, of an n^{th} order linear differential equation

    $$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n \, y = 0$$

    with constant coefficients $a_1, \ldots, a_{n-1}, a_n$ having n roots $m_1, m_2, \ldots, m_n$, when
    i) roots are real and distinct, is
    $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$
    ii) roots are real and repeated, say $m_1 = m_2 = \cdots = m_r$, is
    $y = (c_1 + c_2 x + \cdots + c_r x^{r-1}) e^{m_1 r} + c_{r+1} e^{m_{r+1} x} + \cdots + c_n e^{m_n x}$.
    iii) roots are complex and one such pair is $\alpha \pm i\beta$, is
    $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$
    corresponding to that pair of roots.

10.6 SOLUTIONS/ANSWERS

E1) i) Non-homogeneous, reduced equation is $y''' + xy'' + x^2 y' + x^3 y = 0$.
ii) Homogeneous.
iii) Non-homogeneous, reduced equation is $(1-x)y'' + xy' - y = 0$.

E2) i) Independent, since for any non-zero c_1 and c_2
$c_1 e^x + c_2 e^{-x} = 0$ for every real x, only when $c_1 = 0$ and $c_2 = 0$.

ii) Dependent. We consider

$$2c_1 \cos 3x + 3c_2 \sin\left(3x + \frac{\pi}{2}\right)$$
$$= 2c_1 \cos 3x + 3c_2 \cos 3x$$
$$= (2c_1 + 3c_2)\cos 3x$$

which is satisfied for any non-zero choice of c_1 and c_2 for which $c_1 = -3/2c_2$.

iii) Independent

E3) Functions $y_1(x) = x^2$ and $y_2(x) = x\,|\,x\,|$ are linearly independent on $-1 < x < 1$. You can draw the graphs of the two functions and check it yourself. Also since $y_1(x) = x^2$ and $y_2(x) = x|\,x\,| = \begin{cases} x^2, & x > 0 \\ -x^2, & x \le 0 \end{cases}$.

$W(y_1(x),\ y_2(x)) = 0$ for all x in $-1 < x < 1$.

E4) i) Functions $f_1(x) = x,\ f_2(x) = x^2$ and $f_3(x) = 4x - 3x^2$ are linearly dependent on $]-\infty, \infty[$ since

$$c_1x + c_2x^2 + c_3(4x - 3x^2) = 0$$

when $c_1 = -8,\ c_2 = 6,\ c_3 = 2$.

ii) Functions $f_1(x) = 5,\ f_2(x) = \cos^2 x,\ f_3(x) = \sin^2 x$ are linearly dependent on $]-\infty, \infty[$ since

$$5c_1 + c_2 \cos^2 x + c_3 \sin^2 x = 0$$

when $c_1 = 1,\ c_2 = c_3 = -5$

iii) $W(x^{1/2},\ x^2) = \frac{3}{2}x^{3/2} \ne 0$ on $]0, \infty[$

iv) $W(\sin x, \operatorname{cosec} x) = -2\cot x = 0$ only at $x = \frac{\pi}{2}$ in the interval $]0, \pi[$.

E5) i) No; $W(2, e^x) = 2e^x \ne 0$ on any interval.

ii) Independent; W is not always zero.

E6) Function defined on the interval $]-\infty, \infty[$ whose graph passes through the point $(0, 2)$ having slope -1 at the point.

E7) Function $y = \frac{1}{4}\sin 4x$ is a solution of the given initial value problem.

From Theorem 1 it follows that on any interval containing $x = 0$ the solution is unique.

E8) Comparing the given equation with Eqn. (1), we obtain

$a_0(x) = x^2 - 3x = x(x-3)$.

Thus the only points of discontinuity of the coefficients of the equation are $x = 0$ and $x = 3$.

The largest interval containing the initial point $x = 1$, in which all the coefficients are continuous is $0 < x < 3$. In the interval Theorem 2 guarantees the existence of the solution.

E9) $a_0(x)(cy_1)'' + a_1(x)(cy_1)' + a_2(x)(cy_1)$
$= c\,[a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1]$
$= c.0$ [since y_1 is a solution of Eqn. (5)] on I
$= 0$
Thus cy_1 is also a solution of Eqn. (5) on I.

E10) $y = \frac{1}{x},\ y' = \frac{-1}{x^2},\ y'' = \frac{2}{x^3}$.

Then $y'' - 2y^3 = \frac{2}{x^3} - \frac{2}{x^3} = 0$. Thus $y = \frac{1}{x}$ is a solution of the non-linear differential equation.

Further, for $y = \frac{c}{x}$

$y'' - 2y^3 = \frac{2c}{x^3} - 2\frac{c^3}{x^3} = \frac{2}{x^3}c\,(1 - c^2) \neq 0$ for $c \neq 0$ and $c \neq \pm 1$.

E11) $c_1y_1 + c_2y_2$ (for arbitrary c_2) is not a solution of the given equation. It is a solution for arbitrary c_1 and $c_2 = 0$ or 1 only.

E12) $y_1(x) = x^{1/2},\ y_1'(x) = \frac{1}{2}x^{-1/2},\ y_1''(x) = -\frac{1}{4}x^{-3/2}$

Substituting these values in the given equation, we have

$$2x^2\left(-\frac{1}{4}x^{-3/2}\right) + 3x\left(\frac{1}{2}x^{-1/2}\right) - x^{1/2}$$
$$= \left(-\frac{1}{2} + \frac{3}{2} - 1\right)x^{1/2} = 0$$

Similarly, $y_2(x) = x^{-1},\ y_2'(x) = x^{-2},\ y_2''(x) = 2x^{-3}$, thus

$2x^2(2x^{-3}) + 3x(-x^{-2}) - x^{-1} = (4 - 3 - 1)x^{-1} = 0$

Thus y_1 and y_2 are the solutions of the given equation.

Now $W(y_1, y_2) = \begin{vmatrix} x^{1/2} & x^{-1} \\ \frac{1}{2}x^{-1/2} & -x^{-2} \end{vmatrix} = \frac{-3}{2}x^{-3/2}$

Since $W \neq 0$ for $x > 0$, y_1 and y_2 form a fundamental set of solutions there.

E13) Check that functions $y_1(x) = e^x$, $y_2(x) = e^{2x}$ and $y_3(x) = e^{3x}$ satisfy the third-order equation

$$y''' - 6y'' + 11y' - 6y = 0 \text{ on } -\infty < x < \infty$$

Further $W(e^x, e^{2x}, e^{3x}) = 2e^{6x} \neq 0$ for every real value of x. Thus y_1, y_2, y_3 form a fundamental set of solutions on $]-\infty, \infty[$.

$y = c_1e^x + c_2e^{2x} + c_3e^{3x}$ is the general solution of the given equation.

E14) Verify that $y_1 = \sin x$ and $y_2 = \cos x$ are the solutions of the given equation.

Also $W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$

Thus, y_1 and y_2 form a fundamental solution set of the given equation on any interval I. Thus $y = c_1 \sin x + c_2 \cos x$ is the general solution of the equation on I.

To find the required particular solution, we solve

$$c_1 \sin 0 + c_2 \cos 0 = 2$$

$$c_1 \cos 0 - c_2 \sin 0 = 3$$

and obtain $c_2 = 2, c_1 = 3$. Thus $y = 3\sin x + 2\cos x$ is the required particular solution.

E15) **Proof:** Let $y_p(x)$ be a solution of the non-homogeneous Eqn. (13), i.e.,

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = b(x)$$

and $y_1(x)$ and $y_2(x)$ are the solutions of the corresponding homogeneous equation on an interval I. Then we have

$$a_0(x)y_p'' + a_1(x)y_p' + a_2(x)y_p = b(x) \quad \text{(i)}$$

$$a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0 \quad \text{(ii)}$$

$$a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0 \quad \text{(iii)}$$

Let $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$, where c_1, c_2 are constants. Then

$$a_0(x)[c_1 y_1 + c_2 y_2 + y_p]'' + a_1(x)[c_1 y_1 + c_2 y_2 + y_p]' + a_2[c_1 y_1 + c_2 y_2 + y_p]$$

$$= c_1[a_0 y_1'' + a_1 y_1' + a_2 y_1] + c_2[a_0 y_2'' + a_1 y_2' + a_2 y_2] + [a_0 y_p'' + a_1 y_p' + a_2 y_p]$$

$= c_1.0 + c_2.0 + b(x)$ [using Eqns. (i), (ii) and (iii)]

$= b(x)$

Thus $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$ is also a solution of Eqn. (13) on I.

E16) i) The auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$\Rightarrow \quad (m-1)\,(m-2)\,(m-3) = 0$$

$$\Rightarrow \quad m = 1, 2, 3.$$

The general solution is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

ii) The auxiliary equation is

$$9m^2 + 18m - 16 = 0$$

$$\Rightarrow \; 9m^2 - 6m + 24m - 16 = 0$$

$$\Rightarrow \; (3m+8)\,(3m-2) = 0$$

$$\Rightarrow \; m = -8/3, 2/3$$

The general solution is

$$y = c_1 e^{\frac{2}{3}x} + c_2 e^{\frac{-8}{3}x}$$

iii) $y = c_1 e^{2x} + c_2 e^{-x} + c_3 e^{-3x}$

E17) i) Roots are $3, -1$

$$y = c_1 e^{3x} + c_2 e^{-x}, \; y' = 3c_1 e^{3x} - c_2 e^{-x}$$

Using given conditions

$c_1 + c_2 = 4$ and $3c_1 - c_2 = 0$

$\therefore\ c_1 = 1,\ c_2 = 3$

$y = e^{3x} + 3e^{-x}$

ii) The roots of the auxiliary equation are
$m = 0,\ 2,\ -2$

$y = c_1 + c_2 e^{2x} + c_3 e^{-2x}$

Using given conditions

$c_1 + c_2 + c_3 = 0$

$2c_2 - 2c_3 = 0$

$4c_2 + 4c_3 = 2$

$\therefore\ c_1 = -\frac{1}{2},\ c_2 = \frac{1}{4},\ c_3 = \frac{1}{4}$

$y = \frac{1}{2}(\cosh 2x - 1)$

E18) i) The auxiliary equation is
$m^3 - m^2 - 8m + 12 = 0$

$\Rightarrow\ (m+3)\ (m-2)^2 = 0$

$\Rightarrow\ m = -3,\ 2,\ 2$

Hence, $y = (A + Bx)\ e^{2x} + Ce^{-3x}$

ii) Roots of the auxiliary equation are $m = -1, -1,\ 2,\ 2$

$y = (A + Bx)\ e^{-x} + (C + Dx)\ e^{2x}$

iii) Roots of the auxiliary equation are $m = 1,\ -1,\ -1$

$y = (c_1 + c_2 x)\ e^{-x} + c_3 e^{x}$

iv) $y = (A + Bx + cx^2)e^{x}$.

E19) i) $y = (1 + x)\ e^{-2x}$

ii) Roots of the auxiliary equation are
$m = -1,\ -1,\ 2$

$y = (c_1 + xc_2)e^{-x} + c_3 e^{2x}$

Using conditions

$c_1 + c_3 = 0$

$-c_1 + c_2 + 2c_3 = 9$

$c_1 - 2c_2 + 4c_3 = 0$

$\therefore\ c_1 = -2,\ c_2 = 3,\ c_3 = 2$

$y = 2e^{2x} + (3x - 2)\,e^{-x}$

iii) Roots of the auxiliary equation are
$m = 0,\ 0,\ -1,\ -2$

and $y = c_1 + xc_2 + c_3 e^{-x} + c_4 e^{-2x}$

using given conditions and solving for $c_1,\ c_2,\ c_3,\ c_4$

$y = 2(x + e^{-x} - e^{-2x})$

E20) i) The auxiliary equation is

$m^2 - 2\alpha m + (\alpha^2 + \beta^2) = 0$

$$\Rightarrow \quad m = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha^2 + \beta^2)}}{2} = \alpha \pm i\beta$$

Hence the general solution is

$y = e^{\alpha x}\, [A\cos\beta x + B\sin\beta x]$

ii) The auxiliary equation is $m^4 + a^4 = 0$

Using De'Moivre's theorem to find the roots, we get

$m^4 = a^4(-1) = a^4(\cos\pi + i\sin\pi)$

$= a^4(\cos(2p+1)\pi + i\sin(2p+1)\pi),\ p = 0, 1, 2, \ldots$

Hence $m = a\left[\cos\frac{(2p+1)}{4}\pi + i\sin\frac{(2p+1}{4}\pi\right]$ *for* $p = 0, 1, 2, 3$.

$$\therefore \quad m = a\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right),\ a\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right),\ a\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right),\ a\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

Hence the solution is

$$y = e^{(a/\sqrt{2})x}\left[c_1\cos\left(\frac{a}{\sqrt{2}}x\right) + c_2\sin\left(\frac{a}{\sqrt{2}}x\right)\right]$$

$$+ e^{-(a/\sqrt{2})x}\left[c_3\cos\left(\frac{a}{\sqrt{2}}x\right) + c_4\sin\left(\frac{a}{\sqrt{2}}x\right)\right]$$

iii) $y = e^{-4x}\ (A\cos 3x + B\sin 3x)$.

E21) i) The auxiliary equation having given roots is

$(m-4)\ (m+5)\ (m+5) = 0$

$\Rightarrow \quad m^3 + 6m^2 - 15m - 100 = 0$

$\therefore$ corresponding differential equation is

$y''' + 6y'' - 15y' - 100y = 0$

ii) The auxiliary equation is

$$\left(m + \frac{1}{2}\right)(m-3-i)\ (m-3+i) = 0$$

$$\Rightarrow \quad \left(m + \frac{1}{2}\right)(m^2 - 6m + 10) = 0$$

$\Rightarrow \quad 2m^3 - 11m^2 + 14m + 10 = 0$

$\therefore\ 2y''' - 11y'' + 14y' + 10y = 0$ is the required equation.

E22) i) $y'' + 16y = 0$.

ii) $y''' - 7y'' = 0$.

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APPENDIX

We here give you a brief review of matrices and determinants which you might have studied in your school mathematics.

Matrix

Consider the system of linear equations

$$\begin{aligned} 3x+2y+z&=7 \\ x-y+3z&=3 \\ 5x+4y-2z&=1 \end{aligned} \tag{A1}$$

We can write the coefficients of $x,\ y,\ z$ in the three equations of the system (A1) above, in a table, as follows:

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 5 & 4 & -2 \end{bmatrix} \tag{A2}$$

In Table (A2) the first, second and third row corresponds respectively, to the coefficients of $x,\ y$ and z in the first, second and third equation of the system (A1).
In the same manner, we can rewrite system (A1) as

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 5 & 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \tag{A3}$$

The numbers or the variables in each of the square brackets ([ ]) represent a matrix. Formally, we can have the following definition.

Definition: A matrix is a rectangular arrangement of numbers in the form of horizontal and vertical lines.

The numbers/ variables occurring in a matrix are its **elements**. The elements in one horizontal line of a matrix is called a **row** of the matrix whereas, the elements in a vertical line is its **column**.

For examples, $\begin{bmatrix} 2 & 4 & -6 \\ 3 & 2 & 1 \end{bmatrix}$ is a matrix with 2 rows and 3 columns and

$\begin{bmatrix} 1 & 4 \\ 6 & -1 \\ 0 & 1 \end{bmatrix}$ is a matrix with 3 rows and 2 columns.

Note that each row of a matrix has the same number of elements. Similarly each column of a matrix has the same number of elements. We usually denote matrices by capital letters. In general, the matrix with m rows and n column is written as

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}=[a_{ij}],\ \begin{smallmatrix} i=1,\,2,\,\dots,\,m \\ j=1,\,2,\,\dots,\,n \end{smallmatrix}$$

Here a_{11} denotes the element in the 1st row and 1st column, a_{21} is the element in the 2nd row and 1st column and in general a_{ij} denotes the element in the ith row and jth column of the matrix A. We say that a_{ij} is the $(i,\ j)^{th}$ entry of A. For example $(1,\ 3)^{th}$ entry of $\begin{bmatrix} 1 & 4 & -2 \\ 2 & 0 & 0 \\ -1 & -4 & 3 \end{bmatrix}$ is -2 and $(3,\ 2)^{th}$ entry is -4.

Matrix A consisting of 3 rows and 2 columns is called 3×2 matrix or we say that matrix A has order 3×2. We represent it as $A_{3\times 2}$ or $A(3,\ 2)$ also. Matrix (A2) is a matrix of order 3×3. Since (A2) has the same number of rows as the number of columns, we call it a **square matrix**.

For instance, $[2]$, $\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ are the examples of square matrices of order 1×1, 2×2 and 3×3, respectively. Associated with the square matrices is a unique number called its determinant.

Determinants

We start with the definition of a determinant for a 1×1 matrix.

Definition: The detrminant of 1×1 matrix $A=[a]$, denoted by $|A|$ or $\det A$, is a.

For example if $A=[2]$, then $|A|$ is 2. If $A=[-3]$ then $|A|=-3$.

We now consider the determinant of a 2×2 matrix.

Definition: The determinant of the 2×2 matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is defined by

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$ and is denoted by $|A|$ or $\det A$.

For example, if $A=\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$ then $|A| = 1\times 5 - 0\times 3 = 5$.

Using the determinants of 2×2 matrices, we now define the determinant of a 3×3 matrix.

Definition: The determinant of the 3×3 matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is}$$

$$\begin{aligned} |A| &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= (-1)^{1+1}a_{11}|A_{11}| + (-1)^{1+2}a_{12}|A_{12}| + (-1)^{1+3}a_{13}|A_{13}| \end{aligned}$$

where $A_{ij}\ (j = 1, 2, 3)$, is the matrix obtained from A after deleting the first row and the j^{th} column.

In determing $|A|$ instead of expanding by the first row, as we have done above, we could also expand by the second row, third row or by any of the three columns. If we expand the above determinant by the second column, we get

$$|A| = (-1)^{1+2} a_{12} |A_{12}| + (-1)^{2+2} a_{22} |A_{22}| + (-1)^{3+2} a_{32} |A_{32}|.$$

All 6 ways (expending by any of the 3 rows or 3 columns) of obtaining $|A|$ would lead to the same result. We shall not be proving it here but we shall illustrate it through the following example.

Example 1: Evalute the determinant of the matrix

$$A = \begin{bmatrix} 2 & 4 & 7 \\ 1 & 2 & 3 \\ 1 & 5 & 3 \end{bmatrix}.$$

Solution: Expanding by the first row, we get

$$\begin{vmatrix} 2 & 4 & 7 \\ 1 & 2 & 3 \\ 1 & 5 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} + 4(-1)\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} + 7\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix}$$

$$= 2(-9) - 4(0) + 7(3) = 3$$

Alternatively, we expand the determinant by say, the second column. Then we have

$$\begin{vmatrix} 2 & 4 & 7 \\ 1 & 2 & 3 \\ 1 & 5 & 3 \end{vmatrix} = 4(-1)\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 7 \\ 1 & 3 \end{vmatrix} + 5(-1)\begin{vmatrix} 2 & 7 \\ 1 & 3 \end{vmatrix}$$

$$= -4(0) + 2(-1) - 5(-1) = 3$$

Thus, we get the same result in both the cases. Similarly, you can check that if you expand by other rows and columns of A you will get the same result.

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Determinants are sometimes useful in solving a linear system of n equations in n unknowns. In 1750 the German mathematician Gabriel Cramer published a rule, called Cramer's rule, for solving a set of n linear equations in n unknowns simultaneously.

Let us discuss this rule.

Cramer's Rule

Consider the following linear system of n equations in n unknowns:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{aligned} \qquad \text{(A4)}$$

Let A be the matrix of coefficients of (A4) and let

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

If

$$\det A_k = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & \boxed{b_1} & a_{1k+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k-2} & b_2 & a_{2k+!} & \cdots & a_{2n} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk-1} & b_n & a_{nk+1} & \cdots & a_{nn} \end{vmatrix}, \ k = 1, 2, \ldots, n \qquad \text{(A5)}$$

(k^{th} column)

is the same as $\det A$ except that its k^{th} column has been replaced by the column

$$\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix}$$

Then (A4) has the **unique solution**

$$x_1 = \frac{\det A_1}{\det A}, \ x_2 = \frac{\det A_2}{\det A}, \ \cdots, \ x_n = \frac{\det A_n}{\det A} \qquad \text{(A6)}$$

provided $\det A \neq 0$. This method of solving system (A4) by determinants is known as Cramer's rule.

Remember that Cramer's rule can be applied only if

i) the number of equations in the linear system equals the number of variables, and

ii) the determinant of the coefficient matrix is non-zero.

Let us solve the system of Eqns. (A1) by Cramer's rule

Example 2: Solve the system of equations

$$3x + 2y + z = 7$$
$$x - y + 3z = 3$$
$$5x + 4y - 2z = 1$$

by Cramer's rule.

Solution: For finding the solution we need to calculate the following four detrminants

$$\det A = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 13, \ \det A_1 = \begin{vmatrix} 7 & 2 & 1 \\ 3 & -1 & 3 \\ 1 & 4 & -2 \end{vmatrix} = -39$$

$$\det A_2 = \begin{vmatrix} 3 & 7 & 1 \\ 1 & 3 & 7 \\ 5 & 1 & -2 \end{vmatrix} = 78, \ \det A_3 = \begin{vmatrix} 3 & 2 & 7 \\ 1 & -1 & 3 \\ 5 & 4 & 1 \end{vmatrix} = 52$$

Hence (A6) gives

$$x = \frac{\det A_1}{\det A} = -3, \; y = \frac{\det A_2}{\det A} = 6, \; z = \frac{\det A_3}{\det A} = 4$$

***

If $b_i = 0, i = 1, 2, \ldots, n$, then the system of Eqns. (A4) is said to be **homogeneous**. If at least one of the b_i is non-zero, the system is **non-homogeneous**.

You may please **note** that for the homogeneous system corresponding to system A(4) **if** $\mathbf{det}\ A \neq \mathbf{0}$, then the only solution of the **homogeneous system** from (A6) is the **trivial solution** i.e., $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. This is because if $b_i = 0, i = 1, 2, \ldots, n$ in A(5) then $\det A_k = 0$ for $k = 1, 2, \ldots, n$ in A(6).
If $\mathbf{det}\ A = \mathbf{0}$, then a **homgeneous system** of n linear equations in n unknowns has **infinitely many non-trivial solutions**. These solutions can be obtained by the method of elimination.
If $\mathbf{det}\ A = \mathbf{0}$, then a **non-homogeneous system** (A4) may either have **infinitely many** solutions or **no** solution at all.

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UNIT 11

METHOD OF UNDETERMINED COEFFICIENTS

Structure Page No.

11.1 INTRODUCTION

In Unit 10, you studied that in order to find the complete integral of a general non-homogeneous linear differential equation, namely

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = b(x) \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are constants, it is necessary to find a general solution of the corresponding homogeneous equation, that is, the complementary function and then add to it any particular integral of Eqn. (1). We also discussed there the methods of finding the complementary functions of linear differential equations with constant coefficients. But how do we find a particular integral of the equations of the form $y'' + y' + y = x^2 + 1,\ y''' - y' = 2\cos x$ etc.? We shall now be considering this problem in this unit.

Variety of methods exist for finding a particular integral of a non-homogeneous equation of the form (1). The simplest of these methods is the method of undetermined coefficients. Basically, this method consists in making a guess to the form of a trial solution and then determining the coefficients involved in the trial solution so that it actually satisfies the given equation. You may recall that we had touched upon this method in Sec. 8.3 of Unit 8 for finding a particular integral of non-homogeneous linear differential equations of the first order having constant coefficients. In this unit in Sec. 11.2 we shall discuss the method in general, for finding a particular integral of second and higher order linear non-homogeneous differential equations with constant coefficients where the non-homogeneous term is a polynomial, an exponential function, a sine/cosine function or a combination of these functions. In Sec. 11.3 we have given some of the constraints of the method.

Objectives

After reading this unit, you should be able to:

- identify the types of non-homogeneous terms for which the method of undetermined coefficients can be successfully applied;
- write the form of trial solutions when the non-homogeneous terms of the given equations are polynomials, exponential functions or their combinations and use the method to obtain their particular integral; and
- describe the constraints of this method.

11.2 TYPES OF NON-HOMOGENEOUS TERMS FOR WHICH THE METHOD IS APPLICABLE

The method of undetermined coefficients, as we have already mentioned in Sec. 11.1, is a procedure for finding a particular integral $y_p(x)$ in the general solution $y(x) = y_c(x) + y_p(x)$ of equations of the form (1). The success of this method is based on our ability to guess the probable form of a particular integral.

Suppose that in Eqn. (1), $b(x) = x^r (r > 0$, an integer) i.e., $b(x)$ is a polynomial in x of degree r. Then what is $\frac{db}{dx}$? $\frac{db}{dx} = rx^{r-1}$, again a polynomial of degree $(r-1)$. Similarly, $\int b(x)\, dx = \int x^r dx = \frac{x^{r+1}}{r+1}$, a polynomial in x of degree $(r+1)$. That is, result of differentiating or integrating a polynomial is again a polynomial. Same is true when $b(x) = e^{mx}$, an exponential function or when $b(x)$ is a sine/cosine function like $b(x) = \sin mx$ or $b(x) = \cos mx$, for m a constant. We know that

$$\int e^{mx} dx = \frac{e^{mx}}{m},\ \frac{d}{dx}(e^{mx}) = me^{mx},\ \int \cos mx\, dx = \frac{1}{m}\sin mx, \int \sin mx\, dx = \frac{-1}{m}\cos mx$$

etc. Hence, if the non-homogeneous term $b(x)$ in Eqn. (1) is a polynomial, an exponential function, or a sine or cosine function then we can choose a particular integral to be a suitable combination of a polynomial, an exponential, or a sinusoidal function with a number of undetermined constants. These constants can then be determined so that the choosen trial solution satisfies the given equation.

A function which is a combination of a sine function (or cosine function) with an exponential function and/or a polynomial is a **sinusoidal** function e.g., $x^2 \sin 3x,\ xe^{2x} \cos x$ etc.

Thus the types of non-homogeneous terms for which the method of undetermined coefficients is successfully applicable are

i) polynomials

ii) exponential functions

iii) sine or cosine functions

iv) a combination of the terms of types (i), (ii) and (iii) above.

We shall now discuss each of these types one by one.

11.2.1 Non-homogeneous Term is a Polynomial

Let us start by considering the following differential equation and try to find its particular integral

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=x^2 \tag{2}$$

Here the non-homogeneous term i.e., x^2 is a polynomial of degree 2. As we have already mentioned above that the differentiation or integration of a polynomial is again a polynomial so we can think of taking a trial solution $y_p(x)$ as a second order polynomial of the form:

$y_p(x)=Ax^2+Bx+C$, where A, B and C are the constants to be determined.

Let us now check whether this choice of a trial solution help us to obtain a particular integral of Eqn. (2). In order to determine the values of A, B and C we substitute in Eqn. (2) the values of y_p, y_p' and y_p'' and obtain

$$2A+(2Ax+B)+(Ax^2+Bx+C)=x^2.$$

Equating the coefficients of the like powers of x on both the sides of the above equation we get

Coefficients of $x^2: A=1$
Coefficients of $x: 2A+B=0$
Coefficients of $x^0: 2A+B+C=0$

Solving the above system for A, B and C, we get

$$A=1, B=-2, C=0$$

For these values of A, B and C the trial solution $y_p(x)$ take the form

$$y_p(x)=x^2-2x$$

and it can be easily checked that it also satisfies Eqn. (2).
We have, $y_p''+y_p'+y_p=2+(2x-2)+(x^2-2x)=x^2$. Thus, our choice of $y_p(x)$ leads us to a particular integral of Eqn. (2).

Let us now see how the method used in the example above can be generalized to make an appropriate choice of a trial solution for any given equation with the non-homogeneous term being a polynomial.

Consider Eqn. (1) and assume that the non-homogeneous term $b(x)$ is a polynomial of degree k and is given by

$b(x)=b_0x^k+b_1x^{k-1}+\cdots+b_{k-1}x+b_k=B_k(x)$, say,

where $b_0, b_1, \ldots, b_k$ are known constants.
With the above form of $b(x)$, Eqn. (1) reduces to

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny$$
$$=b_0x^k+b_1x^{k-1}+\cdots+b_{k-1}x+b_k \tag{3}$$

Now since the right hand side of Eqn. (3) is a polynomial of degree k, we can think of a particular integral in the form

$$y_p(x) = A_0 x^k + A_1 x^{k-1} + \cdots + A_{k-1} x + A_k = P_k(x),$$

where $A_0, A_1, \ldots, A_k$ are constants to be determined.

In order to determine the constants $A_0, A_1, \ldots, A_k$, we substitute the values of y_p and its derivatives in Eqn. (3) and compare the coefficients of the like powers of x on both the sides of Eqn. (3). Comparing the coefficients of x^k, we get

$$a_n A_0 = b_0$$

If $a_n \neq 0$, then we get $A_0 = \dfrac{b_0}{a_n}$. Similarly, the constants $A_1, A_2, \ldots, A_k$ in $y_p(x)$ are determined by comparing the coefficients of $x^{k-1}, x^{k-2}, \ldots, x^0$ on both sides.

If $a_n = 0$, i.e., the coefficient of y in Eqn. (3) is zero then obviously zero will be the root of the corresponding auxiliary equation, i.e., the root of the homogeneous differential equation corresponding to Eqn. (3) and hence $y =$constant, will be its solution. In that case $A_0 = \dfrac{b_0}{a_n}$ will be undetermined and we will not be able to determine all the coefficients $A_0, A_1, \ldots, A_k$. Consider for example the equation

$$y''' + y'' + y' = x^2 + x \tag{4}$$

Here there is no y term on the l.h.s. of Eqn. (4), i.e., the coefficient of y is zero. Zero is a root of the homogeneous equation corresponding to Eqn. (4). Further, $b(x) = x^2 + x$ is a polynomial of degree 2. Now, if we take a trial solution as $y_p(x) = A_0 x^2 + A_1 x + A_2$ then substituting for y'_p, y''_p and y'''_p in Eqn. (4) we get $2A_0 + 2A_0 x + A_1 = x^2 + x$. You may **note** here that comparing the coefficients of various powers of x in this equation we get $A_0 = \dfrac{1}{2}$ and $A_1 = -1$. We cannot obtain the value of the coefficient A_2. However, if we consider a trial solution of the form $x(A_0 x^2 + A_1 x + A_2)$ then you may check that we obtain the values of the three coefficients as $A_0 = \dfrac{1}{3}$, $A_1 = -\dfrac{1}{2}$ and $A_2 = -1$. A particular integral of Eqn. (4) is then obtained as $y_p(x) = \dfrac{1}{3}x^3 - \dfrac{1}{2}x^2 - x$.

Coming back to Eqn. (3) we can thus say that when $a_n = 0$, a trial solution of the form $xP_k(x)$ of degree $(k+1)$ is taken instead of k^{th} degree polynomial $P_k(x)$. This trial solution, when substituted in Eqn. (3), gives us a term on the left hand side of Eqn. (3) which balances the term $b_0 x^k$ on the right hand side of Eqn. (3), provided $a_{n-1} \neq 0$.

If $a_n = 0$ and $a_{n-1} = 0$, then zero will be the repeated root of the homogeneous differential equaton corresponding to Eqn. (3) and consequently, $y =$constant and $y = x$ will be its solutions. In that case, we take a trial solution of the form $x^2 P_k(x)$ of degree $(k+2)$ to balance against $b_0 x^k$, provided $a_{n-2} \neq 0$.

In general, let us assume that out of n coefficients

$a_0, a_1, a_2, \ldots, a_r, a_{r+1}, \ldots, a_n$ in Eqn. (3) the last r coefficients, i.e., $a_n, a_{n-1}, a_{n-2}, \ldots, a_{n-(r-1)}$ are zero but $a_{n-r} \neq 0$. The zero will then be r times repeated root of the auxiliary equation corresponding to Eqn. (3). In such a case we multiply the polynomial $P_k(x)$ by x^r and consider a trial solution $x^r P_k(x)$ of degree $(k+r)$.

We illustrate the method discussed above with the help of a few examples.

Example 1: Find a particular integral of the differential equation

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^3 + 2x + 1$$

using the method of undetermined coefficients.

Solution: The auxiliary equation corresponding to the given equation is

$$m^4 + m^2 + m + 1 = 0.$$

Here zero is not a root of the auxiliary equation.
Also the non-homogenous term in the given differential equation, i.e., $x^3 + 2x + 1$ is a polynomial of degree 3. Hence an appropriate form of a trial solution will be a polynomial of degree 3. We take it as

$$y_p(x) = Ax^3 + Bx^2 + Cx + D,$$

where A, B, C and D are the constants to be determined.
Substituting the values of y_p and its derivatives in the given differential equation, we get

$$(Ax^3 + Bx^2 + Cx + D) + (3Ax^2 + 2Bx + C) + (6Ax + 2B) + 0$$
$$= x^3 + 2x + 1 \qquad (5)$$

Equating the coefficients of the like powers of x on both the sides of Eqn. (5), we get

$$\left.\begin{array}{l} \text{Coefficients of } x^3 : A = 1 \\ \text{Coefficients of } x^2 : B + 3A = 0 \\ \text{Coefficients of } x : C + 2B + 6A = 2 \\ \text{Coefficients of } x^0 : D + C + 2B = 1. \end{array}\right] \qquad (6)$$

Solving the set of Eqns. (6) for A, B, C and D, we get

$$A = 1,\ B = -3,\ C = 2,\ D = 5$$

Hence a particular integral of the given equation is

$$y_p(x) = x^3 - 3x^2 + 2x + 5.$$

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Let us now look at examples in which zero is the root of auxiliary equations.

Example 2: Find a particular integral of the differential equation
$\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 + 2x.$

Solution: The auxiliary equation for the given differential equation is

$$m(m+1) = 0$$

$\Rightarrow \quad m = 0, -1$

Thus, the complementary function can be expressed as

$y_c = c_1 + c_2 e^{-x}$, c_1 and c_2 being constants.

Since 0 is a root of the auxiliary equation, the constant c_1 is one of the solutions of the corresponding homogeneous differential equation. Also, the non-homogeneous term in this case is of degree 2. We take the form of a trial solution as

$$y_p(x) = x(Ax^2 + Bx + C) = Ax^3 + Bx^2 + Cx.$$

Substituting the values of y_p and its derivatives in the given differential equation, we get

$$(3Ax^2 + 2Bx + C) + (6Ax + 2B) = x^2 + 2x$$

Equating the coefficients of the like powers of x on both the sides, we get

Coefficients of $x^2 : 3A = 1$
Coefficients of $x : 2B + 6A = 2$
Coefficients of $x^0 : C + 2B = 0$

Solving the above set of linear equations for A, B and C, we get

$$A = \frac{1}{3}, B = 0, C = 0$$

Hence a particular integral for the given differential equation is

$$y_p(x) = \frac{1}{3} x^3.$$

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Example 3: Find a particular integral of the equation

$$\frac{d^3 y}{dx^3} - \frac{d^2 y}{dx^2} = 2x^3.$$

Solution: The auxiliary equation for the given equation is

$m^3 - m^2 = 0$

$\Rightarrow \quad m^2(m-1) = 0$

$\Rightarrow \quad m = 0, 0, 1$

Hence zero is a double root of the auxiliary equation.
The complementary function can be expressed as

$$y_c = c_1 + xc_2 + c_3 e^x,$$

c_1, c_2 and c_3 being constants.

Since the non-homogeneous term is of degree 3 and 0 is a double root of the equation, we take the form of a trial solution as

$$y_p(x) = x^2(Ax^3 + Bx^2 + Cx + D) = Ax^5 + Bx^4 + Cx^3 + Dx^2$$

Substituting for y_p and its derivatives in the given equation, we get

$$(60Ax^2 + 24Bx + 6C) - (20Ax^3 + 12Bx^2 + 6Cx + 2D) = 2x^3$$

Equating the coefficients of the like powers of x on both the sides, we get

Coefficients of $x^3 : -20A = 2$

Coefficients of $x^2 : 60A - 12B = 0$

Coefficients of $x : 24B - 6C = 0$

Coefficients of $x^0 : 6C - 2D = 0$

Solving the above set of linear equations for A, B, C and D, we get

$$A = \frac{-1}{10}, B = \frac{-1}{2}, C = -2, D = -6$$

Hence, a particular integral of the given differential equation is

$$y_p = \frac{-x^5}{10} - \frac{x^4}{2} - 2x^3 - 6x^2$$
$$= -\left(\frac{x^5}{10} + \frac{x^4}{2} + 2x^3 + 6x^2\right)$$

***

Using the method discussed above you may now try to solve the following exercises.

E1) Write a suitable form of a particular integral for the following differential equations

i) $$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2 + 1$$

ii) $$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2$$

E2) Determine the general solution of the following differential equations.

i) $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x^2$$

ii) $$\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = x$$

We next take up the case when $b(x)$ is an exponential function. In this case also the logic used in making an appropriate choice of a trial solution is same as that used in Sub-sec.11.2.1. Let us see how the method works.

11.2.2 Non-homogeneous term is an Exponential Function

Let us start with a simple example and try to find a particular integral of the equation

$$y'' - y = e^x \tag{7}$$

As we mentioned earlier the differentiation or integration of an exponential function is again exponential we can think of a trial solution of Eqn. (7) of the form $y_p(x) = Ae^x$ and then try to determine the value of the coefficient A.

Substituting for $y_p(x)$ and $y_p''(x)$ in Eqn. (7) we obtain $Ae^x - Ae^x = e^x$ which is absurd. Thus the trial solution considered does not serve our purpose. You may **notice** here that since 1 is a root of the homogeneous equation corresponding to Eqn. (7), $y(x) = e^x$ will be the solution of the homogeneous equation. In order to find a particular integral of Eqn. (7) we thus need to take a trial solution of Eqn. (7) of the form $y_p(x) = Axe^x$. Substituting for y_p and y_p'' in Eqn. (7) we obtain

$$Axe^x + 2Ae^x - Axe^x = e^x$$

which yields $A=\frac{1}{2}$ giving $y_p(x)=\frac{x}{2}e^x$ as a particular integral of Eqn. (7).

The method above holds true, in general, for Eqn. (1) when $b(x)$ is an exponential function of the form $e^{\alpha x}$ (α a constant) and Eqn. (1) is of the form

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_n y=e^{\alpha x} \tag{8}$$

To solve Eqn. (8), the appropriate form of a trial solution can be taken as

$$y_p(x)=Ae^{\alpha x} \tag{9}$$

provided, $e^{\alpha x}$ is not a solution of the homogeneous differential equation corresponding to Eqn. (8). That is, α is not a root of the corresponding auxiliary equation.

If α is a root of the auxiliary equation corresponding to Eqn. (8), then the choice (9) would lead us to a relation of the form $Ae^{\alpha x}.0=e^{\alpha x}$ from which the value of A cannot be determined.

In that case, we take $y_p(x)=Axe^{\alpha x}$ as a trial solution which gives us a relation for determining the value of A. Similarly, if α is r-times repeated root of the auxiliary equation, then the suitable form of a trial solution for determining a particular integral will be

$$y_p(x)=Ax^r e^{\alpha x}$$

The value of A can then be determined by substituting for y_p and its derivatives in the given equation and then equating the coefficients of $e^{\alpha x}$ on both the sides of the equation.

Let us take up a few examples to illustrate the method.

Example 4: Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=3e^x.$$

Solution: The auxiliary equation is

$$m^2+3m+2=0$$

$$\Rightarrow\ (m+1)\,(m+2)=0$$

$$\Rightarrow\ m=-1,\,-2$$

$\therefore$ C.F. $=c_1e^{-x}+c_2e^{-2x}$

Since e^x is not a part of the complementary function, a trial solution for finding a particular integral can be taken as

$y_p(x)=Ae^x$, where A is a constant to be determined.

Substituting this value of y_p in the given differential equation, we get

$$2Ae^x+3Ae^x+Ae^x=3e^x$$

$$\Rightarrow\ 6Ae^x=3e^x$$

$$\Rightarrow\ 6A=3 \text{ or } A=\frac{1}{2}$$

Hence

$$\text{P.I.} = \frac{1}{2} e^x$$

$\therefore$ The general solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2} e^x .$$

***

Example 5: Find the general solution of the differential equation

$$\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 3e^{2x}$$

Solution: The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow \ (m-1)(m-2) = 0$$

$$\Rightarrow \ m = 1, 2$$

$$\therefore \ \text{C.F.} = c_1 e^x + c_2 e^{2x}$$

Since 2 is a root of the auxiliary equation and the non-homogenous term e^{2x} is a part of the complementary function, we take the form of a trial solution as

$$y_p(x) = Axe^{2x}$$

Substituting the value of $y_p(x)$ and its derivatives in the given equation, we get

$$4Ae^{2x} + 4Axe^{2x} - 3(Ae^{2x} + 2Axe^{2x}) + 2Axe^{2x} = 3e^{2x}$$

Equating the coefficients of e^{2x} on both the sides of the equation, we get

$$4A - 3A = 3 \Rightarrow A = 3$$

Hence, $\text{P.I.} = y_p(x) = 3xe^{2x}$.

$\therefore$ The general solution of the given differential equation is

$$y = c_1 e^x + c_2 e^{2x} + 3xe^{2x}.$$

***

Let us consider another example which illustrate the case of repeated roots of an auxiliary equation.

Example 6: Solve $\dfrac{d^3 y}{dx^3} - 3\dfrac{d^2 y}{dx^2} + 3\dfrac{dy}{dx} - y = 12e^x$.

Solution: The auxiliary equation is

$$(m-1)^3 = 0$$

$$\Rightarrow \ m = 1, 1, 1$$

$$\therefore \ \text{C.F.} = (c_1 + c_2 x + c_3 x^2)\, e^x$$

Since the non-homogeneous term of the given differential equation is e^x which is present in the complementary function and moreover 1 is 3-times repeated root of the auxiliary equation, we take the form of a trial solution to be

$$y_p(x) = Ax^3 e^x .$$

You may check here that in the selection of a trial solution $y_p(x)$ above no smaller power of x will give us a particular integral. Also, the form of a trial solution is not similar to any term of the complementary function of the given equation.

On substituting this value of y_p in the given differential equation, we get

$$-Ax^3e^x+3A[x^3e^x+3x^2e^x]-3A[x^3e^x+6x^2e^x+6xe^x]$$

$$+A[x^3e^x+9x^2e^x+18xe^x+6e^x]=12e^x$$

Equating the coefficients of e^x on both the sides, we get

$$6A=12, \Rightarrow A=2$$

Thus, P.I. $=2x^3e^x$

$\therefore$ The general solution of the given differential equation is

$$y=(c_1+c_2x+c_3x^2)\ e^x+2x^3e^x.$$

***

And now you can check your understanding of the method while doing the following exercise.

E3) Find a particular integral of the following differential equations

i) $\dfrac{d^3y}{dx^3}-4\dfrac{dy}{dx}=e^{-2x}$

ii) $\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=e^{-x}$

You may also come across the situation when $b(x)$ in Eqn. (1) is a sum of two or more functions. Suppose $b(x)=b_1(x)+b_2(x)$. If $y_p(x),\ y_{p_1}(x)$ and $y_{p_2}(x)$ are respectively, the P.I's of Eqn. (1) corresponding to the non-homogeneous terms $b(x), b_1(x)$ and $b_2(x)$ then from the superposition principle we have $y_p=y_{p_1}+y_{p_2}$. This enables us to decompose the problem of solving linear Eqn. (1) into simpler problems as illustrated in the following examples.

