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BMTC-132
DIFFERENTIAL EQUATIONS

Block

4

FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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BLOCK 4 FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

In Blocks 2 and 3, we confined our attention to the discussion of ODEs. These are the equations involving one dependent and one independent variable. But there exists physical situations for which the governing DE may be simultaneous equation or total differential equation or PDE, involving more than one independent variables or more than one dependent variables. For instance, the diffusion/heat equation governing the flow of electricity in a long insulated cable, wave equation dealing with high-frequency phenomena on a cable etc. are PDEs.

Lagrange (1736-1813) an Italian mathematician, in 1769, provided the method of finding the general solution of linear first order PDEs. He also classified the integrals of first order PDEs, as complete integral, general integral and singular integral. In the case of non-linear PDEs of first order, the complete integral is partly due to Lagrange. Lagrange's results were later perfected by a French mathematician Charpit in 1784.

In Blocks 2 and 3, you must have noticed that the integrals of ODEs are plane curves but in the case of PDEs the solution may be space curves or surfaces. In order to understand the methods of solving these equations and interpret the solutions, the knowledge of space curves and surfaces is essential. Accordingly, we have divided the material in this block into four units.

In Unit 14, which is the first unit of this block, we have started by discussing briefly the two geometrical objects viz., curves and surfaces in space. Parametric representation of some simple curves and surfaces in space, envelopes of one and two parameter family of surfaces, characteristic curves and characteristic points have been introduced in this unit. We have presented in this unit the origin and formation of simultaneous differential equations. The methods of solving simultaneous differential equations have been discussed and illustrated with the help of examples. Some of the applications of simultaneous DEs such as orthogonal trajectories of system of curves on a given surface, particle motion in phase-space and electric circuits are also discussed in this unit.

In Unit 15, we have defined first order total differential equations. We have concentrated mainly on the total DEs in three variables. Given their integrability condition and various methods of solving them, when integrable.

Unit 16 has been devoted to the study of linear first order PDEs. We have begun the unit by giving the origin of first order PDEs. We have discussed and illustrated various situations which give rise to first order PDEs. After classifying the first order PDEs into linear, semi-linear, quasi-linear and non-linear PDEs, we have discussed the various types of solutions/integrals of PDEs of first order and given the relation between these different integrals.

Unit 17, which is the last unit of this block is devoted to the methods of finding solutions of both linear and non-linear PDEs of the first order. Lagrange's method of solving linear PDEs of the first order is discussed and illustrated through various examples. For solving non-linear PDEs of first order we have discussed the Charpit's method. We have also defined here the compatible systems of first order PDEs and obtained the conditions under which two systems are compatible.

NOTATIONS AND SYMBOLS

$D^n y$ where $D = \dfrac{d}{dx}$	:	nth order derivative of y w.r.t. x
$f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-2} D^2 + a_{n-1} D + a_n$	:	nth order polynomial in D or nth order linear differential operator
$W(y_1(x),\ y_2(x),\ \ldots,\ y_n(x))$	:	wronskian of the n functions $y_1(x),\ y_2(x),\ \ldots,\ y_n(x)$
C.F.	:	complementary function
P.I.	:	particular integral
l.h.s.	:	left hand side
r.h.s.	:	right hand side
kg.	:	kilogram
m.	:	meter
sec.	:	second
cm.	:	centimeter
E	:	electromotive force
R	:	resistance
L	:	inductance
C	:	capacitance
q	:	charge
i	:	current