Example 7: Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x+4.$$

Solution: The auxiliary equation is

$$m^2-2m+1=0$$

$$\Rightarrow \quad (m-1)^2=0$$

$$\Rightarrow \quad m=1,1.$$

$\therefore$ C.F. $=(c_1+xc_2)\ e^x$

To find a particular solution we first consider the equation

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x \tag{10}$$

Since 1 is a repeated root of the auxiliary equation and e^x is present in the complementary solution we consider a trial solution

$$y_{p_1}=Ax^2e^x$$

On substituting for y_{p_1} and its derivatives in Eqn. (10), we find that

$$(2Ae^x+4xAe^x+x^2Ae^x)-2(2xAe^x+x^2Ae^x)+Ax^2e^x=e^x$$

Comparing the coefficients of e^x on both the sides, we have

$$2Ae^x=e^x$$

$$\Rightarrow \quad A=\frac{1}{2}$$

$\therefore \quad y_{P_1} = \frac{x^2}{2} e^x .$

Now consider the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 4 \tag{11}$$

Since the non-homogeneous term is a constant, we try $y_{p_2} = A$ and find that $A = 4$ satisfies (11). Hence $y_{p_2} = 4$ and a particular solution of the given equation is

$$y_p = y_{p_1} + y_{p_2} = \frac{x^2}{2} e^x + 4$$

The required general solution will then be

$$y = c_1 e^x + c_2 x e^x + 4 + \frac{x^2}{2} e^x .$$

***

Let us now consider an example where the non-homogeneous term $b(x)$ is a sum of a polynomial term and an exponential term.

Example 8: Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + 3e^{2x}$$

Solution: The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-1)(m-2) = 0$$

$$\Rightarrow m = 1, 2$$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$

To find a particular solution we first consider the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 \tag{12}$$

Here the non-homogeneous term, $2x^2$, is a polynomial of degree 2 hence, we take a trial solution of the form

$$y_{p_1} = Ax^2 + Bx + C$$

Substituting for y_{p_1} and its derivatives in Eqn. (12), we get

$$2(Ax^2 + Bx + C) - 3(2Ax + B) + 2A = 2x^2$$

Equating the coefficients of like powers of x on both the sides and solving, we obtain

$$A = 1,\ B = 3 \text{ and } C = \frac{7}{2}$$

Putting these values of A, B and C in y_{p_1}, we get

$$y_{p_1} = x^2 + 3x + \frac{7}{2}$$

Now consider the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3e^{2x} \tag{13}$$

In Example 5 we have already obtained a particular integral of Eqn. (13) as $y_{p_2} = 3xe^{2x}$.

Hence, a particular solution of the given equation is

$$y_p = y_{p_1} + y_{p_2}$$
$$= x^2 + 3x + \frac{7}{2} + 3xe^{2x}.$$

The general solution of the equation will then be

$$y = c_1 e^x + c_2 e^{2x} + x^2 + 3x + \frac{7}{2} + 3xe^{2x}.$$

***

You may now try these exercises.

E4) Find the general solution of the following differential equations

i) $\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x$

ii) $2\frac{d^2 y}{dx^2} + 8y = x^3 + e^{2x}$.

E5) Solve the following initial value problems:

i) $\frac{d^2 y}{dx^2} - y = e^{2x},\ y(0) = -1,\ y'(0) = 1$

ii) $\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y + e^{2x} = 0,\ y(0) = y'(0) = 0$.

We now take up the case when the $b(x)$ in Eqn. (1) is either a sine or a cosine function.

11.2.3 Non-homogeneous Term is a Sine or a Cosine Function

After going through Sub-secs. 11.2.1 and 11.2.2 and attempting the exercises given there you know how to handle $b(x)$ when it is a polynomial, an exponential function or a sum of both. We shall now discuss the case when $b(x)$ is a sine or a cosine function.

We know that the derivative of a sine or a cosine function $\sin\beta x$ or $\cos\beta x$ is again a sine or a cosine function or their linear combination. Therefore, if the non-homogeneous term $b(x)$ of differential Eqn. (1) is of the form

$$b(x) = \alpha_1 \sin\beta x \text{ or, } \alpha_2 \cos\beta x \text{ or, } \alpha_1 \sin\beta x + \alpha_2 \cos\beta x,$$

we take a trial solution in the form

$$y_p(x) = A\cos\beta x + B\sin\beta x \qquad (14)$$

provided, $\pm i\beta$ are not the roots of the auxiliary equation corresponding to the given differential equation because if they are the roots, than the terms $\sin\beta x$ and $\cos\beta x$ will appear in the complenetary solution of the equation.

If $\pm i\beta$ are the roots, say, r-times repeated roots of the auxiliary equation, then we take the form of a trial solution to be

$$y_p(x) = x^r \ (A\cos\beta x + B\sin\beta x) \qquad (15)$$

where, A and B are the constants to be determined. In order to obtain A and B we proceed as we have been doing above and substitute the value of $y_p(x)$ of the form (14) or (15), whichever is applicable, in Eqn. (1). Equating the coefficients of $\sin\beta x$ and $\cos\beta x$ on both the sides of the resulting equation values of A and B in terms of known quantities are obtained. Knowing the values of A and B, a particular integral of Eqn. (1) is obtained from relations (14) or (15).

We now illustrate the method with the help of a few examples.

Example 9: Find the general solution of the differential equation

$$\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = \sin x$$

Solution: The auxiliary equation is

$$(m^4 - 2m^2 + 1) = 0$$

$$\Rightarrow \quad (m^2 - 1)^2 = 0$$

$$\Rightarrow \quad m = 1,\ 1,\ -1,\ -1$$

$\therefore$ C.F. $= (c_1 + c_2x)\ e^x + (c_3 + c_4x)\ e^{-x}$

Since $\pm\, i$ is not a root of the auxiliary equation the term $\sin x$, which is the non-homogeneous term in this case, does not appear in the complementary function. We can thus take a trial solution in the form

$y_p(x) = A\sin x + B\cos x$.

Substituting this value of y_p and its derivatives in the given differential equation, we get

$$(A\sin x + B\cos x) - 2(-A\sin x - B\cos x) + (A\sin x + B\cos x) = \sin x$$

$$\Rightarrow \quad 4A\sin x + 4B\cos x = \sin x$$

Equating the coefficients of $\sin x$ and $\cos x$ on both the sides, we get

$$4A = 1 \Rightarrow A = \frac{1}{4}$$

and $4B = 0 \Rightarrow B = 0$

Thus, $y_p(x) = \dfrac{1}{4}\sin x$

And the complete solution of the given differential equation is

$$y = (c_1 + c_2x)\,e^x + (c_3 + c_4x)\ e^{-x} + \frac{1}{4}\ \sin x.$$

***

Let us look at another example.

Example 10: Solve an initial value problem

$$\frac{d^2y}{dx^2} + y = 2\cos x,\ \ y(0) = 1,\ \ y'(0) = 0$$

Solution: The auxiliary equation is

$$m^2 + 1 = 0$$

$$\Rightarrow \quad m = \pm\, i$$

$\Rightarrow$ C.F. $= c_1\cos x + c_2\sin x$

Now since $\pm i$ is a root of the auxiliary equation $\cos x$ itself appears in the complementary function. We take the form of a trial solution as

$$y_p(x) = x\ (A\sin x + B\cos x)$$

Substituting the values of $y_p(x)$ and its derivatives in the given equation, we get

$$2(A\cos x - B\sin x) + x(-A\sin x - B\cos x) + x(A\sin x + B\cos x) = 2\cos x$$

$$\Rightarrow \quad 2A\cos x - 2B\sin x = 2\cos x$$

Comparing the coefficients of $\sin x$ and $\cos x$ on both the sides, we get

$2A = 2 \Rightarrow A = 1$ and $B = 0$

Therefore,

$$y_p(x) = x\sin x$$

and the general solution of the given equation is

$$y(x) = c_1\cos x + c_2\sin x + x\sin x.$$

We now use initial conditions to determine c_1 and c_2.

$y(0) = 1$ gives $c_1 = 1$

and $y'(0) = 0$ gives $c_2 = 0$

Thus, $y(x) = \cos x + x\sin x$.

***

You may now try the following exercises.

E6) Solve the following differential equations:

i) $\dfrac{d^4y}{dx^4} + 4\dfrac{d^2y}{dx^2} = \sin 2x$

ii) $\dfrac{d^3y}{dx^3} - \dfrac{dy}{dx} = 2\cos x$

E7) Solve the following initial value problems:

i) $\dfrac{d^2y}{dx^2} + 4y = \sin x,\ y(0) = 2,\ y'(0) = -1$

ii) $\dfrac{d^2y}{dx^2} + y = \cos 2x - \sin 2x,\ y(0) = \dfrac{-7}{20},\ y'(0) = \dfrac{1}{5}$

In the examples considered so far, you must have noticed that the function $b(x)$ itself suggests the form of a particular solution $y_p(x)$. This suggests that we can expand the list of functions $b(x)$, to which the method of undetermined coefficients can be applied, by including the product of these functions as well. We now discuss such cases.

11.2.4 Non-homogeneous Term is a product of an Exponential, a Polynomial and a Sinusoidal Function

Let us first consider the following differential equation and try to solve it

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = -xe^{4x} \tag{16}$$

The auxiliary equation corresponding to Eqn. (16) is

$m^2 + 6m + 9 = 0 = (m+3)^2$.

Thus -3 is a repeated root of the equation and the complementary function for Eqn. (16) can be written as

$$\text{C.F.} = (c_1 + xc_2)e^{-3x}, c_1 \text{ and } c_2 \text{ being constants.}$$

Let us look at the non-homogeneous term in Eqn. (16). It is a product of x, a polynomial of degree one, and an exponential term e^{4x}. Now since e^{4x} does not appear in the complementary solution our understanding of the previous methods suggests us to take a trial solution of the form:

$$y_p(x) = (A + xB)e^{4x} \tag{17}$$

where A and B are the constants to be determined.

Let us now put the values of $y_p(x)$ and its derivatives in Eqn. (16) and try to determine A and B. We have

$$y'_p = 4Ae^{4x} + Be^{4x} + 4Bxe^{4x}$$

$$y''_p = 16A\,e^{4x} + 8B\,e^{4x} + 16Bxe^{4x}$$

Substituting these values of y_p, y'_p and y''_p in Eqn. (16), we get

$$49Ae^{4x} + 14Be^{4x} + 49xB\,e^{4x} = -x\,e^{4x}$$

Comparing the coefficients of e^{4x} and xe^{4x} in the above equation, we obtain

$$B = -\frac{1}{49} \text{ and } A = \frac{2}{343}$$

Thus, from Eqn. (17), we get a particular solution of Eqn. (16) as

$$y_p(x) = \frac{2e^{4x}}{343} - \frac{x}{49}e^{4x}.$$

The general solution of Eqn. (16) can then be written as

$$y = (c_1 + xc_2)e^{-3x} - \frac{x}{49}e^{4x} + \frac{2}{343}e^{4x}.$$

The above method can be generalised to find the solution of Eqn. (1) when $b(x)$ is of the form

$$b(x) = e^{\alpha x}\,[b_0x^k + b_1x^{k-1} + \cdots + b_{k-1}x + b_k] = e^{\alpha x}\,P_k(x).$$

With this form of $b(x)$, Eqn. (1) reduces to

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny$$
$$= e^{\alpha x}[b_0x^k + b_1x^{k-1} + \cdots + b_{k-1}x + b_k] \tag{18}$$

and a trial solution can be taken in the form

$$y_p(x) = e^{\alpha x}\,[A_0x^k + A_1x^{k-1} + \cdots + A_{k-1}x + A_k\,] \tag{19}$$

provided α is not a root of the auxiliary equation corresponding to Eqn. (18). If α is a root of the auxiliary equation, say, it is r-times repeated root of the auxiliary equation then we modify the trial solution as

$$y_p(x) = x^re^{\alpha x}[A_0x^k + A_1x^{k-1} + \cdots + A_{k-1}x + A_k\,] \tag{20}$$

where $A_0, A_1, \ldots, A_k$ are all constants to be determined.

Remember that in Eqn. (20) no smaller power of x will yield a particular integral. Here r is the smallest positive integer for which every term in the trial solution (20) will differ from every term occurring in the complementary function corresponding to Eqn. (18).

In order to determine the constants $A_0, A_1, \ldots, A_k$ substitute $y_p(x)$ of the form (19) or (20), as the case may be, in Eqn. (18) and then compare the coefficients of $e^{\alpha x}, xe^{\alpha x}, x^2e^{\alpha x}, \ldots$ etc. on both the sides of the resulting equation. For a better understanding of whatever we have discussed above, let us consider the following examples.

Example 11: Solve $\dfrac{d^3y}{dx^3} - \dfrac{dy}{dx} = xe^{-x}$.

Solution: The auxiliary equation is

$$m^3 - m = 0$$

$$\Rightarrow \quad m(m^2 - 1) = 0$$

$$\Rightarrow \quad m = 0, -1, 1$$

$\therefore$ C.F. $= c_1 + c_2e^{-x} + c_3e^x$

Here the non-homogeneous term is xe^{-x}. Also e^{-x} appears in the complementary function. Further, (-1) is a non-repeated root of the auxiliary equation. Thus, we take the form of a trial solution as

$$y_p(x) = [Ax + B]\, xe^{-x} = Ax^2e^{-x} + Bxe^{-x}$$

Substituting this value of y_p and its derivatives in the given differential equation, we get

$$-A[-x^2e^x + 2xe^{-x}] + A[-x^2e^{-x} + 6xe^{-x} - 6e^{-x}] - B(-xe^{-x} + e^{-x}) +$$
$$B(-xe^{-x} + 3e^{-x}) = xe^{-x}$$

Further simplifying the above equation and comparing the coefficients of xe^{-x} and e^{-x} on both the sides, we get

$$4A = 1 \Rightarrow A = \frac{1}{4}$$

and $\quad -6A + 2B = 0 \Rightarrow B = \dfrac{3}{4}$

Hence $y_p(x) = \dfrac{1}{4}x^2e^{-x} + \dfrac{3}{4}xe^{-x} = \dfrac{e^{-x}}{4}(x^2 + 3x)$

And the general solution of the given differential equation is

$$y = c_1 + c_2e^{-x} + c_3e^x + \frac{e^{-x}}{4}(x^2 + 3x).$$

***

Let us consider another example.

Example 12: Solve the differential equation

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^{-x}(2 - x^2)$$

Solution: The auxiliary equation is

$$m^3 + 3m^2 + 3m + 1 = 0$$

$$\Rightarrow \quad (m+1)(m^2 + 2m + 1) = 0$$

$$\Rightarrow \quad (m+1)^3 = 0$$

$$\Rightarrow \quad m = -1, -1, -1$$

$\therefore$ C.F. $= (c_1 + xc_2 + x^2c_3)\,e^{-x}$.

Here the root -1 is repeated 3 times and the term e^{-x} appears in the complementary solution. Thus we take the form of a trial solution as

$$y_p(x) = x^3e^{-x}(Ax^2 + Bx + C) = e^{-x}(Ax^5 + Bx^4 + Cx^3)$$

Substituting for y_p and its derivatives in the given equation, we get

$$e^{-x}(60Ax^2 + 24Bx + 6C) = 2e^{-x} - x^2e^{-x}$$

Equating the coefficients of $e^{-x}x^2 : 60A = -1 \Rightarrow A = -1/60$

Equating the coefficients of $e^{-x}x : 24B = 0 \Rightarrow B = 0$

Equating the coefficients of $e^{-x} : 6C = 2 \Rightarrow C = \dfrac{1}{3}$

Substituting the values of A, B and C in $y_p(x)$, we get

$$y_p(x) = \frac{x^3e^{-x}}{60}(20 - x^2).$$

The general solution of the given equation is

$$y = (c_1 + xc_2 + x^2c_3)e^{-x} + \frac{x^3e^{-x}}{60}(20 - x^2).$$

***

You may now try the following exercise.

E8) Solve the following differential equations:

i) $$\frac{d^2y}{dx^2} + 9y = x^2e^{3x}$$

ii) $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4xe^{2x}$$

We now take up an example in which $b(x)$ is a product of a polynomial, an exponential and a sinusoidal function.

Example 13: Write down the form of a trial solution for the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = x^2e^{-x}\sin x$$

Solution: The auxiliary equation is

$$m^2 + 2m + 5 = 0$$

$$\Rightarrow \quad m = -1 \pm 2i$$

$$\Rightarrow \quad \text{C.F.} = e^{-x}(c_1\cos 2x + c_2\sin 2x)$$

The non-homogeneous term in this case is a product of x^2, a polynomial of degree two, an exponential function e^{-x} and a sine function, $\sin x$. The product $e^{-x}\sin x$ which appears in the non-homogenous term in not a part of the complementary solution of the given equation. Thus, from our understanding of the methods discussed in Sub-secs. 11.2.1-11.2.3, the appropriate form of a trial solution is

$$y_p = (A_0 + A_1x + A_2x^2)e^{-x}(B_0\sin x + B_1\cos x)$$

where A_0, A_1, A_2, B_0, B_1 are the constants to be determined. Equivalently, the trial solution y_p can be written in the form

$$y_p = (A + Bx + Cx^2)e^{-x}\sin x + (D + Ex + Fx^2)e^{-x}\cos x$$

for constants A, B, C, D, E and F. The constants can be determined following the usual process of substituting for y_p and its derivatives in the given equation and then comparing the coefficients on both the sides of the resulting equation.

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In general, suppose that the non-homogeneous term $b(x)$ in Eqn. (1) has one of the following two forms:

$$b(x) = e^{\alpha x} B_k(x) \sin \beta x \text{ or } b(x) = e^{\alpha x} B_k(x) \cos \beta x, \tag{21}$$

where $B_k(x)$, is a polynomial of degree k or less and α and β are any real numbers. A trial solution can then be taken in the form

$$y_p(x) = (A_0 x^k + A_1 x^{k-1} + \cdots + A_k)\, e^{\alpha x} \cos \beta x +$$
$$(B_0 x^k + B_1 x^{k-1} + \cdots + B_k)\, e^{\alpha x} \sin \beta x,$$

provided, $\alpha \pm i\beta$ is not a root of the auxiliary equation otherwise, the terms $e^{\alpha x} \cos \beta x$ or $e^{\alpha x} \sin \beta x$ would already be present in the complementary solution of the equation. Here $A_0, A_1, \ldots, A_k, B_0, B_1, \ldots, B_k$ are constants to be determined.

In such cases, i.e., when $(\alpha \pm i\beta)$ is a root, say, it is r-times repeated root of the auxiliary equation then a trial solution is modified by multiplying it by x^r.

We illustrate the case with the help of the following examples.

Example 14: Write the appropriate form of a trial solution for the differential equaton $\frac{d^4 y}{dx^4} + 2\frac{d^3 y}{dx^3} + 2\frac{d^2 y}{dx^2} = 3e^x + 2xe^{-x} + e^{-x} \sin x$.

Solution: The auxiliary equation is

$$m^4 + 2m^3 + 2m^2 = 0$$
$$\Rightarrow \quad m^2(m^2 + 2m + 2) = 0$$
$$\Rightarrow \quad m = 0, 0, -1 \pm i$$

$\therefore$ C.F. $= c_1 + c_2 x + e^{-x}(c_3 \sin x + c_4 \cos x)$

Here the non-homogeneous term is $3e^x + 2xe^{-x} + e^{-x} \sin x$

Since the term $e^{-x} \sin x$ also appear in C.F., the appropriate form of the trial solution is

$$y_p = Ae^x + (Bx + C)\, e^{-x} + xe^{-x}\,(D \cos x + E \sin x)$$

where A, B, C, D and E are the constants to be determined.

***

Example 15: Write the appropriate form of a trial solution for the differential equation

$$\frac{d^4 y}{dx^4} + 2\frac{d^2 y}{dx^2} + y = x \sin x$$

Solution: The auxiliary equation is

$$m^4 + 2m^2 + 1 = 0$$
$$\Rightarrow \quad (m^2 + 1)^2 = 0$$
$$\Rightarrow \quad m = \pm i, \pm i$$

$\therefore$ C.F. $= (c_1 + xc_2) \cos x + (c_3 + xc_4) \sin x$

Here $\pm i$ is a double root of the equation and the non-homogeneous term $x\sin x$ is also a part of the complementary solution. The appropriate form of a trial solution will then be

$$y_p = x^2[(Ax+B)\cos x + (Cx+D)\sin x]$$

where A, B, C and D are the constants to be determined.

***

And now some exercises for you.

E9) Write the form of a trial solution for each of the following differential equations:

i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = x\cos 3x - \sin 3x$

ii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = xe^{-x}\cos 2x$

iii) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = xe^{x}\cos 2x$

iv) $\frac{d^2y}{dx^2} + y = x^2\sin x$

v) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = xe^{2x}\sin x$

E10) Find the general solution of the following differential equations:

i) $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = x + 3\cos x + e^{-2x}$

ii) $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 4 + x\sin x$

iii) $\frac{d^3y}{dx^3} + \frac{dy}{dx} = x^3 + \cos x$

After going through the methods discussed in Sub-secs.11.2.1-11.2.4 and attempting the exercises given at the end of each sub-section you must have observed some of the advantages of using the method. You must have felt at times, the complexities involved in the application of the method. Let us now summarise the observations and the constraints of this method.

11.3 OBSERVATIONS AND CONSTRAINTS OF THE METHOD

1. Method is straight forward in application. Once a trial solution is assumed, the method merely involves the differentiation of algebraic functions and solving of simultaneous equations to obtain the values of undetermined coefficients involved in the trial solution.

2. It can be used by any learner who is not familiar with more elegant techniques of finding the solutions of the differential equations such as

variation of parameters and inverse differential operators which involve integrations. We shall be discussing these techniques in the subsequent units.

3. Success of this method depends to a certain extent on the ability to guess an appropriate form of a trial solution. As illustrated through an example in the case of Eqn. (7), if the form of a trial solution assumed is not proper, the method fails to yield the solution.

4. If the non-homogeneous term is complicated and the trial solution involves a large number of terms as is the case with Examples 14 and 15, then determination of coefficients in the trial solution becomes laborious.

5. This method is not a general method of finding a particular solution of differential equations. It is applicable to linear non-homogeneous equations with **constant coefficients** with non-homogeneous terms restricted to certain particular forms. More general methods of finding a particular solution will be discussed in the units to follow.

We now end this unit by giving a summary of what we have coverd in it.

11.4 SUMMARY

In this unit, we have coverd the following:

1. Method of undertermined coefficients is applicable if
   i) the equation is a linear equation with constant coefficients.
   ii) the non-homogeneous term is either a polynomial, an exponential function, a sinusoidal function or a sum/product of these functions.

2. The results giving the form of a trial solution $y_p(x)$ for different non-homogeneous term $b(x)$ for the cases where the corresponding auxiliary equation has ***r*-times repeated root** are summarised in the following table.

Non-homogeneous term, $\mathrm{b(x)}$	Trial solution, $\mathrm{y_p(x)}$
$p_k(x) = b_0 x^k + b_1 x^{k-1} + \cdots + b_{k-1}x + b_k$	$x^r(A_0x^k + A_1x^{k-1} + \cdots + A_k)$
$e^{\alpha x}$	$x^r(Ae^{\alpha x})$
$\begin{cases}\sin\beta x\\ \cos\beta x\end{cases}$	$x^r(A\sin\beta x + B\cos\beta x)$
$e^{\alpha x}P_k(x)$	$x^r e^{\alpha x}(A_0x^k + \cdots + A_k)$
$e^{\alpha x}P_k(x)\begin{cases}\sin\beta x\\ \cos\beta x\end{cases}$	$x^r[(A_0x^k + \cdots + A_k)e^{\alpha x}\sin\beta x$ $+(B_0x^k + \cdots + B_k)e^{\alpha x}\cos\beta x]$

3. Observations and constraints of the method.

6.5 SOLUTIONS/ANSWERS

E1) i) The auxiliary equation is $m^2 + m + 1 = 0$

$$\Rightarrow \quad m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Here zero is not a root of the auxiliary equation. Also non-homogeneous term for the given equation is x^2+1, a polynomial of degree 2. Hence a suitable form of a P.I. is

$y_p(x) = Ax^2 + Bx + C$, where A, B, C are the constants to be determined.

ii) The auxiliary equation is

$m(m^3 - m^2 - m + 1) = 0$

Since zero is a non-repeated root of the auxiliary equation and the non-homogeneous term in the given equation is x^2, the suitable form of P.I. is $y_p(x) = x(Ax^2 + Bx + C)$, A, B, C are the constants to be determined.

E2) i) The auxiliary equation is

$m^2 + 3m + 2 = 0$

$\Rightarrow m = -1, -2$

Hence C.F. $= c_1 e^{-x} + c_2 e^{-2x}$

The form of a trial solution considered is

$y_p(x) = Ax^2 + Bx + C$.

Substituting for $y_p(x)$ and its derivatives in the given equation, we get

$2(Ax^2 + Bx + C) + 3(2Ax + B) + 2A = 4x^2$

Equating the coefficients of like powers of x on both the sides and solving for A, B and C, we obtain

$A = 2, B = -6$ and $C = 7$

Hence, $y_p(x) = 2x^2 - 6x + 7$ and the general solution is

$y = c_1 e^{-x} + c_2 e^{-2x} + 7 - 6x + 2x^2$.

ii) The given equation is

$$\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = x$$

The auxiliary equation is

$m^3 + 4m = 0 \Rightarrow m = 0, \pm 2i$

Hence $\text{C.F.} = c_1 + c_2 \cos 2x + c_3 \sin 2x$

Since zero is a solution of the auxiliary equation and the non-homogeneous term in this case is x, which is of degree 1, the appropriate form of a trial solution is

$y_p(x) = x(Ax + B) = Ax^2 + Bx$.

Substituting this value of y_p and its derivatives in the given differential equation and comparing the coefficients of like powers of x, we get

$A = \frac{1}{8}$ and $B = 0$

Hence $\text{P.I.} = \frac{1}{8}x^2$ and the general solution is

$$y = c_1 + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{8}x^2.$$

E3) i) $\text{C.F.} = c_1 + c_2e^{2x} + c_3e^{-2x}$

Form of a trial solution is $y_p(x) = Axe^{-2x}$

By substituting this value of y_p in the given equation, we get

$A = \frac{1}{8}$. Hence, P.I. is $\frac{1}{8}xe^{-2x}$.

ii) $\text{C.F.} = c_1e^{-x} + c_2\cos x + c_3\sin x$

Since the non-homogeneous term of the given equation is e^{-x} which is present in C.F. and -1 is a non-repeated root of the auxiliary equation, a suitable form of the trial solution is $y_p(x) = Axe^{-x}$.

Substituting this value of y_p in the given equation, we get $A = \frac{1}{2}$.

Hence $P.I. = \frac{1}{2}xe^{-x}$.

E4) i) Auxiliary equation is

$m^2 - 4m + 4 = 0$

$\Rightarrow m = 2, 2$

$\therefore$ C.F. $= (c_1 + xc_2)e^{2x}$

First consider the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2$$

Take a trial solution of the form

$y_{p_1} = Ax^2 + Bx + C$

Substituting for y_{p_1} and its derivatives in the above equation and solving for A, B, C, we obtain

$A = \frac{1}{4}, B = \frac{1}{2}, C = \frac{3}{8}$

$\therefore \; y_{p_1} = \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$

For the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$$

consider $y_{p_2} = Ae^x$ and obtain $A = 1$ $y_{p_2} = e^x$

$\therefore \; y_p = y_{p_1} + y_{p_2} = \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} + e^x$.

The general solution of the given equation is

$y = (c_1 + xc_2)e^{2x} + \frac{1}{8}(2x^2 + 4x + 3) + e^x$.

ii) $y = c_1\cos 2x + c_2\sin 2x - \frac{3}{16}x + \frac{1}{8}x^3 + \frac{1}{16}e^{2x}$

Hint: Take trial solutions in the form

$y_{p_1} = A_0 + A_1x + A_2x^2 + A_3x^3$

$y_{p_2} = Be^{2x}$.

E5) i) The auxiliary equation is

$m^2 - 1 = 0$

$\Rightarrow m = 1, -1$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{-x}$

Take a trial solution of the form

$y_p = Ae^{2x}$

and obtain $A = \frac{1}{3}$

$\therefore \quad y_p = \frac{1}{3} e^{2x}$

The general solution of the given equation is

$y = c_1 e^x + c_2 e^{-x} + \frac{1}{3} e^{2x}$

Using given initial conditions

$y(0) = -1 \Rightarrow c_1 + c_2 + \frac{1}{3} = -1$

$y'(0) = 1 \Rightarrow c_1 - c_2 + \frac{2}{3} = 1.$

Solving for c_1 and c_2, we get

$c_1 = \frac{-1}{2}, c_2 = \frac{-5}{6}$

$\therefore \quad y = -\frac{e^x}{2} - \frac{5e^{-x}}{6} + \frac{e^{2x}}{3}$

ii) C.F. $= (c_1 + xc_2) e^{2x}$

Take $y_p = Ax^2 e^{2x}$ and obtain $A = \frac{-1}{2}$

General solution is

$y = (c_1 + xc_2) e^{2x} - \frac{x^2}{2} e^{2x}$

Using given initial conditions obtain

$c_1 = c_2 = 0$

$\therefore \quad y = \frac{-x^2 e^{2x}}{2}.$

E6) i) C.F. $= (c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x)$

Since term $\sin 2x$ appears in C.F. and it is the non-homogeneous term of the given equation, the trial solution may be taken as

$y_p(x) = x(A \sin 2x + B \cos 2x)$

Substituting this value of y_p in the given equation, we get

$A = 0$ and $B = \frac{1}{16}.$

$\therefore \quad$ P.I. $= \frac{1}{16} x \cos 2x.$

Hence the solution of the given equation is

$y = (c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x) + \frac{1}{16} x \cos 2x.$

ii) $y = c_1 + c_2 e^x + c_3 e^{-x} - \sin x$

E7) i) C.F. $= c_1 \cos 2x + c_2 \sin 2x$.

Form of a trial solution is

$y_p = A\cos x + B\sin x$

Substituting for y_p in the given equation and simplifying

$$y_p = \frac{1}{3}\sin x$$

$\therefore \quad y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3}\sin x$ is the general solution.

Using initial conditions, we get

$$c_1 = 2,\ c_2 = \frac{-2}{3}$$

$$\therefore \quad y = 2\cos 2x - \frac{2}{3}\sin 2x + \frac{1}{3}\sin x$$

ii) C.F. $= c_1 \cos x + c_2 \sin x$

Form of a trial solution is

$y_p = A\cos 2x + B\sin 2x$

Substituting for y_p in the given equation and simplifying

$$y_p = \frac{1}{3}(\sin 2x - \cos 2x)$$

General solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{3}(\sin 2x - \cos 2x)$$

Using I.Cs, we get

$$c_1 = \frac{-1}{60},\ c_2 = \frac{-7}{15}$$

$$\therefore \quad y = \frac{-1}{60}\cos x - \frac{7}{6}\sin x + \frac{1}{3}(\sin 2x - \cos 2x)$$

E8) i) C.F. $= \mathrm{c_1 \cos 3x + c_2 \sin 3x}$

The appropriate form of a trial solution is

$y_p(x) = e^{3x}(A + Bx + Cx^2)$

On substituting this value of y_p in the given equation and comparing the coefficients of $x^2e^{3x}, x\,e^{3x}$ and e^{3x} on both the sides of the resulting equation, we get

$A = \frac{1}{162},\ B = \frac{-1}{27}$ and $C = \frac{1}{18}$.

Hence $y_p(x) = e^{3x}\left(\frac{1}{162} - \frac{1}{27}x + \frac{1}{18}x^2\right)$

$\therefore$ The solution of the given equation is

$$y = c_1 \cos 3x + c_2 \sin 3x + e^{3x}\left(\frac{1}{162} - \frac{1}{27}x + \frac{1}{18}x^2\right).$$

ii) C.F. $= \mathrm{(c_1 + c_2 x)\, e^{2x}}$

Here the non-homogenenous term is $4xe^{2x}$. Also e^{2x} appears in the C.F. Further 2 is repeated root of the auxiliary equation. Thus appropriate form of a trial solution is

$y_p(x) = x^2 e^{2x}(Ax + B) = (Ax^3 + Bx^2)\ e^{2x}$.

Substituting this value of y_p in the given equation and equating the coefficients of xe^{2x} and e^{2x} on both the sides of the resulting equation, we get $A = \frac{2}{3}$ and $B = 0$. Hence $P.I. = \frac{2}{3}x^3 e^{2x}$ and the general solution is

$y = (c_1 + c_2 x) e^x + \frac{2}{3}x^3 e^{2x}$.

E9) i) $y_p = (A_0 + A_1 x)\cos 3x + (B_0 + B_1 x)\sin 3x$

ii) $y_p = x[(A_0 + A_1 x)\ e^{-x}\cos 2x + (B_0 + B_1 x) e^{-x}\sin 2x]$

iii) $y_p = e^x[(A_x + B)\cos 2x + (Cx + D)\sin 2x]$

iv) $y_p = x[(A_0 + A_1 x + A_2 x^2)\ \cos x + (B_0 + B_1 x + B_2 x^2)\sin x]$

v) $y_p = xe^{2x}[(A + Bx)\cos x + (C + Dx)\sin x]$

E10) i) $\text{C.F.} = c_1 + c_2 e^{2x} + c_3 e^{-2x}$

Here e^{-2x} is present in the non-homogeneous term as well as in the C.F. Also zero is a root of auxiliary equation. Hence a suitable form of a trial solution is

$y_p(x) = x(Ax + B) + (C\cos x + D\sin x) + Exe^{-2x}$

Substituting this value of y_p and its derivatives in the given equation and equating the coefficients of x, constant term, $\cos x$, $\sin x$ and e^{-2x} on both the sides of the resulting equation, we get

$A = -\frac{1}{8},\ B = 0,\ C = 0,\ D = -\frac{3}{5},\ E = \frac{1}{8}$

Hence $y_p(x) = -\frac{1}{8}x^2 - \frac{3}{5}\sin x + \frac{1}{8}xe^{-2x}$

$\therefore$ The general solution of the given equation is

$y = c_1 + c_2 e^{2x} + c_3 e^{-2x} - \frac{1}{8}x^2 - \frac{3}{5}\sin x + \frac{1}{8}xe^{-2x}$.

ii) $\text{C.F.} = c_1 x + (c_2 + c_3 x)e^x + c_4 e^{-x}$

Appropriate form of a trial solution is

$y_p(x) = x(Ax^2 + Bx + C) + (Dx\sin x + Ex\cos x + F\sin x + G\cos x)$

Substituting this value of y_p and its derivatives in the given equation and equating the coefficients of x^2, x, constant term, $x\sin x$, $x\cos x$, $\sin x$ and $\cos x$ on both the sides of a resulting equation, we have

$A = \frac{1}{3},\ B = 1,\ C = 8,\ D = 0,\ E = -\frac{1}{2},\ F = 1,\ G = -\frac{1}{2}$

$\therefore\ y_p(x) = x\left(\frac{1}{3}x^2 + x + 8\right) + \left(-\frac{1}{2}x\cos x\right) + \left(\sin x - \frac{1}{2}\cos x\right)$

The general solution of the given equation is

$$y = c_1 x + (c_2 + c_3 x)e^x + c_4 e^{-x} + x\left(\frac{1}{3}x^2 + x + 8\right)$$
$$-\frac{1}{2}x\cos x + \sin x - \frac{1}{2}\cos x.$$

iii) $\text{C.F.} = c_1 + c_2 \cos x + c_3 \sin x$

Since zero is a root of A.E. and also since $\cos x$ appears in C.F., therefore a suitable form of a trial solution is

$$y_p(x) = x(Ax^3 + Bx^2 + Cx + D) + x(E\cos x + F\sin x)$$

Substituting this value of y_p and its derivatives in the given equation and comparing the coefficients of x^3, x^2, x, constant term, $\sin x$ and $\cos x$ in the resulting equation, we get

$$A = \frac{1}{4},\ B = 0,\ C = -3,\ D = 0,\ E = -\frac{1}{2},\ F = 0$$

Hence $y_p(x) = x\left(\frac{1}{4}x^3 - 3x\right) + x\left(-\frac{1}{2}\cos x\right)$

$$= \frac{1}{4}x^4 - 3x^2 - \frac{1}{2}x\cos x$$

$\therefore$ The general solution of the given equation is

$$y = c_1 + c_2\cos x + c_3\sin x + \frac{1}{4}x^4 - 3x^2 - \frac{1}{2}x\cos x.$$

– x –

UNIT 12

DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Structure **Page No.**

12.1 INTRODUCTION

In Unit 11, we discussed the method of undetermined coefficients for determining a particular solution of the non-homogeneous differential equations with constant coefficients when the non-homogeneous term is of a particular form (viz., a polynomial, an exponential, a sinusoidal function or a sum or a product of these functions).

In this unit we familiarise you with a method of determining a particular solution that can be applied even when the coefficients of the differential equation are functions of the independent variable and the non-homogeneous term may not be restricted to the particular forms mentioned above. Such an approach as discussed in Sec. 12.2 is due to the French mathematician Joseph Louis Lagrange (1736-1813) and is termed as **variation of parameters**. Even though the approach is quite general but is limited in its scope in the sense that it can be utilised in situations where the fundamental solution set for the reduced equation is known. Also, it can be used for first and higher order equations alike though its appreciation can be well understood for the later set of equations. The method requires for its applicability the complete knowledge of the fundamental solution set of the reduced equation and for equations with variable coefficients the determination of this set may be extremely difficult. In the case of linear differential equations with variable coefficients, at times, it may not be possible to find all linearly independent solutions of the reduced equation but at least one or more may be obtainable. For such situations Jean le Rond d'Alembert (1717-1783), a French mathematician and a physicist, developed a method that is often called the method of **reduction of order**. When one or more

D'Alembert (1717-1783)

solutions of the reduced equation are known then D'Alembert's method can be used to derive an equation of order lower than that of a given equation and obtain the rest of the solutions of a reduced equation as well as a particular integral of the non-homogeneous equation. We shall be discussing the method of reduction of order in Sec. 12.3 of the unit.

However, there exist linear differential equations with variable coefficients of second and higher order for which we may not be able to guess any integral of its complementary function. But, among such equations is a class of equation known as Cauchy-Euler equation or more commonly as Euler's equation, where, by certain transformation of the independent variable, it is possible to find all the integrals of its complementary function. In Sec. 12.4, we shall be discussing the methods of solving Euler's equation and those equations which are reducible to Euler's form. Leonhard Euler (1707-1783) born in Basel, Switzerland, was a physicist, astronomer, linguist, physiologist and primarily, a mathematician. He made contributions to algebra, trigonometry, analytic geometry, calculus, differential equations, complex variables, number theory and topology.

Objectives

After reading this unit, you should be able to:

- use the method of variation of parameters to find a particular integral of the non-homogeneous linear differential equations with constant or variable coefficients;
- use the method of reduction of order to find the complete integral of linear non-homogeneous equation of second order when one integral of the corresponding homogeneous equation is known; and
- solve Euler's equation.

12.2 VARIATION OF PARAMETERS

Before we discuss the method of variation of parameters in general, we illustrate it through the following example.

Example 1: Find a particular solution of the differential equation

$$y'' + 4y = 3\operatorname{cosec} 2x, \ \ 0 < x < \pi/2. \qquad (1)$$

Solution: You may observe here that the problem does not fall within the scope of the method of undetermined coefficients since the non-homogeneous term $b(x) = \dfrac{3}{\sin 2x} = 3\operatorname{cosec} 2x$ involves a quotient rather than a sum or a product of $\sin x$ or $\cos x$. Thus, we need to use a different method for solving the equation. The homogeneous equation corresponding to Eqn. (1) is

$$y'' + 4y = 0 \qquad (2)$$

The general solution of Eqn. (2) or the complementary function of differential Eqn. (1) is given by

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x \qquad (3)$$

where c_1 and c_2 are constants.

For finding a particular integral of Eqn. (1) the basic idea in the method of variation of parameters, as the name suggests, is to vary the constants c_1 and

c_2 in Eqn. (3) and replace them by functions $u_1(x)$ and $u_2(x)$, respectively. These functions are then determined so that the function

$$y_p(x) = u_1(x)\cos 2x + u_2(x)\sin 2x \tag{4}$$

is a solution of the non-homogeneous Eqn. (1).

In order to determine u_1 and u_2 we need to substitute for y_p from Eqn. (4) in Eqn. (1). We know that the substitution of y_p and its derivatives from Eqn. (4) in Eqn. (1) will give us a single equation involving some combination of u_1 and u_2 and their first two derivatives. Since there is only one equation and the two unknown functions u_1 and u_2, we may expect a few possible choices of u_1 and u_2 that would serve our purpose. In other words, we may impose the second condition of our own choice and obtain two equations for the two unknowns u_1 and u_2. We shall now show that following Lagrange's approach the second condition on u_1 and u_2 may be chosen so that the computations are simplified. Let us see how it is done.
Differentiating Eqn. (4), we obtain

$$y_p' = -2u_1(x)\sin 2x + 2u_2(x)\cos 2x + u_1'(x)\cos 2x + u_2'(x)\sin 2x \tag{5}$$

Keeping in mind the possibility of choosing the second condition on u_1 and u_2, let us choose the last two terms of Eqn. (5) to be zero, that is,

$$u_1'(x)\cos 2x + u_2'(x)\sin 2x = 0 \tag{6}$$

Then from Eqn. (5), we get the simplified expression for y_p' as

$$y_p' = -2u_1 \sin 2x + 2u_2 \cos 2x \tag{7}$$

Differentiating Eqn. (7) once again, we obtain

$$y_p'' = -4u_1 \cos 2x - 4u_2 \sin 2x - 2u_1' \sin 2x + 2u_2' \cos 2x \tag{8}$$

Substituting for y_p' and y_p'' in Eqn. (1) from Eqns. (7) and (8), respectively, we obtain

$$-2u_1' \sin 2x + 2u_2' \cos 2x = 3\operatorname{cosec} 2x \tag{9}$$

We can now sum up by saying that we want to choose u_1 and u_2 satisfying Eqns. (6) and (9). Eqns. (6) and (9) form a pair of linear algebraic equations for the two unknown quantities $u_1'(x)$ and $u_2'(x)$. Solving these equations, we obtain

$$u_1'(x) = \frac{-3}{2} \text{ and } u_2'(x) = \frac{3}{2}\cot 2x \tag{10}$$

Integrating $u_1'(x)$ and $u_2'(x)$, we obtain

$$u_1(x) = \frac{-3}{2}x \text{ and } u_2(x) = \frac{3}{4}\ln|\sin 2x|$$

Substituting the above values of u_1 and u_2 in Eqn. (4), we obtain a particular integral of Eqn. (1) as

$$y_p(x) = \frac{-3}{2}x\cos 2x + \frac{3}{2}\ln|\sin 2x|\sin 2x \tag{11}$$

Then from Eqns (3) and (11), we get the general solution of Eqn. (1) as

$$y = y_c + y_p$$
$$= c_1 \cos 2x + c_2 \sin 2x - \frac{3}{2} x \cos 2x + \frac{3}{2} \ln |\sin 2x| \sin 2x$$

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Let us now see how the method above works in general.