Greek Alphabets

σ	:	Sigma
ζ	:	Zeta
ε	:	Epsilon
τ	:	Tao
π	:	Pi
ω	:	Omega
δ	:	Delta
ξ	:	Xi

UNIT 14

SIMULTANEOUS DIFFERENTIAL EQUATIONS

Structure **Page No.**

14.1 INTRODUCTION

So far in Blocks 2 and 3 you have studied ordinary differential equations, that is, equations involving one independent and one dependent variable. You must have noticed that the solutions of ordinary differential equations are plane curves (ref. Sec. 6.4 of Unit 6). You may recall that in Unit 6 we introduced you to total differential equations i.e., equations of the form $Pdx+Qdy+Rdz=0$. These equations involve one independent and more than one dependent variables. Other equations of similar type are simultaneous differential equations. We shall study simultaneous differential equations in this unit and total differential equations in Unit 15. Partial differential equations of the type $f(x,\ y,\ z,\ z_x,\ z_y)=0$ which involve one dependent and more than one independent variables will be discussed in Unit 16. You will see that the solutions of all these differential equations are space curves and surfaces. For understanding the methods of solving such equations and for interpretation of their solutions, knowledge of curves and surfaces in three dimensional space is essential. Accordingly, we shall start the unit by discussing in Sec. 14.2 the two geometrical objects, viz., curves and surfaces, in space. In Sec. 14.3 we shall take up the formation of simultaneous differential equations and discuss the methods of solving these equations in Sec. 14.4. Finally, in Sec. 14.5 we shall discuss a few interesting applications of simultaneous differential equations in geometry and mathematical physics.

Objectives

After studying this unit, you should be able to:

- identify the equations of space curves;
- identify the equation of a surface and relation between a surface, a curve and a point;
- state the meaning of envelope of a one-parameter and two-parameter family of surfaces, characteristic curve and characteristic point;
- describe the origin of simultaneous differential equations;
- state that the solution set of simultaneous differential equations is a two parameter family of space curves;
- use various methods of solving simultaneous differential equations; and
- find the orthogonal trajectories of a system of curves on a given surface.

14.2 CURVES AND SURFACES IN SPACE

We shall discuss in this section briefly the two geometrical objects viz., curves and surfaces.

Let us first consider curves in space.

Curves in Space

We start by considering some simple examples of space curves.

The simplest curve in space is a straight line. The equation of a straight line passing through a given point $P_0(x_0,\ y_0,\ z_0)$ and making angles $\alpha,\ \beta,\ \gamma$ with OX, OY, OZ axes respectively, is (see Fig. 1).

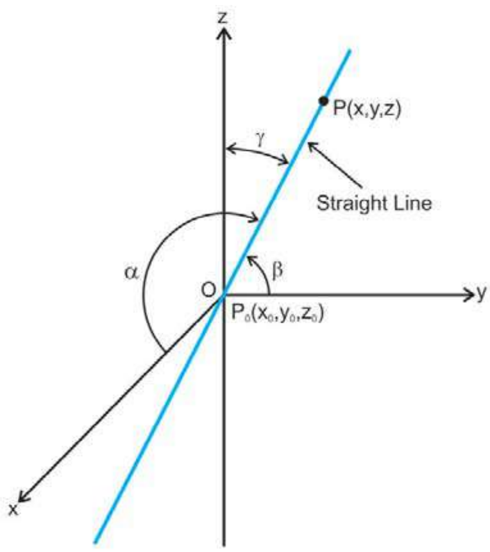

Fig. 1: The Straight Line.

$$\frac{x-x_0}{\cos\alpha}=\frac{y-y_0}{\cos\beta}=\frac{z-z_0}{\cos\gamma}=s,\ s=\widehat{P_0P} \qquad (1)$$

where $s=\widehat{P_0P}$ is the distance P_0P of P from fixed point P_0 measured along the straight line.
Therefore,

$$\left.\begin{aligned} x&=x_0+s\cos\alpha\\ y&=y_0+s\cos\beta\\ z&=z_0+s\cos\gamma \end{aligned}\right\},\ -\infty<s<\infty . \qquad (2)$$

are the coordinates of the point $P(x,\ y,\ z)$ on the straight line (1) at a distance s from $P_0(x_0,\ y_0,\ z_0)$. They are expressed in terms of a single parameter s. By varying s, we obtain different points on the line. On differentiating Eqn. (2), we obtain

$$\frac{dx}{ds}=\cos\alpha,\ \frac{dy}{ds}=\cos\beta,\ \frac{dz}{ds}=\cos\gamma$$

or,

$$\frac{dx}{\cos\alpha}=\frac{dy}{\cos\beta}=\frac{dz}{\cos\gamma}=ds \qquad (3)$$

Eqns. (3) are simultaneous differential equations which define the straight line. We shall discuss simultaneous differential equations in detail later in this unit.

We next consider another curve in space. Let the parametric equations of the curve be

$$\left.\begin{aligned} x&=a\cos\omega t,\\ y&=a\sin\omega t,\\ z&=Wt \end{aligned}\right\}\ -\infty<t<\infty . \qquad (4)$$

where a, ω and W are constants.
Then the point $P(x,\ y,\ z)$ describes a space curve called the **helix** on the surface $x^2+y^2=a^2$, (obtained by eliminating t from $x=a\cos\omega t,\ y=a\sin\omega t$) of a circular cylinder of radius a (see Fig. 2).

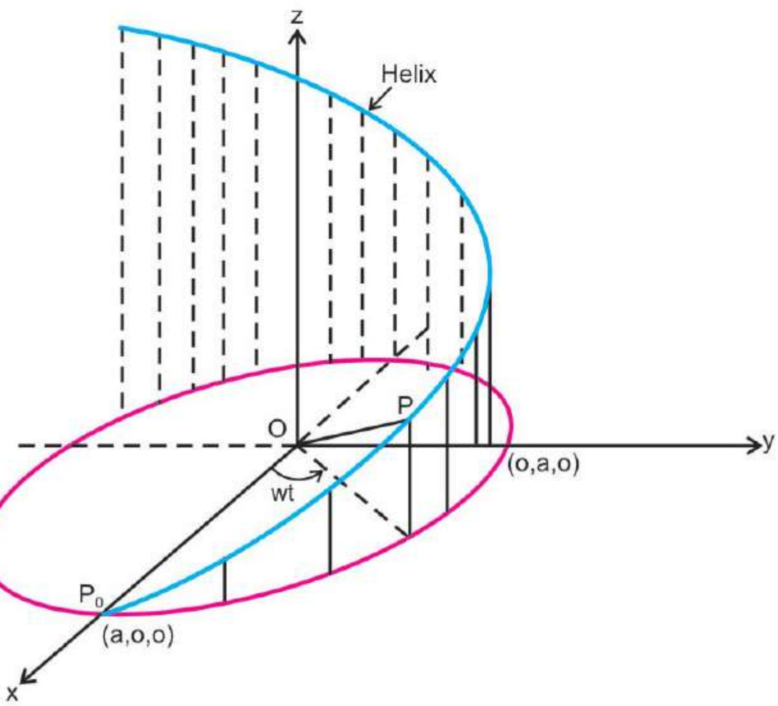

Fig. 2: The Helix.

The pitch of a helix is the height of one complete helix turn, measured parallel to the axis of the helix.

Let us call the point P as P_0 when $t = 0$, then the coordinates of the point P_0 are $(a, 0, 0)$. Then at any time t, we get the length of the arc $s = \widehat{P_0P}$ along the curve as

$$\int_0^s ds = \int_{P_0}^{P} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right]^{1/2} dt$$

or

$$s = \int_{P_0}^{P} (a^2\omega^2 \sin^2 \omega t + a^2\omega^2 \cos^2 \omega t + W^2)^{1/2} dt = \int_0^t (a^2\omega^2 + W^2)^{1/2} dt$$

$$= (a^2\omega^2 + W^2)^{1/2} t$$

If the parameter t is interpreted as time, then in time $t = 2\pi / \omega$, the point P describes a complete circle of radius a and also moves parallel to OZ a distance $2\pi W / \omega$, called the **pitch** of the helix. From Eqn. (4) you can see that the differential equations of the helix are the simultaneous differential equations

$$\frac{dx}{-\omega y} = \frac{dy}{\omega x} = \frac{dz}{W} = dt .$$

The examples considered above suggest that we may take the parametric equations of a space curve as

$$\left.\begin{aligned} x &= \phi(t), \\ y &= \psi(t), \\ z &= \theta(t), \end{aligned}\right\} t_1 \le t \le t_2, \tag{5}$$

where ϕ, ψ, θ, are functions having continuous derivatives w.r.t. t. We eliminate t from the first two to get

$$C_1 : f_1(x, y) = 0$$

Similarly, from the last two equations, we get

$$C_2 : f_2(y, z) = 0 .$$

Hence, C_1 and C_2 are the equations of a cylinder which intersect in the space curve given by Eqn. (5).