Consider the non-homogeneous second order linear equation

$$y'' + a_1(x)y' + a_2(x)y = b(x) \tag{12}$$

where we have taken the coefficient of y'' to be 1 and assumed that a_1, a_2 and b are defined and continuous on some interval I. Let $\{y_1(x), y_2(x)\}$ be a fundamental solution set for the corresponding homogeneous equation

$$y'' + a_1(x)y' + a_2(x)y = 0. \tag{13}$$

Then we know that the general solution of Eqn. (13) is given by

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x), \tag{14}$$

where c_1 and c_2 are constants. As we have mentioned in Example 1, the idea associated with the method of variation of parameters is to replace the constants c_1 and c_2 in Eqn. (14) by functions $u_1(x)$ and $u_2(x)$ and then determine $u_1(x)$ and $u_2(x)$ so that $y_p(x)$ given by the equation

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{15}$$

satisfies Eqn. (12).
That is, we seek a particular integral of Eqn. (12) of the form (15) where u_1 and u_2 are unknown functions to be determined. Since we have introduced two unknowns, we need two equations involving these functions for their determination. Thus, we impose two conditions on the functions u_1 and u_2 in order that relation (15) is a solution of Eqn. (12). We call these conditions the **auxiliary conditions.** These conditions are imposed so that the calculations are simplified. Let us see how this is done.

If $y_p(x)$ given by Eqn. (15) is a solution of Eqn. (12), then it must satisfy it. We compute $y_p'(x)$ and $y_p''(x)$ from Eqn. (15), and obtain

$$y_p' = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2') \tag{16}$$

To simplify the computation and to avoid second order derivatives for the unknowns u_1, u_2 in the expression for y_p'', let us choose the **first auxiliary condition** as

$$u_1' y_1 + u_2' y_2 = 0 \tag{17}$$

Eqn. (16) then reduces to

$$y_p' = u_1 y_1' + u_2 y_2' \tag{18}$$

and we have:

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' \tag{19}$$

Substituting in Eqn. (12), the expressions for y_p, y_p' and y_p'' as given by Eqns. (15), (18) and (19), respectively, we get

$$b(x) = (u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + a_1(u_1y_1' + u_2y_2') + a_2(u_1y_1 + u_2y_2)$$

$$= (u_1'y_1' + u_2'y_2') + u_1(y_1'' + a_1y_1' + a_2y_1) + u_2(y_2'' + a_1y_2' + a_2y_2) \tag{20}$$

Since y_1 and y_2 are the solutions of the homogeneous Eqn. (13), we have

$$y_1'' + a_1y_1' + a_2y_1 = 0$$

and

$$y_2'' + a_1y_2' + a_2y_2 = 0.$$

Thus, Eqn. (20) becomes

$$u_1'y_1' + u_2'y_2' = b(x) \tag{21}$$

which is the **second auxiliary condition**.

Now if we can find u_1 and u_2 satisfying the two auxiliary conditions given by Eqns. (17) and (21), viz.,

$$\left.\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= b(x) \end{aligned}\right] \tag{22}$$

then $y_p(x)$ given by Eqn. (15) will be a particular solution of Eqn. (12). In order to determine u_1, u_2 we first solve the linear system of Eqns. (22) for u_1' and u_2' using Cramer's rule (ref. appendix Unit 10). Algebraic manipulations yield

$$u_1'(x) = \frac{-b(x)\ y_2(x)}{W(y_1,\ y_2)},\ u_2'(x) = \frac{b(x)\ y_1(x)}{W(y_1,\ y_2)}, \tag{23}$$

where

$$W(y_1,\ y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

is the **Wronskian** of y_1 and y_2.

Note that the division by W in Eqn. (23) is permissible since y_1 and y_2 form a fundamental set of solutions and therefore their Wronskian is non-zero on I. On integrating $u_1'(x)$ and $u_2'(x)$ given by Eqns. (23), we obtain

$$u_1(x) = \int \frac{-b(x)\ y_2(x)}{W(y_1,\ y_2)}\, dx,\ u_2(x) = \int \frac{b(x)\ y_1(x)}{W(y_1,\ y_2)}\, dx \tag{24}$$

Hence

$$y_p(x) = y_1(x) \int \frac{-b(x)\ y_2(x)}{W(y_1,\ y_2)}\, dx + y_2(x) \int \frac{b(x)\ y_1(x)}{W(y_1,\ y_2)}\, dx \tag{25}$$

is a particular integral of Eqn. (12).

We now sum up the various steps involved in determining a particular solution of Eqn. (12).

Step I: Given Eqn. (12), find a fundamental solution set $\{y_1(x),\ y_2(x)\}$ for the corresponding homogeneous Eqn. (13).

Step II: Assume a particular integral of Eqn. (12) in the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)\ y_2(x)$$

and determine $u_1(x)$ and $u_2(x)$ by using the formula (24) directly or by first solving the system of Eqns. (22) for $u_1'(x)$ and $u_2'(x)$ and then integrating.

Step III: Substitute $u_1(x)$ and $u_2(x)$ into the expression for $y_p(x)$ in Eqn. (15) to obtain a particular integral.

We now illustrate these steps with the help of the following examples.

Example 2: Determine the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \sec x,\ 0 < x < \frac{\pi}{2}$$

Solution: Step I: The auxiliary equation corresponding to the given equation is

$$m^2 + 1 = 0$$
$$\Rightarrow \quad m = \pm i$$

and the two solutions of the reduced equation are

$$y_1(x) = \cos x$$

and

$$y_2(x) = \sin x$$

Hence the complementary function is given by

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

Step II: To find a particular integral, we write

$$y_p(x) = u_1(x)\cos x + u_2(x)\sin x \tag{26}$$

$$\therefore \quad \frac{dy_p}{dx} = [-u_1(x)\sin x + u_2(x)\cos x] + \frac{du_1}{dx}\cos x + \frac{du_2}{dx}\sin x$$

Let us take the **first auxiliary condition** as

$$\frac{du_1}{dx}\cos x + \frac{du_2}{dx}\sin x = 0 \tag{27}$$

so that

$$\frac{dy_p}{dx} = -u_1(x)\sin x + u_2(x)\cos x$$

Differentiating the above equation once again, we get

$$\frac{d^2y_p}{dx^2} = -u_1(x)\cos x - u_2(x)\sin x - \sin x\frac{du_1}{dx} + \cos x\frac{du_2}{dx} \tag{28}$$

Since $y_p(x)$ must satisfy the given equation, we substitute in the given equation the expressions for y_p and y_p'' from Eqns. (26) and (28), respectively, and obtain the **second auxiliary condition as**

$$-\sin x\frac{du_1}{dx} + \cos x\frac{du_2}{dx} = \sec x \tag{29}$$

On solving Eqns. (27) and (29) for $\frac{du_1}{dx}$ and $\frac{du_2}{dx}$, we get

$$\frac{du_1}{dx} = -\tan x,\ \frac{du_2}{dx} = 1,$$

which on integration yields

$$u_1(x) = \ln\ (\cos x) \text{ and } u_2(x) = x.$$

Step III: Substituting the values of $u_1(x)$ and $u_2(x)$ in Eqn. (26) we obtain a particular integral of the given equation in the form

$$y_p(x) = \cos x\ \ln\ (\cos x) + x\sin x$$

and the general solution is

$$y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln\ (\cos x).$$

***

Note that in Eqn. (12) we have taken the coefficient of y'' to be 1. If the given equation is of the form $a_0(x)y'' + a_1(x)y' + a_2(x)y = b(x)$, where $a_0(x) \neq 1$ then before applying the method, particularly, if you are using the formula (24), the equation must be put in the form $y'' + p(x)y' + q(x)y = b(x)$; otherwise, the non-homogeneous term $b(x)$ will not be correctly identified. Let us consider the following example to understand the point made above.

Example 3: Find the general solution of the differential equation

$$(1-x^2)\,\frac{d^2y}{dx^2} - \frac{1}{x}\,\frac{dy}{dx} = f(x) \quad x \neq \pm 1$$

Solution: Step I: To make the coefficient of $y'' = 1$, we rewrite the given equation in the form

$$\frac{d^2y}{dx^2} - \frac{1}{x(1-x^2)}\,\frac{dy}{dx} = \frac{f(x)}{(1-x^2)}. \tag{30}$$

The homogeneous equation corrsesponding to this equation is

$$\frac{d^2y}{dx^2} - \frac{1}{x(1-x^2)}\,\frac{dy}{dx} = 0 \tag{31}$$

To solve Eqn. (31) we put $\frac{dy}{dx} = p$ and obtain the equation

$$\frac{dp}{dx} - \frac{1}{x(1-x^2)}\ p = 0$$

or, $$\frac{1}{p}\,dp = \frac{dx}{x(1-x^2)} \tag{32}$$

Now Eqn. (32) is in variable separable form and can be expressed as

$$\frac{dp}{p} = \left[\frac{1}{x} + \frac{x}{1-x^2}\right] dx$$

Integrating the above equation, we get

$\ln p = \ln x - \frac{1}{2}\ln(1-x^2) + \ln c_1$, c_1 is a constant.

$$\Rightarrow \quad \ln p = \ln \frac{c_1 x}{(1-x^2)^{1/2}}$$

$$\Rightarrow \quad p = \frac{c_1 x}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{c_1 x}{\sqrt{1-x^2}}. \tag{33}$$

Integrating Eqn. (33), once again, we get the solution of Eqn. (31) i.e., the C.F. of Eqn. (30) in the form

$$y_c(x) = -c_1\,\sqrt{1-x^2} + c_2 \tag{34}$$

where c_1 and c_2 are arbitrary constants.

Step II: For the given differential Eqn. (30), assume a particular integral in the form

$$y_p(x) = u_1(x)\sqrt{1-x^2} + u_2(x) \tag{35}$$

$$\therefore \quad \frac{dy_p}{dx} = \frac{-x}{\sqrt{1-x^2}}u_1 + \sqrt{1-x^2}\,\frac{du_1}{dx} + \frac{du_2}{dx}$$

We choose the **first auxiliary condition** as

$$\sqrt{1-x^2}\,\frac{du_1}{dx} + \frac{du_2}{dx} = 0 \tag{36}$$

Then

$$\frac{dy_p}{dx} = \frac{-x}{\sqrt{1-x^2}}u_1$$

and $\dfrac{d^2y_p}{dx^2} = \dfrac{-1}{\sqrt{1-x^2}}u_1 - \dfrac{x^2}{(1-x^2)^{3/2}}u_1 - \dfrac{x}{\sqrt{1-x^2}}\dfrac{du_1}{dx}$

$$\Rightarrow \quad \frac{d^2y_p}{dx^2} = -\frac{1}{(1-x^2)^{3/2}}u_1 - \frac{x}{\sqrt{1-x^2}}\frac{du_1}{dx}$$

Substituting, from above, the expressions for y_p' and y_p'' in Eqn. (30), we get

$$\frac{-1}{(1-x^2)^{3/2}}u_1 - \frac{x}{\sqrt{1-x^2}}\frac{du_1}{dx} - \frac{1}{x(1-x^2)}\left(\frac{-x}{\sqrt{1-x^2}}u_1\right) = \frac{f(x)}{1-x^2}$$

$$\Rightarrow \quad -x\sqrt{1-x^2}\,\frac{du_1}{dx} = f(x) \tag{37}$$

which is our **second auxiliary condition**.

Solving Eqns. (36) and (37) for $u_1'(x)$ and $u_2'(x)$ and integrating, we get

$$u_1(x) = -\int \frac{f(x)}{x\sqrt{1-x^2}}\,dx \text{ and } u_2(x) = \int \frac{f(x)}{x}\,dx$$

Step III: The expressions for $u_1(x)$ and $u_2(x)$ when substituted in Eqn. (35) gives a particular integral of the given equation in the form

$$y_p(x) = -\sqrt{1-x^2}\int \frac{f(x)}{x\sqrt{1-x^2}}\,dx + \int \frac{f(x)}{x}\,dx$$

Hence the general solution of the given differential equation is

$$y = y_c(x) + y_p(x)$$

$$= -c_1\sqrt{1-x^2} + c_2 - \sqrt{1-x^2}\int \frac{f(x)}{x\sqrt{1-x^2}}\,dx + \int \frac{f(x)}{x}\,dx.$$

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You may now try the following exercises.

E1) Determine a particular integral, using the method of variation of parameters, for the following differential equations:

i) $y'' + y = \operatorname{cosec} x,\ 0 < x < \dfrac{\pi}{2}$

ii) $y'' - 2y' + y = xe^x \ln x,\ x > 0$

iii) $y'' + y = \tan x,\ 0 < x < \dfrac{\pi}{2}$.

E2) Find the general solution of the following differential equations, given that the functions $y_1(x)$ and $y_2(x)$ for $x > 0$ are linearly independent solutions of the corresponding homogeneous equations

i) $x^2 y'' - 2xy' + 2y = x + 1;\ y_1(x) = x,\ y_2(x) = x^2$

ii) $x^2 y'' + xy' - y = x^2 e^x;\ y_1(x) = x,\ y_2(x) = \dfrac{1}{x}$

iii) $xy'' - (x+1)\ y' + y = x^2;\ y_1(x) = e^x,\ y_2(x) = x + 1$

We now state the result obtained above as a theorem.

Theorem 1: If the functions $a_0,\ a_1,\ a_2$ and b of variable x are continuous on some interval I, and if the functions y_1 and y_2 are linearly independent solutions of the homogeneous equation associated with the differential equation

$$a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = b(x) \tag{38}$$

then a particular integral of Eqn. (38) is given by

$$y_p(x) = -y_1(x)\int \frac{y_2(x)\ b(x)}{a_0(x)\ W(y_1,\ y_2)}\ dx + y_2(x)\int \frac{y_1(x)b(x)}{a_0(x)W(y_1,\ y_2)}\ dx \tag{39}$$

and the general solution is then

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x) \tag{40}$$

where $W(y_1,\ y_2)$ is the Wronskian of $y_1(x)$ and $y_2(x)$.

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Remark: In using the method of variation of parameters for finding a particular integral of a given equation, it is advisable to choose a particular integral $y_p(x) = u_1(x)\ y_1(x) + u_2(x)\,y_2(x)$, and then proceed to find $u_1(x)$ and $u_2(x)$ as we have done in Examples 1 to 3 above. It is usually avoided to memorise formulas given by Eqns. (24) and (39). Since the procedure is highly involved and complex and moreover, it may not always be easy or even possible to evaluate the integrals involved, these formulas turn out to be useful. In such cases, these formulas provide a starting point for the numerical evaluation of $y_p(x)$.

The method of variation of parameters which we have discussed for non-homogeneous second order Eqn. (12) can be easily generalised to nth order non-homogeneous equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n(x)y = b(x)$$

where $a_0,\ a_1,\ \ldots,\ a_n,\ b$ are continuous in some interval I. The learners interested in the details of the method for a higher order equation may refer to the Appendix at the end of the unit. We shall not be giving the details at this stage but, however, illustrate it through examples.

Example 4: Using the method of variation of parameters find the general solution of the differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^{2x}.$$

Solution: Step I: The auxiliary equation corresponding to the given equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$\Rightarrow \quad (m-1)\,(m^2 - 5m + 6) = 0$$

$$\Rightarrow \quad (m-1)\,(m-2)\,(m-3) = 0$$

Thus the linearly independent solutions are

$$y_1(x) = e^x,\ y_2(x) = e^{2x},\ y_3(x) = e^{3x},$$

and the complementary function is given by

$$y_c(x) = c_1e^x + c_2e^{2x} + c_2e^{3x} \tag{41}$$

Step II: To find a particular integral, we write

$$y_p(x) = u_1(x)e^x + u_2(x)e^{2x} + u_3(x)e^{3x} \tag{42}$$

$$\therefore \quad \frac{dy_p}{dx} = (u_1'e^x + u_2'e^{2x} + u_3'e^{3x}) + (u_1e^x + 2u_2e^{2x} + 3u_3e^{3x})$$

Let the **first auxiliary condition** be

$$u_1'e^x + u_2'e^{2x} + u_3'e^{3x} = 0 \tag{43}$$

Thus

$$y_p' = u_1e^x + 2u_2e^{2x} + 3u_3e^{3x}$$

and

$$y_p'' = (u_1'e^x + 2u_2'e^{2x} + 3u_3'e^{3x}) + (u_1e^x + 4u_2e^{2x} + 9u_3e^{3x})$$

Let us choose the **second auxiliary condition** as

$$u_1'e^x + 2u_2'e^{2x} + 3u_3'e^{3x} = 0 \tag{44}$$

Then

$$y_p'' = u_1e^x + 4u_2e^{2x} + 9u_3e^{3x}$$

$$\therefore \quad y_p''' = (u_1'e^x + 4u_2'e^{2x} + 9u_3'e^{3x}) + (u_1e^x + 8u_2e^{2x} + 27u_3e^{3x})$$

Substituting the values of y_p, y_p', y_p'' and y_p''' from above in the given equation, we get

$$(u_1'e^x + 4u_2'e^{2x} + 9u_3'e^{3x}) + (u_1e^x + 8u_2e^{2x} + 27u_3e^{3x})$$

$$-6(u_1e^x + 4u_2e^{2x} + 9u_3e^{3x}) + 11(u_1e^x + 2u_2e^{2x} + 3u_3e^{3x})$$

$$-6(u_1e^x + u_2e^{2x} + u_3e^{3x}) = e^{2x}$$

$$\Rightarrow \quad u_1'e^x + 4u_2'e^{2x} + 9u_3'e^{3x} = e^{2x}, \tag{45}$$

which is our **third auxiliary condition**

Thus, we get the following system of equations for determining u_1', u_2' and u_3'

$$\left.\begin{aligned} u_1'e^x + u_2'e^{2x} + u_3'e^{3x} &= 0 \\ u_1'e^x + 2u_2'e^{2x} + 3u_3'e^{3x} &= 0 \\ u_1'e^x + 4u_2'e^{2x} + 9u_3'e^{3x} &= e^{2x} \end{aligned}\right] \tag{46}$$

Solving Eqns. (46) for u_1', u_2' and u_3', we get

$$u_1' = \frac{1}{2}e^x,\ u_2' = -1 \text{ and } u_3' = \frac{1}{2}e^{-x}$$

Integrating, we get

$$u_1 = \frac{1}{2}e^x,\ u_2 = -x \text{ and } u_3 = \frac{-1}{2}e^{-x}$$

Step III: We get from Eqn. (42) a particular integral of the given equation in the form

$$y_p(x) = \frac{1}{2}e^{2x} - xe^{2x} - \frac{1}{2}e^{2x} = -xe^{2x},$$

and the general solution is then given by

$$\begin{aligned} y &= y_c(x) + y_p(x) \\ &= c_1e^x + c_2e^{2x} + c_3e^{3x} - xe^{2x}. \end{aligned}$$

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Example 5: Using the method of variation of parameters solve the differential equation

$$y^{IV} - y = x$$

given that the functions $\sin x, \cos x, \sinh x, \cosh x$ form a fundamental set of solutions of the homogeneous equation.

Solution: Step I: The complementary solution is

$$y_c(x) = c_1 \sin x + c_2 \cos x + c_3 \sinh x + c_4 \cosh x \tag{47}$$

Step II: Consider a particular integral of the form

$$y_p(x) = u_1(x)\sin x + u_2(x)\cos x + u_3(x)\sinh x + u_4(x)\cosh x \tag{48}$$

$$\begin{aligned} \therefore\quad y_p'(x) &= (u_1 \cos x - u_2 \sin x + u_3 \cosh x + u_4 \sinh x) \\ &\quad + (u_1' \sin x + u_2' \cos x + u_3' \sinh x + u_4' \cosh x) \end{aligned}$$

The **first auxiliary condition** is

$$u_1' \sin x + u_2' \cos x + u_3' \sinh x + u_4' \cosh x = 0 \tag{49}$$

Thus $y_p' = u_1 \cos x - u_2 \sin x + u_3 \cosh x + u_4 \sinh x$

and $y_p'' = (-u_1 \sin x - u_2 \cos x + u_3 \sinh x + u_4 \cosh x)$

$$+ (u_1' \cos x - u_2' \sin x + u_3' \cosh x + u_4' \sinh x)$$

Let the **second auxiliary condition** is

$$u_1' \cos x - u_2' \sin x + u_3' \cosh x + u_4' \sinh x = 0 \tag{50}$$

Then $y_p''' = (-u_1 \cos x + u_2 \sin x + u_3 \cosh x + u_4 \sinh x)$

$$+(-u_1' \sin x - u_2' \cos x + u_3' \sinh x + u_4' \cosh x)$$

Let the **third auxiliary condition** is

$$-u_1' \sin x - u_2' \cos x + u_3' \sinh x + u_4' \cosh x = 0 \tag{51}$$

and $y_p''' = -u_1 \cos x + u_2 \sin x + u_3 \cosh x + u_4 \sinh x$

$$\therefore \quad y_p^{IV} = u_1 \sin x + u_2 \cos x + u_3 \sinh x + u_4 \cosh x$$
$$-u_1' \cos x + u_2' \sin x + u_3' \cosh x + u_4' \sinh x$$

Substituting the values of y_p^{IV} and y_p in the given equation, we get

$$u_1 \sin x + u_2 \cos x + u_3 \sinh x + u_4 \cosh x - u_1' \cos x + u_2' \sin x$$
$$+u_3' \cosh x + u_4' \sinh x - u_1 \sin x - u_2 \cos x - u_3 \sinh x - u_4 \cosh x = x$$

$$\Rightarrow \quad -u_1' \cos x + u_2' \sin x + u_3' \cosh x + u_4' \sinh x = x \tag{52}$$

which is our **fourth auxiliary condition**

Thus, we get the system of equations

$$\left.\begin{aligned} u_1' \sin x + u_2' \cos x + u_3' \sinh x + u_4' \cosh x = 0 \\ u_1' \cos x - u_2' \sin x + u_3' \cosh x + u_4' \sinh x = 0 \\ -u_1' \sin x - u_2' \cos x + u_3' \sinh x + u_4' \cosh x = 0 \\ -u_1' \cos x + u_2' \sin x + u_3' \cosh x + u_4' \sinh x = x \end{aligned}\right\} \tag{53}$$

Solving Eqns. (53) for u_1', u_2', u_3' and u_4', we get

$$u_1' = \frac{-x}{2}\cos x,\ u_2' = \frac{x}{2}\sin x,\ u_3' = \frac{x}{2}\cosh x,\ u_4' = \frac{-x}{2}\sinh x$$

Integrating, we get

$$u_1 = \frac{-1}{2}(x \sin x - \cos x),\ u_2 = \frac{1}{2}(-x \cos x + \sin x)$$

$$u_3 = \frac{1}{2}(x \sinh x - \cosh x),\ u_4 = \frac{1}{2}(-x \cosh x + \sinh x)$$

Step III: We get a particular integral from Eqn. (48) of the form

$$y_p(x) = -x$$

and the general solution of the given equation is

$$y = c_1 \sin x + c_2 \cos x + c_3 \sinh x + c_4 \cosh x - x.$$

***

You may now try the following exercise.

E3) Using the method of variation of parameters, find the general solution of the following differential equations:

i) $y''' - y' = x^2$

ii) $y''' - 2y'' - y' + 2y = e^{3x}$

iii) $x^3 y''' + x^2 y'' - 2xy' + 2y = 2x^4,\ x > 0$; given $y_1 = x,\ y_2 = x^2$ and $y_3 = \dfrac{1}{x}$ are the solutions of the corresponding homogeneous equation.

You must have **noticed** that the method of variation of parameters has an advantage over the method of undetermined coefficients in the sense that it always yields a particular integral y_p, provided all the solutions of the corresponding homogeneous equation are known. Moreover, its application is not restricted to equations with constant coefficient having particular forms of the non-homogeneous term. In the next section we shall discuss a technique which is very similar to the method of variation of parameters and is called the method of reduction of order. As we mentioned in the introduction, for equations with variable coefficients, determination of a fundamental solution set of a reduced equation corresponding to a given equation may be extremely difficult. For such equations the method of reduction of order has an advantage over the method of variation of parameters particularly, for equations of second order where knowing just one linearly independent solution of the reduced equation, second linearly independent solution as well as the general solution of the given equation can be obtained by this method.

12.3 REDUCTION OF ORDER

The method of reduction of order, as the name suggests, reduces the order of the given homogenenous/non-homogeneous equation with variable coefficients by one if one non-trivial solution of the corresponding homogeneous equation is known. For instance, in the case of a second order non-homogeneous differential equation of the form (12), viz,

$$y'' + a_1(x)y' + a_2(x)y = b(x)$$

if one non-trivial solution $y = y_1(x)$ say, of its corresponding homogeneous equation is known then method of reduction of order reduces the given equation to a first order equation. In general, if k linearly independent solutions of a homogeneous linear equation corresponding to an nth order non-homogeneous equation are known, where $k < n$, then the technique of reduction of order can be used to obtain a linear equation of order $(n-k)$. However, if $n \geq 3$, the reduced equation is itself at least of second order, and solving it may not be simpler than solving the original equation. But the technique is particularly interesting when $n = 2$, i.e., for the second order equation since the resulting first order equation can always be solved by the methods we have learnt in Block-2. Here we shall be restricting our discussion to the second order linear non-homogeneous equations. In the case of second order linear non-homogeneous equation we shall show that knowing one solution of the corresponding homogeneous equation how the method of reduction of order yields both a particular solution and a second linearly independent solution of the given equation.

We first illustrate the method through a simple example.

Consider the differential equation

$$x^2y'' - 2y = 0, \; x > 0 \tag{54}$$

It may be easily verified that $y_1 = x^2$ satisfies Eqn. (54). We proceed as we do in the case of method of variations of parameters and try to determine a solution of Eqn. (54) of the form $y = v(x)x^2$. We have

$$y' = 2xv + x^2v'$$

$$y'' = 2v + 4xv' + x^2v''$$

and thus

$$x^2y'' - 2y = x^3(4v' + xv'') = 0 \tag{55}$$

Since $x \neq 0$, we obtain from Eqn.(55)

$$xv'' + 4v' = 0 \tag{56}$$

Now let $w = v'$, then Eqn. (56) reduces to a first order linear equation

$$xw' + 4w = 0$$

which can be easily integrated to obtain

$$w = c_1 x^{-4} \text{ or } v' = c_1 x^{-4}.$$

Thus, we get

$$v = \frac{c_1 x^{-3}}{-3} + c_2$$

$$\therefore \quad y = vx^2 = \frac{-c_1}{3}\frac{1}{x} + c_2 x^2 \tag{57}$$

where c_1 and c_2 are arbitrary constants.

If we choose $c_2 = 0$ and $c_1 = -3$, we obtain the second solution $y_2 = \frac{1}{x}$. Also since $W\left(x^2, \frac{1}{x}\right) = -3 \neq 0 \ \forall x > 0$, the solutions y_1 and y_2 are linearly independent for $0 < x < \infty$. Thus $y_1 = x^2$ and $y_2 = \frac{1}{x}$ form a fundamental set of solutions of Eqn. (54) for $0 < x < \infty$.

You may **notice** here that the expression for y given by Eqn. (57) is actually the general solution of the given equation.

Let us now consider the general second order non-homogeneous Eqn. (12)

$$\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\ y = b(x)$$

where a_1, a_2 and b are continuous on some interval I and see how the method above, works for it.

Suppose that $y = y_1(x)$ is a non-trivial solution of the corresponding homogeneous equation

$$\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\ y = 0 \tag{58}$$

Then $y = cy_1(x)$ is also a solution of Eqn. (58) for some constant c. To obtain the second solution of Eqn. (58) replace the constant c by an unknown function $v(x)$ and take a trial solution in the form

$$y = v(x)\ y_1(x).$$

Now,

$$y' = v'y_1 + vy_1'$$

$$y'' = v''y_1 + 2v'y_1' + vy_1''$$

Substituting from above the expressions for y, y' and y'' in the given non-homogeneous equation, we get

$$(v''y_1 + 2v'y_1' + vy_1'') + a_1(v'y_1 + vy_1') + a_2 v y_1 = b(x)$$

$$\Rightarrow \quad v''y_1 + v'(2y_1' + a_1 y_1) + v(y_1'' + a_1 y_1' + a_2 y_1) = b(x) \tag{59}$$

Since y_1 is a solution of Eqn. (58), the last term on the l.h.s. of Eqn. (59) is zero. Therefore Eqn. (59) reduces to

$$v''y_1 + v'(2y_1' + a_1 y_1) = b(x). \tag{60}$$

Let $v' = \frac{dv}{dx} = p(x)$, so that Eqn. (60) becomes

$$\frac{dp}{dx} + \frac{2y_1' + a_1 y_1}{y_1} p = \frac{b(x)}{y_1}. \tag{61}$$

Eqn. (61) is a first order linear differential equation with integrating factor

$$\text{I.F.} = \exp\left[\int \frac{2y_1' + a_1 y_1}{y_1} dx\right]$$

Now $\int \frac{2y_1' + a_1 y_1}{y_1} dx = 2\ln y_1 + \int a_1(x)\, dx$

$\therefore$ I.F. $= y_1^2 e^{\int a_1(x)\, dx} = y_1^2 h(x)$, where $h(x) = e^{\int a_1(x)\, dx}$.

Thus the solution of Eqn. (61) can be written as

$$y_1^2 h(x)\ p(x) = c_1 + \int b(x)\ y_1 h(x)\ dx.$$

$$\Rightarrow\ \frac{dv}{dx} = p(x) = \frac{1}{y_1^2 h(x)}\left[c_1 + \int b(x)\ y_1 h(x)\ dx\right].$$

Integrating the above equation once again, we obtain

$$v(x) = c_2 + c_1 \int \frac{1}{y_1^2 h(x)} dx + \int \frac{1}{y_1^2 h(x)} \left[\int b(x)\ y_1 h(x)\ dx\right] dx.$$

Thus the general solution of the given equation can be expressed as

$$y = v(x) y_1(x) = c_2 y_1(x) + c_1 y_1(x) \int \frac{dx}{y_1^2 h(x)}$$

$$+\, y_1(x) \int \frac{1}{y_1^2 h(x)} \left[\int b(x)\ y_1 h(x)\ dx\right] dx. \tag{62}$$

Note that the function $y_1(x) \int \frac{dx}{y_1^2 h(x)} = y_1(x) \int \frac{e^{-\int a_1(x) dx}}{y_1^2} dx$, which is the second term on the r.h.s. of Eqn. (62), is the 2nd linearly independent solution of Eqn. (58) and the last term on the r.h.s. is a particular integral of the given non-homogeneous equation.

Thus Eqn. (62) can be written as

$$y = c_2 y_1(x) + c_1 y_2(x) + y_p(x)$$

where

$$y_2(x) = y_1(x) \int \frac{dx}{y_1^2 h(x)} = y_1(x) \int \frac{e^{-\int a_1(x) dx}}{y_1^2} dx \tag{63}$$

is the **second linearly independent solution** of Eqn. (58) and

$$y_p(x) = y_1(x) \int \frac{e^{-\int a_1(x) dx}}{y_1^2} \left(\int b(x) y_1 e^{\int a_1(x) dx} dx\right) dx \tag{64}$$

is a **particular integral** of Eqn. (12).

We now take up examples to illustrate the method discussed above.

Example 6: Find the general solution of the differential equation

$$x^2y'' - xy' + y = x^{1/2}, \quad 0 < x < \infty$$

given that $y_1 = x$ is a solution of the corresponding homogeneous equation.

Solution: The given equation is

$$x^2y'' - xy' + y = x^{1/2} \tag{65}$$

Let us take $y = xv(x)$ as a trial solution for Eqn. (65), so that

$$y' = v + xv'$$

$$y'' = 2v' + xv''$$

Substituting for y, y' and y'' from above in Eqn. (65), we obtain

$$x^2(2v' + xv'') - x(v + xv') + xv = x^{1/2}$$

$$\Rightarrow \quad x^3v'' + x^2v' = x^{1/2}$$

$$\Rightarrow \quad v'' + \frac{1}{x}v' = x^{-5/2} \tag{66}$$

Eqn. (66) is a linear differential equation in v'. Its integrating factor is

$$I.F. = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Therefore, Eqn. (66) yields

$$v'x = \int x.x^{-5/2}dx + c_1$$

$$\Rightarrow \quad v' = c_1x^{-1} - 2x^{-3/2}$$

Integrating once again, we have

$$v = c_1 \ \ln x + 4x^{-1/2} + c_2$$

Thus,

$$y = xv = c_1x \ln x + c_2x + 4x^{1/2}$$

is the general solution of Eqn. (65). Here the term $x \ln x$ is the second linearly independent solution of Eqn. (65) and $4x^{1/2}$ is its particular integral.

Let us now see how formulas (63) and (64) can be used directly to obtain the second linearly independent solution $y_2(x)$ and a particular solution $y_p(x)$, respectively, of the given non-homogeneous Eqn. (65).

Consider Eqn. (65), namely,

$$x^2y'' - xy' + y = x^{1/2}, \quad 0 < x < \infty.$$

It can be written in the form

$$y'' - \frac{y'}{x} + \frac{y}{x^2} = x^{-3/2}$$

Comparing the above equation with Eqn. (58), we find

$$a_1(x) = -\frac{1}{x} \text{ and } a_2(x) = \frac{1}{x^2}$$

From formula (63), we have

$$y_2(x) = y_1(x)\int \frac{e^{-\int a_1(x)dx}}{y_1^2}dx = x\int \frac{e^{\int \frac{1}{x}dx}}{x^2}dx$$

$$= x\int \frac{x}{x^2}dx = x\ln x \text{ [given } y_1(x) = x]$$

From formula (64), we have

$$y_p(x) = y_1(x)\int \frac{e^{-\int a_1(x)dx}}{y_1^2}\left(\int b(x)\,y_1 e^{\int a_1(x)dx}dx\right)dx$$

$$= x\int \frac{x}{x^2}\left(\int x^{-3/2}\, x.\frac{1}{x}dx\right)dx$$

$$= x\int \frac{1}{x}(-2x^{-1/2})dx = 4x^{1/2}.$$

***

Example 7: Find the general solution of the differential equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0, \qquad 0 < x < \infty$$

given that the function $y_1(x) = \dfrac{\sin x}{\sqrt{x}}$ is a solution of the equation.

Solution: The equation can be rewritten as

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0.$$

From formula (63), we get

$$y_2(x) = \frac{\sin x}{\sqrt{x}}\int \frac{e^{-\int \frac{dx}{x}}}{\left(\dfrac{\sin x}{\sqrt{x}}\right)^2}dx$$

$$= \frac{\sin x}{\sqrt{x}}\int cosec^2 x\, dx$$

$$= \frac{\sin x}{\sqrt{x}}(-\cot x) = -\frac{\cos x}{\sqrt{x}}$$

Hence, the general solution of the given equation is

$$y = c_1\frac{\sin x}{\sqrt{x}} + c_2\frac{\cos x}{\sqrt{x}}.$$

***

Remember that while using the formulas (63) or (64) for finding the second solution or a particular integral, the equation must be put in the form (58) as we have done in Examples 6 and 7 above.

And now some exercises for you.

E4) Solve the following differential equations:

i) $x^2y'' - 2xy' + 2y = 4x^2, \;\; x > 0; \;\; y_1(x) = x$

ii) $x^2y'' + 5xy' - 5y = x^{-1/2}, \;\; x > 0; \;\; y_1(x) = x$

E5) A solution of the differential equation

$$x^2(1-x^2)\frac{d^2y}{dx^2}-x^3\frac{dy}{dx}-2y=0, \quad 0<|x|<1$$

is $y_1=\dfrac{\sqrt{1-x^2}}{x}$. Use the method of reduction of order to find its general solution.

E6) Solve the equation

$$x(x\cos x-2\sin x)\frac{d^2y}{dx^2}+(x^2+2)\sin x\frac{dy}{dx}-2(x\sin x+\cos x)\ y=0,\ x>0,$$

given that $y=x^2$ is a solution of the equation.

So far you have seen that the method of variation of parameters can be used only for those non-homogeneous linear differential equations with variable coefficients for which we know all the linearly independent solutions of the corresponding homogeneous equation. For second order linear non-homogeneous equations with variable coefficients, the method of reduction of order is helpful for finding the complete solution even if **one** solution of the corresponding homogeneous equation is known. As we mentioned in the introduction to this unit, there exists a class of linear differential equation with variable coefficients known as **Cauchy-Euler equation** or **Euler's equation** for which it is possible to find all the linearly independent integrals of the complementary function. The equation is first reduced to an equation with constant coefficients through a transformation of the independent variable and then its solutions are obtained. In the next section we discuss the method of solving Euler's Equation.

12.4 EULER'S EQUATION

Consider the following differential equations

$$4x^2\frac{d^2y}{dx^2}+8x\frac{dy}{dx}+y=0 \tag{67}$$

$$x^3\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x \tag{68}$$

$$x\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+2y=e^x \tag{69}$$

$$(2x-1)^3\frac{d^3y}{dx^3}+(2x-1)\frac{dy}{dx}-2y=\sin x \tag{70}$$

Out of the four equations above, Eqns. (67) and (68) are such that the **power of x in the coefficients are equal to the orders of the derivatives associated with them**. These type of equations are known as **Euler's Equation** or **equidimensional equation**. Eqn. (70) is not of Euler's form but can be reduced to Euler's form by the substitution $X=2x-1$. We shall also consider such equations later in this section. Eqn. (69) as you can see is neither Euler's equation nor can it be reduced to Euler's form.

The general form of Euler's equation of nth order is

$$a_0x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\cdots+a_{n-1}x\frac{dy}{dx}+a_ny=f(x), \tag{71}$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants and right hand side is a constant or a function of x alone.

Let us start by considering the Euler's Eqn. (67), namely,

$$4x^2\frac{d^2y}{dx^2}+8x\frac{dy}{dx}+y=0$$

Now we have to reduce this equation to an equation with constant coefficients. We transform the independent variable x to another variable z in Eqn. (67) by means of the transformation $x=e^z$. We write

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=e^{-z}\frac{dy}{dz} \quad \left(\text{since } \frac{dx}{dz}=e^z\right) \tag{72}$$

and

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(e^{-z}\frac{dy}{dz}\right)=-e^{-z}\frac{dz}{dx}\frac{dy}{dz}+e^{-z}\frac{d^2y}{dz^2}.\frac{dz}{dx}$$

$$=-e^{-2z}\frac{dy}{dz}+e^{-2z}\frac{d^2y}{dz^2}$$

$$=e^{-2z}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right) \tag{73}$$

Substituting from Eqns. (72) and (73) in Eqn. (67), we obtain

$$4e^{2z}e^{-2z}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+8e^ze^{-z}\frac{dy}{dz}+y=0$$

$$\Rightarrow \quad 4\frac{d^2y}{dz^2}+4\frac{dy}{dz}+y=0 \tag{74}$$

Eqn. (74) is an equation with constant coefficient having z as the independent variable and can be solved by the known methods. Its auxiliary equation is

$$4m^2+4m+1=0$$

$$\Rightarrow \quad (2m+1)^2=0$$

$$\Rightarrow \quad m=-\frac{1}{2}, -\frac{1}{2}$$

The general solution of Eqn. (74) can be written as

$$y=(c_1+zc_2)e^{-z/2} \tag{75}$$

Substituting $x=e^z$ or $z=\ln x$ in Eqn. (75), the general solution of Eqn. (67) is obtained as

$$y=(c_1+c_2\ln x)x^{-1/2}.$$

To see how the method works, in general, we consider the second order Euler's equation

$$a_0\,x^2\frac{d^2y}{dx^2}+a_1x\frac{dy}{dx}+a_2y=f(x). \tag{76}$$

You may **note** here that the coefficient of $\frac{d^2y}{dx^2}$ is zero at $x=0$. Hence we confine our attention to finding the general solution on the interval $]0, \infty[$. Consider the substitution

$$z=\ln x \text{ or } x=e^z$$

With this substitution, we have

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}\quad\left(\because\frac{dz}{dx}=\frac{1}{x}\right)$$

$$\Rightarrow\ x\frac{dy}{dx}=\frac{dy}{dz}\tag{77}$$

Also $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{1}{x}\dfrac{dy}{dz}\right)=-\dfrac{1}{x^2}\dfrac{dy}{dz}+\dfrac{1}{x}\dfrac{d^2y}{dz^2}\dfrac{dz}{dx}$

$$=\frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

$$\Rightarrow\ x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}\tag{78}$$

Thus, substituting from Eqns. (77) and (78), Eqn. (76) is transformed to the equation

$$a_0\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+a_1\frac{dy}{dz}+a_2y=f(e^z)$$

or, $$A_0\frac{d^2y}{dz^2}+A_1\frac{dy}{dz}+A_2y=Q(z)\tag{79}$$

where $A_0=a_0,\ A_1=a_1-a_0,\ A_2=a_2$ and $Q(z)=f(e^z)$.

Eqn. (79) is an equation with constant coefficients and its complementary function can be determined by the methods discussed in Unit 5. For obtaining its particular integral either the method of undetermined coefficients subject to the form of $f(e^z)$, or the method of variation of parameters can be utilised. If the solution of Eqn. (79) is

$$y=g(z),$$

then the solution of Eqn. (76) will be

$$y=g(\ln x)$$

We take up some examples to illustrate the method.

Example 8: Solve $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+y=\ln x,\ 0<x<\infty$.

Solution: Given equation is an Euler's equation of order 2. To solve it, let

$x=e^z$ or $z=\ln x$

Then we know that

$$x\frac{dy}{dx}=\frac{dy}{dz},$$

and $$x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$$

Substituting for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in the given equation, we get

$$\frac{d^2y}{dz^2}-2\frac{dy}{dz}+y=z\tag{80}$$

The auxiliary equation of Eqn. (80) is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow \quad m = 1, 1$$

$\therefore$ C.F. $= (c_1 + c_2 z)\, e^z$

To find a P.I. of Eqn. (80), let us assume that

$$y_p = u_1(z)\, e^z + u_2(z)\, ze^z \tag{81}$$

$$\therefore \quad \frac{dy_p}{dz} = u_1' e^z + u_2' ze^z + u_1 e^z + u_2 (ze^z + e^z)$$

As the first auxiliary condition, assume that

$$u_1' e^z + u_2' ze^z = 0 \tag{82}$$

so that

$$\frac{dy_p}{dz} = u_1 e^z + u_2 (z+1)\, e^z \tag{83}$$

Differentiating Eqn. (83) once again, we have

$$\frac{d^2 y_p}{dz^2} = u_1' e^z + u_2' (z+1)\, e^z + u_1 e^z + u_2 e^z (z+1) + u_2 e^z \tag{84}$$

If $y_p(z)$ is a solution of Eqn. (80), it must satisfy it. Hence substituting the expressions for y_p, y_p' and y_p'' from Eqns. (81), (83) and (84), respectively, in Eqn. (80), we obtain the second auxiliary condition as

$$u_1' e^z + u_2' (z+1)\, e^z = z \tag{85}$$

Solving Eqns. (82) and (85) for u_1' and u_2', we get

$$u_2' e^z = z \text{ and } e^z u_1' = -z^2$$

$$\Rightarrow \quad u_1' = -z^2 e^{-z} \text{ and } u_2' = ze^{-z}$$

Integrating the above equation, we get

$$u_1 = -\int z^2 e^{-z}\, dz$$

$$= -\left[z^2 \frac{e^{-z}}{-1} + 2 \int z\, e^{-z}\, dz \right]$$

$$= +z^2 e^{-z} - 2\left[z\frac{e^{-z}}{-1} + \int e^{-z}\, dz \right]$$

$$= z^2 e^{-z} + 2z e^{-z} + 2e^{-z}$$

and $u_2 = \int ze^{-z}\, dz = -ze^{-z} + \int e^{-z}\, dz = -z\, e^{-z} - e^{-z}$.

Substituting the values of $u_1(z)$ and $u_2(z)$ in Eqn. (81), a particular integral of Eqn. (80) can be expressed in the form

$$y_p(z) = (z^2 + 2z + 2)\, e^{-z}.e^z + (-z-1)\, e^{-z}\, ze^z$$

$$= (z^2 + 2z + 2) - z(z+1)$$

$$= z^2 + 2z + 2 - z^2 - z$$

$$= z + 2$$

and the general solution of Eqn. (80) is

$$y = (c_1 + c_2 z)\, e^z + z + 2$$

Replacing z by $\ln x$, the general solution of the given equation is

$$y = (c_1 + c_2 \ln x)\; x + \ln x + 2$$

***

Example 9: Solve $x^3 \dfrac{d^3 y}{dx^3} - x^2 \dfrac{d^2 y}{dx^2} - 6x\dfrac{dy}{dx} + 18y = 0,\ 0 < x < \infty$.