For instance, if S is the sphere with equation $x^2 + y^2 + z^2 = a^2$, then the points of S in the plane $z = k$ have

$$z = k,\ x^2 + y^2 = a^2 - k^2$$

and lies on a circle of radius $\sqrt{(a^2 - k^2)}$ if $k < a$. Corresponding to each point of the sphere is one such circle with k varying between $-a$ and $+a$ (see Fig. 3). The surface of the sphere can thus be considered as being generated by such circles.

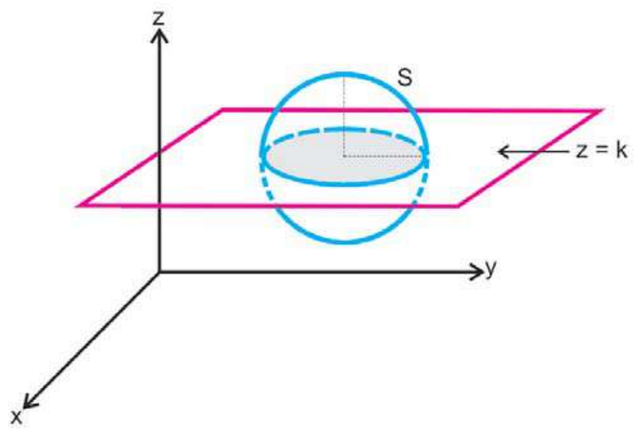

Fig. 3

Thus a curve in space can be interpreted as intersection of a surface and a plane.

We next consider surfaces in 3-dimensions.

Surfaces in Space

The simplest example of a 3-dimensional surface is the plane. The general equation of a plane is

$$ax + by + cz + d = 0$$

where a, b, c, d are given constants.
In above equation we can choose x, y arbitrarily; let us say that

$$x = u,\ y = v,\ -\infty < u,\ v < \infty$$

and then

$$z = -(au + bv + d)/c,\ c \neq 0\,.$$

Thus, we have expressed the Cartesian coordinates of any point on the plane in terms of **two parameters**, u and v.
Also you know that a sphere is the locus of a point $P(x,\ y,\ z)$ whose distance from the centre of the sphere, say $O(0,0,\ 0)$ is equal to the radius a. Thus,

$$(OP)^2 = a^2 \Rightarrow x^2 + y^2 + z^2 = a^2 \qquad (6)$$

From your knowledge of Unit 1 you know the relationship between $(x,\ y,\ z)$, the cartesian coordinate of a point and its spherical polar coordinates $(r,\ \theta,\ \phi)$. When $r = a$, then we can write

$$\left.\begin{aligned} x &= a\sin\theta\cos\phi \\ y &= a\sin\theta\sin\phi \\ z &= a\cos\theta \end{aligned}\right\} \qquad (7)$$

where $0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi,\ 0 \le a < \infty$.
Eqn. (7) is the equation of the sphere in the parametric form.

Again in this case you may note that the cartesian coordinates of any point on the surface which is sphere in this case are expressed in terms of **two parameters**, θ and ϕ. Eqn. (7) gives the relation connecting these coordinates.
We have,

$$\begin{aligned} x^2 + y^2 + z^2 &= a^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + a^2\cos^2\theta \\ &= a^2(\sin^2\theta + \cos^2\theta) \\ &= a^2 \end{aligned}$$

Also, the first two equations in (7) when solved for parameter θ and ϕ in terms of x and y, yield

$$\theta = \sin^{-1}\frac{\sqrt{x^2 + y^2}}{a},\ \phi = \sin^{-1}\frac{y}{\sqrt{x^2 + y^2}}$$

The value of θ obtained above, when substituted in the third equation of (7) gives

$$\begin{aligned} z = a\cos\theta &= a\frac{\sqrt{a^2 - x^2 - y^2}}{a} \\ &= \sqrt{a^2 - x^2 - y^2} \end{aligned}$$

or, $$z^2 = a^2 - x^2 - y^2.$$

Thus, from the above two special cases, it is seen that the equation of a surface is a relation connecting coordinates (x, y, z) of a point in 3-dimensional space.