Solution: Let $x = e^z$ or $z = \ln x$

Then, we have

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz},\ \frac{d^2 y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) \tag{86}$$

$$\frac{d^3 y}{dx^3} = \frac{-2}{x^3}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) + \frac{1}{x^2}\left(\frac{d^3 y}{dz^3}\frac{dz}{dx} - \frac{d^2 y}{dz^2}\frac{dz}{dx}\right)$$

$$= \frac{1}{x^3}\left(\frac{d^3 y}{dz^3} - 3\frac{d^2 y}{dz^2} + 2\frac{dy}{dz}\right) \tag{87}$$

Substituting from Eqns. (86) and (87) in the given equation, we get

$$\frac{d^3 y}{dz^3} - 3\frac{d^2 y}{dz^2} + 2\frac{dy}{dz} - \frac{d^2 y}{dz^2} + \frac{dy}{dz} - 6\frac{dy}{dz} + 18y = 0$$

$$\Rightarrow\ \frac{d^3 y}{dz^3} - 4\frac{d^2 y}{dz^2} - 3\frac{dy}{dz} + 18y = 0$$

The auxiliary equation corresponding to the above equation is

$$m^3 - 4m^2 - 3m + 18 = 0$$

$$\Rightarrow\ (m+2)\,(m^2 - 6m + 9) = 0$$

$$\Rightarrow\ (m+2)\,(m-3)^2 = 0$$

$$\Rightarrow\ m = -2, 3, 3$$

$\therefore$ C.F. $= c_1 e^{-2z} + (c_2 + c_3 z) e^{3z}$

Substituting $z = \ln x$ and $e^z = x$ in the above equation the general solution of the given equation is

$$y = c_1 x^{-2} + (c_2 + c_3 \ln x)\, x^3$$

***

We now take up an example where the auxiliary equation has complex roots.

Example 10: Solve $x^2 y'' + xy' + 4y = 0,\ 0 < x < \infty$.

Solution: Letting $x = e^z$ or $z = \ln x$ and substituting the values of the derivatives from Eqn. (86), the given equation reduces to

$$\frac{d^2 y}{dz^2} + 4y = 0$$

The auxiliary equation corresponding to the above equation is

$$m^2 + 4 = 0$$

$$\Rightarrow\ m = \pm 2i$$

Hence the general solution of the given equation is

$$y = c_1 \cos 2z + c_2 \sin 2z$$

or, $$y = c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x).$$

***

We now take up an example of an initial value problem where initial conditions are prescribed for negative values of x and we are required to find the general solution in the interval $]-\infty, 0[$. In such cases, we first transform the given equation and the initial conditions to the interval $]0, \infty[$ by using the substitution $t = -x$ and then obtain the general solution by using the method discussed above. Let us see how this is done.

Example 11: Solve the given differential equation

$$4x^2 y'' + y = 0, \ -\infty < x < 0;$$
$$y(-1) = 2, \ y'(-1) = 4.$$

Solution: Let $t = -x$, then $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -\frac{dy}{dt} \left(\because \frac{dt}{dx} = -1 \right)$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-dy}{dt}\right) = \frac{-d^2y}{dt^2}\frac{dt}{dx} = \frac{d^2y}{dt^2}$

with above substitutions the given equation reduces to

$$4t^2 \frac{d^2y}{dt^2} + y = 0, \ y(1) = 2, \ y'(1) = -4 \left(\because \frac{dy}{dx} = \frac{-dy}{dt} \right)$$

To solve the above equation we use the substitution $t = e^z$ or $z = \ln t$ and obtain the equation

$$4\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + y = 0$$

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$\Rightarrow \ (2m-1)\ (2m-1) = 0$

$\Rightarrow \ m = \frac{1}{2}, \frac{1}{2}$

$\therefore \ y = e^{z/2}(c_1 + zc_2)$

$\quad = t^{1/2}(c_1 + c_2 \ln t)$

Now $y(1) = 2 \Rightarrow c_1 = 2$

and $y'(1) = -4 \Rightarrow c_2 = -5$

Thus $y = t^{1/2}(2 - 5\ln t)$

$\quad = (-x)^{1/2}[2 - 5\ln(-x)]$

is the required general solution.

***

You may now try the following exercises.

E7) Solve the following differential equations

i) $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = 0, \ 0 < x < \infty.$

ii) $x^3 \frac{d^3y}{dx^3} + 5x^2 \frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y = 0, \ 0 < x < \infty.$

E8) Solve the following differential equations on the interval $]0, \infty[$

i) $x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} = \dfrac{1}{x}$

ii) $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^n$

iii) $\dfrac{d^3y}{dx^3} - \dfrac{4}{x}\dfrac{d^2y}{dx^2} + \dfrac{5}{x^2}\dfrac{dy}{dx} - \dfrac{2y}{x^3} = 1$

E9) Show that a solution of the Euler's equation $r^2 \dfrac{d^2R}{dr^2} + r\dfrac{dR}{dr} - n^2R = 0$ satisfying $R(b) = 0$, where b is a constant, is given by

i) $R = C\left[\left(\dfrac{b}{r}\right)^n - \left(\dfrac{r}{b}\right)^n\right]$ for $n = 1, 2, 3, \ldots,$ C a constant;

ii) $R = C \ln\left(\dfrac{r}{b}\right)$ for $n = 0,$ C a constant.

E10) Solve the given differential equations subject to the indicated initial conditions

i) $x^2y'' + xy' + y = 0,\ y(1) = 1,\ y'(1) = 2,\ 0 < x < \infty$.

ii) $x^2y'' - 4xy' + 6y = 0,\ y(-2) = 8,\ y'(-2) = 0, -\infty < x < 0$.

Earlier we mentioned that Eqn. (70) is not Euler's equation, but can be reduced to Euler's form by the substitution $X = 2x - 1$. We now consider such equations which are reducible to Euler's form.

Equations Reducible to Euler's Form

Consider Eqn. (70), namely,

$$(2x-1)^3 \frac{d^3y}{dx^3} + (2x-1)\frac{dy}{dx} - 2y = \sin x$$

Let $X = 2x - 1$, then we have

$$\frac{dy}{dx} = \frac{dy}{dX}\frac{dX}{dx} = 2\frac{dy}{dX},\ \frac{d^2y}{dx^2} = 4\frac{d^2y}{dX^2} \text{ and } \frac{d^3y}{dx^3} = 8\frac{d^3y}{dX^3} \quad (88)$$

Substituting from Eqn. (88) in Eqn. (70), it reduces to

$$8X^3 \frac{d^3y}{dX^3} + 2X\frac{dy}{dX} - 2y = \sin\left(\frac{X+1}{2}\right) \quad (89)$$

Which is in the Euler's form.

In the same way for the general nth order equation

$$(ax+b)^n \frac{d^ny}{dx^n} + (ax+b)^{n-1}a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + (ax+b)a_{n-1}\frac{dy}{dx} + a_ny = f(x), \quad (90)$$

where $a, b, a_1, a_2, \ldots, a_n$ are all constants, consider the substitution $X = ax + b$.

With this substitution

$$\frac{dy}{dx}=\frac{dy}{dX}\frac{dX}{dx}=a\frac{dy}{dX},\ \frac{d^2y}{dx^2}=a^2\frac{d^2y}{dX^2},\cdots,\ \frac{d^ny}{dx^n}=a^n\frac{d^ny}{dX^n}.$$

and Eqn. (90) reduces to the equation

$$a^nX^n\frac{d^ny}{dX^n}+a^{n-1}X^{n-1}a_1\frac{d^{n-1}y}{dX^{n-1}}+\cdots+aXa_{n-1}\frac{dy}{dX}+a_ny=g(X) \qquad (91)$$

where g is the transformed form of the function f.

Both the Eqns. (89) and (91), in the Euler's form, can now be solved by reducing them to equations with constant coefficients by the method discussed above. However, substitution $ax+b=e^z$ reduce Eqn. (90) directly to an equation with constant coefficients. Similarly, substitution $2x-1=e^z$ can be used in Eqn. (70) to reduce it to an equation with constant coefficients.

We illustrate the method above with the help of the following example.

Example 12: Solve

$$(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=3x^2+4x+1,\ 0<x<\infty.$$

Solution: The given equation is an equation reducible to Euler's equation. We can, however, reduce it to an equation with constant coefficients by a single substitution.

$$3x+2=e^z \text{ or } z=\ln(3x+2)$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{1}{3x+2}.3\frac{dy}{dz}\Rightarrow(3x+2)\frac{dy}{dx}=3\frac{dy}{dz}\left(\because\frac{dz}{dx}=\frac{3}{3x+2}\right)$$

$$\text{and } \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{3}{3x+2}\frac{dy}{dz}\right]=\frac{-3^2}{(3x+2)^2}\frac{dy}{dx}+\frac{3}{3x+2}\frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$=\frac{3^2}{(3x+2)^2}\left[\frac{d^2y}{dz^2}-\frac{dy}{dz}\right]$$

Substituting for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from above in the given equation, we get

$$9\left[\frac{d^2y}{dz^2}-\frac{dy}{dz}\right]+3.3\frac{dy}{dz}-36y=\frac{1}{3}[e^{2z}-1]$$

$$\Rightarrow\ \frac{d^2y}{dz^2}-4y=\frac{1}{27}(e^{2z}-1) \qquad (92)$$

The auxiliary equation is

$$m^2-4=0\Rightarrow m=\pm2$$

Hence C.F. $=y_c=c_1e^{2z}+c_2e^{-2z}$

To find a particular integral, we write

$$y_p(z)=u_1(z)\,e^{2z}+u_2(z)\,e^{-2z} \qquad (93)$$

$$\therefore\ \frac{dy_p}{dz}=u_1'e^{2z}+u_2'e^{-2z}+2(u_1e^{2z}-u_2e^{-2z})$$

As the first auxiliary condition, let

$$u_1' e^{2z} + u_2' e^{-2z} = 0 \tag{94}$$

so that

$$\frac{dy_p}{dz} = 2(u_1 e^{2z} - u_2 e^{-2z}) \tag{95}$$

Differentiating Eqn. (95) once again, we get

$$\frac{d^2 y_p}{dz^2} = 2(u_1' e^{2z} - u_2' e^{-2z}) + 4u_1 e^{2z} + 4u_2 e^{-2z} \tag{96}$$

Since $y_p(z)$ must satisfy Eqn. (92), we combine Eqns. (93), (95) and (96), and obtain the second auxiliary condition as

$$2(u_1' e^{2z} - u_2' e^{-2z}) = \frac{1}{27}(e^{2z} - 1)$$

$$\Rightarrow \quad u_1' e^{2z} - u_2' e^{-2z} = \frac{1}{54}(e^{2z} - 1) \tag{97}$$

Solving Eqns. (94) and (97) for u_1' and u_2', we get

$$u_1' = \frac{1}{108}(1 - e^{-2z}) \text{ and } u_2' = -\frac{1}{108}(e^{4z} - e^{2z})$$

Integrating u_1' and u_2', we get

$$u_1(z) = \frac{1}{108}\left(z + \frac{e^{-2z}}{2}\right) \text{ and } u_2(z) = -\frac{1}{108}\left(\frac{e^{4z}}{4} - \frac{e^{2z}}{2}\right)$$

On substituting the values of $u_1(z)$ and $u_2(z)$ in relation (93), a particular solution of Eqn. (92) is obtained in the form

$$y_p = \frac{1}{108}\left(z + \frac{e^{-2z}}{2}\right)e^{2z} - \frac{1}{108}\left(\frac{e^{4z}}{4} - \frac{e^{2z}}{2}\right)e^{-2z}$$

$$= \frac{1}{108}e^{2z}\left(z - \frac{1}{4}\right) + \frac{1}{108}$$

$\therefore$ The general solution of Eqn. (92) is

$$y = c_1 e^{2z} + c_2 e^{-2z} + \frac{1}{108}e^{2z}\left(z - \frac{1}{4}\right) + \frac{1}{108}$$

and the required solution of the given equation is

$$y = c_1(3x+2)^2 + \frac{c_2}{(3x+2)^2} + \frac{1}{108}(3x+2)^2\left[\ln\,(3x+2) - \frac{1}{4}\right] + \frac{1}{108}.$$

***

You may now try the following exercise.

E11) Solve the following differential equations in the interval $]0, \infty[$

i) $(x+a)^2 \dfrac{d^2y}{dx^2} - 4(x+a)\dfrac{dy}{dx} + 6y = x$

ii) $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos[\ln\,(x+1)]$

We now end this unit by giving a summary of what we have covered in it.

12.5 SUMMARY

In this unit we have studied the details concerning the following results:

1. Let y_1 and y_2 be the linearly independent solutions of the reduced equation corrsesponding to a non-homogeneous second order linear differential equation of the Form (12) with constant or variable coefficients. Then a particular integral $y_p(x)$ of the equation is obtained by substituting $y_p(x) = y_1u_1(x) + y_2u_2(x)$ in the given equation and determining $u_1(x)$ and $u_2(x)$ by using the formulas
$$u_1(x) = \int \frac{-b(x)y_2(x)}{W(y_1,\, y_2)}dx,\; u_2(x) = \int \frac{b(x)y_1(x)}{W(y_1,\, y_2)}dx$$
where $W(y_1,\, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$ is the Wronskian of $y_1(x)$ and $y_2(x)$.

2. If $y = y_1(x)$ is one solution of the reduced equation, then on substituting $y = y_1(x)\,v(x)$ the second solution of the reduced equation and a particular integral of the corresponding non-homogeneous equation can be determined.

3. Differential equation with variable coefficient of the form
$x^n \dfrac{d^ny}{dx^n} + a_1x^{n-1}\dfrac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\dfrac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}x\dfrac{dy}{dx} + a_ny = f(x)$ where $a_1, a_2, \ldots, a_n$ are constants and in which the powers of x in the coefficients are equal to the order of the derivatives associated with them, is known as Euler's equation. This equation can be reduced to an equation with constant coefficients by using the substitution $x = e^z$ and then it can be solved by the known methods.

12.6 SOLUTIONS/ANSWERS

E1) i) Complementary function of the given equation is given by
$$y_c(x) = c_1\cos x + c_2\sin x$$
Let $y_p(x) = u_1(x)\cos x + u_2(x)\sin x$

Then the two auxiliary conditions are
$$\frac{du_1}{dx}\cos x + \frac{du_2}{dx}\sin x = 0$$
and $-\sin x\dfrac{du_1}{dx} + \cos x\dfrac{du_2}{dx} = \operatorname{cosec} x$

Solving the above system for $\frac{du_1}{dx}$ and $\frac{du_2}{dx}$, we get

$$\frac{du_1}{dx} = -1 \text{ and } \frac{du_2}{dx} = \cot x$$

Integrating u_1' and u_2', we get

$$u_1(x) = -x \text{ and } u_2(x) = \ln \sin x$$

Hence $y_p(x) = -x\cos x + \ln(\sin x).\sin x$

and the required general solution is

$$y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \ln(\sin x).$$

ii) The given equation is

$$y'' - 2y' + y = xe^x \ln x,\ x > 0$$

Complementary function is

$$y_c = (c_1 + c_2 x)\, e^x$$

Assume

$$y_p(x) = [u_1(x) + u_2(x).x]\ e^x$$

Then the auxiliary conditions are

$$\frac{du_1}{dx} + x\frac{du_2}{dx} = 0$$

and $\frac{du_1}{dx} + (1+x)\frac{du_2}{dx} = x \ln x$

Solving the above system for $\frac{du_1}{dx}$ and $\frac{du_2}{dx}$, we get

$$\frac{du_1}{dx} = -x^2 \ln x \text{ and } \frac{du_2}{dx} = x \ln x$$

Integrating u_1' and u_2', we get

$$u_1 = -\int x^2 \ln x\, dx = -\frac{x^3}{3}\ln x + \int \frac{x^3}{3}\frac{1}{x}\, dx$$

$$= -\frac{x^3}{3}\ln x + \frac{1}{9}x^3,$$

$$u_2 = \int x \ln x\, dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\, dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4}$$

Hence $y_p(x) = e^x\left[-\frac{x^3}{3}\ln x + \frac{1}{9}x^3 + \frac{x^3}{2}\ln x - \frac{x^3}{4}\right]$

$$= e^x\left[\frac{x^3}{6}\ln x - \frac{5}{36}x^3\right].$$

iii) Complementary function is

$$y_c(x) = c_1 \cos x + c_2 \sin x$$

Assume $y_p(x) = u_1(x)\cos x + u_2(x)\sin x$

The two auxiliary conditions are

$$\frac{du_1}{dx}\cos x + \frac{du_2}{dx}\sin x = 0$$

and $-\dfrac{du_1}{dx}\sin x+\dfrac{du_2}{dx}\cos x=\tan x$

Solving, we get

$$\frac{du_1}{dx}=\cos x-\sec x \text{ and } \frac{du_2}{dx}=\sin x$$

Integrating u_1' and u_2', we get

$$u_1=\sin x-\ln\,(\sec x+\tan x) \text{ and } u_2=-\cos x.$$

Thus $y_p(x)=-\cos x\ \ln\,(\sec x+\tan x)$.

E2) i) Complementary function is

$$y_c(x)=c_1x+c_2x^2$$

Assume that a particular integral is $y_p(x)=xu_1(x)+x^2u_2(x)$.

The two auxiliary conditions are

$$xu_1'+x^2u_2'=0$$

and $u_1'+2xu_2'=\dfrac{x+1}{x^2}$

Solving the above system for u_1' and u_2', we get

$$u_1'=-\frac{1}{x}-\frac{1}{x^2} \text{ and } u_2'=\frac{1}{x^2}+\frac{1}{x^3}$$

Thus, $u_1=-\ln\ x+\dfrac{1}{x}$ and $u_2=-\dfrac{1}{x}-\dfrac{1}{2x^2}$

Thus, $y_p(x)=\dfrac{1}{2}-x-x\ \ln\ x$

Therefore, the general solution is

$$y=c_1x+c_2x^2+\frac{1}{2}-x-\ln\ x.$$

ii) $y=c_1x+\dfrac{c_2}{x}+e^x-\dfrac{1}{x}e^x.$

iii) The given equation can be rewritten as

$$y''-\frac{(x+1)}{x}y'+\frac{y}{x}=x$$

Complementary solution is

$$y_c(x)=c_1e^x+c_2(x+1)$$

Assume a particular integral as $y_p(x)=u_1e^x+u_2(x+1)$.

The two auxiliary conditions are

$$e^xu_1'+(x+1)u_2'=0$$

$$e^xu_1'+u_2'=x$$

Solving the above system for u_1' and u_2', we get

$$u_1'=(x+1)e^{-x},\ u_2'=-1$$

Thus $u_1=-e^{-x}(x+2),\ u_2=-x$

$$y_p(x)=-(x+2)-x(x+1)=-(x^2+2x+2)$$

$\therefore$ The general solution is

$$y=c_1e^x+c_2(x+1)-(x^2+2x+2).$$

E3) i) $y = c_1 + c_2 e^x + c_3 e^{-x} - 2x - \dfrac{x^3}{3}$.

ii) $y = c_1 e^x + c_2 e^{2x} + c_3 e^{-x} + \dfrac{1}{8} e^{3x}$.

iii) Rewrite the given equation as

$$y''' + \frac{y''}{x} - \frac{2}{x^2} y' + \frac{2}{x^3} y = 2x,\ x > 0$$

$$y_c(x) = c_1 x + c_2 x^2 + \frac{c_3}{x}$$

Let $y_p(x) = x u_1 + x^2 u_2 + \dfrac{1}{x} u_3$

The three auxiliary conditions are

$$x u_1' + x^2 u_2' + \frac{1}{x} u_3' = 0$$

$$u_1' + 2x u_2' - \frac{1}{x^2} u_3' = 0$$

$$u_2' + \frac{1}{x^3} u_3' = 0$$

Solving the above system for u_1', u_2' and u_3', we get

$$u_1' = -x^2,\ u_2' = \frac{2}{3} x,\ u_3' = \frac{x^4}{3}$$

Integrating, we get

$$u_1 = \frac{-x^3}{3},\ u_2 = \frac{x^2}{3},\ u_3 = \frac{x^5}{15}$$

$$\therefore\ y_p(x) = \frac{-x^4}{3} + \frac{x^4}{3} + \frac{x^4}{15} = \frac{x^4}{15}$$

and the required general solution is

$$y = c_1 x + c_2 x^2 + \frac{c_3}{x} + \frac{x^4}{15}.$$

E4) i) The given equation is

$$x^2 y'' - 2xy' + 2y = 4x^2$$

Here one of the solutions of the corresponding homogeneous equation is $y_1(x) = x$.

$\therefore$ Assume $y = vx$.

Substituting for y, y' and y'' in the given equation, we get

$$x^2(2v' + xv'') - 2x(v + xv') + 2vx = 4x^2$$

$$\Rightarrow\ x^3 v'' = 4x^2$$

$$\Rightarrow\ v'' = \frac{4}{x},\ x > 0$$

Integrating, we get

$$v' = 4 \ln x + c_1$$

Integrating again, we get

$$v = c_1 x + c_2 + \int 4 \ln x \, dx$$

$$= c_1 x + c_2 + 4x \ln x - 4 \int x \frac{1}{x} \, dx$$

$$= c_1 x + c_2 + 4x \ln x - 4x$$

Thus, $y = vx = c_2 x + c_1 x^2 + 4x^2 \ln x - 4x^2$

ii) $y = -\dfrac{c_1}{6} x^{-5} + c_2 x - \dfrac{4}{27} x^{-1/2}$.

E5) The given equation is $x^2(1-x^2)y'' - x^3 y' - 2y = 0$

Since one of the solutions of this equation is $y_1 = \dfrac{\sqrt{1-x^2}}{x}$, assume that

$$y = v\frac{\sqrt{1-x^2}}{x}$$

Substituting for y, y' and y'' in the given equation, we get

$$v'' - \frac{2+x^2}{x(1-x^2)} v' = 0$$

It is a linear homogeneous equation in v' and x and its I.F is

$$\text{I.F.} = e^{-\int \frac{2+x^2}{x(1-x^2)} dx}$$

Now $-\displaystyle\int \frac{2+x^2}{x(1-x^2)} dx = -\int \frac{2}{x} dx - \frac{3}{2}\int \frac{1}{1-x} dx + \frac{3}{2}\int \frac{1}{1+x} dx$

[using partial fractions]

$$= -2\ln x + \frac{3}{2}\ln(1-x) + \frac{3}{2}\ln(1+x)$$

$$= \ln \frac{(1-x^2)^{3/2}}{x^2}$$

$\therefore$ I.F. $= \dfrac{(1-x^2)^{3/2}}{x^2}$

and $v'.\dfrac{(1-x^2)^{3/2}}{x^2} = c_1$

$\Rightarrow v' = \dfrac{c_1 x^2}{(1-x^2)^{3/2}}$

Integrating the above equation once again, we get

$$v = c_1 \int \frac{x}{1-x^2} \frac{x}{\sqrt{1-x^2}} dx$$

Substitute $\sqrt{1-x^2} = \sin\theta$

$$v = -c_1 \int \cot^2\theta \, d\theta$$

$$= -c_1[-\cot\theta - \theta] + c_2$$

$$= c_2 + c_1\left(\frac{x}{\sqrt{1-x^2}} + \sin^{-1}\sqrt{1-x^2}\right)$$

Hence $y = v\dfrac{\sqrt{1-x^2}}{x} = c_2\dfrac{\sqrt{1-x^2}}{x} + c_1\left(1 + \dfrac{\sqrt{1-x^2}}{x}\sin^{-1}\sqrt{1-x^2}\right)$

Note that the in the above problem you can obtain the second linearly independent solution directly by using formula (63) and solving

$$y_2 = y_1(x)\int \frac{e^{-\int a_1(x)dx}}{y_1^2}dx = \sqrt{\frac{1-x^2}{x}}\int \frac{x^2}{1-x^2}e^{\int \frac{x}{1-x^2}dx}dx = \sqrt{\frac{1-x^2}{x}}\int \frac{x^2}{(1-x^2)^{3/2}}dx$$

E6) The given equation is

$$x(x\cos x - 2\sin x)\frac{d^2y}{dx^2} + (x^2+2)\sin x\frac{dy}{dx} - 2(x\sin x + \cos x)y = 0$$

Since $y_1 = x^2$ is a solution of the given equation, assume that $y = vx^2$

Substituting for y, y' and y'' in the given equation, we have

$$v'' + \frac{x^2\sin x + 4x\cos x - 6\sin x}{x^2\cos x - 2x\sin x}v' = 0$$

It is a linear homogenous equation and its I.F is

$$\text{I.F.} = e^{\int (x^2\sin x + 4x\cos x - 6\sin x)/(x^2\cos x - 2x\sin x)\, dx}$$

Now, $\displaystyle\int \frac{x^2\sin x + 4x\cos x - 6\sin x}{x^2\cos x - 2x\sin x}dx$

$$= \int\left[-\frac{(-x^2\sin x - 2\sin x}{x^2\cos x - 2x\sin x} + \frac{4(x\cos x - 2\sin x)}{x(x\cos x - 2\sin x)}\right]dx$$

$$= -\ln(x^2\cos x - 2x\sin x) + 4\ln x$$

$$= \ln\left[\frac{x^4}{(x^2\cos x - 2x\sin x)}\right]$$

$$\therefore\ \text{I.F.} = \frac{x^4}{(x^2\cos x - 2x\sin x)}$$

Thus, $v'\dfrac{x^4}{x^2\cos x - 2x\sin x} = c_1$

$$\Rightarrow\ v' = c_1\left(\frac{x^2\cos x - 2x\sin x}{x^4}\right)$$

$$= c_1\left(\frac{\cos x}{x^2} - \frac{2}{x^3}\sin x\right)$$

$$= c_1\ d\left(\frac{\sin x}{x^2}\right)$$

$$\Rightarrow\ v = c_2 + c_1\left(\frac{\sin x}{x^2}\right)$$

Therefore, $y = c_2x^2 + c_1\sin x$

E7) i) Substituting $x = e^z$ or $z = \ln x$ the given equation reduces to

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + 3\frac{dy}{dz} + 3y = 0$$

Auxiliary equation corresponding to above equation is

$$m^2 + 2m + 3 = 0$$

$$\Rightarrow\ m = -1 \pm \sqrt{2}\,i$$

The general solution is

$$y = e^{-z}[c_1\cos(\sqrt{2}z) + c_2\sin(\sqrt{2}z)]$$

$$= x^{-1}[c_1\cos(\sqrt{2}\ln x) + c_2\sin(\sqrt{2}\ln x)]$$

ii) Substituting $x = e^z$ or $z = \ln x$, the given equation reduces to

$$\frac{d^3y}{dz^3}+2\frac{d^2y}{dz^2}+4\frac{dy}{dz}+8y=0$$

Auxiliary equation is

$$m^3+2m^2+4m+8=0$$

$$\Rightarrow \quad (m+2)\,(m^2+4)=0$$

$$\Rightarrow \quad m=-2,\ 2i,\ -2i$$

The general solution is

$$y=\frac{c_1}{x^2}+c_2\cos(2\ln x)+c_3\sin(2\ln x)$$

E8) i) Let $x=e^z$. Substituting for its derivatives the given equation becomes

$$\frac{d^2y}{dz^2}+2\frac{dy}{dz}=e^{-z} \qquad (98)$$

A.E. is $m^2+2m=0 \Rightarrow m=0,-2$

C.F. is $y=c_1+c_2e^{-2z}$

To find particular integral of Eqn. (98), let us assume that

$$y_p(z)=u_1(z)+u_2(z)\,e^{-2z} \qquad (99)$$

$$\therefore \quad \frac{dy_p}{dz}=u_1'+u_2'e^{-2z}+u_2(-2)\,e^{-2z}$$

As first auxiliary condition, assume that

$$u_1'+u_2'e^{-2z}=0, \qquad (100)$$

So that

$$\frac{dy_p}{dz}=-2u_2\,e^{-2z} \qquad (101)$$

$$\therefore \quad \frac{d^2y_p}{dz^2}=-2u_2'e^{-2z}+4u_2e^{-2z} \qquad (102)$$

Since $y_p(z)$ must satisfy Eqn. (98), hence substituting y_p, y_p' and y_p'' from relations (99), (101) and (102) in Eqn. (98), we get

$$-2u_2'e^{-2z}+4u_2e^{-2z}+2[-2u_2e^{-2z}]=e^{-z}$$

$$\Rightarrow \quad u_2'=-\frac{1}{2}e^{z} \qquad (103)$$

Solving Eqns. (100) and (103) for u_1' and u_2', we get

$u_1'=\frac{1}{2}e^{-z}$ and $u_2'=-\frac{1}{2}e^{z}$

Hence, on substituting the values of $u_1(z)$ and $u_2(z)$ in Eqn. (99) a particular integral of Eqn. (98) can be written in the form

$$y_p(z)=-\frac{1}{2}e^{-z}-\frac{1}{2}e^{-z}=-e^{-z}$$

The general solution of Eqn. (98) assumes the following form

$y=c_1+c_2e^{-2z}-e^{-z}$,

Thus, the general solution for the given equation becomes

$y=c_1+\frac{c_2}{x^2}-\frac{1}{x}$.

ii) $$y = \begin{cases} Ax + \dfrac{B}{x} + \dfrac{1}{m^2-1}x^m & ; \quad if \ m \neq \pm 1 \\ Ax + \dfrac{B}{x} + \dfrac{\ln x}{2}x^m & ; \quad if \ m = \pm 1 \end{cases}$$

iii) The given equation can be written as

$$x^3\frac{d^3y}{dx^3} - 4x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} - 2y = x^3$$

which is an Euler's equation and its general solution is

$$y = c_1x^2 + x^{5/2}(c_2x^{\sqrt{21}/2} + c_3x^{-\sqrt{21}/2}) - \frac{x^3}{5}.$$

E9) Substituting $r = e^z$ or $z = \ln r$, the given equation reduces to

$$\frac{d^2R}{dz^2} - n^2R = 0$$

The auxiliary equation is

$$m^2 - n^2 = 0$$

$$\Rightarrow \quad m = \pm n$$

The general solution is

$$R = c_1e^{nz} + c_2e^{-nz}$$

or $$R = c_1r^n + c_2r^{-n}$$

Given that $R(b) = 0$

$\therefore \quad c_1b^n + c_2b^{-n} = 0$, for $n = 1, 2, \ldots$

$$\Rightarrow \quad c_1 = \frac{-c_2b^{-n}}{b^n}$$

$$\therefore \quad R = \frac{-r^nc_2b^{-n}}{b^n} + c_2r^{-n}$$

$$= -\left(\frac{r}{b}\right)^n c_2b^{-n} + \frac{r^{-n}}{b^{-n}}c_2b^{-n}$$

$$= C\left[-\left(\frac{r}{b}\right)^n + \left(\frac{b}{r}\right)^n\right]$$, where $C = c_2b^{-n}$

For $n = 0$, auxiliary equation reduces to

$$m^2 = 0 \Rightarrow m = 0, 0$$

The general solution is

$$R = c_1 + c_2z$$

or $$R = c_1 + c_2\ln r$$

$$R(b) = 0 \Rightarrow c_1 + c_2\ln b = 0$$

$$\Rightarrow c_1 = -c_2\ln b$$

$$\therefore \quad R = c_2[-\ln b + \ln r] = c_2\ln(r/b).$$

E10) i) Substitution $x = e^z$, $z = \ln x$ reduces the given equation to

$$\frac{d^2y}{dz^2} + z = 0$$

The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore$ The general solution is $y = c_1\cos(\ln x) + c_2\sin(\ln x)$

$y(1) = 1 \Rightarrow c_1 = 1$ and $y'(1) = 2 \Rightarrow c_2 = 2$

Thus the general solution is

$$y = \cos(\ln x) + 2\sin(\ln x).$$

ii) Substituting $t = -x$, the given equation reduces to

$$t^2 \frac{d^2 y}{dt^2} - 4t\frac{dy}{dt} + 6y = 0,\ y(2) = 8,\ y'(2) = 0$$

Substituting $t = e^z$, $z = \ln t$ in the above equation, we obtain

$$\frac{d^2 y}{dz^2} - 5\frac{dy}{dz} + 6y = 0$$

The auxiliary equation is $m^2 - 5m + 6 = 0 \Rightarrow m = 2,3$

$\therefore$ The general solution is $y = c_1 e^{2z} + c_2 e^{3z} = c_1 t^2 + c_2 t^3$

$$y(2) = 8 \Rightarrow 4c_1 + 8c_2 = 8$$

and $y'(2) = 0 \Rightarrow 4c_1 + 12c_2 = 0$

Solving the above system for c_1 and c_2, we get $c_1 = 6$ and $c_2 = -2$.

Thus $y = 6x^2 + 2x^3$ is the required general solution.

E11) i) The given equation is

$$\left[(x+a)^2 \frac{d^2 y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6\right] y = x \qquad (104)$$

To solve it, assume that $x + a = e^z$ or $z = \ln(x+a)$

$$\therefore\ \frac{dy}{dx} = \frac{1}{x+a}\frac{dy}{dz}$$

and $\dfrac{d^2 y}{dx^2} = -\dfrac{1}{(x+a)^2}\dfrac{dy}{dz} + \dfrac{1}{(x+a)^2}\dfrac{d^2 y}{dz^2} = \dfrac{1}{(x+a)^2}\left(\dfrac{d^2 y}{dz^2} - \dfrac{dy}{dz}\right)$

With this substitution, the given equation is reduced to

$$\left[\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) - 4\frac{dy}{dz} + 6y\right] = e^z - a$$

$$\Rightarrow \left(\frac{d^2 y}{dz^2} - 5\frac{dy}{dz} + 6y\right) = e^z - a \qquad (105)$$

The general solution of Eqn. (105) can be expressed as

$$y = c_1 e^{2z} + c_2 e^{3z} + \frac{1}{2}e^z - \frac{a}{6}$$

and the general solution of Eqn. (104) is

$$y = c_1 (x+a)^2 + c_2 (x+a)^3 + \frac{1}{2}(x+a) - \frac{a}{6}.$$

ii) $y = a\cos\{\ln(1+x)\} + b\sin\{\ln(1+x)\} + 2\ln(1+x)\sin\{\ln(1+x)\}$

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APPENDIX

Let us consider the linear differential equation in the form

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1} y}{dx^{n-1}}+\cdots+a_n(x)\,y=b(x)\,, \tag{A1}$$

where $a_0(x)\neq 0$ and $a_0,\, a_1,\, \ldots,\, a_n$ and b are continuous in some interval I.

We shall assume that the solution of the homogenous differential equation corresponding to A(1) is

$$y_c(x)=c_1 y_1(x)+c_2 y_2(x)+\cdots+c_n y_n(x)$$

$$=\sum_{i=1}^{n} c_i y_i(x)\,,$$

(A2)

where y_i's are linearly independent solutions of the homogeneous differential equation and c_i's are constants.

Let us assume that a particular integral of Eqn. (A1) has the form

$$y_p(x)=\sum_{i=1}^{n} u_i(x)\; y_i(x) \tag{A3}$$

This relation contains n unknown functions $u_1,\, u_2,\, \ldots,\, u_n$. The condition that relation (A3) satisfies Eqn. (A1) is the only necessary condition which must be satisfied and this leaves considerable freedom in the choice of u_i's. In fact, we can impose $(n-1)$ conditions, which along with the given differential equation, gives us n conditions to determine n unknown functions $u_1,\, u_2,\, \ldots,\, u_n$.

Differentiating relation (A3), we obtain

$$\frac{dy_p}{dx}=\sum_{i=1}^{n} u_i \frac{dy_i}{dx}+\sum_{i=1}^{n} \frac{du_i}{dx}\, y_i \tag{A4}$$

We choose our first auxiliary condition as the vanishing of the second term on the right hand side of Eqn. (A4), i.e.,

$$\sum_{i=1}^{n} \frac{du_i}{dx}\, y_i=0 \tag{A5}$$

Then the expression for $\dfrac{dy_p}{dx}$ reduces to

$$\frac{dy_p}{dx}=\sum_{i=1}^{n} u_i \frac{dy_i}{dx} \tag{A6}$$

Differentiating Eqn. (A6), we get

$$\frac{d^2 y_p}{dx^2}=\sum_{i=1}^{n} u_i \frac{d^2 y_i}{dx^2}+\sum_{i=1}^{n} \frac{du_i}{dx}\frac{dy_i}{dx} \tag{A7}$$

As a second auxiliary condition, we choose

$$\sum_{i=1}^{n} \frac{du_i}{dx}\frac{dy_i}{dx}=0\,, \tag{A8}$$

so that $\dfrac{d^2 y_p}{dx^2}=\displaystyle\sum_{i=1}^{n} u_i \frac{d^2 y_i}{dx^2}$ (A9)

We continue this process of differentiation y_p, $(n-1)$ times and obtain $(n-1)$ auxiliary conditions. Then $(n-1)th$ auxiliary condition is

$$\sum_{i=1}^{n} \frac{du_i}{dx}\frac{d^{n-2}y_i}{dx^{n-2}} = 0 \qquad \text{(A10)}$$

and $$\frac{d^{n-1}y_p}{dx^{n-1}} = \sum_{i=1}^{n} u_i \frac{d^{n-1}y_i}{dx^{n-1}} \qquad \text{(A11)}$$

Differentiating Eqn. (A11) one again, we get

$$\frac{d^n y_p}{dx^n} = \sum_{i=1}^{n} u_i \frac{d^n y_i}{dx^n} + \sum_{i=1}^{n} \frac{du_i}{dx}\frac{d^{n-1}y_i}{dx^{n-1}} \qquad \text{(A12)}$$

Substituting in Eqn. (A1) the values of y_p and its derivatives from equations like (A3), (A6), (A9), (A11) and (A12) etc., we get

$$\sum_{i=1}^{n} u_i \left[a_0 \frac{d^n y_i}{dx^n} + a_1 \frac{d^{n-1}y_i}{dx^{n-1}} + \cdots + a_n y_i \right]$$

$$+ a_0(x) \sum_{i=1}^{n} \frac{d^{n-1}y_i}{dx^{n-1}} . \frac{du_i}{dx} = b(x) \qquad \text{(A13)}$$

Since the y_i's are the solutions of the homogeneous equation corresponding to Eqn. (A1), the first expression on the left hand side of Eqn. (A13) must vanish and then Eqn. (A13) reduces to

$$a_0(x) \sum_{i=1}^{n} \frac{d^{n-1}y_i}{dx^{n-1}} \frac{du_i}{dx} = b(x)\,, \qquad \text{(A14)}$$

which is our nth auxiliary condition.

Writing the various conditions imposed on u_1's, we get the following set of simultaneous linear differential equations in u_1's:

$$\left.\begin{array}{ccccccc}
y_1 u_1' & + & y_2 u_2' & + \cdots + & y_n u_n' = 0 \\
\frac{dy_1}{dx} u_1' & + & \frac{dy_2}{dx} u_2' & + \cdots + & \frac{dy_n}{dx} u_n' = 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{d^{n-2}y_1}{dx^{n-2}} u_1' & + & \frac{d^{n-2}y_2}{dx^{n-1}} u_2' & + \cdots + & \frac{d^{n-2}y_n}{dx^{n-2}} u_n' = 0 \\
\frac{d^{n-1}y_1}{dx^{n-1}} u_1' & + & \frac{d^{n-1}y_2}{dx^{n-1}} u_2' & + \cdots + & \frac{d^{n-1}y_n}{dx^{n-1}} u_n' = \frac{1}{a_0(x)} b(x)
\end{array}\right\} \qquad \text{(A15)}$$

Since the y_i's are known functions of x, we can solve the set of simultaneous Eqns. (A15) for the u_i's by using the Cramer's rule

Cramer's rule gives

$$u_k = \frac{W_k}{W},\ k = 1, 2, \ldots, n$$

where W is the wronskian of $y_1, y_2, \ldots, y_n$, which is non-vanishing since y_i's are linearly independent. W_k is the determinant obtained by replacing the k^{th} column of the wronskian by the column

$$\begin{array}{c} 0 \\ 0 \\ \vdots \\ \vdots \\ \dfrac{b(x)}{a_0(x)} \end{array}$$

For details of the Cramer's rule, ref. appendix of Unit 10.

Thus we are lead to n first order linear differential equations in the $u_i's$, which can always be expressed as simple integrals capable of numerical integration even when they cannot be integrated explicitly.

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UNIT 13

METHOD OF DIFFERENTIAL OPERATORS

Structure **Page No.**

13.1 INTRODUCTION

In Unit 10, we saw that the general solution of a non-homogeneous linear differential equation consists of two parts, namely, the complementary function and a particular integral. In Unit 11, we developed the method of undetermined coefficients for finding a particular integral of non-homogeneous linear differential equations with constant coefficients for certain particular forms of the non-homogeneous terms. We dealt with the method of variation of parameters in Unit 12. This method provides a particular integral of non-homogeneous linear differential equations with constant as well as variable coefficients provided, all the linearly independent solutions of the corresponding homogeneous equation are known. In the case of non-homogeneous equations with constant coefficients, the constraints on the use of both these methods namely, the method of undetermined coefficients and the method of variation of parameters can be overcome to a large extent when we use the method of **differential operators**. The notion of differential operator can be traced back to Barnabé Brisson (1777-1820), a French mathematician and civil engineer and its use was carried out by another French mathematician Louis Cauchy (1789-1857), a reputed pioneer of analysis.

An operator is a function that takes a function as an argument instead of numbers i.e., a function defined with domain as set of functions.

Differential operators are a generalization of the operation of differentiation. The most commonly used differential operator is the action of taking the derivative itself. Common notation used for the differential operator is

$D = \dfrac{d}{dx}$. Thus, if y is an n^{th} order differentiable function, then

$$D^o y = y,\ Dy = y',\ D^2 y = y'',\ \ldots,\ D^n y = y^{(n)} \tag{1}$$

and $D^m D^n = D^{m+n}$, for positive integers m and n..

In this unit we shall make use of the method of differential operator to find a particular integral of a non-homogeneous linear differential equation with constant coefficient. The determination of a particular integral of a non-homogeneous equation depends upon the properties of the operators inverse to D, that is, D^{-1}. The problem of inverse operator has been the subject of investigation and was studied by Rehuel Lobatto (1797-1866), a Dutch mathematician and George Boole (1815-1864) an English mathematician, philosopher and logician.

In this unit we are mainly concerned with the polynomial differential operators with constant coefficients. In Sec.13.2 we shall start by defining a polynomial differential operator of order n and give the fundamental laws of operation for the polynomial operators. We have also defined the inverse differential operators and given some general properties of the polynomial operators and the inverse operators in this section. In Sec.13.3 we shall discuss some general methods of finding a particular integral of non-homogeneous differential equtions with constant coefficients using differential operators. In certain cases, depending on the form of the non-homogeneous terms in the differential equations, there are methods available which are shorter than the general methods. We shall be discussing these shorter methods in Sec.13.4. Finally, in Sec.13.5 we shall discuss the applications of non-homogeneous differential equations with constant coefficients in the study of vibrations in mechanics and the theory of electric circuits.

Objectives

After going through this unit you should be able to:

- define a differential operator and inverse differential operator;
- state properties of differential operators and inverse operators;
- obtain a particular integral of a given non-homogeneous differential equation using the method of differential operators;
- use shorter operator methods of finding a particular integral when non-homogeneous term is of the form $\exp(\alpha x)$, $\sin(ax+b)$ or $\cos(ax+b)$, polynomial $V(x)$ in x, $\exp(\alpha x).V(x)$; and
- derive differential equations for some physical problems and obtain their solutions.

13.2 DIFFERENTIAL OPERATORS

Consider a linear non-homogeneous differential equation of order n with constant coefficients, viz.,

$$a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = b(x),\ a_0 \neq 0 \tag{2}$$

Using Eqn. (1), Eqn. (2) can be written as

$$(a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_n y) = b(x)$$

or, $\quad (a_0 D^n + a_1 D^{n-1} + \cdots + a_n)\, y = b(x)$.

If we write

$$L(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-2} D^2 + a_{n-1} D + a_n,\ a_0 \neq 0 \tag{3}$$

then $L(D)$ is a **polynomial differential operator** of order n. Eqn. (2) can than be written in the form

$$L(D)\,y = b(x) \tag{4}$$

and is read as "$L(D)$ operating on y equals $b(x)$". It may be noted that $L(D)$ has meaning only when applied to some function.

Let us consider two polynomial differential operators L_1 and L_2 with constant co-efficients where

$L_1 = D + 2$ and $L_2 = 3D - 1$

Then, $L_1(L_2(y)) = (D+2)\left(3\dfrac{dy}{dx} - y\right)$

$$= 3\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6\frac{dy}{dx} - 2y$$

$$= (3D^2 + 5D - 2)\,y \tag{5}$$

Similarly $L_2(L_1(y)) = (3D-1)\left(\dfrac{dy}{dx} + 2y\right)$

$$= 3\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - \frac{dy}{dx} - 2y$$

$$= (3D^2 + 5D - 2)\,y \tag{6}$$

From Eqns. (5) and (6), we get

$$L_1(L_2(y)) = L_2(L_1(y)) \tag{7}$$

We can thus say that the product L_1L_2 of two polynomial differential operators L_1 and L_2 is defined as that operator which produces the same result as is obtained by using the operators L_2 followed by the operator L_1.
The product of two polynomial differential operators always exists and is again a polynomial differential operator. Moreover, if L_1 and L_2 are polynomial differential **operators with constant coefficients then $L_1L_2 = L_2L_1$**, but it is usually **not true** for polynomial differential operators with **variables coefficients**. For instance, if $L_1 = xD + 2$ and $L_2 = D - 1$, then it can be checked that $L_2L_1 = xD^2 + (3-x)\,D - 2$, whereas $L_1L_2 = xD^2 + (2-x)\,D - 2$. This is because L_1 is an operator with variable coefficients whose product is dependent on the order of the factors. In this unit we shall mainly be dealing with polynomial differential operators with constant coefficients.

Further, in addition to L_1 and L_2 above, if we have $L_3 = D + 1$, then,

$$L_1L_2[L_3(y)] = (3D^2 + 5D - 2)\left(\frac{dy}{dx} + y\right)$$

$$= 3\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 5\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2\frac{dy}{dx} - 2y$$

$$= 3\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y \qquad (8)$$

and $L_1[L_2L_3(y)] = (D+2)\left[3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y\right] \left(\because\ L_2L_3 = 3D^2 + 2D - 1\right)$

$$= 3\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y$$

$$= 3\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y \qquad (9)$$

From Eqns. (8) and (9) we conclude that

$$\boldsymbol{L_1L_2(L_3(y)] = L_1[L_2L_3(y)]}. \qquad (10)$$

Also the **sum** of any two polynomial differential operators is obtained by adding their corresponding coefficients. For instance, if $L_1 = 3D^2 - D + x - 2$ and $L_2 = x^2D + 4D + 7$ then

$$\boldsymbol{L_1 + L_2} = 3D^2 + (x^2+3)D + x + 5 = \boldsymbol{L_2 + L_1} \qquad (11)$$

Similarly, if $L_3 = 2D^2 + 2xD + 2$, then

$$\begin{aligned}(L_1 + L_2) + L_3 &= [3D^2 + (x^2+3)D + x + 5] + 2D^2 + xD + 2 \\ &= 3D^2 + x + (x^2D + 3D + 7 + 2D^2 + xD) \\ &= 3D^2 + x - D - 2 + (x^2D + 3D + 7 + 2D^2 + xD + D + 2) \\ &= 3D^2 - x - D - 2 + [(x^2 + x + 4)D + 2D^2 + 9] \\ &= L_1 + (L_2 + L_3)\end{aligned}$$

i.e., $\boldsymbol{(L_1 + L_2) + L_3 = L_1 + (L_2 + L_3)}$ (12)

Let us now consider an operator $L = 3D^2 - 2xD$ and functions $y_1 = x^2 + 4$ and $y_2 = 2x + x^3$.

For constants c_1 and c_2, we then have

$$\begin{aligned}L(c_1y_1 + c_2y_2) &= (3D^2 - 2xD)\,[c_1(x^2+4) + c_2(2x+x^3)] \\ &= 3D^2[c_1(x^2+4) + c_2(2x+x^3)] - 2xD[c_1(x^2+4) + c_2(2x+x^3)] \\ &= (6c_1 + 18c_2x) - (4x^2c_1 + 4xc_2 + 6x^3c_2) \\ &= c_1(6 - 4x^2) + c_2(18x - 4x - 6x^3) \qquad (13)\end{aligned}$$

and $c_1L(y_1) + c_2L(y_2) = c_1(3D^2 - 2xD)\,(x^2+4) + c_2(3D^2 - 2xD)\,(2x + x^3)$

$$= c_1(6 - 4x^2) + c_2(18x - 4x - 6x^3) \qquad (14)$$

From Eqns. (13) and (14), we conclude that

$$\boldsymbol{L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2)} \qquad (15)$$

In words, we say from Eqn. (15) that the **polynomial differential operators are linear operators**. That is, if L is any polynomial differential operator, c_1 and c_2 are constants and y_1 and y_2 are any functions of x each possessing the derivatives of the required order, then

$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$.

You may **check** here that in general, equality (10) holds true for every polynomial differential operators of the type L_1, L_2 and L_3 having **constant coefficients**. Whereas equalities (11), (12) and (15) are valid in general, for every polynomial differential operators of the type L_1, L_2, L_3, which may have **constant or variable coefficients**.

Summing up the discussion above, we can say that the polynomial differential operators satisfy the following fundamental laws of operation.

Fundamental Laws of Operation

If L_1, L_2 and L_3 be any three polynomial differential operators, then

i) $L_1 + L_2 = L_2 + L_1$ **(addition is commutative)**

ii) $(L_1 + L_2) + L_3 = L_1 + (L_2 + L_3)$ **(addition is associative)**

iii) $L_1(L_2 + L_3) = L_1 L_2 + L_1 L_3$ **(multiplication is distributive with respect to addition)**

If L_1 and L_2 are operators with **constant coefficients**, then we also have

iv) $L_1 L_2 = L_2 L_1$ **(multiplication is commutative)**

v) $(L_1 L_2) L_3 = L_1(L_2 L_3)$ **(multiplication is associative)**

Note that under the operation of addition and multiplication the polynomial differential operators with constant coefficients behave like algebraic polynomials. We can, therefore, use the tools of elementary algebra while dealing with these operators. In particular, multiplication may be used to factor operators with constant coefficients. For instance, we can write

$D^3 - 3D^2 + 4 = (D+1)\,(D^2 - 4D + 4) = (D+1)\,(D-2)^2$ and

$D^3 + 2D^2 - D - 2 = (D-1)\,(D^2 + 3D + 2) = (D-1)\,(D+1)\,(D+2)$.

You may now try this exercise.

E1) Factor each of the following operators:

i) $2D^2 + 3D - 2$

ii) $D^3 - 2D^2 - 5D + 6$

iii) $2D^4 + 12D^3 + 18D^2 + 4D - 8$.

iv) $D^3 - 11D - 20$

Operations i) – v) above are very useful for obtaining certain properties of the polynomial differential operators and the inverse differential operators, which in turn are useful in finding the solutions of non-homogeneous linear differential equations. Before discussing the properties of the polynomial differential operators let us define inverse differential operators.

Consider Eqn. (4), namely,

$$L(D)\,y = b(x)$$

where $L(D)$ is a polynomial differential operator of order n with constant coefficients only. In order to find a particular solution of Eqn. (4), we write it as

$$y = \frac{1}{L(D)} b(x) \tag{16}$$

and then try to define an inverse operator of $L(D)$, written as $\frac{1}{L(D)}$ so, that the function y in relation (16) have a meaning and satisfy Eqn. (4). In other words, what all we require is that

$$L(D).\frac{1}{L(D)} b(x) = b(x) \tag{17}$$

In particular, if we have $L(D) = D$ then Eqn. (4) reduces to $Dy = b(x)$

$\Rightarrow \quad y = D^{-1}b(x)$

and $b(x) = DD^{-1}b(x)$

so that $DD^{-1} = 1$.

Thus, D^{-1} represents such an operation on any quantity that if the operation D is subsequently performed, the quantity is left unaltered. **Thus, D^{-1} is an operator inverse to D.** Moreover, we know that differentiation and integration are inverse operations. Therefore, D^{-1} is an operation of simple indefinite integration. Similarly, D^{-p} is the operation of p-times integration. You may note here that these inverse operations yield a particular integral but not the complete integral, and therefore we can omit the arbitrary constant which arises in integration.

From relations (16) and (17) we can thus say that **$\frac{1}{L(D)} b(x)$ is that function of x which when operated upon by $L(D)$ gives $b(x)$.**

For example, $\frac{1}{D^2 + 2D}(6x + 6x^2) = x^3$, because $(D^2 + 2D)x^3 = 6x + 6x^2$.

Thus **the inverse differential operator of $L(D)$**, written as $L^{-1}(D)$ or $\frac{1}{L(D)}$, is an operator which, when operating on $b(x)$, yields a particular integral y_p of $L(D)y = b(x)$, i.e.,

$$y_p = \frac{1}{L(D)} b(x). \tag{18}$$

We know that the general solution of Eqn. (4) is $y = y_c + y_p$. With the above form of y_p, it reduces to

$$y = y_c + \frac{1}{L(D)} b(x) \tag{19}$$

If $g(D)$ and $h(D)$ are two polynomial differential operators with constant coefficients then the **sum** and **difference** of the inverse operators is defined as

$$\left[\frac{1}{g(D)} \pm \frac{1}{h(D)}\right] b(x) = \frac{1}{g(D)} b(x) \pm \frac{1}{h(D)} b(x) \tag{20}$$

When we apply two or more inverse operators in succession to a function (an operand), then the operator immediately next to the operand in the left is applied first, then the next and so on. Thus,

$$\frac{1}{g(D)} \cdot \frac{1}{h(D)} b(x) = \frac{1}{g(D)} \left[\frac{1}{h(D)} b(x) \right]. \qquad (21)$$

We now give some general properties of the linear polynomial differential operators and the inverse differential operators in the form of theorems.

Let us start by considering a differential equation

$$y'' - 5y' + 6y = e^{3x}$$

or, $\quad (D^2 - 5D + 6)y = e^{3x}$

It can be checked that $y_1 = xe^{3x}$ and $y_2 = (x-1)e^{3x}$ are two particular integrals of the equation. We have

$[D^2 - 5D + 6]\ (xe^{3x}) = 9xe^{3x} + 6e^{3x} - 5(3xe^{3x} + e^{3x}) + 6xe^{3x} = e^{3x}$ and

$[D^2 - 5D + 6]\ (x-1)e^{3x} = 9xe^{3x} - 3e^{3x} - 5(3xe^{3x} - 2e^{3x}) + 6xe^{3x} - 6e^{3x} = e^{3x}$.

Thus both y_1 and y_2 satisfy the given equation. Further, it can checked that the difference of two particular solutions $y_1 - y_2 = xe^{3x} - (x-1)e^{3x} = e^{3x}$ is a solution of the corresponding homogenous equation i.e., $(D^2 - 5D - 6)y = 0$. Obviously, we can see that $(D^2 - 5D - 6)e^{3x} = 9e^{3x} - 15e^{3x} + 6e^{3x} = 0$, i.e., $y_1 - y_2 = e^{3x}$ satisfies the corresponding homogeneous equation.

We now give this result in general in the form of the following theorem.

Theorem 1: If y_1 and y_2 are two particular integrals of the equation $L(D)\ y = b(x)$, then their difference is a solution of the corresponding homogeneous equation.

Proof: Since y_1 and y_2 are particular integrals of $L(D)\ y = b(x)$, we have $L(D)\ y_1 = b(x)$ and $L(D)\ y_2 = b(x)$

Now, $L(D)$ is a linear polynomial differential operator, therefore

$$\begin{aligned} L(D)\ (y_1 - y_2) &= L(D)y_1 - L(D)y_2, \\ &= b(x) - b(x) \\ &= 0. \end{aligned}$$

– ■ –

In equations of the form (4) we can use any particular integral of the given equation to obtain the general solution of the equation.

Let us now consider a differential equation of the form.

$$\frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta\ y = X \qquad (22)$$

or, $\quad [D^2 - (\alpha+\beta)D + \alpha\beta]\ y = X$,

where α, β are constants.

Its particular integral is obtained as

$$\frac{1}{D^2 - (\alpha+\beta)D + \alpha\beta} X$$

We can equivalently write it as

$$\frac{1}{(D-\alpha)\ (D-\beta)} X \qquad (23)$$

Applying the operator $(D-\alpha)\,(D-\beta)$ to Eqn. (23) and using the fundamental laws of operations for the polynomial differential operators with constant coefficients, we have

$$(D-\alpha)\,(D-\beta)\,\frac{1}{(D-\alpha)\,(D-\beta)}X$$

$$=(D-\beta)\,(D-\alpha)\,\frac{1}{(D-\alpha)}.\frac{1}{(D-\beta)}X$$

$$=(D-\beta)\left[D-\alpha\frac{1}{D-\alpha}\right]\frac{1}{D-\beta}X$$

$$=(D-\beta)\,\frac{1}{D-\beta}X=X$$

From this reduction you might have observed that we can as well write a particular integral (23) as

$$\frac{1}{(D-\beta)\,(D-\alpha)}X \tag{24}$$

Therefore, we can say that the **inverse differential operators with constant coefficients are commutative**. This property of inverse differential operators hold true in general also. We shall give this property in the form of the Theorem 2. However, we shall not be proving the theorem here.

Theorem 2: If $g(D)$ and $h(D)$ are two polynomial differential operators with constant coefficients then

$$\frac{1}{g(D)\,h(D)}=\frac{1}{g(D)}.\frac{1}{h(D)}=\frac{1}{h(D)}.\frac{1}{g(D)}.$$

– ■ –

Let us now suppose that we are dealing with two differential equations, namely

$$\frac{d^2y}{dx^2}-y=2 \text{ or, } (D^2-1)y=2 \tag{25}$$

and $$\frac{d^2y}{dx^2}-y=5x \text{ or, } (D^2-1)y=5x \tag{26}$$

Let y_{1p} and y_{2p} be particular integrals of Eqns. (25) and (26), respectively.

Since $(D^2-1)\,(-2)=2$, therefore $y_{1P}=\frac{1}{D^2-1}2=-2$.

Also $(D^2-1)\,(-5x)=5x$, thus $y_{2P}=\frac{1}{D^2-1}5x=-5x$.

Now if we add the non-homogeneous terms of Eqns. (25) and (26) and consider an equation

$$\frac{d^2y}{dx^2}-y=2+5x \tag{27}$$

Then, $(D^2-1)\,(-2-5x)=2+5x$

$\therefore$ P.I.. of Eqn. (27) is $y_p=-2-5x=y_{1p}+y_{2p}$.

Thus we see that if y_{1p} and y_{2p} are particular integrals of Eqns. (25) and (26), respectively, then P.I. of Eqn. (27) is $y_{1p}+y_{2p}$. This is known as **superposition** of solutions.

We now give the result above in the form of the following theorem.

Theorem 3: If $y_1, y_2, \ldots, y_m$ are particular solutions of the respective equations $L(D)y = b_1(x), L(D)y = b_2(x), \ldots, L(D)y = b_m(x)$, then $y = y_1 + y_2 + \cdots + y_m$ is a particular solution of $L(D)y = b_1(x) + b_2(x) + \cdots + b_m(x)$.

Proof: We know that a derivative of a sum is the sum of their derivatives. Therefore, it follows that

$$L(D)[y_1 + y_2 + \cdots + y_m] = L(D)y_1 + L(D)y_2 + \cdots + L(D)y_m \tag{28}$$

We are given that $y_1, y_2, \ldots, y_m$ are particular solutions of the equations $L(D)y = b_1(x), L(D)y = b_2 x, \ldots, L(D)y = b_m(x)$, respectively.
Hence under this hypothesis, the right-hand side of Eqn. (28) equals $b_1(x) + b_2(x) + \cdots + b_m(x)$, which proves the required result.

– ■ –

E2) Using Theorem 3, find a particular integral of the equation
$$\left(\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - y\right) = e^x + 2.$$

We now consider another property of the polynomial differential operators.

Let us consider

$$(D^2 + 3D + 4)e^{ax}y \tag{29}$$

It can be equivalently written as

$$\begin{aligned}
&D[De^{ax}y] + 3D\,[e^{ax}y] + 4e^{ax}y \\
&= D[e^{ax}Dy + ae^{ax}y] + 3[e^{ax}Dy + ae^{ax}y] + 4e^{ax}y \\
&= e^{ax}D^2y + ae^{ax}Dy + ae^{ax}Dy + a^2e^{ax}y + 3e^{ax}Dy + 3ae^{ax}y + 4e^{ax}y \\
&= e^{ax}[D^2y + 2aDy + a^2y + 3Dy + 3ay + 4y] \\
&= e^{ax}[(D^2y + 2aDy + a^2y) + (3Dy + 3ay) + 4y] \\
&= e^{ax}[(D+a)^2 + 3(D+a) + 4]\ y
\end{aligned} \tag{30}$$

Thus, if $L(D) = D^2 + 3D + 4$, then from Eqns. (29) and (30), we have

$$L(D)e^{ax}y = e^{ax}L(D+a)\ y$$

This is know as the **shift formula** for the polynomial differential operator $L(D)$.

Relation (30) shows us how to shift an exponential factor from the right to the left of a polynomial differential operator. The formula is very useful in finding the solutions of differential equations. We shall be illustrating it in Sec. 13.3. However, in the next theorem we shall prove this formula in general.

Theorem 4: Suppose $L(D)$ is a polynomial differential operator of order n. If the first n derivatives of y w.r.t. x exist and are finite and a is any constant, then

$$\boldsymbol{L(D)[e^{ax}y] = e^{ax}L(D+a)\ y} \tag{31}$$

Proof: We have

$$D(e^{ax}y) = e^{ax}Dy + ae^{ax}y = e^{ax}(D+a)\ y$$

Suppose that for some positive integer k, we have

$$D^k(e^{ax}y) = e^{ax}(D+a)^k y \tag{32}$$

Differentiating both sides of Eqn. (32) w.r. to x, we get

$$\begin{aligned} D^{k+1}(e^{ax}y) &= D[e^{ax}(D+a)^k y] \\ &= e^{ax}D(D+a)^k y + ae^{ax}(D+a)^k y \\ &= e^{ax}[(D+a).\,(D+a)^k y] \\ &= e^{ax}(D+a)^{k+1} y \end{aligned}$$

Thus, if relation (32) is true for k, it is also true for $k+1$. We have already verified it for $k=1$. Hence by induction, we conclude that relation (32) is true for every positive integer k.

Since $L(D)$ is a polynomial in D, using the superposition Theorem 3, and relation (32), result (31) is proved.
Thus, in general, we have

$$L(D)\,(e^{ax}y) = e^{ax}\,L(D+a)\,y$$

– ■ –

So far, we have given in Theorems 1-4 certain properties of the polynomial differential operators and the inverse differential operators. So far, we have not discussed the methods of finding a particular integral using the differential operators. In the next section we shall give the general method of finding a particular integral of the given differential equation.

13.3 GENERAL METHOD OF FINDING A PARTICULAR INTEGRAL

We start by considering the following example.

Example 1: Solve the differential equation

$$y''' + 2y'' - y' - 2y = e^{2x} \tag{33}$$

Solution: In the operator notation, Eqn. (33) can be written as

$$(D^3 + 2D^2 - D - 2)y = e^{2x}$$

or, $$(D-1)\,(D+1)\,(D+2)y = e^{2x} \tag{34}$$

Let $$u = (D+1)\,(D+2)y \tag{35}$$

Eqn. (34) then reduces to $(D-1)u = e^{2x}$, which is a linear equation. Its solution is

$$u = e^{2x} + c_1e^x,\ c_1 \text{ a constant.} \tag{36}$$

Putting from Eqn. (36) in Eqn. (35), we get

$$(D+1)\,(D+2)y = e^{2x} + c_1e^x \tag{37}$$

Let $$(D+2)y = v \tag{38}$$

then Eqn. (37) becomes $(D+1)v = e^{2x} + c_1e^x$, which is linear equation with I.F. $e^{\int dx}$ and its solution is

$$ve^x = \int (e^{2x} + c_1 e^x) e^x dx + c_2$$

$$= \frac{e^{3x}}{3} + \frac{c_1 e^{2x}}{2} + c_2$$

$$\Rightarrow \quad v = \frac{e^{2x}}{3} + \frac{c_1}{2} e^x + c_2 e^{-x}. \tag{39}$$

Putting from Eqn. (39) in Eqn. (38), we obtain

$$(D+2)y = \frac{e^{2x}}{3} + \frac{c_1 e^x}{2} + c_2 e^{-x}$$

which is again a linear differential equation with the solution as

$$y = \frac{1}{12} e^{2x} + \frac{c_1}{6} e^x + c_2 e^{-x} + c_3 e^{-2x} \tag{40}$$

where c_1, c_2 and c_3 are arbitrary constants.

You may **note** that in Eqn. (40), $(c_1/6)e^x + c_2 e^{-x} + c_3 e^{-2x}$ is the complementary function of Eqn. (33) and $(1/12)e^{2x}$ is a particular integral (free from arbitrary constants).

***

Let us now see how the method used in Example 1 can be generalised for finding a particular integral of an n^{th} order equation.
Let us consider Eqn. (2), which when written in terms of differential operator D reduces to

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_n y = b(x) \tag{41}$$

or, $L(D)y = b(x)$,
where $L(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_n$.

If $m_1, m_2, \ldots, m_n$ are the n distinct roots of the auxiliary equation corresponding to differential Eqn. (41), then we can write it in the form

$$L(D)y = (D - m_1)(D - m_2) \ldots (D - m_n)\, y = b(x) \tag{42}$$

Putting $(D - m_2)(D - m_3) \ldots (D - m_n) = \eta_1$
Eqn. (42) reduces to

$$(D - m_1)\,\eta_1 = b(x) \tag{43}$$

Eqn. (43) is a linear differential equation of the first order and we can write its solution in the form

$$\eta_1 = \frac{1}{(D - m_1)} b(x) = e^{m_1 x} \int e^{-m_1 x} b(x)\, dx + c_1 e^{m_1 x} \tag{44}$$

Since we are looking for a particular integral of Eqn. (41), we may simplify the expression (44) by putting $c_1 = 0$. Next, we put

$$(D - m_3)(D - m_4), \ldots, (D - m_n)\, y = \eta_2 \tag{45}$$

so that,

$$(D - m_2)\eta_2 = \eta_1 = \frac{1}{(D - m_1)} b(x) = e^{m_1 x} \int e^{-m_1 x}\, b(x)\, dx \tag{46}$$

Solving linear differential Eqn. (46), we get

$$\eta_2 = \frac{1}{(D - m_2)(D - m_1)} b(x) = e^{m_2 x} \int e^{(m_1 - m_2)x} \left(\int e^{-m_1 x} b(x)\, dx \right) dx \tag{47}$$

Continuing this process n-times, we get

$$y_p(x) = \frac{1}{(D-m_n)\ldots(D-m_2)\,(D-m_1)} b(x) = e^{m_n x} \int e^{(m_{n-1}-m_n)\,x} \left(\int e^{(m_{n-2}-m_{n-1})\,x} \int \ldots \left(\int e^{(m_1-m_2)x} \left(\int e^{-m_1 x} b(x)\,dx\right) dx \ldots dx\right) dx\right. \quad (48)$$

Relation (48) yields a particular integral of Eqn. (41).

We now illustrate the theory above, through the following examples.

Example 2: Find a particular integral of the differential equation

$$(D^2 - 5D + 6)\ y = e^{3x} \quad (49)$$

Solution: Here P.I. $= y_p = \dfrac{1}{D^2 - 5D + 6}\ e^{3x}$

$$= \frac{1}{(D-3)\,(D-2)}\ e^{3x}$$

$$= \frac{1}{D-3}\ \frac{1}{D-2}\ e^{3x}$$

$$= \frac{1}{D-3}\ e^{2x} \int e^{-2x} e^{3x}\ dx$$

$$= \frac{1}{D-3} e^{2x} \int e^{x}\ dx$$

$$= \frac{1}{D-3}\ e^{2x}\ e^{x}$$

$$= \frac{1}{D-3} e^{3x}$$

$$= e^{3x} \int e^{-3x}\ e^{3x}\ dx$$

$$= e^{3x} \int 1\ dx$$

$$= x\ e^{3x}$$

Hence, $y_p = xe^{3x}$ is a particular integral of Eqn. (49).

***

Let us consider the case of repeated roots.

Example 3: Find a particular integral of the differential equation

$$y'' + 4y' + 4y = -x^{-2} e^{-2x},\ x > 0. \quad (50)$$

Solution: Writing Eqn. (50) in the operator form, we get

$$(D^2 + 4D + 4)\ y = -x^{-2}\ e^{-2x}$$

$\therefore$ A particular Integral of Eqn. (50) is

$$y_p = \frac{1}{D^2 + 4D + 4}\ (-x^{-2}\ e^{-2x})$$

$$= -\frac{1}{(D+2)^2}\ x^{-2}\ e^{-2x}$$

$$= -\frac{1}{D+2}\ \frac{1}{D+2}\ x^{-2}\ e^{-2x}$$

$$= -\frac{1}{D+2}\ e^{-2x} \int x^{-2}\ e^{2x}\ e^{-2x}\ dx$$

$$= \frac{1}{D+2} x^{-1}\ e^{-2x}$$

$$= e^{-2x} \int x^{-1} e^{-2x}\, e^{2x}\, dx$$

$$= e^{-2x} \ln |x|$$

***

Sometimes the actual integrations of the form as involved in Eqn. (48) turn out to be extremely tedious. In such situations we use a method in which the repeated integration can be avoided. In this method the polynomial differential operator $\frac{1}{L(D)}$ is resolved into partial fractions.

For instance, in Example 2, we can obtain a particular integral y_p of Eqn. (49) by writing it in the form

$$y_p = \frac{1}{(D-3)\,(D-2)} e^{3x}$$

$$= \left(\frac{1}{D-3} - \frac{1}{D-2} \right) e^{3x}$$

$$= \frac{1}{D-3} e^{3x} - \frac{1}{D-2} e^{3x} \qquad (51)$$

and then applying the method above to solve each term of Eqn. (51).

Similarly, if the n factors of an n^{th} order polynomial differential operator $L(D)$ are distinct (corresponding to the distinct roots of the auxiliary equation), then we can write a particular integral in the form

$$y_p = \frac{1}{L(D)} b(x) = \left(\frac{\alpha_1}{D-m_1} + \frac{\alpha_2}{D-m_2} + \cdots + \frac{\alpha_n}{D-m_n} \right) b(x) \qquad (52)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are constants and for a particular problem we can obtain these constants by simple algebraic manipulations.

On applying the method above to solve each term of Eqn. (52), a particular integral can be obtained in the form

$$y_p = \alpha_1 e^{m_1 x} \int e^{-m_1 x} b(x)\, dx + \alpha_2 e^{m_2 x} \int e^{-m_2 x}\, b(x)\, dx + \cdots$$

$$+ \alpha_n e^{m_n x} \int e^{-m_n x}\, b(x)\, dx \qquad (53)$$

In case a root m_1 of the auxiliary equation corresponding to the differential Eqn. (52) is repeated r-times, then the corresponding partial fractions of $\frac{1}{L(D)}$ will be of the form

$$\frac{\alpha_1}{D-m_1} + \frac{\alpha_2}{(D-m_1)^2} + \cdots + \frac{\alpha_r}{(D-m_1)^r} + \frac{\alpha_{r+1}}{(D-m_{r+1})} + \cdots + \frac{\alpha_n}{(D-m_n)}$$

and a particular integral of Eqn. (52) will then be given by

$$y_p = \alpha_1 e^{m_1 x} \int e^{-m_1 x} b(x)\, dx + \alpha_2 e^{m_1 x} \int \left(\int e^{-m_1 x} b(x)\, dx \right) dx + \cdots$$

$$+ \alpha_r e^{m_1 x} \int \left(\int \ldots \int \left(\int e^{-m_1 x}\, b(x)\, dx \right) dx \ldots\, dx \right) dx + \cdots$$

$$\cdots + \alpha_n\, e^{m_n x} \int e^{-m_n x}\, b(x)\, dx \qquad (54)$$

To have a better understanding of what we have discussed above, let us consider a few examples.

Example 4: Find a particular integral of the differential equation

$$(D^2 - 5D + 6)\ y = \ln x,\ x > 0. \tag{55}$$

Solution: We have

$$\frac{1}{D^2 - 5D + 6} = \frac{1}{(D-3)(D-2)}$$

$$= \frac{1}{D-3} - \frac{1}{D-2}$$

Hence a particular integral of Eqn. (55) is

$$y_p = \frac{1}{D^2 - 5D + 6} \ln x = \left(\frac{1}{D-3} - \frac{1}{D-2}\right) \ln x$$

$$= \frac{1}{D-3} \ln x - \frac{1}{D-2} \ln x$$

$$= e^{3x} \int e^{-3x}\ (\ln x)\ dx - e^{2x} \int e^{-2x} (\ln x)\ dx$$

Since the integrals on the r.h.s. of the above equation cannot be evaluated in terms of the elementary functions we have left the solution in terms of these integrals only.

***

We now consider the case of repeated roots.

Example 5: Find a particular integral of the following differential equation

$$(D-1)^2 (D+1)^2\ \ y = e^x$$

Solution: A particular integral is

$$y_p = \frac{1}{(D-1)^2 (D+1)^2}\ e^x$$

$$= \frac{1}{4}\left[\frac{-1}{D-1} + \frac{1}{(D-1)^2} + \frac{1}{D+1} + \frac{1}{(D+1)^2}\right] e^x$$

$$= \frac{1}{4}\left[\frac{-1}{D-1} e^x + \frac{1}{(D-1)^2} e^x + \frac{1}{D+1} e^x + \frac{1}{(D+1)^2} e^x\right]$$

$$= \frac{1}{4}\left[-e^x \int e^{-x} e^x dx + e^x \int\left(\int e^{-x} e^x\ dx\right) dx\right.$$

$$\left. + e^{-x} \int e^x\ e^x\ dx + e^{-x} \int\left(\int e^x\ e^x\ dx\right) dx\right]$$

$$= \frac{1}{4}\left[-xe^x + \frac{x^2}{2} e^x + \frac{e^x}{2} + \frac{e^x}{4}\right]$$

***

We now take up an example in which the non-homogeneous term of the given equation is a trigonometric function.

Example 6: Solve $\dfrac{d^2 y}{dx^2} + y = \sec^2 x$.

Solution: The given differential equation can be written as

$$(D^2 + 1)\ y = \sec^2 x$$

The auxiliary equation is

$$m^2 + 1 = 0$$

$$\Rightarrow\ \ m = \pm i$$

$\therefore$ C.F. $= y_c = c_1 \cos x + c_2 \sin x$

$$\text{P.I.} = y_p = \frac{1}{D^2+1}\sec^2 x = \frac{1}{(D+i)\,(D-i)}\sec^2 x$$

$$= \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\sec^2 x$$

$$= \frac{1}{2i}\left[e^{ix}\int e^{-ix}\sec^2 x\, dx - e^{-ix}\int e^{ix}\sec^2 x\, dx\right]$$

$$= \frac{1}{2i}\left[e^{ix}\int \frac{\cos x - i\sin x}{\cos^2 x}\, dx - e^{-ix}\int \frac{\cos x + i\sin x}{\cos^2 x}\, dx\right]$$

$$= \frac{1}{2i}\left[e^{ix}\int (\sec x - i\sec x\,\tan x)\, dx - e^{-ix}\int (\sec x + i\sec x\,\tan x)\, dx\right]$$

$$= \frac{1}{2i}\left[(e^{ix} - e^{-ix})\int \sec x\, dx - i\,(e^{ix} + e^{-ix})\int \tan x\,\sec x\, dx\right]$$

$$= \frac{1}{2i}\,[\,(2i\sin x)\ \ln|(\sec x + \tan x)| - (2i\,\cos x).\,\sec x]$$

$$= \sin x\,\ln|(\sec x + \tan x)| - 1$$

$$e^{ix} = \cos x + i\sin x$$
$$e^{-ix} = \cos x - i\sin x$$

$\therefore$ The general solution of the given differential equation is

$$y = y_c + y_p = c_1\cos x + c_2\sin x + \sin x\,\ln|(\sec x + \tan x)| - 1.$$

***

You may now try the following exercise.

E3) Find a particular integral of the following differential equations.

i) $(D^2 + n^2)\ y = \sec nx$

ii) $(D^2 - 3D + 2)\ y = \sin x\, e^{-x}$

iii) $(D^2 + 2D + 1)\ y = 2e^{2x}$

iv) $(D^3 - D^2 - 8D + 12)\ y = X(x)$

The general method of computing a particular integral as discussed in Sec. 13.3 requires a lot of calculations. In certain cases, a P.I. can be obtained by the methods which are shorter than the general methods. We shall discuss such methods in the next section.

13.4 SHORT METHODS OF FINDING A PARTICULAR INTEGRAL

Consider the general nth order linear differential equation of the form (41) namely,

$$L(D)\,y = (a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n)\ y = b(x)$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants and $a_0 \neq 0$. For certain particular forms of the non-homogeneous term $b(x)$ in the equation above, there do exist shorter methods of finding particular integrals.

Let us take up these methods for these various particular forms of $b(x)$ one by one.

I. $b(x) = e^{\alpha x}$, α constant

We know that

$$D\, e^{\alpha x} = \alpha\, e^{\alpha x}$$
$$D^2 e^{\alpha x} = D(\alpha\, e^{\alpha x}) = \alpha^2\, e^{\alpha x}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$D^n e^{\alpha x} = \alpha^n\, e^{\alpha x}$$

$$\therefore\ L(D)\, e^{\alpha x} = (a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)\, e^{\alpha x}$$
$$= (a_0 \alpha^n + a_1 \alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n) e^{\alpha x}$$
$$= L(\alpha) e^{\alpha x} \qquad (56)$$

Further, if $L(\alpha) \neq 0$, i.e., α is not a root of polynomial $L(D)$, then

$$L(D)\left[\frac{1}{L(\alpha)}\, e^{\alpha x}\right] = \frac{1}{L(\alpha)}\left[L(D)\, e^{\alpha x}\right]$$
$$= \frac{1}{L(\alpha)}.\, L(\alpha)\, e^{\alpha x} \text{ (using Eqn. (56))}$$
$$= e^{\alpha x}.$$

That is, $\frac{1}{L(\alpha)}\, e^{\alpha x}$ is a particular integral of $L(D)\ y = e^{\alpha x}$, whenever $L(\alpha) \neq 0$.

Thus, $$\boxed{\frac{1}{L(D)} e^{\alpha x} = \frac{e^{\alpha x}}{L(\alpha)},\ L(\alpha) \neq 0} \qquad (57)$$

Now suppose that $L(\alpha) = 0$. Then $L(D)$ contains the factor $(D-\alpha)$. Suppose that the factor occurs p times in $L(D)$, that is, let

$$L(D) = (D-\alpha)^p\ \phi(D),\ \phi(\alpha) \neq 0,\ p \geq 1, \qquad (58)$$

where $\phi(D)$ is a polynomial in D of order $(n-p)$.

Now, $$\frac{1}{L(D)} e^{\alpha x} = \frac{1}{(D-\alpha)^p\ \phi(D)}\ e^{\alpha x} \text{ (using Eqn. (58))}$$
$$= \frac{1}{(D-\alpha)^p}\left[\frac{1}{\phi(D)}\, e^{\alpha x}\right]$$
$$= \frac{1}{(D-\alpha)^p}\left[\frac{e^{\alpha x}}{\phi(\alpha)}.1\right] \text{ (using Eqn. (57))}$$
$$= \frac{1}{\phi(\alpha)}\ \frac{1}{(D-\alpha)^p}\,[e^{\alpha x}.1]$$
$$= \frac{1}{\phi(\alpha)} e^{\alpha x}\, \frac{x^p}{p!}$$

where $\frac{1}{(D-\alpha)^p} e^{\alpha x}$ has been evaluated by the general method.

Also consider, $$L(D)\left[\frac{e^{\alpha x}}{\phi(\alpha)}.\frac{x^p}{p!}\right] = (D-\alpha)^p\ \phi(D)\left[\frac{e^{\alpha x}}{\phi(\alpha)}.\frac{x^p}{p!}\right]$$
$$= \phi(D)\left\{\ (D-\alpha)^p\left[\frac{e^{\alpha x}}{\phi(\alpha)}.\frac{x^p}{p!}\right]\right\}$$

$$= \phi(D) \left\{ \frac{e^{\alpha x}}{\phi(\alpha)} . D^p \left(\frac{x^p}{p!} \right) \right\} \text{ (using Theorem 4)}$$

$$= \phi(D) \left\{ \frac{e^{\alpha x}}{\phi(\alpha)} . 1 \right\}$$

$$= \frac{1}{\phi(\alpha)} \phi(D) \; e^{\alpha x}$$

$$= \frac{1}{\phi(\alpha)} \phi(\alpha) \; e^{\alpha x} \text{ (using Eqn. (56)}$$

$$= e^{\alpha x}$$

Hence $\left[\frac{1}{\phi(\alpha)} e^{\alpha x} \frac{x^p}{p!} \right]$ is a particular integral of $L(D) y = e^{\alpha x}$, where $L(D) = (D - \alpha)^p \; \phi(D)$ and $\phi(\alpha) \neq 0$.

That is,

$$\boxed{\frac{1}{(D-\alpha)^p \; \phi(D)} e^{\alpha x} = \frac{x^p e^{\alpha x}}{p! \; \phi(\alpha)}, \; \phi(\alpha) \neq 0} \tag{59}$$

We now illustrate the method above with the help of the following examples.

Example 7: Solve $(D^2 - 4D + 3) \; y = e^{2x}$.

Solution: A.E. is

$$m^2 - 4m + 3 = 0$$

$$\Rightarrow \; (m-3)\,(m-1) = 0$$

$$\Rightarrow \; m = 1, 3$$

$\therefore$ $\text{C.F.} = c_1 e^x + c_2 e^{3x}$, c_1 and c_2 are arbitrary constants.

$$\text{P.I.} = \frac{1}{(D-1)\,(D-3)} e^{2x} \text{ (here 2 is not a root of A.E.)}$$

$$= \frac{1}{(2-1)\,(2-3)} e^{2x} \text{ (using relation (57))}$$

$$= -e^{2x}$$

$\therefore$ The complete solution of the given differential equation is

$$y = c_1 e^x + c_2 e^{3x} - e^{2x}$$

***

Let us consider another example.

Example 8: Solve $\frac{d^3 y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$ (60)

Solution: In the operator form, Eqn. (60) reduces to

$$(D^3 + 1) \; y = 3 + e^{-x} + 5e^{2x}$$

Its A.E. is

$$m^3 + 1 = 0$$

$$\Rightarrow \; (m+1)\,(m^2 - m + 1) = 0$$

$$\Rightarrow \; m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

Hence C.F. $= c_1 e^{-x} + e^{x/2}\left[c_2 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_3 \sin\left(\frac{x\sqrt{3}}{2}\right)\right]$

where c_1, c_2 and c_3 are arbitrary constants.

P.I. $= \frac{1}{D^3+1}\,[3 + e^{-x} + 5e^{2x}]$

$= \frac{1}{D^3+1}\,[3e^{ox} + e^{-x} + 5e^{2x}]$

$= 3\frac{1}{D^3+1}e^{ox} + \frac{1}{D^3+1}e^{-x} + 5\frac{1}{D^3+1}\,e^{2x}$

$= 3\frac{1}{0+1}e^{0x} + \frac{1}{(D+1)}\cdot\frac{1}{D^2-D+1}e^{-x} + 5\frac{1}{2^3+1}e^{2x}$ [$\alpha = -1$ is a root of the A.E.]

$= 3 + \frac{1}{D+1}\cdot\frac{1}{(-1)^2-(-1)+1}e^{-x} + \frac{5}{9}e^{2x}$

$= 3 + \frac{1}{3}e^{-x}\frac{x}{1!} + \frac{5}{9}e^{2x}$ (using relation (59) with $p = 1,\ \alpha = -1,\ \phi(D) = 1$)

$= 3 + \frac{1}{3}e^{-x}.x + \frac{5}{9}e^{2x}$

Hence the complete solution of Eqn. (60) is

$$y = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_3 \sin\left(\frac{x\sqrt{3}}{2}\right)\right] + 3 + \frac{e^{-x}x}{3} + \frac{5}{9}e^{2x}.$$

***

You may now check your understanding of the method above, while doing the following exercise.

E4) Solve the following differential equations.

i) $(D^2 - 2D + 1)\ y = 3e^{(5/2)x}$

ii) $(D^2 - 1)\ y = (e^x + 1)^2$

iii) $(D^3 + 5D^2 + 7D - 3)\ y = e^{2x}\cosh x$

iv) $(D^3 - 6D^2 + 11D - 6)\ y = e^{2x}$

We now consider the case when $b(x)$ is a sine or a cosine function.

II. $\boldsymbol{b(x) = \cos(ax+b)}$ **or** $\boldsymbol{\sin(ax+b)}$

Successive differentiation of $\cos(ax+b)$ gives

$D\cos(ax+b) = -a\sin(ax+b)$

$D^2\cos(ax+b) = -a^2\cos(ax+b)$

$D^3\cos(ax+b) = a^3\sin(ax+b)$

$D^4\cos(ax+b) = (D^2)^2\cos(ax+b) = a^4\cos(ax+b) = (-a^2)^2\cos(ax+b)$

……………………

Therefore, in general

$(D^2)^n\cos(ax+b) = (-a^2)^n\cos(ax+b)$

Thus, if $\phi(D^2)$ is a polynomial function of D^2, then

$$\left.\begin{aligned}&\phi(D^2)\cos(ax+b)=\phi(-a^2)\cos(ax+b)\\ &\text{Similarly, } \phi(D^2)\sin(ax+b)=\phi(-a^2)\sin(ax+b)\end{aligned}\right\} \qquad (61)$$

Now the following two possibilities arise.

Case I: $\phi(-a^2)\neq 0$

In this case $\phi(D^2)\left[\dfrac{\cos(ax+b)}{\phi(-a^2)}\right]=\dfrac{1}{\phi(-a^2)}\phi(D^2)\,[\cos(ax+b)]$

$$=\frac{1}{\phi(-a^2)}\phi(-a^2)\cos(ax+b) \text{ (using Eqn. (61))}$$

$$=\cos(ax+b)$$

Similarly, $\phi(D^2)\left[\dfrac{\sin(ax+b)}{\phi(-a^2)}\right]=\sin(ax+b)$

Hence,

$$\frac{1}{\phi(-a^2)}\cos(ax+b) \text{ is a P.I. of } \phi(D^2)\ y=\cos(ax+b)$$

and

$$\frac{1}{\phi(-a^2)}\sin(ax+b) \text{ is a P.I. of } \phi(D^2)y=\sin(ax+b)$$

whenever $\phi(-a^2)\neq 0$.

That is, we have the following results

$$\boxed{\begin{aligned}&\boldsymbol{\frac{1}{\phi(D^2)}\cos(ax+b)=\frac{1}{\phi(-a^2)}\cos(ax+b),\ \phi(-a^2)\neq 0}\\ &\boldsymbol{\frac{1}{\phi(D^2)}\sin(ax+b)=\frac{1}{\phi(-a^2)}\sin(ax+b),\ \phi(-a^2)\neq 0}\end{aligned}} \qquad (62)$$

Case II: $\phi(-a^2)=0$

Let $\phi(D^2)=(D^2+a^2)^p\psi(D^2)$, where $\psi(-a^2)\neq 0$.

$$\therefore\quad \frac{1}{\phi(D^2)}\cos(ax+b)=\frac{1}{(D^2+a^2)^p\psi(D^2)}\cos(ax+b),$$

$$=\frac{1}{(D^2+a^2)^p}\,\frac{1}{\psi(-a^2)}\cos(ax+b)$$

$$=\frac{1}{\psi(-a^2)}\,\frac{1}{(D^2+a^2)^p}\cos(ax+b),$$

where $\dfrac{1}{(D^2+a^2)^p}\cos(ax+b)$ can then be evaluated by the general method.

Similarly, $\dfrac{1}{\phi(D^2)}\sin(ax+b)=\dfrac{1}{\psi(-a^2)}.\dfrac{1}{(D^2+a^2)^p}\sin(ax+b)$,

where $\dfrac{1}{(D^2+a^2)^p}\sin(ax+b)$ is evaluated by the general method.

We now consider a few examples, to illustrate the method above.

Example 9: Find a particular integral of the differential equation $(D^4+10D^2+9)\ y=\cos(2x+3)$.

Solution: A particular integral is given by

$$
\begin{aligned}
y_p &= \frac{1}{(D^4+10D^2+9)}\cos(2x+3) \\
&= \frac{1}{D^2.D^2+10D^2+9}\cos(2x+3) \\
&= \frac{1}{(-2^2)\ (-2^2)+10(-2^2)+9}\cos(2x+3) \\
&= \frac{1}{(-4)(-4)+10(-4)+9}\cos(2x+3) \\
&= \frac{1}{16-40+9}\cos(2x+3) \\
&= -\frac{1}{15}\cos(2x+3).
\end{aligned}
$$

***

Let us look at another example.

Example 10: Find a particular integral of the differential equation $(D^4-1)\ y=\sin x$.

Solution: A particular integral of the given equation is

$$
\begin{aligned}
y_p &= \frac{1}{(D^4-1)}\sin x \\
&= \frac{1}{(D^2-1)\ (D^2+1)}\sin x \\
&= \frac{1}{(-1^2-1)\ (D^2+1)}\sin x\ \ (\because D^2+1=0\ \ for\ D^2=-1^2) \\
&= -\frac{1}{2}\ \frac{1}{D^2+1}\sin x \\
&= -\frac{1}{2}\left[\frac{1}{(D+i)\ (D-i)}\right]\sin x \\
&= -\frac{1}{2}\ \frac{1}{2i}\left[\frac{1}{D-i}-\frac{1}{D+i}\right]\sin x \\
&= -\frac{1}{4i}\left[\frac{1}{D-i}\sin x-\frac{1}{D+i}\sin x\right] \\
&= -\frac{1}{4i}\left[e^{ix}\int e^{-ix}\sin x\ dx-e^{-ix}\int e^{ix}\sin x\ dx\right] \\
&= -\frac{1}{4i}\left[e^{ix}\int e^{-ix}\left(\frac{e^{ix}-e^{-ix}}{2i}\right)dx-e^{-ix}\int e^{ix}\left(\frac{e^{ix}-e^{-ix}}{2i}\right)dx\right] \\
&= \frac{1}{8}\left[e^{ix}\int(1-e^{-2ix})dx-e^{-ix}\int(e^{2ix}-1)\ dx\right] \\
&= \frac{1}{8}\left[e^{ix}\left(x+\frac{e^{-2ix}}{2i}\right)-e^{-ix}\left(\frac{e^{2ix}}{2i}-x\right)\right]
\end{aligned}
$$

$$= \frac{1}{8}\left[x(e^{ix} + e^{-ix}) + \frac{1}{2i}(e^{-ix} - e^{ix}) \right]$$

$$= \frac{1}{8}[2x\cos x - \sin x]$$

***

We can alternatively deal with sine and cosine functions in II above by considering $L(D)y = e^{i(ax+b)} = \cos(ax+b) + i\sin(ax+b)$ and then considering the real part or the imaginary part on the r.h.s., as the case may be.
By superposition Theorem 3, we then have

$$\boxed{\begin{aligned} \frac{1}{L(D)}\cos(ax+b) &= \mathrm{Re}\left[\frac{1}{L(D)}e^{i(ax+b)}\right] \\ \frac{1}{L(D)}\sin(ax+b) &= \mathrm{Im}\left[\frac{1}{L(D)}e^{i(ax+b)}\right] \end{aligned}} \qquad (63)$$

Symbols Re and Im are read as 'real part of' and 'imaginary part of', respectively.

In particular, suppose we want to solve

$$V = \frac{1}{D^2 + a^2}\sin ax$$

Then let us consider

$$U = \frac{1}{D^2 + a^2}e^{iax}$$

$$= \frac{1}{(D+ai)(D-ai)}e^{iax}$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right]e^{iax}$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai}e^{iax} - \frac{e^{iax}}{2ai}\right] \text{ (here } D - ai = 0 \text{ } for \text{ } D = ai)$$

$$= \frac{1}{2ai}\left[e^{iax}\int e^{-iax}e^{iax}dx - \frac{e^{iax}}{2ai}\right]$$

$$= \frac{1}{2ai}\left[xe^{iax} - \frac{e^{iax}}{2ai}\right]$$

$$= \frac{ix}{-2a}(\cos ax + i\sin ax) + \frac{1}{4a^2}(\cos ax + i\sin ax)$$

Now, $V = \mathrm{Im}\, U = \dfrac{-x\cos ax}{2a} + \dfrac{\sin ax}{4a^2}$ (64)

Similarly, $\dfrac{1}{D^2 + a^2}\cos ax = \mathrm{Re}\, U = \dfrac{x\sin ax}{2a} + \dfrac{\cos ax}{4a^2}$ (65)

Remark: You must have felt that by using the above alternative approach, the term $\dfrac{1}{D^2+1}\sin x$ in Example 10, could have been evaluated very easily thus avoiding long manipulations. Here we would like to remark that choosing an appropriate method for the evaluation of a particular integral for a given equation is a skill, which comes through practice only.

You may now try this exercise.

E5) Solve the following differential equations for integers m and n:

i) $(D^4 + 2n^2D^2 + n^4)\ y = \cos\ mx,\ m \neq n$

ii) $(D^2 + m^2)\ (D^2 + n^2)\ y = \cos\left\{(m+n)\dfrac{x}{2}\right\}\cos\left\{(m-n)\dfrac{x}{2}\right\},\ m \neq n.$

In many problems involving the sine or the cosine functions, you may find that the polynomial $L(D)$ is an odd order function. Consider for instance, the differential equation

$$(D^3 + 2D^2 - 5D - 10)\,y = 2\sin x \tag{66}$$

A particular integral of Eqn. (66) is then obtained as follows

$$\begin{aligned} y_p &= 2\frac{1}{D^3 + 2D^2 - 5D - 10}\sin x \\ &= 2\frac{1}{(D+2)(D^2-5)}\sin x \end{aligned} \tag{67}$$

Here $L(D) = (D^2 - 5)\ (D+2) = g(D)\ h(D)$ say, where

$g(D) = (D^2 - 5)$ is an even order and $h(D) = (D+2)$ is an odd order polynomial function.

In order to make $h(D)$ an even order polynomial, we multiply and divide Eqn. (67) by $(D-2)$.

Eqn. (67) can than be solved by writing it in the form

$$\begin{aligned} y_p &= 2\frac{(D-2)}{(D^2-4)\ (D^2-5)}\sin x \\ &= \frac{2(D-2)}{(-1^2-4)\ (-1^2-5)}\sin x = \frac{2}{30}(D-2)\sin x \\ &= \frac{1}{15}(\cos x - 2\sin x) \end{aligned}$$

Thus, in situations when

$L(D) = g(D)\ h(D)$, where $g(D)$ is an even order polynomial factor and $h(D)$ is an odd order polynomial factor, we can write

$$\frac{1}{L(D)}\sin(ax+b) = \frac{h(-D)}{g(D)\,h(D)\ h(-D)}\sin(ax+b) \tag{68}$$

and, $$\frac{1}{L(D)}\cos(ax+b) = \frac{h(-D)}{g(D)\ h(D)\ h(-D)}\cos(ax+b) \tag{69}$$

and obtain P.I. of Eqns. (68) or (69) as in Case II above.

Let us consider another example to illustrate the method above.

Example 11: Solve $(D^3 + D^2 - D - 1)\,y = \cos 2x$. (70)

Solution: The A.E. corresponding to Eqn. (70) is

$$\begin{aligned} & m^3 + m^2 - m - 1 = 0 \\ \Rightarrow\ & m^2(m+1) - (m+1) = 0 \\ \Rightarrow\ & (m^2-1)\ (m+1) = 0 \\ \Rightarrow\ & m = 1,\ -1,\ -1. \end{aligned}$$

$\therefore$ C.F. $= y_c = c_1 e^x + (c_2 + c_3 x)\, e^{-x}$

$$\text{P.I.} = y_p = \frac{1}{D^3 + D^2 - D - 1} \cos 2x$$

$$= \frac{1}{(D+1)\,(D^2-1)} \cos 2x$$

$$= \frac{(D-1)}{(D-1)\,(D+1)(D^2-1)} \cos 2x$$

$$= \frac{(D-1)}{(D^2-1)^2} \cos 2x$$

$$= (D-1) \left[\frac{1}{(D^2-1)^2} \cos 2x \right]$$

$$= (D-1) \left[\frac{1}{(-2^2-1)^2} \cos 2x \right] = (D-1)\, \frac{\cos 2x}{25}$$

$$= -\frac{2 \sin 2x}{25} - \frac{\cos 2x}{25}$$

Hence the complete solution of Eqn. (70) is given by

$$y = y_c + y_p = c_1 e^x + (c_2 + c_3 x)\, e^{-x} - \frac{2\sin 2x}{25} - \frac{\cos 2x}{25}.$$

***

You may now try the following exercise.

E6) Solve the following differential equations:

i) $(D^2 + D + 1)\; y = \sin 2x$

ii) $(D^2 + 2n \cos \alpha D + n^2)\; y = a \cos nx$

We now discuss the case when the non-homogeneous term $b(x)$ is a polynomial in x.

III. $b(x) = Ax^n$, n integer, A constant

We start by considering a simple situation when the polynomial operator $L(D) = D - a,\; a \neq 0$.

Then $L(D)\, y = b(x)$ becomes

$$(D - a) y = Ax^n \tag{71}$$

A particular integral of Eqn. (71) is obtained as

$$y_p = \frac{Ax^n}{D-a} = \frac{-1}{a}\left(1 - \frac{D}{a}\right)^{-1} (Ax^n) = \frac{-1}{a}\left(1 + \frac{D}{a} + \frac{D^2}{a^2} + \cdots + \frac{D^n}{a^n} + \cdots\right)(Ax^n) \tag{72}$$

Since $D^{n+1} x^n = 0$, differentiating x^n, n-times in Eqn. (72), we obtain

$$y_p = \frac{-A}{a}\left(x^n + \frac{nx^{n-1}}{a} + \cdots + \frac{n!}{a^n}\right), \quad a \neq 0.$$

We illustrate the above situation through an example.

Example 12: Solve the differential equation

$$(D^2-4)y = x^2 \tag{73}$$

Solution: The auxiliary equation is

$$m^2-4=0$$

$$\Rightarrow \quad m=\pm 2$$

Thus, C.F. $= y_c = c_1e^{2x}+c_2e^{-2x}$

Also, a particular integral is

$$\begin{aligned} y_p &= \frac{1}{D^2-4}x^2 \\ &= \frac{-1}{4}\left(1-\frac{D^2}{x}\right)^{-1}x^2 \\ &= \frac{-1}{4}\left(1+\frac{D^2}{4}+\frac{D^4}{4^2}+\cdots\right)x^2 \\ &= \frac{-1}{4}\left(x^2+\frac{1}{2}\right) \end{aligned}$$

Therefore, the general solution of Eqn. (73) is

$$y = y_c + y_p = c_1e^{2x}+c_2e^{-2x}-\frac{1}{4}\left(x^2+\frac{1}{2}\right).$$

***

The method above can be generalised when

$$L(D)y = (a_0D^n + a_1D^{n-1}+\cdots+a_{n-1}D+a_n)y = Ax^n,\ a_n \neq 0. \tag{74}$$

A particular integral is then given by

$$\begin{aligned} y_p &= \frac{A}{L(D)}x^n \\ &= \frac{A}{a_n}\left(1+\frac{a_{n-1}}{a_n}D+\cdots+\frac{a_0}{a_n}D^n\right)^{-1}x^n,\ a_n\neq 0 \\ &= \frac{A}{a_n}(1+b_1D+b_2D^n+\cdots+b_nD^n)x^n \end{aligned} \tag{75}$$

$b_1, b_2, \ldots, b_n$ being constant. Here $(1+b_1D+\cdots+b_nD^n)$ is the binomial expansion of $\dfrac{1}{L(D)}$ upto n^{th} term.

Let us consider some more examples to illustrate the method above.

Example 13: Find a particular integral of the differential equation

$$y''' + y'' + y = x^4+2x+1.$$

Solution: Here P.I. $= \dfrac{1}{1+D^2+D^3}(x^4+2x+1)$

$$\begin{aligned} &= [1+(D^2+D^3)]^{-1}(x^4+2x+1) \\ &= (1-D^2-D^3+D^4+2D^5+\cdots)\ (x^4+2x+1) \\ &= (x^4+2x+1)-12x^2-24x+24 \\ &= x^4-12x^2-22x+25 \end{aligned}$$

***

We now illustrate a situation where certain manipulations with $L(D)$ simplify the binomial expansion of $\frac{1}{L(D)}$.

Example 14: Find a particular integral of the differential equation

$$y''' + y'' + y' + y = x^4 + 2x + 1$$

Solution: Here P.I. $= \frac{1}{1 + D + D^2 + D^3}(x^4 + 2x + 1)$

$$= \frac{(1-D)}{(1-D)\,(1+D+D^2+D^3)}\,(x^4 + 2x + 1)$$

$$= \frac{1}{1-D^4}\left[(1-D)\,(x^4 + 2x + 1)\right]$$

$$= \frac{1}{1-D^4}\left[x^4 + 2x + 1 - 4x^3 - 2\right]$$

$$= (1-D^4)^{-1}\,(x^4 - 4x^3 + 2x - 1)$$

$$= (1 - D^4 + D^8 + \cdots)\,(x^4 - 4x^3 + 2x - 1)$$

$$= (x^4 - 4x^3 + 2x - 1) + 24$$

$$= x^4 - 4x^3 + 2x + 23$$

***

When in Eqn. (74) $a_n = 0$ then $L(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D$ can be written in the form

$L(D) = D(a_0 D^{n-1} + a_1 D^{n-2} + \cdots + a_{n-1})$ where $a_{n-1} \neq 0$.

If both $a_n = 0$ and $a_{n-1} = 0$ then D^2 is a factor of $L(D)$ and $L(D)$ can be written as

$$L(D) = D^2(a_0 D^{n-2} + a_1 D^{n-1} + \cdots + a_{n-2}).$$

In general, if D^r is a factor of $L(D)$, then $L(D)y = Ax^n$ has the form

$$L(D)y = D^r(a_0 D^{n-r} + \cdots + a_{r+1}D + a_r)y = Ax^n,\ a_r \neq 0 \tag{76}$$

and we have

$$y_p = \frac{1}{D^r(a_0 D^{n-r} + \cdots + a_{r+!}D + a_r)}Ax^n \tag{77}$$

The above situation is illustrated in the following example.

Example 15: Solve $(D^3 + 3D^2 + 2D)\ y = x^2$.

Solution: A.E. is

$$m^3 + 3m^2 + 2m = 0$$

$\Rightarrow\ m(m+2)\,(m+1) = 0$

$\Rightarrow\ m = 0, -1, -2$

$\therefore$ C.F. $= y_c = c_1 + c_2 e^{-x} + c_3 e^{-2x}$

P.I. $= y_p = \frac{1}{D^3 + 3D^2 + 2D}x^2 = \frac{1}{D(D^2 + 3D + 2)}x^2$

$$= \frac{1}{2D}\left(\frac{1}{1 + \frac{3}{2}D + \frac{1}{2}D^2}\right)x^2$$

$$= \frac{1}{2D}\left[1 - \frac{3}{2}D + \frac{7}{4}D^2 + \cdots\right]x^2$$
$$= \frac{1}{2D}\left[x^2 - \frac{3}{2}2x + \frac{7}{4}.2\right]$$
$$= \frac{1}{2}\left[\frac{x^3}{3} - 3\frac{x^2}{2} + \frac{7}{2}x\right]$$
$$= \frac{1}{6}x^3 - \frac{3}{4}x^2 + \frac{7}{4}x$$

Hence the complete solution is

$$y = y_c + y_p = c_1 + c_2e^{-x} + c_3e^{-2x} + \frac{1}{12}(2x^3 - 9x^2 + 21x).$$

***

And now an exercise for you.

E7) Solve the following differential equations:

i) $(D^4 - 2D^3 + D^2)\ y = x$

ii) $(D^3 - 3D^2 - 6D + 8)\ y = x$

iii) $(D^2 + D - 2)\ y = 2(1 + x - x^2)$

iv) $(D^4 + 2D^3 - 3D^2)\ y = x^2 + 3e^{2x} + 4\sin x$

We now take up the case which is a combination of I-III above. The non-homogeneous term of the differential equation being the product of an exponential function and a function of x. The function of x could be a polynomial or a sine/cosine function. For example, terms of the type $e^{ax}x^2$ or $e^{ax}\cos x$ or $e^{ax}\sin x$ or their linear combinations.

IV. $b(x) = e^{\alpha x}\,V(x)$, α constant

Consider an equation of the form

$$L(D)\ y = e^{\alpha x}\ V(x) \tag{78}$$

where α is a constant and V is a function of x and
let us find its particular integral

$$y_p = \frac{1}{L(D)}e^{\alpha x}\ V(x).$$

By the shift formula for the polynomial differential operator (Theorem 4), we know that

$$L(D)\ e^{\alpha x}\ V(x) = e^{\alpha x}L(D + \alpha)\ V(x) \tag{79}$$

Now put

$$L(D + \alpha)\ V = V_i \tag{80}$$

then $V = \dfrac{1}{L(D + \alpha)}\ V_1$ (81)

Since V is a function of x, V_1 will also be a function of x.
On substituting from Eqns. (80) and (81) in Eqn. (79), we get

$$L(D)e^{\alpha x}\frac{1}{L(D + \alpha)}\ V_1 = e^{\alpha x}V_1 \tag{82}$$

Operating on both sides of Eqn. (82) with $\frac{1}{L(D)}$, we get

$$\boxed{\frac{1}{L(D)} e^{\alpha x}\, V_1(x) = e^{\alpha x}\, \frac{1}{L(D+\alpha)} V_1(x)} \tag{83}$$

where V_1 is any function of x. Thus using relation (83) the problem of finding a particular integral of Eqn. (78) is now reduced to a problem which can be solved by the methods discussed in I-III above. We illustrate the method through the following examples.

Example 16: Find a particular integral of the differential equation $(D^2+1)\ y = xe^{2x}$.

Solution: P.I. $= \frac{1}{D^2+1}\ xe^{2x}$

$$= e^{2x} \frac{1}{(D+2)^2+1} x \text{ (using relation (83))}$$

$$= e^{2x} \frac{1}{D^2+4D+5} x$$

$$= e^{2x} \frac{1}{5} \left(1+\frac{4}{5}D+\frac{1}{5}D^2\right)^{-1} x$$

$$= e^{2x} \frac{1}{5}\left(1-\frac{4}{5}D-\frac{1}{5}D^2+\frac{16}{25}D^2+\cdots\right) x$$

$$= e^{2x} \frac{1}{5}\left(x-\frac{4}{5}\right)$$

$$= \frac{e^{2x}}{25}\ (5x-4)$$

***

Let us look at another example.

Example 17: Solve $\frac{d^2y}{dx^2}+2y = x^2e^{3x}+e^x \cos 2x$

Solution: A.E. is

$$m^2+2=0$$

$$\Rightarrow\ m = \pm i\sqrt{2}$$

$\therefore$ C.F. $= y_c = c_1 \cos\sqrt{2}\ x + c_2 \sin\sqrt{2}\ x$

P.I. $= y_p = \frac{1}{D^2+2}\ (x^2e^{3x}+e^x \cos 2x)$

$$= \frac{1}{D^2+2} x^2e^{3x} + \frac{1}{D^2+2} e^x \cos 2x$$

$$= e^{3x} \frac{1}{(D+3)^2+2} x^2 + e^x \frac{1}{(D+1)^2+2} \cos 2x$$

$$= e^{3x} \frac{1}{D^2+6D+11} x + e^x \frac{1}{D^2+2D+3} \cos 2x$$

$$= e^{3x} \frac{1}{11}\left(1-\frac{6}{11}D+\cdots\right) x + e^x \frac{1}{-2^2+2D+3} \cos 2x$$

$$= e^{3x}\frac{1}{11}\left(x-\frac{6}{11}\right)+e^{x}\frac{(2D+1)}{(2D-1)\,(2D+1)}\cos 2x$$

$$= \frac{1}{121}e^{3x}(11x-6)+e^{x}(2D+1).\frac{1}{4D^2-1}\cos 2x$$

$$= \frac{1}{121}e^{3x}(11x-6)+e^{x}(2D+1)\frac{1}{4(-2^2)-1}\cos 2x$$

$$= \frac{1}{121}e^{3x}(11x-6)+\frac{e^{x}}{-17}\,[2\,(-2\sin x)+\cos 2x)]$$

$$= \frac{1}{121}e^{3x}(11x-6)+\left(\frac{4}{17}\sin 2x-\frac{1}{17}\cos 2x\right)e^{x}$$

Hence the complete solution is

$$y = c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x+\frac{1}{121}e^{3x}(11x-6)+\frac{1}{17}e^{x}(4\sin 2x-\cos 2x)$$

***

We now take up an example where $b(x)$ is a product of an exponential, a polynomial and a sine function.

Example 18: Find a particular integral of differential equation $(D^2-2D+1)\ y = xe^x\sin x$.

Solution: P.I. is

$$y_p = \frac{1}{D^2-2D+1}xe^x\sin x$$

$$= \mathrm{Im}\left\{\frac{1}{D^2-2D+1}xe^xe^{ix}\right\}$$

$$= \mathrm{Im}\left\{\frac{1}{D^2-2D+1}xe^{(1+i)x}\right\}$$

$$= \mathrm{Im}\left\{e^{(1+i)x}\frac{1}{(D+1+i)^2-2\ (D+1+i)+1}x\right\}$$

$$= \mathrm{Im}\left\{e^{(1+i)x}\frac{1}{D^2+2iD-1}x\right\}$$

$$= \mathrm{Im}\ \{-e^{(1+i)x}[1-(D^2+2iD)]^{-1}x\}$$

$$= \mathrm{Im}\{-e^{(1+i)x}(1+2iD+D^2+\cdots)\ x\}$$

$$= \mathrm{Im}\ \{-e^{(1+i)x}(x+2i)\}$$

$$= \mathrm{Im}\{-e^{x}(\cos x+i\sin x)\ (x+2iD)\ \}$$

$$= -e^{x}(2\cos x+x\sin x)$$

***

You may now try the following exercises.

E8) Solve the following differential equations:

i) $(D^2+3D+2)\ y = e^{2x}\sin x$

ii) $(D^2-2D+1)\ y = x^2e^{3x}$

iii) $(D^3-2D^2-19D+20)\ y = xe^x+2e^{-4x}\sin x$

iv) $(D^3 - 3D^2 + 3D - 1)\ y = xe^x + e^x$

E9) Solve the following differential equations:

i) $(D^2 - 1)\ y = x^2 \cos x$

ii) $(D^2 - 4D + 4)\ y = 8x^2 e^{2x} \sin 2x$

iii) $(D^2 - 1)\ y = x \sin x + (1 + x^2)\ e^x$

iv) $(D^4 - 1)\ y = x^2 \sin x$

Note that the methods of polynomial differential operators for finding a particular solution are applicable only to differential equations with constant coefficients. They do not work for equations with variable coefficients. Quite often operators with variable coefficients are not factorisable. Even if so, factors do not commute as you have already seen in Sec. 13.2. However, operator methods can be applied to certain particular types of differential equations with variable coefficients namely, Euler's equation and equations reducible to Euler's form which you have already studied in Unit 12. This is because such equations can be reduced to equations with constant coefficients by using certain transformation of the independent variables and hence can be dealt by the polynomial differential operator methods. We shall now take up these equations.

Euler's Equations

Consider the n^{th} order Euler's equation, namely,

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = b(x), \quad x > 0 \tag{84}$$

where $a_0, a_1, a_2, \ldots, a_n$ are real constants and $a_0 \neq 0$.
In the operator notation Eqn. (84) can be written as

$$(a_0 x^n D^n y + a_1 x^{n-1} D^{n-1} y + \cdots + a_{n-1} x\, Dy + a_n y) = b(x) \tag{85}$$

On substituting $x = e^z$ or $z = \ln x$, we can reduce Eqn. (85) to the form

$$[a_0 D'(D'-1)\ldots(D' - \overline{n-1}) + a_1 D'(D'-1)\ldots(D' - \overline{n-2}) + \cdots$$
$$+ a_{n-1} D' + a_n]\ y = b(e^z)$$

or $[A_0 D'^n + A_1 D'^{n-1} + \cdots + A_{n-1} D' + A_n]\ y = b(e^z)$ (86)

where $D' = \dfrac{d}{dz}$ and $A_0, A_1, \ldots, A_n$ are constants.

Eqn. (86) is a linear differential equation with constant coefficients and can be treated by the differential operator methods. If its solution is $y = g(z)$, then the solution of Eqn. (86) is $y = g(\ln x)$.

Let us consider the following example to illustrate the method above.

Example 19: Solve $x^2 \dfrac{d^2 y}{dx^2} - x \dfrac{dy}{dx} + y = 2 \ln x,\ x > 0$. (87)

Solution: Putting $z = \ln x$, and denoting $\frac{d}{dz}$ by D', Eqn. (87) can be written as

$$[D'(D'-1) - D' + 1]\ y = 2z$$

$$\Rightarrow \quad (D'^2 - 2D' + 1)\ y = 2z$$

A.E. is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow \quad m = 1, 1$$

$\therefore$ C.F. $= (c_1 + c_2 z) e^z$

$$\text{P.I} = \frac{1}{1 - 2D' + D'^2} 2z$$

$$= 2\ [1 - 2D' + D'^2]^{-1} z$$

$$= 2\ [1 + 2D' + \cdots]\ z$$

$$= 2\ (z + 2) = 2z + 4$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 z)\ e^z + 2z + 4$$

$$= (c_1 + c_2 \ln x)\ x + 2 \ln x + 4$$

***

You may now try the following exercise.

E10) Solve the following differential equations:

i) $(x^2 D^2 - 3xD + 4)\ y = 2x^2,\ x > 0$

ii) $(x^2 D^2 + 3xD + 1)\ y = \dfrac{1}{(1-x)^2}, \quad x > 1$

iii) $\left(D^3 - \dfrac{4}{x} D^2 + \dfrac{5D}{x^2} - \dfrac{2}{x^3}\right) y = 1,\ x > 0$

iv) $(x^2 D^2 - xD + 4)\ y = \cos\ (\ln x) + x \sin\ (\ln x),\ x > 0$

There are some differential equations that are easily reducible to the Euler's form and hence to the equations with constant coefficients. We shall now take up such equations.

Equations reducible to the Euler's Form.

Consider an equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = f(x),\ x > 0, \tag{88}$$

where a and b are positive real constants and the coefficients $a_1, a_2, \ldots, a_n$ are constants.

We can transform equations of the form (88) to Euler's equation when the independent variable x is changed to z by means of the substitution $z = ax + b$. Eqn. (88), under this substitution, reduces to

$$z^n \frac{d^n y}{dx^n} + \frac{a_1}{a} z^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + \frac{a_{n-1}}{a^{n-1}} z \frac{dy}{dz} + \frac{a_n}{a^n} y = \frac{1}{a^n} f\left(\frac{z-b}{a}\right) \quad (89)$$

Eqn. (89) can then be reduced to an equation with constant coefficients by the substitution $t = \ln z$.

In practice, instead of making two substitutions, we make only one substitution, viz., $e^t = ax + b$, and then from Eqn. (88) we can directly derive an equation with constant coefficients.

We now illustrate through an example how this is achieved.

Example 20: Solve

$$(3x+2)^2 \frac{d^2 y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1,\ x > 0. \quad (90)$$

Solution: Putting $3x + 2 = e^z$ and denoting $\frac{d}{dz} = D'$, we have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{3}{3x+2} D'y \quad (91)$$

$$\frac{d^2 y}{dx^2} = \frac{-3.3}{(3x+2)^2} D'y + \frac{3}{3x+2} (D'^2 y)\frac{3}{3x+2}$$

$$= \frac{9}{(3x+2)^2} (D'^2 - D')\ y \quad (92)$$

Substituting from Eqns. (91) and (92) into Eqn. (90), we get

$$[D'(D'-1) + D' - 4]\ y = \frac{1}{27}(e^{2z} - 1) \quad (93)$$

You may **note** here that the r.h.s. of Eqn. (90) can be written as
$\frac{1}{3}[(3x+2)^2 - 1]$.
From Eqn. (93), we get

$$(D'^2 - 4)\ y = \frac{1}{27}(e^{2z} - 1)$$

A.E. is

$$m^2 - 4 = 0$$

$$\Rightarrow \quad m = \pm 2$$

$\therefore$ C.F. $= c_1 e^{2z} + c_2 e^{-2z}$

$$\text{P.I.} = \frac{1}{D'^2 - 4} \cdot \frac{1}{27} (e^{2z} - 1)$$

$$= \frac{1}{27}\left[\frac{1}{D'^2 - 4}.e^{2z} - \frac{1}{D'^2 - 4} e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{1}{(D'+2)\ (D'-2)} e^{2z} - \frac{1}{D'^2 - 4} e^{0z}\right]$$

$$= \frac{1}{27}\left[e^{2z}\frac{z}{4} + \frac{1}{4}\right]$$

$\therefore$ The complete solution of Eqn. (90) is

$$y = c_1 e^{2z} + c_2 e^{-2z} + \frac{1}{27}\left[z\frac{e^{2z}}{4} + \frac{1}{4}\right]$$

$$= c_1 (3x+2)^2 + \frac{c_2}{(3x+2)^2} + \frac{1}{108}\,[(3x+2)^2 \ln(3x+2) + 1]$$

***

And now an exercise for you.

E11) Solve the following differential equations:

i) $(2x-1)^2 \dfrac{d^2y}{dx^2} + (2x-1)\dfrac{dy}{dx} - 2y = 0,\ x > \dfrac{1}{2}$

ii) $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\,[\ln\,(1+x)],\ x > 0$

In the next section we discuss a few applications of non-homogeneous differential equations in physical models.

13.5 APPLICATIONS IN PHYSICAL MODELS

In this section, we discuss some physical models where the governing differential equations can be approximated to second order non-homogeneous linear differential equations.

Generally, vibrations occur whenever a physical system in stable equilibrium is disturbed, for then it is subjected to forces in order to restore its equilibrium. We shall see how situations of this kind can lead to differential equations and how the study of these equations can be used to draw inference about the physical situations.

We first consider different aspects of mechanical vibrations.

13.5.1 Mechanical Vibrations

Each day we encounter many types of mechanical vibrations. The bouncing motion of an automobile due to the bumps and cracks in the pavement, the vibrations of a bridge caused by traffic and wind are some common examples. To study mechanical vibrations, we shall start with the simple mechanical system consisting of a coil spring suspended from a rigid support with a mass, m attached to the end of the spring.

Newton's second law states that when a body is subjected to one or more external forces, the time rate of change of the body's momentum is equal to the vector sum of the external forces acting on it.

To analyse this spring-mass system, you need to recall two laws of physics: Hooke's law and Newton's second law of motion. Robert Hooke (1635-1703), an English physicist, published Hooke's law in 1658. The law states that the spring exerts a restoring force opposite to the direction of elongation of the spring with a magnitude directly proportional to the amount of elongation. That is, the spring exerts a restoring force F whose magnitude is ks, where s is the amount of elongation and $k(>0)$ is the **spring constant**.

For example, if a $20\ kg.$ weight stretches a spring $\dfrac{1}{2}m$, then Hooke's law gives

$$20 = |F| = ks = k\left(\frac{1}{2}\right) \text{ (in the kg/m. system)}$$

Hence, the spring constant is $k = 40\ kg/m$.
Newton's second law enables us to formulate the equations of motion for a moving body. When the mass remains constant, this law can be expressed as

$$m\frac{d^2x}{dt^2} = ma = F\left(t,\ x,\ \frac{dx}{dt}\right) \qquad (94)$$

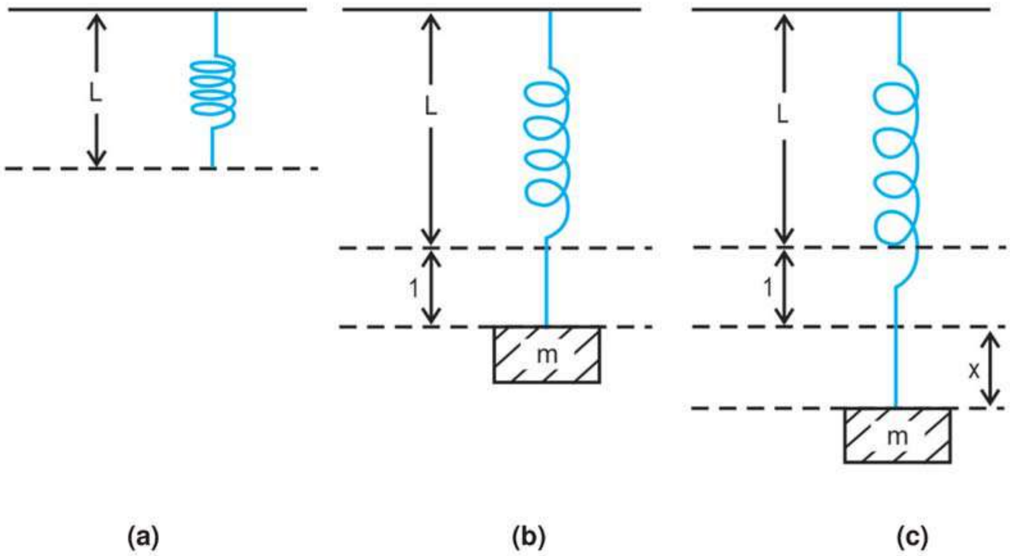

Fig. 1

A first step in the analysis of the spring-mass system is to choose a coordinate axis to represent the motion of the mass. Let the spring have length L when hanging from its support (see Fig. 1(a)). Let the mass m when attached to it elongate the spring, and when it comes to rest (equilibrium) the spring has been stretched by a length, say, 1 unit (Fig. 1(b)). Therefore, let's choose a vertical coordinate axis passing through the spring, with the origin at the equilibrium position of the mass. Let x denote the displacement of the mass from its equilibrium position and is positive when the mass is below its equilibrium position, as shown in Fig. 1(c).
We now consider the various forces acting on the mass m.

Gravity: The force of gravity F_1 is a downward force with magnitude mg, where g is the acceleration due to gravity. Hence

$$F_1 = mg\,.$$

$g = 32\,ft/\sec^2$, $g = 9.8m/\sec^2$ or $980cm/\sec^2$

Restoring Force: The spring exerts a restoring force F_2 whose magnitude is proportional to the elongation of the spring. From Fig. 1(c), you can see that the spring is stretched $x+1$ units beyond its natural length. Hence the magnitude of F_2 is $k(x+1)$, where k is the spring constant. Since the spring pulls upward (in the negative x direction), we have

$$F_2 = -k(x+1)\,.$$

Note that k has the units of force/length.

When $x = 0$, that is, when the system is at equilibrium, the force of gravity and the force due to the spring balance each other. Thus, $mg = kl$ and F_2 can be expressed as

$$F_2 = -kx - mg\ .$$

Damping Force: There is a damping or frictional force F_3 acting on the mass. For example, this force may be air resistance or friction due to a shock absorber. In either case we assume that the damping force is proportional to the magnitude of the velocity of the mass, but opposite in direction. That is,

$$F_3 = -c\frac{dx}{dt},\ c > 0$$

where c is the **damping constant** given in units of mass/time.

External Forces: Any external force acting on the mass (for example, a magnetic force or the forces exerted on a car by bumps in the pavement) will be denoted by $F_4 = f(t)$. Let us assume that these forces depend only on time and not on the location of the mass or its velocity.

Then the **total force** F acting on the mass m is the sum of the four forces $F_1,\ F_2,\ F_3$ and F_4 above. That is,

$$F\left(t,\ x,\ \frac{dx}{dt}\right) = mg - kx - mg - c\ \frac{dx}{dt} + f(t) \tag{95}$$

Applying Newton's second law to the system, we obtain the equation of motion of the mass as

$$m\frac{d^2x}{dt^2} = mg - kx - mg - c\frac{dx}{dt} + f(t)$$

$$\Rightarrow\quad m\frac{d^2x}{dt^2} + c\ \frac{dx}{dt} + kx = f(t) \tag{96}$$

When $c = 0$, we say that the system is **undamped**; otherwise it is **damped**. When $f(t) = 0$ we call the motion to be **free**, otherwise the motion is **forced**.

Note that Eqn. (96) can be solved by the methods you have studied so far.

Let us solve this equation for the following cases:

(i) Undamped free vibrations

Let us begin with the simple system in which $c = 0$ and $f(t) = 0$. In this case, Eqn. (96) reduces to

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{97}$$

On solving Eqn. (97), we get

$$x = A\cos\omega_0 t + B\sin\omega_0 t\ , \tag{98}$$

where $\omega_0^2 = k/m$.

Here ω_0 is called the **angular frequency** of the vibration. For a particular problem, the constants A and B are determined by the prescribed initial conditions.

Let $A = R\cos\delta$ and $B = R\sin\delta$, that is,

$$R\cos(\omega_0 t-\delta)=R\cos\omega_0 t\cos\delta+R\sin\omega_0 t\sin\delta$$
$$=A\cos\omega_0 t+B\sin\omega_0 t$$

where R and δ are constants.
Then we can write Eqn. (98) in a more convenient form as

$$x(t)=R\cos(\omega_0 t-\delta) \quad (99)$$

where $R=\sqrt{A^2+B^2}$ and $\tan\delta=\frac{B}{A}$.

Because of the periodic character of the cosine function. Eqn. (99) represents a **periodic motion**, or a **simple harmonic motion** of **period** T where,

$$T=\frac{2\pi}{\omega_0}=2\pi\left(\frac{m}{k}\right)^{1/2}. \quad (100)$$

That means that the graph of $x(t)$ repeats every $\frac{2\pi}{\omega_0}$ units.

Since $|\cos(\omega_0 t-\delta)|\le 1$, hence x always lies between the lines $x=\pm R$. The maximum displacement occurs at the times $\omega_0 t-\delta=0, \pm\pi, \pm 2\pi, \ldots$. Here the constant R is the maximum displacement of the mass from equilibrium and is called the **amplitude** of the motion. The constant δ is called the **phase angle** and it measures the displacement (in time) of the wave from its normal position corresponding to $\delta=0$. In this case, the periodic motion does not tend to zero as the time increases. A sketch of the motion, represented by Eqn. (99) is shown in Fig. 2.

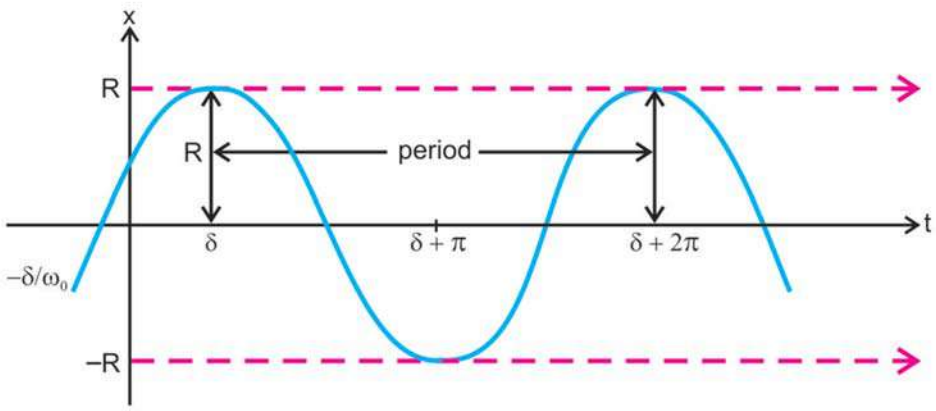

Fig. 2: Simple Harmonic Motion

Let us consider the following example.

Example 21: Interpret and solve the initial value problem in terms of a spring-mass system

$$\frac{d^2x}{dt^2}+16x=0, \quad (101)$$

with $x(0)=10, \left.\frac{dx}{dt}\right|_{t=0}=0$.

Solution: The problem is equivalent to pulling a mass on a spring down 10 units below the equilibrium position. This position is considered as the initial position, $t = 0$. At this position, the system is at rest and then mass is released to cause motion in the spring (see Fig. 3).

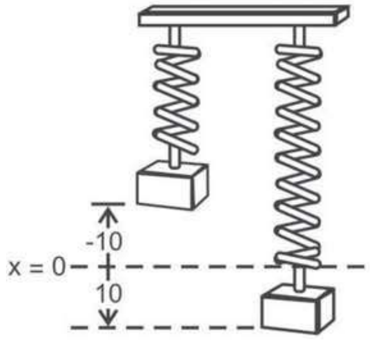

Fig. 3

The solution of Eqn. (101) is

$$x(t) = c_1 \cos 4t + c_2 \sin 4t \tag{102}$$

In order to determine c_1 and c_2 in Eqn. (102), we apply initial conditions.

Now, $x(0) = 10$

$\Rightarrow \quad c_1.1 + c_2.0 = 10$

$\Rightarrow \quad c_1 = 10$

From Eqn. (102), we have

$$\frac{dx}{dt} = -4c_1 \sin 4t + 4c_2 \cos 4t$$

Applying 2nd initial condition, we have

$$\left.\frac{dx}{dt}\right|_{t=0} = 0 = 4c_2.1$$

$\Rightarrow \quad c_2 = 0$

Thus, Eqn. (102) of motion reduces to

$$x(t) = 10\cos 4t$$

The solution clearly shows that once the system is set in motion, it stays in motion, with the mass bouncing back and forth 10 units on either side of the equilibrium position $x = 0$. The period of motion is $\frac{2\pi}{4} = \frac{\pi}{2}$ (see Fig. 4).

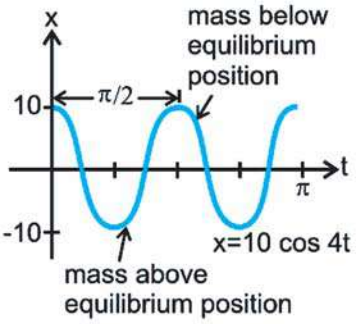

Fig. 4

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And now an exercise for you.

E12) A mass weighing 19.6 kg. stretches a spring $\frac{1}{2}m$. At $t = 0$, the mass is released from a point $\frac{2}{3}m$, below the equilibrium position with an upward velocity $4/3\,m/\text{sec}$. Determine the function $x(t)$ that describes the subsequent free motion.

In the undamped free case we considered vibrations in an ideal settings, that is, when no external or frictional forces were assumed to be present. In most applications, however, there is at least some type of frictional or damping force that plays a significant role. This force may be due to a component in the system, such as a shock absorber in a car, or due to the medium that surrounds the system, such as air or some liquid.

We now study the effect of a damping force on free vibrations.

(ii) Damped Free Vibrations

If we include the effect of damping and assume that $f(t) = 0$, then Eqn. (96) governing the motion of the mass reduces to

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{103}$$

The roots of the corresponding auxiliary equation are

$$r_1, r_2 = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c^2}{4m^2} - \frac{k}{m}\right)} \tag{104}$$

Since c, k, m are positive, therefore $(c^2 - 4km)$ is always less than c^2. Hence if $c^2 - 4km \geq 0$, then the values of r_1 and r_2 given by Eqn. (104) are negative. Further, if $(c^2 - 4km) < 0$, then the values of r_1 and r_2 are complex, but real part is negative. Hence we can write the solutions of Eqn. (104) as follows:

$$c^2 - 4km > 0,\ x = Ae^{r_1 t} + Be^{r_2 t},\ r_1, r_2 < 0 \tag{105}$$

$$c^2 - 4km = 0,\ x = (A + Bt)\ e^{-(c/2m)t} \tag{106}$$

$$c^2 - 4km < 0,\ x = e^{-(c/2m)t}\ [A\cos\mu t + B\sin\mu t] \tag{107}$$

where $\mu = \sqrt{(4km - c^2)}/2m > 0$

You may **note** here that in all the three cases, whatever may be the initial conditions and the values of A and B, $x \to 0$ as $t \to \infty$. In other words we can say that the motion dies out with increasing time, which is in contrast with the solutions of Eqn. (97).

Thus, without damping, the motion always continues and with damping the motion must tend to zero with increasing time.

We call the first case, i.e., when $c^2 - 4km > 0$, as **over damped** (see Fig. 5) and the second case $c^2 - 4km = 0$ as **critically damped** (Fig. 6).

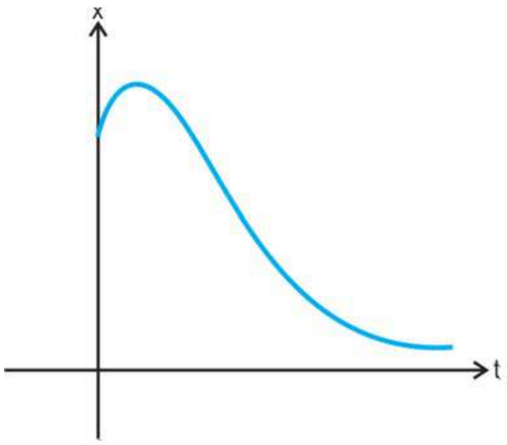

Fig. 5: Over damped motion

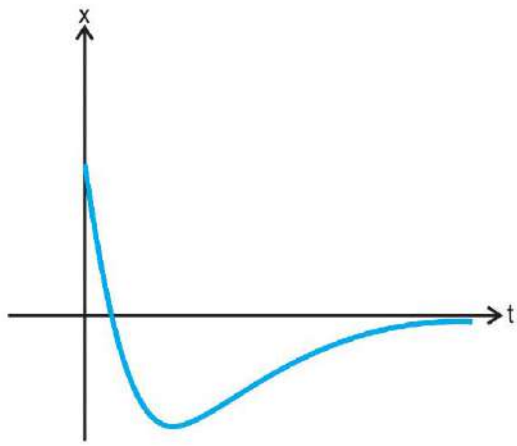

Fig. 6: Critically damped motion

In the third case, i.e., $c^2 - 4km < 0$, the system is said to be **under damped** since the damping coefficient is small compared to the spring constant. In this case we can write Eqn. (107) in the form,

$$x = R\ e^{-(c/2m)t} \cos\ (\mu t - \delta), \tag{108}$$

where $R = \sqrt{A^2 + B^2}$ and $\tan\delta = \frac{B}{A}$.

In Eqn. (108), the exponential factor $R\,r^{-(c/2m)t}$ is the **damping factor** and $\cos\ (\mu t - \delta)$ accounts for the oscillatory motion. Since the cosine factor varies between -1 and 1, with period $2\pi/\mu$, the displacement $x(t)$ lies

Quasi-frequency is the frequency with which the mass oscillates back and forth its equilibrium position.

Quasi-period is the time between successive maxima or successive minima of the position of mass.

between the curves $x = \pm \mathrm{Re}^{-ct/2m}$; hence it resembles a cosine wave whose amplitude decreases as t increases. The graph of a typical solution $x(t)$ is shown in Fig. (7). Although the motion is not periodic, the parameter μ determines the frequency with which the mass oscillates back and forth and is called the **quasi-frequency**. The quantity $p = 2\pi/\mu = \dfrac{4m\pi}{\sqrt{4mk - c^2}}$ is called the **quasi-period**. We call the system as **under damped** because there is not enough damping present (c is too small) to prevent the system from oscillating.

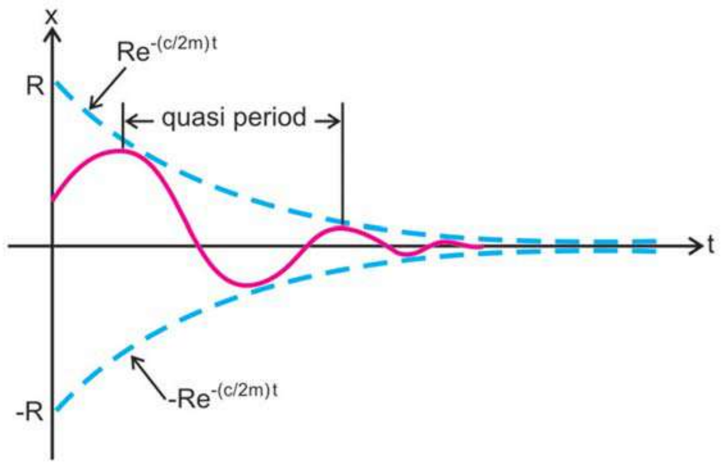

Fig. 7: Under Damped Vibration

We illustrate the above theory with the help of the following examples.

Example 22: A $9.8kg.$ weight stretched a spring $2.45m.$ Assume that the damping constant c for the system is $4\ kg/m$. Determine the equation of motion if weight is released from the equilibrium position with an upward velocity of $3\ m/\sec$.

Solution: From Hooke's Law, we have

$$9.8 = k(2.45) \Rightarrow k = 4\ kg/m.$$

From $m = w/g$, we have

$$m = \frac{9.8}{9.8} = 1\ kg.$$

Thus, the differential equation governing the motion can be expressed as

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0 \tag{109}$$

The initial conditions are $x(0) = 0,\ \left.\dfrac{dx}{dt}\right|_{t=0} = -3$ (110)

A.E. of Eqn. (109) is

$$m^2 + 4m + 4 = 0$$

$$\Rightarrow \quad m = -2, -2$$

Here, the system is **critically damped** and

$$x(t) = (c_1 + c_2 t)e^{-2t} \tag{111}$$

Using conditions (110), we get $c_1 = 0$ and $c_2 = -3$

Thus, Eqn. (111) of motion assumes the following form

$$x(t) = -3t\, e^{-2t} \tag{112}$$

Clearly $x'(t) = -3e^{-2t}(1-2t)$ and $x'(t) = 0$ when $t = \frac{1}{2}$.

The corresponding extreme displacement is

$$x\left(\frac{1}{2}\right) = -3\left(\frac{1}{2}\right)e^{-1} = -0.552m.$$

This means that the weight reaches a maximum heights of $0.552m$ above the equilibrium position. The graph of the motion is shown in Fig. 8.

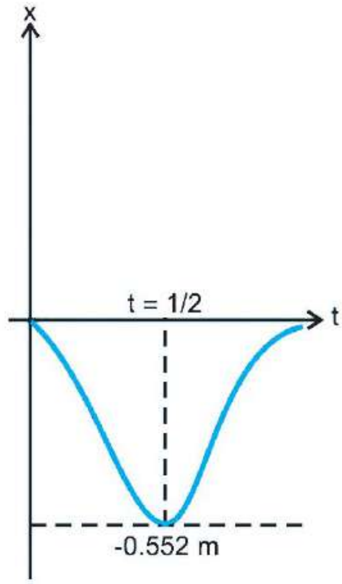

Fig. 8

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Example 23: The motion of a certain spring-mass system is governed by the differential equation

$$\frac{d^2x}{dt^2} + \frac{1}{8}\frac{dx}{dt} + x = 0. \tag{113}$$

If $x(0) = 2$ and $x'(0) = 0$, determine the position of the mass at any time. Also find the quasi-frequency, the quasi-period and the time at which the mass first passes through its equilibrium position.

Solution: A.E. of Eqn. (113) is

$$8m^2 + m + 8 = 0$$

$$\Rightarrow \quad m = -\frac{1}{16} \pm \frac{i\sqrt{255}}{16}.$$

Hence, the system is under damped and

$$x(t) = e^{-t/16}\left(A\cos\frac{\sqrt{255}}{16}t + B\sin\frac{\sqrt{255}}{16}t\right) \tag{114}$$

Using initial conditions

$$x(0) = 2 \Rightarrow A = 2 \text{ and } x'(0) = 0 \Rightarrow B = \frac{2}{\sqrt{255}}.$$

Thus, Eqn. (114) of motion assumes the following form

$$x(t) = e^{-t/16}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right) \tag{115}$$

Letting $2 = R\cos\delta$ and $\frac{2}{\sqrt{255}} = R\sin\delta$ in Eqn. (115) it can be written as

$$x(t) = e^{-t/16}R\cos\left(\frac{\sqrt{255}}{16}t - \delta\right) \tag{116}$$

where $R = \sqrt{4 + \frac{4}{255}} = \frac{32}{\sqrt{255}} \approx 2.004$ and $\delta = \tan^{-1}\left(\frac{1}{\sqrt{255}}\right) \approx 0.663$.

The quasi-frequency is $\mu = \frac{\sqrt{255}}{16} \approx 0.998$ and quasi-period

$$p = \frac{2\pi}{\mu} = \frac{2\times 22/7}{\mu} = \frac{44\times 16}{7\sqrt{255}} \approx 6.298 \text{ sec.}$$

The time at which the mass passes through its equilibrium position is obtained when

$$\cos\left(\frac{\sqrt{255}}{16}t-\delta\right)=0$$

or, $\dfrac{\sqrt{255}}{16}t-\delta=(2n+1)\,\dfrac{\pi}{2},\ n=0,1,2,\ldots.$ Thus the first time at which the mass passes through the equilibrium position is obtained as

$$t=\frac{16}{\sqrt{255}}\left(\frac{\pi}{2}+\delta\right)\approx 1.637 \text{ sec.}$$

To graph $x(t)$ given by Eqn. (116), the intercepts $t_1, t_2, \ldots, t_k, \ldots$ are obtained as

$$t_n=\frac{16}{\sqrt{255}}\left((2n+1)\frac{\pi}{2}+\delta\right),\ n=0,1,2,\ldots.$$

Also we have $|x(t)|\le \mathrm{Re}^{-t/16}$ since

$$\left|\cos\left(\frac{\sqrt{255}}{16}t-\delta\right)\right|\le 1$$

Thus the graph of Eqn. (116) touches the graphs of $\pm\mathrm{Re}^{-t/16}$ at values $t_1^*, t_2^*, \ldots, t_k^*, \ldots$ for which

$$\cos\left(\frac{\sqrt{255}}{16}t-\delta\right)=\pm 1.$$

That is, $\dfrac{\sqrt{255}}{16}t-\delta$ must be an even multiple of π

or, $\dfrac{\sqrt{255}}{16}t-\delta=2n\pi$

or, $t=\dfrac{16(2n\pi+\delta)}{\sqrt{255}}$

The graph of solution $x(t)$ is shown in Fig. 9.

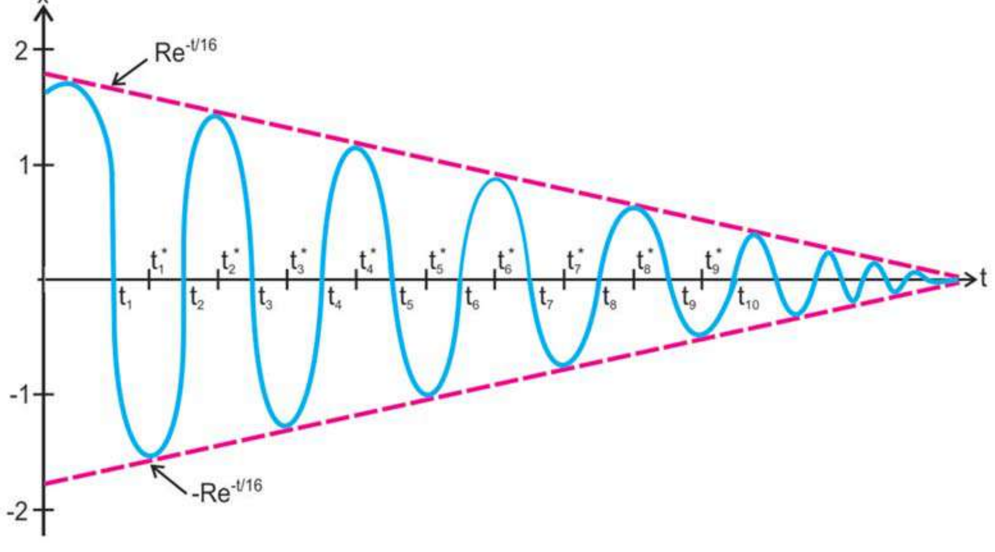

Fig. 9

You may now try the following exercise.

E13) In each of the following problems determine ω_0, R and δ so as to write the given expression in the form $x(t) = R\cos(\omega_0 t - \delta)$

i) $x(t) = 3\cos 2t + 4\sin 2t$

ii) $x(t) = -\cos t + \sqrt{3}\sin t$

iii) $x(t) = -2\cos \pi t - 3\sin \pi t$.

E14) A $4.9\,kg.$ weight is attached to a $1\,m.$ long spring. At equilibrium the spring measures $1.98\,m.$ If the weight is pushed up and released from rest at a point $2\,m.$ above the equilibrium position, find the displacement $x(t)$, if it is further known that the surrounding medium offers a resistance equal to the instantaneous velocity.

We now consider the vibrations of a spring mass system when an external force is applied. Of particular interest is the response of the system to a **periodic** forcing term.

(iii) Forced Vibrations

Let us consider the case in which a force $F = F_0 \cos\omega t$ is applied to a spring-mass system. In this case the equation of motion (96) takes the form

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0\cos\omega t, \qquad (117)$$

where F_0, ω are non-negative constants.

When there is no damping effect, Eqn. (117) reduces to

$$m\frac{d^2x}{dt^2} + kx = F_0\cos\omega t \qquad (118)$$

or, $$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{F_0\cos\omega t}{m}.$$

In the absence of damping force, the two types of phenomena occur in nature. We call them **beats** and **resonance**, which we shall study now.
Roots of the auxiliary equation, corresponding to Eqn. (118), are

$$r_1, r_2 = \pm\sqrt{k/m}$$

Let $\sqrt{k/m} = \omega_0$;

The C.F. $= A\cos\omega_0 t + B\sin\omega_0 t$

and P.I. $= \dfrac{1}{D^2+\omega_0^2}\dfrac{F_0}{m}\cos\omega t$.

For determining a particular integral, two cases arise, namely,

$\omega_0 \neq \omega$ and $\omega_0 = \omega$

When $\omega_0 \neq \omega$,

P.I. $= \dfrac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t$

and when $\omega_0 = \omega$

$$\text{P.I.} = \text{Re}\frac{1}{(D+i\omega_0)\,(D-i\omega_0)}\,\frac{F_0}{m}e^{i\omega_0 t}$$

$$= \text{Re}\frac{F_0}{m}\,\frac{1}{2i\omega_0}\,\frac{1}{D-i\omega_0}e^{i\omega_0 t}$$

$$= \text{Re}\frac{F_0}{m}\,\frac{1}{2i\omega_0}te^{i\omega_0 t}$$

$$= \frac{F_0}{2m\omega_0}\,t\,\sin\omega_0 t$$

In case $\omega_0 \neq \omega$, we can write the solution of Eqn. (118) as

$$x = A\cos\omega_0 t + B\sin\omega_0 t + \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t \tag{119}$$

and for $\omega = \omega_0$, solution of Eqn. (118) is

$$x = A\cos\omega_0 t + B\sin\omega_0 t + \frac{F_0}{2m\omega_0}\,t\sin\omega_0 t \tag{120}$$

The constants A and B in Eqns. (119) and (120) are determined by the initial conditions.
If in Eqn. (119) we assume that the mass is initially at rest then

$$x(0) = 0 \text{ and } \left(\frac{dx}{dt}\right)_{t=0} = 0 \tag{121}$$

and with these conditions, we get

$$A = -\frac{F_0}{m(\omega_0^2-\omega^2)} \text{ and } B = 0$$

With these values of A and B, solution (119) takes the form

$$x = \frac{F_0}{m(\omega_0^2-\omega^2)}(\cos\omega t - \cos\omega_0 t) \tag{122}$$

$$\Rightarrow\quad x = \frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\left(\frac{\omega_0-\omega}{2}\right)t\,\sin\left(\frac{\omega_0+\omega}{2}\right)t \tag{123}$$

Eqn. (123) can be written in the form

$$x = \left[\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{(\omega_0-\omega)t}{2}\right]\sin\frac{(\omega_0+\omega)t}{2}$$

If $|\omega_0-\omega|$ is small then $\omega_0+\omega > |\omega_0-\omega|$, and consequently $\sin\frac{(\omega_0+\omega)t}{2}$ is a rapidly oscillating function compared to $\sin\frac{(\omega_0-\omega)t}{2}$. Thus motion is a rapid oscillation with frequency $\frac{(\omega_0+\omega)}{2}$ and amplitude $\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{(\omega_0-\omega)t}{2}$.

This type of motion, possessing a periodic variation of amplitude, exhibits what we call a **beat**. The phenomenon of beats, as given by Eqn. (123), is shown in Fig. 10.

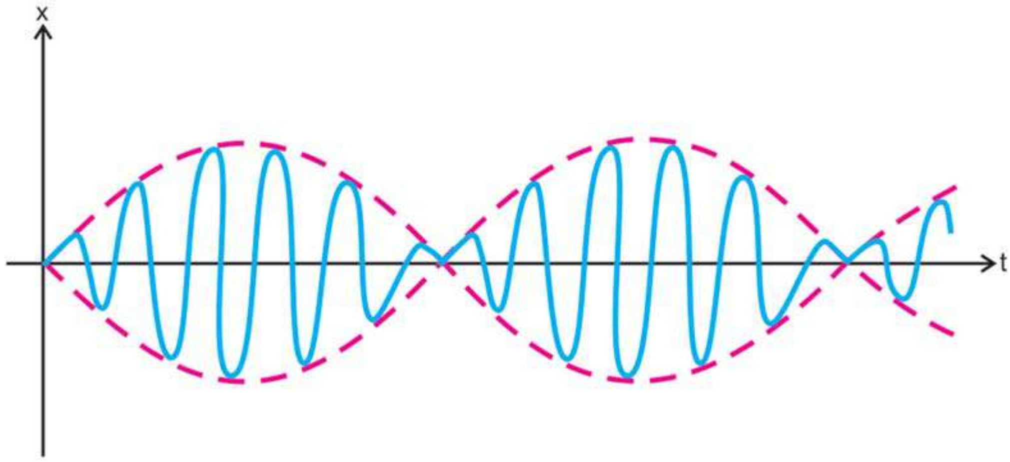

Fig. 10: The phenomenon of beats

Consider Eqn. (120), for $\omega_0 = \omega$, that is, when the frequency of the forcing motion is the same as the natural frequency of the system. You may **note** that as $t \to \infty$, the term $t \sin \omega_0 t$ increase without bound and the motion becomes unbounded regardless of the values of A and B (see Fig. 11). In this case we say that the external force is in **resonance** with the vibrating mass. The displacement here becomes so large that the elastic limit of the spring is exceeded. This leads to a fracture or to a permanent distortion in the spring.

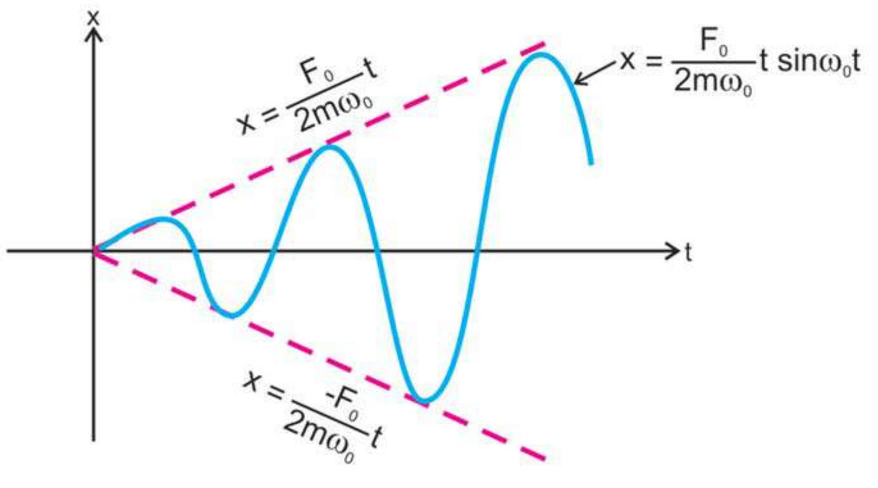

Fig. 11: The phenomenon of resonance

Resonance can create serious difficulties in the design of structures, where it can produce instabilities leading to the failure of the structure. For example, soldiers usually do not march in step when crossing a bridge in order to eliminate the periodic force of their marching that could resonate a natural frequency of the bridge. On the other hand the phenomenon of resonance is not always destructive. It is the resonance of an electric circuit that enables a radio to be tuned to a specific station.

You may now try the following exercises:

E15) Solve the initial value problem

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \gamma t, \; F_0 = \text{constant}; \; (\omega \neq \gamma),$$

$$x_0(0)=0, \left.\frac{dx}{dt}\right|_{t=0}=0.$$

E16) Consider the forced vibration of an undamped mechanical spring-weight system, where the external force is $F_0 \sin \omega t$ newtons. Show that if $\omega \neq \omega_0 \left(=\sqrt{k/m}\right)$, then the solution is given by

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t + \frac{F_0}{k-m\omega^2}\sin\omega t.$$

In designing seismic instruments to detect periodic forces in a narrow frequency range, there is a damping effect when forced vibrations take place. In this case we call the vibrations as damped force vibrations and the motion of the system is governed by Eqn. (117).

We now take up an example to illustrate the situation above.

Example 24: Interpret and solve the initial value problem

$$\frac{1}{5}\frac{d^2x}{dt^2}+1.2\frac{dx}{dt}+2x=5\cos 4t,$$

$$x(0)=\frac{1}{2}, \left.\frac{dx}{dt}\right|_{t=0}=0.$$

Solution: We can interpret the problem to represent a vibrational system consisting of a mass $\left(m=\frac{1}{5}kg.\right)$ attached to a spring $(k=2kg/m.)$. The mass is released from rest $\frac{1}{2}m.$ below the equilibrium position. The motion is damped $(c=1.2)$ and is being driven by an external force $5\cos 4t$, beginning at $t=0$.

As the problem is given, external force $f(t)=5\cos 4t$ will always act on the system and the system represents damped forced vibrations.
The given differential equation can be written as

$$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+10x=25\cos 4t \qquad (124)$$

A.E. corresponding to Eqn. (124) is

$$m^2+6m+10=0$$

$$\Rightarrow \quad m=-3\pm i$$

$\therefore$ C.F. $=x_c(t)=e^{-3t}(c_1\cos t+c_2\sin t)$

$$\text{P.I}=\frac{1}{D^2+6D+10}25\cos 4t$$

$$=\frac{25}{-(4^2)+6D+10}\cos 4t=\frac{25}{6}\frac{1}{D-1}\cos 4t$$

$$=\frac{25}{6}\text{Re}\frac{1}{D-1}e^{4it}$$

$$=\frac{25}{6}\text{Re}\frac{1}{4i-1}e^{4it}$$

$$= \frac{25}{6\times(-17)} \mathrm{Re}(4i+1)\ (\cos 4t + i \sin 4t)$$

$$= \frac{-25}{102}\cos 4t + \frac{50}{51}\sin 4t\ .$$

Hence the solution of Eqn. (124) is

$$x(t) = e^{-3t}(c_1 \cos t + c_2 \sin t) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t \qquad (125)$$

Constants c_1 and c_2 can be determined using initial conditions.

$$x(0) = \frac{1}{2} \Rightarrow c_1 = \frac{38}{51}$$

and $x'(0) = 0 \Rightarrow c_2 = \dfrac{-86}{51}$

Substituting the values of c_1 and c_2 in Eqn. (125), we obtain the equation of motion as

$$x(t) = e^{-3t}\left(\frac{38}{51}\cos t - \frac{86}{51}\sin t\right) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t\ .$$

***

And now an exercise for you.

E17) In the case of underdamped vibrations, show that the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F_0 \sin \gamma t$$

is $x(t) = Ae^{-\lambda t} \sin\left(\sqrt{\omega^2 - \lambda^2}\,t + \phi\right) + \dfrac{F_0}{\sqrt{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}} \sin(\gamma t + \theta)$

where $A = \sqrt{c_1^2 + c_2^2}$ and the phase angles ϕ and θ are respectively, defined by $\sin\phi = \dfrac{c_1}{A}, \cos\phi = \dfrac{c_2}{A}$.

and $\sin\theta = \dfrac{-2\lambda\gamma}{\sqrt{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}}$, $\cos\theta = \dfrac{\omega^2 - \gamma^2}{\sqrt{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}}$.

If $g(\gamma) = \dfrac{F_0}{\sqrt{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}}$ then show that $g(\gamma)$ is maximum at $\gamma = \sqrt{\omega^2 - 2\lambda^2}$.

We now consider the application of the second order non-homogeneous linear differential equations to an elementary electric circuit consisting of an electromotive force (e.g. a battery or a generator), resistor, inductor and a capacitor in series. We call these circuits, **RLC Series Circuits.**

13.5.2 Electric Circuits

Consider RLC circuit as shown in Fig. 12.

Physcial principles governing RLC series circuits are

i) conservation of charge and

ii) conservation of energy.

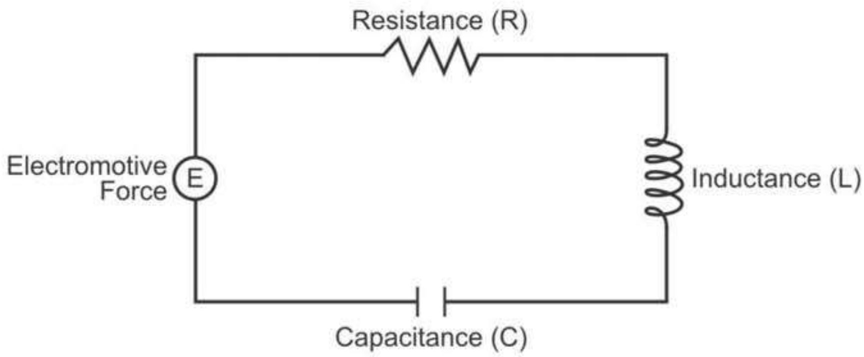

Fig. 12: RLC Series Circuit

These conservation laws were formulated for electric circuits by Gustav Kirchhoff (1824-1887), a German scientist in 1859 and we call them **Kirchhoff's laws**. These laws state

i) The current i passing through each of the elements (resistor, inductor, capacitor or electromotive force) in the series circuit must be the same.

ii) The algebraic sum of the instantaneous changes in potential (voltage drops) around a closed circuit must be zero.

Let $i(t)$ denote current in an RLC series circuit. In order to apply Kirchhoff's laws, we need to know the voltage drop across each element of the circuit. We now state these voltage formulas

- According to **Ohm's law** given by Georg Simon Ohm (1787-1854), a German physicist, the voltage drop E_R across a resistor is proportional to the current i passing through the resistor. That is,

  $$E_R = Ri$$

  The proportionality constant R is called the **resistance.**

- **Faraday's law**, named after an English physicist Michael Faraday, states that the voltage drop E_L across an inductor is proportional to the instantaneous rate of change of the current i, that is,

  $$E_L = L\frac{di}{dt}$$

  The proportionality constant L is called the **inductance.**

- The voltage drop E_c across a capacitor is proportional to the electric charge q on the capacitor, that is

  $$E_C = \frac{1}{C}q.$$

  The proportionality constant $1/C$ is called the **elastance** and C the **capacitance**. An electromotive force is assumed to add voltage or potential energy to the circuit. If we let $E(t)$ denote the voltage supplied to the circuit at time t, then Kirchhoff's second law gives

  $$E_L + E_R + E_C = E(t) \tag{126}$$

  Substituting the expression for E_L, E_R and E_C in Eqn. (126), we get

$$L\frac{di}{dt}+Ri+\frac{1}{C}q=E(t) \tag{127}$$

Now current is just the instantaneous rate of change in charge; that is, $i=\frac{dq}{dt}$. Therefore, we can express Eqn. (127) in terms of the charge q as

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t) \tag{128}$$

The initial conditions are $q(t_0)=q_0,\ q'(t_0)=i(t_0)=i_0$. Thus we must know the charge on the capacitor and the current in the circuit at some initial time t_0.

If $E(t)=0$, we say that the **electric vibrations** are **free**.

Since the A.E. of Eqn. (128) is

$$Lm^2+Rm+\frac{1}{C}=0,$$

there will be three forms of the solution, when $R\neq 0$, depending upon the value of the discriminant $R^2-\frac{4L}{C}$. We say that the circuit is

Overdamped if $R^2-\frac{4L}{C}>0$;

Critically damped if $R^2-\frac{4L}{C}=0$

and **Underdamped** if $R^2-\frac{4L}{C}<0$

In each of these three cases, the general solution of Eqn. (128) contains the factor $e^{-Rt/2L}$ and so $q(t)\to 0$ as $t\to\infty$.

In the underdamped case when $q(0)=q_0$, the charge on the capacitor will oscillate as it decays. In other words, the capacitor is charging and discharging as $t\to\infty$. When $E(t)=0$ and $R=0$, we say that the circuit is undamped and the electric vibrations do not approach zero as t increase without bounds; the response of the circuit is **simple harmonic**.

Let us consider the following example to understand the theory above.

Example 25: Find the charge $q(t)$ on the capacitor in an RLC series circuit when $L=0.25\,Henrys,\ R=10\ ohms,\ C=0.001\ farad,\ E(t)=0,\ q(0)=q_0$ coulombs and $i(0)=0$.

Solution: Since $C=0.001$, thus $\frac{1}{C}=1000$ and Eqn. (128) becomes

$$\frac{1}{4}q''+10q'+1000q=0$$

$$\Rightarrow\quad q''+40q'+4000q=0$$

A.E. is

$$m^2+40m+4000=0$$

$$\Rightarrow\quad m=-20\pm i60$$

Thus the circuit is underdamped and

$$q_c(t) = e^{-20t}(c_1 \cos 60t + c_2 \sin 60t)$$

Applying the initial conditions, we have

$$q(0) = q_0 = 1(c_1 + 0.c_2) \Rightarrow c_1 = q_0$$

$$i(0) = 0 \Rightarrow q'(0) = 0$$

$$\Rightarrow \quad 0 = -20e^{20t}(c_1 \cos 60t + c_2 \sin 60t)\,|_{t=0}$$
$$+ e^{-20t}(-60c_1 \sin 60t + 60c_2 \cos 60t)\,|_{t=0}$$

$$\Rightarrow \quad 0 = -20q_0 + 60c_2$$

$$\Rightarrow \quad c_2 = \frac{q_0}{3}$$

Thus, the solution is given by

$$q_c(t) = q_0\ e^{-20t}\left(\cos 60t + \frac{1}{3}\sin 60t\right)$$

In the case when $R \neq 0$ we call the complementary function $q_c(t)$ of Eqn. (128) a **transient solution**. If $E(t)$ is periodic or constant, then a particular solution $q_p(t)$ of Eqn. (128) is a **steady-state solution**.

You may now try the following exercise where the voltage supplied $E(t)$ is a periodic function.

E18) Find the steady-state solution and the steady-state current in an RLC series circuit when the impressed voltage is $E(t) = E_0 \sin \gamma t$.
(**Hint:** Steady-state current is given by $i_p(t) = q'_p(t)$.)

We now end this unit by giving a summary of what we have covered in it.

13.6 SUMMARY

In this unit we have covered the following:

1. The symbol D is used for $\frac{d}{dx}$ and an n^{th} degree polynomial $L(D)$ in D is called the **polynomial differential operator.**

2. $\frac{1}{L(D)}b(x)$ is that function of x which when operated upon by $L(D)$ gives $b(x)$ and is a particular integral of $L(D)\ y = b(x)$. Here $\frac{1}{L(D)}$ is called the **inverse operator** of $L(D)$.

3. If $y_1, y_2, \ldots, y_m$ are the solutions of equations $L(D)y = b_1(x), L(D)\,y = b_2(x), \ldots, L(D)\,y = b_m(x)$ respectively, then $y = y_1 + y_2 + \cdots + y_m$ is a solution of $L(D)y = b_1(x) + \cdots + b_m(x)$.

4. $L(D)\,[e^{ax}y] = e^{ax}\,L(D + a)\ y$.

5. $$\frac{1}{(D-m_1)(D-m_2)\ldots(D-m_n)}y$$
$$= e^{m_n x}\int e^{(m_{n-1}-m_n)x}\left(\int e^{(m_{n-2}-m_{n-1})x}\int\ldots\left(\int e^{(m_1-m_2)x}\right.\right.$$
$$\left.\left.\left(\int e^{m_1 x}b(x)\,dx\right)\ldots dx\right)dx$$

6. $$\frac{1}{D-a}\,b(x) = e^{ax}\int e^{-ax}\,b(x)\,dx.$$

7. $$\frac{1}{L(D)}e^{ax} = \frac{1}{L(a)}e^{ax} \text{ if } L(a)\neq 0.$$

8. If $L(D)=(D-a)^p\phi(D),\ \phi(a)\neq 0,\ p\geq 1$
$$\frac{1}{L(D)}e^{ax} = \frac{1}{(D-a)^p\phi(D)}e^{ax} = \frac{x^p}{p!}\frac{e^{ax}}{\phi(a)},\ \ \phi(a)\neq 0.$$

9. $$\frac{1}{\phi(D^2)}\cos(ax+b) = \frac{1}{\phi(-a^2)}\cos(ax+b), \text{ if } \phi(-a^2)\neq 0.$$

10. $$\frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b), \text{ if } \phi(-a^2)\neq 0.$$

11. $$\frac{1}{\phi(D^2)}\cos(ax+b) = \frac{1}{(D^2+a^2)^p}\,\frac{1}{\psi(D^2)}\cos(ax+b), \text{ if } \phi^{(k)}(-a^2)=0$$
for $k=1, 2, \ldots, p-1$ but $\psi(-a^2)\neq 0$.

12. $$\frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{(D^2+a^2)^p}\,\frac{1}{\psi(D^2)}\sin(ax+b), \text{ if } \phi^{(k)}(-a^2)=0$$
for $k=1, 2, \ldots, p-1$ but $\psi(-a^2)\neq 0$.

13. $$\frac{1}{L(D)}e^{ax}V(x) = e^{ax}\frac{1}{L(D+a)}V(x).$$

14. Euler's equation $(x^n D^n + a_1 x^{n-1}D^{n-1} + \cdots + a_{n-1}xD + a_n)\ y = b(x),\ x>0$, where $a_1, a_2, \ldots, a_n$ are real constants can be reduced to an equation with constant coefficients with the help of the substitution $x=e^z$.

15. Differential equation $[(ax+b)^n D^n + a_1(ax+b)^{n-1}D^{n-1} + \cdots + a_{n-1}xD + a_n)y = b(x),\ x>0$, where a, b are positive real constants and coefficients $a_1, a_2, \ldots, a_n$ are constants can be reduced to Euler's equation by using the substitution $ax+b=z$ and then to an equation with constant coefficient by means of a substitution $t=\ln z$. Also, it can be reduced directly to an equation with constant coefficients by means of the transformation $ax+b=e^t$.

16. The application of second order non-homogeneous linear differential equations have been studied for

i) **Mechanical Vibrations:** Here a mass m is attached to a spring which stretches to a position where the restoring force of the spring is balanced by the weight mg. Any subsequent motion is above or below this **equilibrium position**. The equation of motion of the system is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t), \text{ (see Eqn. (96))},$$

where c is the damping constant, k is spring constant and $f(t)$ is the external force acting on the system.
When $c = 0$, the system is **undamped**, otherwise it is **damped**. Moreover, when $f(t) = 0$, the motion is **free**, otherwise the motion is **forced**. When $c = 0,\ f(t) = 0$, the mass exhibits **simple harmonic motion**. When $c \neq 0,\ f(t) = 0$, then if

a) $c^2 - 4km < 0$, motion is called **under damped**.

b) $c^2 - 4km = 0$, motion is called **critically damped**.

c) $c^2 - 4km > 0$, motion is called **overdamped**.

In the absence of a damping force, periodic force can cause the amplitudes of vibration to become very large. In this case we say that the system is in a state of **resonance**. Further, if the motion possesses a periodic variation of amplitude, it is called **beat**.

ii) **Electric Circuits:** When a series circuit containing an inductor, resistor and capacitor is driven by an electromagnetic force, the resulting differential equation for the charge is given by

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \text{ (see Eqn. (128))}.$$

Analysis of such circuits is the same as outlined for mechanical vibrations and have been illustrated by examples.

13.7 SOLUTIONS/ANSWERS

E1) i) $(D+2)\ (2D-1)$

ii) $(D-1)\ (D+2)\ (D-3)$

iii) $(D+2)^3\ (2D-1)$

iv) $(D-4)\ (D^2+4D+5)$

E2) The given equation can be written as

$$(D^3 - 3D^2 + 4D - 1)y = e^x + 2.$$

Since $(D^3 - 3D^2 + 4D - 1)(e^x) = e^x$

$\therefore$ P.I. $y_1 = \dfrac{1}{D^3 - 3D^2 + 4D - 1}e^x = e^x$

Also since $(D^3 - 3D^2 + 4D - 1)\ (-2) = 2$

P.I. $y_2 = \dfrac{1}{D^3 - 3D^2 + 4D - 1}(2) = -2$

$\therefore\ y_p = y_1 + y_2 = e^x - 2.$

E3) i) $$\text{P.I} = \frac{1}{D^2+n^2}\sec nx.$$

$$= \frac{1}{2ni}\left(-\frac{1}{D+in}+\frac{1}{D-in}\right)\sec nx$$

$$= \frac{1}{2ni}\left[e^{nix}\int e^{-inx}\sec nx\ dx - e^{-inx}\int e^{inx}\sec nx\ dx\right]$$

$$= \frac{1}{2ni}\left[e^{nix}\int(1-i\tan nx)\ dx - e^{-nix}\int(1+i\tan nx)\ dx\right]$$

$$= \frac{1}{2ni}\left[e^{nix}\left(x-\frac{i}{n}\ln(\sec nx\right)-e^{-nix}\left(x+\frac{i}{n}\ln(\sec nx\right)\right]$$

$$= \frac{1}{n}\left[x\sin nx - \frac{1}{n}\cos nx\ \ln\ (\sec nx)\right]$$

ii) $$\text{P.I.} = \left(\frac{1}{D-2}-\frac{1}{D-1}\right)e^{-x}\sin x$$

$$= \frac{1}{D-2}e^{-x}\sin x - \frac{1}{D-1}e^{-x}\sin x$$

$$= e^{2x}\int e^{-2x}e^{-x}\sin x\ dx - e^{x}\int e^{-x}e^{-x}\sin x\ dx$$

$$= -\frac{e^{-x}}{10}(3\sin x+\cos x)+\frac{1}{5}e^{-x}\ (2\sin x+\cos x).$$

$$= \frac{e^{-x}}{10}(\sin x+\cos x).$$

iii) $\frac{2}{9}e^{2x}$

iv) $$\text{P.I.} = \frac{1}{(D-2)^2\ (D+3)}\ X(x)$$

$$= \left[-\frac{1}{25(D-2)}+\frac{1}{5(D-2)^2}+\frac{1}{25(D+3)}\right]X(x)$$

$$= \frac{1}{25}\left[-e^{2x}\int e^{-2x}X(x)\ dx + 5e^{2x}\int\left(\int e^{-2x}X(x)\ dx\right)dx + e^{-3x}\int e^{3x}X(x)\,dx\right]$$

E4) i) $y = (c_1+c_2x)e^{x} + \frac{4}{3}\ e^{(5/2)x}$

ii) The auxiliary equation is

$m^2-1=0$

$\Rightarrow\quad m = 1,\ -1$

$y_c = c_1e^{x}+c_2e^{-x}$

$$\text{P.I.} = \frac{1}{D^2-1}(e^{x}+1)^2$$

$$= \frac{1}{D^2-1}(e^{2x}+1+2e^{x})$$

$$= \frac{1}{D^2-1}e^{2x}+\frac{1}{D^2-1}e^{2x}+\frac{2}{(D+1)\ (D-1)}e^{x}$$

$$= \frac{1}{3}e^{2x} - 1 + \frac{1}{D-1}e^{x}$$

$$= \frac{1}{3}e^{2x} - 1 + xe^{x}$$

$$\therefore \quad y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{3}e^{2x} + xe^{x} - 1$$

iii) Given equation can be written as

$$(D^3 - 5D^2 + 7D - 3)\ y = e^{2x}\left(\frac{e^{x} + e^{-x}}{2}\right), \left(\cosh x = \frac{1}{2}(e^{x} + e^{-x})\right)$$

$$= \frac{1}{2}(e^{3x} + e^{x})$$

$$\Rightarrow \quad (D-1)^2 (D-3) y = \frac{1}{2}(e^{3x} + e^{x})$$

Roots of the auxiliary equation is

$m = 1, 1, 3$

$$\therefore \quad y_c = (c_1 + xc_2)e^{x} + c_3 e^{3x}$$

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{(D-1)^2 (D-3)}e^{3x} + \frac{1}{(D-1)^2 (D-3)}e^{x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}\,\frac{1}{D-3}e^{3x} - \frac{1}{2(D-1)^2}e^{x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}x\,e^{3x} - \frac{1}{4}x^2 e^{x}\right]$$

$$\therefore \quad y = (c_1 + c_2 x)\ e^{x} + c_3 e^{3x} + \frac{x}{8}e^{3x} - \frac{x^2}{8}e^{x}$$

iv) $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - xe^{x}$

E5) i) $y = (c_1 + c_2 x)\cos nx + (c_3 + c_4 x)\ \sin nx + \dfrac{1}{(n^2 - m^2)^2}\cos mx$

ii) The roots of the auxiliary equation are

$\pm mi, \pm ni$

$\therefore$ C.F. $= y_c = (c_1 \cos mx + c_2 \sin mx) + (c_3 \cos nx + c_4 \sin nx)$

$$\text{P.I.} = y_p = \frac{1}{(D^2 + m^2)\ (D^2 + n^2)}\cos\left\{(m+n)\frac{x}{2}\right\}\cos\left\{(m-n)\frac{x}{2}\right\}$$

$$= \frac{1}{(D^2 + m^2)(D^2 + n^2)}\,\frac{1}{2}(\cos mx + \cos nx)$$

$$\left[\text{since } \cos A \cos B = \frac{1}{2}\cos(A+B) + \cos(A-B)\right]$$

$$= \frac{1}{2}\left[\frac{1}{n^2 - m^2}\,\frac{1}{D^2 + m^2}\cos mx + \frac{1}{m^2 - n^2}\,\frac{1}{D^2 + n^2}\cos nx\right]$$

$$= \frac{1}{2(n^2 - m^2)}\left[\frac{x \sin mx}{2m} + \frac{\cos mx}{4m^2} - \frac{x \sin nx}{2n} - \frac{\cos nx}{4n^2}\right]$$

[ref. Eqn. (65)]

$$\therefore y = y_c + y_p = (c_1 \cos mx + c_2 \sin mx) + (c_3 \cos nx + c_4 \sin nx)$$

$$+\frac{x}{4(n^2-m^2)}\left(\frac{\sin mx}{m} - \frac{\sin nx}{n}\right) + \frac{1}{8(n^2-m^2)}\left(\frac{\cos mx}{m^2} - \frac{\cos nx}{n^2}\right)$$

E6) i) C.F. $= e^{-(1/2)x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

$$\text{P.I.} = \frac{1}{D^2+D+1}\sin 2x$$

$$= \frac{1}{-2^2+D+1}\sin 2x$$

$$= \frac{D+3}{(D-3)\,(D+3)}\sin 2x$$

$$= \frac{D+3}{D^2-9}\sin 2x$$

$$= -\frac{1}{13}\,(2\cos 2x + 3\sin 2x)$$

$\therefore$ The complete solution is

$$y = e^{(-1/2)x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$-\frac{2}{13}\cos 2x - \frac{3}{13}\sin 2x$$

ii) Roots of the auxiliary equation are given by

$$m = \frac{-2n\cos\alpha \pm \sqrt{4n^2\cos^2\alpha - 4n^2}}{2}$$

$$= -n\cos\alpha \pm n\sqrt{\cos^2\alpha - 1}$$

$$= -n\cos\alpha \pm ni\sin\alpha$$

$$\therefore y_c = e^{-(n\cos\alpha)x}[c_1\cos(n\sin\alpha)x + c_2\sin(n\sin\alpha)x]$$

$$\text{P.I.} = y_p = \frac{1}{D^2 + 2n\cos\alpha D + n^2}a\cos nx$$

$$= \frac{a}{-n^2 + 2n\cos\alpha\ D + n^2}\cos nx$$

$$= \frac{a}{2n\cos\alpha}\ \frac{\sin nx}{n}$$

$$\therefore y = e^{-(n\cos\alpha)x}[c_1\cos(n\sin\alpha)x + c_2\sin(n\sin\alpha)x]$$

$$+\frac{a}{2n^2\cos\alpha}\sin nx.$$

E7) i) $y = c_1 + c_2 x + (c_3 + c_4 x)\ e^x + x^2 + \frac{x^3}{6}$

ii) $y = c_1 e^x + c_2 e^{-2x} + c_3 e^{4x} + \frac{x}{8} + \frac{3}{32}$

iii) C.F. $= c_1 e^x + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2+D-2}\ 2(1+x-x^2)$$

$$= -\left[1 - \frac{D}{2} - \frac{D^2}{2}\right]^{-1} (1 + x - x^2)$$

$$= -\left[1 + \frac{D}{2} + \frac{D^2}{2} + \frac{D^2}{4} + \text{order of } (D^3) + \cdots\right] (1 + x - x^2)$$

$$= -(1 + x - x^2) + \frac{-1}{2}(1 - 2x) - \frac{3}{4}(-2)$$

$$= x^2$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-2x} + x^2$$

iv) $y_c = c_1 + c_2 x + c_3 e^x$

$$y_p = \frac{1}{D^2(D^2 + 2D - 3)}(x^2 + 3e^{2x} + 4\sin x)$$

$$= \frac{1}{D^2}\left[\frac{1}{D^2 + 2D - 3}x^2 + \frac{3}{D^2 + 2D - 3}e^{2x} + \frac{4}{D^2 + 2D - 3}\sin x\right]$$

$$= \frac{1}{D^2}\left[-\frac{1}{3}\left[1 - \frac{(2D + D^2)}{3}\right]^{-1} x^2 + \frac{3}{5}e^{2x} + \frac{2}{D-2}\sin x\right]$$

$$= \frac{1}{D^2}\left[-\frac{1}{3}\left(1 + \frac{2}{3}D + \frac{7}{9}D^2\right)x^2 + \frac{3}{5}e^{2x} - \frac{2}{5}(\cos x + 2\sin x)\right]$$

$$= \frac{1}{D^2}\left[\frac{-x^2}{3} - \frac{4}{9}x - \frac{14}{27} + \frac{3}{5}e^{2x} - \frac{2}{5}\cos x - \frac{4}{5}\sin x\right]$$

$$= \frac{-x^4}{36} - \frac{2}{27}x^3 - \frac{7}{27}x^2 + \frac{3}{20}e^{2x} + \frac{2}{5}\cos x + \frac{4}{5}\sin x$$

$\therefore$ $y = y_c + y_p$ is the required solution.

E8) i) $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{170}e^{2x}(11\sin x - 7\cos x)$

ii) $y = (c_1 + c_2 x)\, e^x + \frac{1}{4}e^{3x}\left(x^2 - 2x + \frac{3}{2}\right)$

iii) C.F. $= c_1 e^x + c_2 e^{5x} + c_3 e^{-4x}$

P.I. $= \frac{1}{D^3 - 2D^2 - 19D + 20}(xe^x + 2e^{-4x}\sin x)$

$$= e^x \frac{1}{(D+1)^3 - 2(D+1)^2 - 19(D+1) + 20}x$$

$$+ 2e^{-4x}\frac{1}{(D-4)^3 - 2(D-4)^2 - 19(D-4) + 20}\sin x$$

$$= e^x \frac{1}{D^3 + D^2 - 20D}x + 2e^{-4x}\frac{1}{D^3 - 14D^2 + 45D}\sin x$$

$$= -e^x \frac{1}{20D}\left[1 - \frac{D + D^2}{20}\right]^{-1} x + 2e^{-4x}\frac{1}{-D + 14 + 45D}\sin x$$

$$= -\frac{1}{20} e^x \frac{1}{D}\left[1 + \frac{D}{20} + \text{order } (D^2)\right] x$$

$$+ e^{-4x} \frac{7-22D}{(7-22D)(7+22D)} \sin x$$

$$= -\frac{1}{20} e^x \frac{1}{D}\left[x + \frac{1}{20}\right] + e^{-4x}(7-22D)\frac{1}{49+484}\sin x$$

$$= -\frac{e^x}{20}\left(\frac{x^2}{2} + \frac{x}{20}\right) + \frac{e^{-4x}}{533}(7\sin x - 22\cos x)$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{5x} + c_3 e^{-4x} - \frac{e^x}{20}\left(\frac{x^2}{2} + \frac{x}{20}\right)$$

$$+ \frac{e^{-4x}}{533}(7\sin x - 22\cos x)$$

iv) $y = (c_1 + c_2 x + c_3 x^2)\, e^x + e^x\left(\frac{x^4}{24} + \frac{x^3}{6}\right)$

E9) i) $y = c_1 e^x + c_2 e^{-x} + \frac{x}{2}\sin x + \left(\frac{1}{2} - \frac{x^2}{2}\right)\cos x$

ii) Roots of the auxiliary equation are
$m = 2, 2$

C.F. $= (c_1 + xc_2)e^{2x}$

P.I. $= \text{Im}\, 8\frac{1}{(D-2)^2} x^2 e^{(2+2i)x}$

$$= \text{Im}\, 8e^{(2+2i)x} \frac{1}{(D+2+2i+2)^2} x^2$$

$$= \text{Im}\, 8e^{(2+2i)x} \frac{1}{D^2 - 4 + 4iD} x^2$$

$$= \text{Im}\, 8e^{(2+2i)x} \frac{1}{-4}\left[1 - \frac{(4iD + D^2)}{4}\right]^{-1} x^2$$

$$= \text{Im}\frac{8e^{(2+2i)x}}{-4}\left[1 + iD + \frac{D^2}{4} - D^2\right] x^2$$

$$= \text{Im} - 2e^{(2+2i)x}\left[x^2 + 2xi - \frac{3}{2}\right]$$

$$= \text{Im} - 2e^{2x}\left(x^2 + 2xi - \frac{3}{2}\right)(\cos 2x + i\sin 2x)$$

$$= -2e^{2x}\left(x^2 \sin 2x + 2x\cos 2x - \frac{3}{2}\sin 2x\right)$$

$$\therefore\ y = (c_1 + c_2 x)\, e^{2x} + e^{2x}(3\sin 2x - 4x\cos 2x - 2x^2 \sin 2x)$$

iii) $y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(\cos x + x\sin x) + \frac{1}{12} xe^x(9 - 3x + 2x^2)$

iv) $y = c_1 e^{-x} + c_2 e^x + c_3 \cos x + c_4 \sin x + \frac{\cos x}{4}\left(\frac{x^3}{3} - \frac{5}{2}x\right).$

E10) i) $y = (c_1 + c_2 \ln x)x^2 + (\ln x)^2 x^2$

ii) $y = (c_1 + c_2 \ln x)\frac{1}{x} + \frac{1}{x}\ln\frac{x}{1-x}$

iii) The given equation can be written as

$$(x^3D^3 - 4x^2D^2 + 5xD - 2)\ y = x^3$$

Let $z = \ln x$ and denoting $\frac{d}{dz}$ by D', the given equation reduces to

$$[D'(D'-1)\ (D'-2) - 4D'\ (D'-1) + 5D' - 2]\ y = e^{3z}$$

$$\Rightarrow \quad (D'^3 - 7D'^2 + 11D' - 2)\ y = e^{3z}$$

$$\therefore \quad \text{C.F.} = c_2e^{2z} + e^{(5/2)z}(c_2e^{(\sqrt{21}/2)z} + c_3e^{(-\sqrt{21}/2)z})$$

$$= c_1x^2 + x^{5/2}(c_2x^{\sqrt{21}/2} + c_3x^{-\sqrt{21}/2})$$

$$\text{P.I.} = \frac{1}{(D'-2)\ (D^2 - 5D + 1)}e^{3z}$$

$$= \frac{1}{(3-2)\ (9 - 15 + 1)}\ e^{3z}$$

$$= -\frac{1}{5}e^{3z} = -\frac{x^3}{5}$$

Hence the general solution is

$$y = c_1x^2 + x^{5/2}(c_2x^{\sqrt{21}/2} + c_3x^{-\sqrt{21}/2}) - \frac{x^3}{5}$$

iv) The given equation is

$$(x^2D^2 - xD + 4)\,y = \cos(\ln x) + \sin(\ln x)$$

Let $z = \ln x$ and denoting $\frac{d}{dz}$ by D', the given equation reduces to

$$[D'(D'-1) - D' + 4]\ y = \cos z + e^z \sin z$$

$$\Rightarrow \quad (D'^2 - 2D' + 4)\ y = \cos z + e^z \sin z$$

A.E. is

$$m^2 - 2m + 4 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}$$

$$\text{C.F.} = e^z(c_1 \cos\sqrt{3}z + c_2 \sin\sqrt{3}z) = x[c_1 \cos\left(\sqrt{3}\ln x\right) + c_2 \sin\left(\sqrt{3}\ln x\right)]$$

$$\text{P.I.} = \frac{1}{D'^2 - 2D' + 4}\cos z + \frac{1}{D'^2 - 2D' + 4}e^z \sin z$$

$$= \frac{1}{-1 - 2D' + 4}\cos z + e^z\,\frac{1}{(D'+1)^2 - 2(D'+1) + 4}\sin z$$

$$= (3 + 2D')\frac{1}{9 - 4D'^2}\cos z + e^z\,\frac{1}{D'^2 + 3}\sin z$$

$$= (3 + 2D')\frac{1}{9+4}\cos z + e^z\,\frac{1}{-1+3}\sin z$$

$$= (3 + 2D')\frac{\cos z}{13} + \frac{e^z}{2}\sin z$$

$$= \frac{3}{13}\cos z - \frac{2}{13}\sin z + \frac{1}{2}e^z \sin z$$

$$= \frac{3}{13}\cos(\ln x) - \frac{2}{13}\sin(\ln x) + \frac{1}{2}x\sin(\ln x)$$

Hence the general solution is

$$y = c_1 x\cos(\sqrt{3}\ln x) + c_2 x\sin(\sqrt{3}\ln x) + \frac{3}{13}\cos(\ln x) - \frac{2}{13}\sin(\ln x)$$
$$+\frac{x}{2}\sin(\ln x)$$

E11) i) The given equation is

$$(2x-1)^2 D^2 + (2x-1)D - 2y = 0$$

Let $(2x-1) = e^z$ and denoting $\frac{d}{dz} = D'$, we have

$$\frac{dy}{dx} = \frac{2}{2x-1} D'y$$
$$\frac{d^2y}{dx^2} = \frac{4}{(2x-1)^2} (D'^2 - D')\ y$$

Substituting them in the given equation, we have

$$[4(D'^2 - D') + 2D' - 2]\ y = 0$$
$$\Rightarrow \quad (4D'^2 - 2D' - 2)\ y = 0$$
$$\Rightarrow \quad (2D' - D' - 1)\ y = 0$$

A.E. is

$$2m^2 - m - 1 = 0$$
$$\Rightarrow \quad m = \frac{1\pm\sqrt{1+8}}{4} = \frac{1\pm 3}{4} = 1, -\frac{1}{2}$$

$\therefore$ The general solution is

$$y = c_1 e^z + c_2 e^{-(1/2)z}$$
$$\Rightarrow \quad y = c_1(2x-1) + c_2(2x-1)^{-1/2}$$

ii) The given equation is

$$[(1+x^2)\ D^2 + (1+x)\ D + 1]\ y = 4\cos\ (\ln(1+x))$$

Let $1+x = e^z$ and denoting $\frac{d}{dz}$ by D', we have

$$\frac{dy}{dx} = \frac{1}{1+x}\ D'y$$
$$\frac{d^2y}{dx^2} = \frac{1}{(1+x)^2} D'(D'-1)\ y$$

Then the given equation reduces to

$$[D'(D'-1) + D' + 1]\ y = 4\cos z$$

A.E. is

$$m^2 + 1 = 0$$
$$\Rightarrow \quad m = \pm i$$

$\therefore$ C.F. $= c_1\cos z + c_2\sin z = c_1\cos(\ln(1+x)) + c_2\sin(\ln(1+x))$

$$\text{P.I.} = 4\frac{1}{D'^2+1}\cos z = 4\left(\frac{z}{2}\sin z + \frac{\cos z}{4}\right)$$
$$= 2\ln(1+x)\sin(\ln(1+x)) + \cos(\ln(1+x))$$

Hence the required solution is

$$y = c_1\cos(\ln(1+x)) + c_2\sin(\ln(1+x)) + 2\ln(1+x)\sin(\ln(1+x)).$$

E12) We know that $mg = w$

$$\Rightarrow \quad m = \frac{w}{g} = \frac{19.6}{9.8} = 2kg.$$

From Hook's law,

$$19.6 = k\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad k = 39.2 \ kg/m.$$

Hence the differential equation governing the motion is

$$2\frac{d^2x}{dt^2} = -39.2x$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + 19.6x = 0$$

$$\Rightarrow \quad x(t) = c_1 \cos\sqrt{19.6}t + c_2 \sin\sqrt{19.6}t$$

The initial displacement and initial velocity are given by

$$x(0) = \frac{2}{3}m.; \left.\frac{dx}{dt}\right|_{t=0} = -\frac{4}{3}\ m/\sec.$$

The negative sign in the second condition is a consequence of the fact that the mass is given an initial velocity in the upward direction.
Applying the above initial conditions, we have

$$x(0) = \frac{2}{3} = c_1.1 + c_2.0$$

$$\Rightarrow \quad c_1 = \frac{2}{3}$$

and $\left.\dfrac{dx}{dt}\right|_{t=0} = -\dfrac{4}{3}$

$$\Rightarrow \quad \left. -c_1\sqrt{19.6}\sin\sqrt{19.6}t + c_2\sqrt{19.6}\cos\sqrt{19.6}t\right|_{t=0} = -\frac{4}{3}$$

$$\Rightarrow \quad c_2\sqrt{19.6} = -\frac{4}{3}$$

$$\Rightarrow \quad c_2 = \frac{-4}{3\sqrt{19.6}}$$

Thus the function that describes the motion is

$$x(t) = \frac{2}{3}\left(\cos\sqrt{19.6}t - \frac{2}{\sqrt{19.6}}\sin\sqrt{19.6}t\right).$$

E13) i) $x(t) = 3\cos 2t + 4\sin 2t$

Let $3 = R\cos\delta$ and $4 = R\sin\delta$
then $x(t) = R\cos(2t-\delta)$

$$R = \sqrt{16+9} = 5,\ \delta = \tan^{-1}\left(\frac{4}{3}\right),\ \omega_0 = 2.$$

ii) Let $-1 = R\cos\delta,\ \sqrt{3} = R\sin\delta$

$$R = 2,\ \tan\delta = -\sqrt{3}$$

$$\therefore \quad \delta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$x(t) = 2\cos\left(t - \frac{2\pi}{3}\right).$$

iii) $x(t) = \sqrt{13}\cos(\pi t - \delta)$

$$\delta = \pi + \tan^{-1}(3/2).$$

E14) After the weight is attached, the elongation of the spring is

$1.98-1=.98m$

From the Hooke's law, it follows that

$4.9=k\,(.98) \Rightarrow \; \mathrm{k}=5$

Also, $m=\dfrac{4.9}{9.8}=\dfrac{1}{2}\,kg$.

So the differential equation governing the motion is

$$\frac{1}{2}\frac{d^2x}{dt^2}=-5x-\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2}+2\frac{dx}{dt}+10x=0$$

$$\therefore \quad x(t)=e^{-t}(c_1\cos 3t+c_2\sin 3t)$$

The constants c_1 and c_2 are determined by using the conditions.

$$x(0)=-2,\ \left.\frac{dx}{dt}\right|_{t=0}=0$$

Now, $x(0)=-2 \Rightarrow \quad c_1=-2$

and $\left.\dfrac{dx}{dt}\right|_{t=0}=0 \Rightarrow \quad c_2=-\dfrac{2}{3}$

Thus, the displacement $x(t)$ at any time t is given by

$$x(t)=e^{-t}\left(-2\cos 3t-\frac{2}{3}\sin 3t\right)$$

E15) In this case, $x_c(t)=c_1\cos\omega t+c_2\sin\omega t$

and $x_p(t)=\dfrac{F_0}{\omega^2-\gamma^2}\sin\gamma t$

Initial conditions yield

$c_1=0$ and $c_2=\dfrac{-\gamma F_0}{\omega(\omega^2-\gamma^2)}\sin\gamma t$

Thus, the required solution is

$x(t)=\dfrac{F_0}{\omega(\omega^2-\gamma^2)}\,[-\gamma\sin\omega t+\omega\sin\gamma t]$, with $\gamma\neq\omega$.

E16) We have

$$m\frac{d^2x}{dt^2}+kx=F_0\sin\omega t$$

$\therefore$ C.F. $=A\cos\omega_0 t-1B\sin\omega_0 t$, $\omega_0=\sqrt{\dfrac{k}{m}}$

and P.I. $=\dfrac{1}{D^2+\omega_0^2}\,\dfrac{F_0}{m}\sin\omega t$

For $\omega\neq\omega_0$

$$\text{P.I.}=\frac{F_0}{m(\omega_0^2-\omega^2)}\sin\omega t$$

$$=\frac{F_0}{k-m\omega^2}\sin\omega t$$

E17) $g(\gamma)=\dfrac{F_0}{\sqrt{(\omega^2-\gamma^2)+4\lambda^2\gamma^2}}$ is maximum if $f(\gamma) = (\omega^2 - \gamma^2) + 4\lambda^2\gamma^2$ is minimum

Now, $f'(\gamma) = 0$

$\Rightarrow \quad \gamma = 0$ or $\gamma = \sqrt{\omega^2 - 2\lambda^2}$

Using maximum and minimum principles you can check that $f(\gamma)$ is minimum for $\gamma = \sqrt{\omega^2 - 2\lambda^2}$.

E18) The steady state solution $q_p(t)$ is a particular solution of the differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}.q = E_0 \sin \gamma t$$

The particular solution, using method of undetermined coefficient is

$q_p(t) = A\sin \gamma t + B\cos \gamma t$

with $A = \dfrac{E_0\left(\dfrac{1}{c\gamma} - L\gamma\right)}{\gamma\left[L^2\gamma^2 - \dfrac{2L}{c} + \dfrac{1}{c^2\gamma^2} + R^2\right]}$

with $B = \dfrac{E_0 R}{\left[L^2\gamma^2 - \dfrac{2L}{c} + \dfrac{1}{c^2\gamma^2} + R^2\right](-\gamma)}$

and $i_p = \dfrac{dq_p}{dt} = \gamma\,[A\cos\gamma t - B\sin\gamma t]$.

– x –

MISCELLANEOUS EXERCISES

1. State whether the following statements are true or false. Justify you answer with the help of a short proof or a counter example.
   i) Equation $yy'' + (y')^2 = 0$ cannot be solved by reducing it to a first order differential equation.
   ii) Equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + (\sin y - 2) = 0$ in $]0, \pi[$ is a linear differential equation.
   iii) The solution $y = cx^2 + x + 3$ of the I.V.P., $x^2y'' - 2xy' + 2y = 6$, $y(0) = 3,\ y'(0) = 1$ is a unique solution of this problem on the interval $]-\infty, \infty[$.
   iv) A particular integral of the differential equation $(D^2 - a^2)y = a^x$ is $\{(\ln a)^2 - a^2\}^{-1}a^x$.
   v) The solution of the differential equation $(D^2 + 4D + 4)y = 0$ satisfying at $x = 0,\ y = 1$ and $y' = -1$, is $(1 - x)e^{-x}$.

2. Determine whether the given functions are linearly dependent or independent on the indicated interval.
   i) $f_1(x) = x,\ f_2(x) = x - 1,\ f_3(x) = x + 3;\ ]-\infty, \infty[$
   ii) $f_1(x) = 1 + x,\ f_2(x) = x,\ f_3(x) = x^2;\ ]-\infty, \infty[$
   iii) $f_1(x) = e^x,\ f_2(x) = e^{-x},\ f_3(x) = e^{4x};\ ]-\infty, \infty[$
   iv) $f_1(x) = \tan x,\ f_2(x) = \cot x;\ ]0, \pi/2[$
   v) $f_1(x) = x,\ f_2(x) = x\ln x,\ f_3(x) = x^2\ln x;\ ]0, \infty[$

3. Show that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Also write the general solution of the equations.
   i) $y'' - 2y' + 5y = 0;\ e^x\cos 2x,\ e^x\sin 2x,\ ]-\infty, \infty[$
   ii) $x^2y'' - 6xy' + 12y = 0;\ x^3,\ x^4,\ ]0, \infty[$
   iii) $x^3y''' + 6x^2y'' + 4xy' - 4y = 0;\ x,\ x^{-2},\ x^{-2}\ln x,\ ]0, \infty[$

4. Consider the differential equation $x^2y'' - 4xy' + 6y = 0$
   i) Verify that $y_1(x) = x^3$ and $y_2(x) = |x|^3$ are linearly independent solutions of the differential equation on $]-\infty, \infty[$
   ii) Show that $W(y_1, y_2) = 0$ for every real number
   iii) Does the result of part ii) violate Theorem 4 of Unit 10?
   iv) Verify that $Y_1 = x^3$ and $Y_2 = x^2$ are also linearly independent solutions of the differential equation on the interval $]-\infty, \infty[$.
   v) Find a solution of the equation satisfying $y(0) = 0,\ y'(0) = 0$.
   vi) By the superposition principle both linear combinations

   $$y = c_1y_1 + c_2y_2 \text{ and } y = c_1Y_1 + c_2Y_2,\ c_1,\ c_2 \text{ constants,}$$

   are solutions of the differential equation. Is one, both, or neither the general solution of the differential equation on $]-\infty, \infty[$?

5. Write the form of a trial solution for the following differential equations.
   i) $y''' - y' = 4e^x + 3e^{2x}$

ii) $y^{iv} + 2y''' + 2y'' = 3e^x + 2xe^{-x} + e^{-x}\sin x$

iii) $y''' - 4y' = x + 3\cos x + e^{-2x}$

iv) $y'' - y = e^x(2 + 3x\cos 2x)$

6. Solve the following differential equations using the method of variation of parameters.

i) $x^2y'' - 4x^2y' + 4x^2y = e^{2x},\ x > 0$

ii) $y'' - y = \dfrac{2}{1+e^x}$

iii) $y'' + 2y' + y = x^2e^{-x}$

iv) $y'' + y = 4x\sin x$

v) $y'' + 2y' + y = e^{-x}\ln x$

7. Solve the following differential equations by the method of reduction of order

i) $y'' + y = \operatorname{cosec} x$

ii) $y'' - y = e^x$

iii) $x^2y'' - 4xy' + 6y = 0$

iv) $y'' - 3(\tan x)y' = 0$

v) $y'' - 3y' + 2y = 5e^{3x}$

8. Solve the following differential equations

i) $x^4y''' + 2x^3y'' - x^2y' + xy = x^2 + 1$

ii) $yy'' + (y')^2 - 2y(y')^3 = 0$

iii) $y'' - 2y' + y = xe^x\sin x$

iv) $(1+x^2)y'' + (1+x)y' + y = 4\cos\{\ln(1+x)\},\ 1 + x > 0$

v) $x^2y'' - 2xy' - 4y = x^2 + 2\ln x,\ x > 0$

vi) $xy'' + y' = 0$.

9. A mass (in kgs), acted on by a constant force p newtons, moves a distance x meters in t seconds and acquires a velocity v meters per second. Show that $x = \dfrac{mv^2}{2gp} = \dfrac{gt^2p}{2m}$, where g is the acceleration due to gravity.

10. A steel bal weighing 39.2 kg is suspended from a spring, due to which the spring is stretched 2 m from its natural length. The ball is started in motion with no initial velocity by displacing it through 0.5 m above the equilibrium position. Assuming no air resistance, find an expression for the position of the ball at any time t.

11. The differential equation satisfied by a beam uniformly loaded (W kg/meter), with one end fixed and the second end subjected to a tensile force P, is given by $EI\dfrac{d^2y}{dx^2} = Py - \dfrac{1}{2}Wx^2$, where E is the modulus of elasticity and I is the moment of inertia. Show that the elastic curve for the beam with conditions $y = 0$ and $\dfrac{dy}{dx} = 0$ at $x = 0$ is given by

$$y = \frac{W}{Pn^2}(1 - \cosh nx) + \frac{Wx^2}{2P}, \text{ where } n^2 = \left(\frac{P}{EI}\right).$$

12. An electric circuit consists of an inductor of 1 Henry, a resistor of 12 ohms, a capacitor of 0.01 farad, and a generator having voltage given by $E(t) = 24\sin 10t$. Find the charge q and the current i at time t, if $q = 0$ and $i = 0$ at $t = 0$.

– x –

SOLUTIONS/ANSWERS TO MISCELLANEOUS EXERCISES

1. i) False, Equation can be reduced to first order differential equation by putting $y' = p$. Equation becomes $yp\frac{dp}{dy} + p^2 = 0$, which is first order DE and can be solved by variable seperable.

   ii) False, it is non-linear due to the presence of the term $\sin y$.

   iii) False, since x^2, the coefficient of $\frac{d^2y}{dx^2}$ vanishes at $x = 0$ on the interval $]-\infty, \infty[$.

   iv) True, as a^x can be written as $e^{\ln a^x} = e^{x\ln a}$ and $\frac{1}{D^2 - a^2}e^{x\ln a} = \frac{1}{(\ln a)^2 - a^2}e^{x\ln a}$.

   v) False, it is $y = (1+x)e^{-2x}$.

2. i) dependent

   ii) independent

   iii) independent, $W(e^x, e^{-x}, e^{4x}) = -30e^{4x} \neq 0$ on $]-\infty, \infty[$

   iv) independent

   v) independent

3. i) The functions satisfy the differential equation and are linearly independent on the interval since

   $$W(e^x \cos 2x, e^x \sin 2x) = 2e^{2x} \neq 0$$

   The general solution is

   $$y = c_1e^x \cos 2x + c_2e^x \sin 2x$$

   ii) The function satisfy the differential equation and are linearly independent on the interval since

   $$W(x^3, x^4) = x^6 \neq 0,\ y = c_1x^3 + c_2x^4.$$

   iii) The functions satisfy the differential equation and are linearly independent on the interval since

   $$W(x, x^{-2}, x^{-2}\ln x) = 9x^{-6} \neq 0;\ y = c_1x + c_2x^{-2} + c_3x^{-2}\ln x.$$

4. i) The graphs of the two functions show that y_1 and y_2 are not multiples of one another. Also for $x \geq 0,\ y_1 = x^3,\ y_2 = x^3$ satisfy the give DE and for $x < 0,\ y_1 = x^3$ and $y_2 = -x^3$ satisfy the given DE.

   ii) For $x \geq 0, W(y_1, y_2) = 3x^5 - 3x^5 = 0$
   for $x < 0, W(y_1, y_2) = -3x^5 + 3x^5 = 0$

thus $W(y_1, y_2) = 0$ for every real value of x.

iii) No, $a_2(x) = x^2$ is zero at $x = 0$.

iv) Y_1 and Y_2 both satisfy the given DE, also since $W(x^3, x^2) = -x^4$, Y_1 and Y_2 are linearly independent solutions on the interval.

v) $Y_1 = x^3, Y_2 = x^2$ or $y_2 = |x|^3$

vi) Neither, for the general solution $a_2(x) \neq 0$ for every x in the interval. The linear combination $y = c_1Y_1 + c_2Y_2$ would be a general solution of the equation on the interval, say, $]0, \infty[$.

5. i) $y_p = Ae^{2x} + (B + Cx)e^{-x}$

ii) $y_p = Ae^x + (Bx + C)e^{-x} + xe^{-x}(D\cos x + E\sin x)$

iii) $y_p = x(A_0x + A_1) + B\cos x + C\sin x + Exe^{-2x}$

iv) $y_p = Axe^x + Be^x\cos 2x + Ce^x\sin 2x + Exe^x\cos 2x + Fxe^x\sin 2x$

6. i) The given equation can be written as

$$y'' - 4y' + 4y = e^{2x}/x^2$$

$$y_c = c_1e^{2x} + c_2xe^{2x}$$

$$y_p = V_1e^{2x} + V_2xe^{2x}$$

$$V_1 = \int \frac{(-xe^{2x})e^{2x}}{x^2e^{4x}}dx = -\int \frac{dx}{x} = -\ln x$$

$$V_2 = \int \frac{e^{2x}.e^{2x}}{x^2e^{4x}}dx = \int \frac{1}{x^2}dx = -\frac{1}{x}$$

$$y = c_1e^{2x} + c_2xe^{2x} - e^{2x}\ln x - e^{2x}$$

ii) $y_c = Ae^x + Be^{-x}$

$$y_p = V_1(x)e^x + V_2(x)e^{-x}$$

$$V_1(x) = \int \frac{1}{2}\frac{2}{1+e^x}e^{-x}dx = \int \frac{dx}{e^x(1+e^x)} = -e^{-x} + \ln(1+e^{-x})$$

$$V_2(x) = \int \frac{e^x}{1+e^x}dx = \ln(1+e^x)$$

$$y = y_c + y_p$$

iii) $y = c_1e^{-x} + c_2xe^{-x} + \dfrac{x^4e^{-x}}{12}$

iv) $y = c_1\cos x + c_2\sin x - x^2\cos x + x\sin x$

v) $y = (c_1 + c_2x)e^{-x} + \dfrac{1}{4}x^2e^{-x}(2\ln x - 3)$

7. i) Associated homogeneous equation is

$$y'' + y = 0$$

$y_1 = \sin x$ is a solution of this equation

Take $y = V(x)\sin x$

The given differential equation reduces to

$$V''\sin x + 2V'\cos x - V\sin x + V\sin x = \operatorname{cosec} x$$

$$\Rightarrow \quad V''\sin x + 2V'\cos x = \operatorname{cosec} x$$

$$\Rightarrow \quad V'' + 2V'\cot x = \operatorname{cosec}^2 x$$

Put $w = V'$

$$w' + 2w\cot x = \operatorname{cosec}^2 x \text{ (linear equation)}$$

$$\therefore \quad we^{\int 2\cot x dx} = \int e^{\int 2\cot x dx} \operatorname{cosec}^2 x dx + c_1$$

$$\Rightarrow \quad w\sin^2 x = x + c_1$$

$$\Rightarrow \quad V' = x\operatorname{cosec}^2 x + c_1 \operatorname{cosec}^2 x$$

$$\Rightarrow \quad V = -\cot x + \ln|\sin x| + c_1(-\cot x) + c_2$$

$$\Rightarrow \quad y = -x\cos x - c_1\cos x + c_2\sin x + \sin x \ln|\sin x|$$

ii) $y_1 = e^x$ is a solution of the given equation

Take $y = Ve^x$, given equation reduces to

$$V'' + 2V' = 1$$

$$\Rightarrow \quad w' + 2w = 1 \text{ (Putting } w = V')$$

$$\Rightarrow \quad we^{\int 2dx} = \int e^{\int 2dx} dx + c_1$$

$$\Rightarrow \quad we^{2x} = \frac{e^{2x}}{2} + c_1$$

$$\Rightarrow \quad V' = \frac{1}{2} + c_1 e^{-2x}$$

$$\Rightarrow \quad V = \frac{x}{2} - \frac{c_1}{2}e^{-2x} + c_2$$

$$\Rightarrow \quad y = \frac{xe^x}{2} - \frac{c_1}{2}e^{-x} + c_2 e^x$$

iii) $y = c_1(x^2 + x^3) + c_2 x^2$

iv) $y = c_1 + \frac{c_2}{2}[\tan x \sec x + \ln|\sec x + \tan x|]$

v) $y = c_1 e^x + c_2 e^{2x} + \frac{5}{2}e^{3x}$

8. i) Dividing the given equation by x, we get

$$x^3 y''' + 2x^2 y'' - xy' + y = x + x^{-1}$$

which is of the Euler's form

Putting $x = e^z$, this equation reduces to

$$[D(D-1)(D-2) + 2D(D-1) - D + 1]y = e^z + e^{-z} \left[D = \frac{d}{dz} \right]$$

$$\Rightarrow \quad (D-1)^2(D+1)y = e^z + e^{-z}$$

The auxiliary equation is $(m-1)^2(m+1) = 0 \Rightarrow m = 1, 1, -1$

$$\therefore \text{ C.F.} = (c_1 + c_2 z)e^z + c_3 e^{-z}$$
$$= (c_1 + c_2 \ln x)x + c_3 x^{-1}$$

$$\text{P.I.} = \frac{1}{(D-1)^2(D+1)}(e^z + e^{-z})$$
$$= \frac{1}{2}\frac{1}{(D-1)^2}e^z + \frac{1}{4}\frac{1}{D+1}e^{-z}$$
$$= \frac{e^z}{2}\frac{1}{D^2}(1) + \frac{e^{-z}}{4}\frac{1}{D}(1) = \frac{z^2 e^z}{4} + \frac{3e^{-z}}{4} = \frac{2(\ln x)^2}{4} + \frac{x^{-1}\ln x}{4}$$
$$\therefore \quad y = \text{C.F} + \text{P.I}$$

ii) $yy'' + (y')^2 - 2y(y')^3 = 0$

Put $\frac{dy}{dx} = p \Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$

$$\therefore \quad yp\frac{dp}{dy} + p^2 - 2yp^3 = 0 \Rightarrow \frac{dp}{dy} + \frac{p}{y} = 2p^2$$

Put $\frac{1}{p} = w \Rightarrow -\frac{1}{p^2}\frac{dp}{dy} = \frac{dw}{dy}$

Then $\frac{1}{p^2}\frac{dp}{dy} + \frac{1}{py} = 2 \Rightarrow \frac{dw}{dy} - \frac{w}{y} = -2$

$$\Rightarrow \quad we^{-\int\frac{dy}{y}} = \int e^{-\int\frac{dy}{y}}(-2)dy + c_1 \Rightarrow \frac{w}{y} = -2\int\frac{dy}{y} + c_1$$
$$\Rightarrow \quad w = -2y\ln y + c_1 y \Rightarrow \frac{dx}{dy} = -2y\ln y + c_1 y$$
$$\Rightarrow \quad x = -2\int y\ln y dy + c_1\int y\, dy$$
$$\Rightarrow \quad x = -2\left(y^2\frac{\ln y}{2} + \frac{y^2}{4}\right) + c_1\frac{y^2}{2}$$
$$\Rightarrow \quad x = y^2\left(\frac{c_1+1}{2}\right) - y^2\ln y$$

iii) $y'' - 2y' + y = xe^{2x}\sin x$

$\text{C.F.} = c_1 e^x + xc_2 e^x$

$$\text{P.I.} = \frac{1}{(D-1)^2}xe^x\sin x = e^x\frac{1}{(D+1-1)^2}x\sin x$$
$$= e^x\frac{1}{D^2}x\sin x$$
$$= e^x\frac{1}{D}(-x\cos x + \sin x) = (-x\sin x - x\cos x - \cos x)e^x$$

$y = \text{C.F.} + \text{P.I.}$

iv) Putting $\ln(1+x) = z$ in the given equation, we get

$$(1+x)\frac{dy}{dx} = \frac{dy}{dz},\ (1+x)^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

The given equation reduces to

$$\frac{d^2y}{dz^2} + y = 4\cos z$$

$$y_c = c_1 \cos z + c_2 \sin z$$

$$y_p = 4\frac{1}{D^2+1}\cos z = 2z \sin z$$

$$y = c_1 \cos(\ln(1+x)) + c_2 \sin(l(1+x)) + 2\ln(1+x)\sin(\ln(1+x))$$

v) $x^2 y'' - 2xy' - 4y = x^2 + 2\ln x,\ x > 0$

Putting $x = e^z$, equation reduced to

$$(D^2 - 3D - 4)y = e^{2z} + 2z$$

C.F. $= c_1 e^{-z} + c_2 e^{4z} = c_1 \frac{1}{x} + c_2 x^4$

P.I. $= \frac{3}{8} - \frac{1}{2}\ln x - \frac{1}{6}x^2$

vi) $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Put $\frac{dy}{dx} = p$, then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$. Equation reduces to $x\frac{dp}{dx} + p = 0$

$\Rightarrow \frac{dp}{dx} + \frac{1}{x}p = 0$

$\Rightarrow p = \frac{dy}{dx} = c_1 e^{-\ln x} = \frac{c_1}{x}$

$\Rightarrow y = c_1 \ln x + c_2$

9. Equation of motion using Newton's second law is

$m\frac{d^2x}{dt^2} = pg$ where at $t = 0,\ x = 0,\ \frac{dx}{dt} = 0$

Initial position of m is taken as origin.
Integrating the equation of motion

$$\frac{dx}{dt} = \frac{pg}{m}t + c$$

at $t = 0,\ \frac{dx}{dt} = 0 \Rightarrow c = 0$

$\therefore\ \frac{dx}{dt} = \frac{pg}{m}t$

Again integrating, we obtain

$$x = \frac{pg}{m}\frac{t^2}{2} + c_1$$

at $t = 0,\ x = 0 \Rightarrow c_1 = 0$, therefore $x = \frac{pgt^2}{2m}$

in terms of $v = \frac{dx}{dt}$, equation of motion can be written as

$mv\frac{dv}{dx} = pg$ at $t = 0,\ v = 0$

Integrating, we get $mv^2 = 2pgx + c_2$

$$t = 0,\ v = 0 \Rightarrow c_2 = 0$$

$\therefore\ x = \frac{mv^2}{2pg} = \frac{pgt^2}{2m}$.

10. The equation of motion is

$$\frac{d^2x}{dt^2}+\frac{a}{m}\frac{dx}{dt}+\frac{k}{m}x=\frac{F(t)}{m}$$

where m is the mass, k is the spring constant, $F(t)$ is the external force and a is damping constant.

$F(t)=0$, since there is no external force.

No resistance from air $\Rightarrow a=0$

Also $m=\frac{w}{g}=\frac{39.2}{9.8}=4$

and $w=39.2=2k \Rightarrow k=19.6$

Thus the equation of motion reduces to

$$\frac{d^2x}{dt^2}+4.9x=0$$

$$\Rightarrow \quad x(t)=c_1\cos\sqrt{4.9}\,t+c_2\sin\sqrt{4.9}\,t$$

$x(0)=-\frac{1}{2}\Rightarrow c_1=-\frac{1}{2}$ and $\left.\frac{dx}{dt}\right|_{t=0}=0\Rightarrow c_2=0$

$$\therefore \quad x(t)=-\frac{1}{2}\cos\sqrt{4.9}\,t$$

11. $\frac{d^2y}{dx^2}-\frac{P}{EI}=0\Rightarrow \text{C.F}=c_1e^{nx}+c_2e^{-nx},\ n^2=\frac{P}{EI}$

$$\text{P.I.}=\frac{1}{D^2-n^2}\left(\frac{-W}{2EI}x^2\right)=\frac{W}{2n^2EI}\left(1-\frac{D^2}{n^2}\right)^{-1}x^2=\frac{W}{2n^2EI}\left(x^2+\frac{2}{n^2}\right)$$

$$\therefore \quad y=c_1e^{nx}+c_2e^{-nx}+\frac{W}{2n^2EI}\left(x^2+\frac{2}{n^2}\right),\ n^2=\frac{P}{EI}$$

Using the condition $\frac{dy}{dx}=0$ at $x=0$, we get $c_1=c_2$

Using $x=0,\ y=0$, we get $c_1=c_2=\frac{-W}{2n^2P},\ P=n^2EI$

$$\therefore \quad y=\frac{-W}{2n^2P}(e^{nx}+e^{-nx})+\frac{W}{2P}\left(x^2+\frac{2}{n^2}\right)$$

$$=\frac{W}{n^2P}(1-\cosh nx)+\frac{Wx^2}{2P},\ n^2=\frac{P}{EI}$$

12. The governing equation is

$$1.\frac{d^2q}{dt^2}+12\frac{dq}{dt}+\frac{1}{0.01}q=24\sin 10t$$

or $$\frac{d^2q}{dt^2}+12\frac{dq}{dt}+100q=24\sin 10t$$

$\text{C.F.}=e^{-6t}(c_1\cos 8t+c_2\sin 8t)$

$$\text{P.I.}=24\frac{1}{D^2+12D+100}\sin 10\,t$$

$$=24\frac{1}{-100+12+100}\sin 10t=24\frac{1}{12D}\sin 10t$$

$$=-\frac{1}{5}\cos 10t$$

$$\therefore \quad q = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) - \frac{1}{5}\cos 10t$$

Using initial conditions $c_1 = \frac{1}{5}$ and $c_2 = \frac{3}{20}$

$$\therefore \quad q = \frac{e^{-6t}}{20}(4\cos 8t + 3\sin 8t) - \frac{1}{5}\cos 10t$$

Current $i = \frac{dq}{dt} = \frac{-5}{2}e^{-6t}\sin 8t + 2\sin 10t$.

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