We, now, give the formal definition of a surface.

Definition: If the Cartesian coordinates (x, y, z) of a point in a 3-dimensional space are connected by a single relation of the type

$$f(x, y, z) = 0 \tag{8}$$

then the collection of all such points is said to determine a **surface**.
Eqn. (8) is the equation of a surface.
In some simple cases, that is, when Eqn. (8) is solvable for z, we can write it as

$$z = F(x, y)$$

By looking at the examples considered above, we can say that the parametric equations of a surface are of the form

$$\left.\begin{aligned} x &= \phi(u, v) \\ y &= \psi(u, v) \\ z &= \theta(u, v) \end{aligned}\right\} \tag{9}$$

where u and v are **two parameters**, $u_1 \le u \le u_2$ and $v_1 \le v \le v_2$.

Thus the expression (9) when substituted in Eqn. (8) must reduce it to an identity. Also we can solve the first two equations in expression (9) and get

$$u = F_1(x, y), v = F_2(x, y)$$

and these, when substituted in the third equation of (9) will give us the equation of the surface as

$$z = F(x, y).$$

In general, if a point P with coordinates (x, y, z) lies on a surface S_1, then the relation of the form $f_1(x, y, z) = 0$ exists. Further, if P also lies on a surface S_2, then we will have another relation of the same type, say $f_2(x, y, z) = 0$. Therefore, the points common to S_1 and S_2 will satisfy a pair of equations

$$f_1(x, y, z) = 0, f_2(x, y, z) = 0 \tag{10}$$

The two surfaces S_1 and S_2 intersect in a curve. The locus of a point whose coordinates satisfy a pair of relations of the type (10) will thus be a curve in space. Fig. 4 shows the intersection of two spheres of different radii which is a circle.

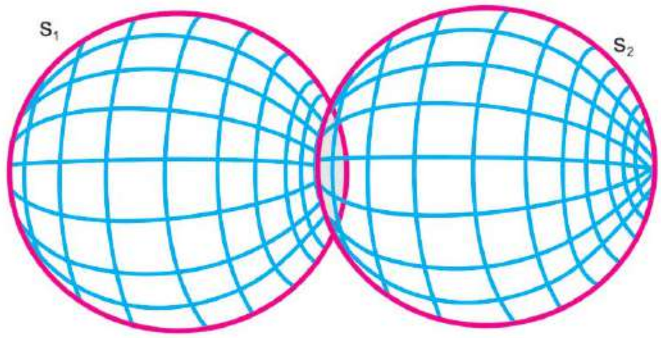

Fig. 4

Thus a curve in space can also be interpreted as intersection of two surfaces.

Remember that the parametric equations of a surface are not unique. As an illustration, you may see that both the set of parametric equations

$$x = a\sin u\cos v,\ y = a\sin u\sin v,\ z = a\cos u$$

and

$$x = a\frac{1-v^2}{1+v^2}\cos u,\ y = a\frac{1-v^2}{1+v^2}\sin u,\ z = \frac{2av}{1+v^2}$$

yield the spherical surface

$$x^2 + y^2 + z^2 = a^2.$$

We now take up another example to illustrate this fact.

Example 1: Find the parametric equation of the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ -\infty < z < \infty.$$

Solution: We can rewrite the given equation of the surface in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{z^2}{c^2},\ -\infty < z < \infty.$$

The intersection of this surface with the plane $z = k$, a constant, is the ellipse

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1, \text{ with } a' = a\sqrt{1+\frac{k^2}{c^2}},\ \ b' = b\sqrt{1+\frac{k^2}{c^2}}$$

You can see here that if we take $x = a'\cos\theta$ and $y = b'\sin\theta$ then

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=\frac{a'^2\cos^2\theta}{a'^2}+\frac{b'^2\sin^2\theta}{b'^2}=1.$$

Thus we can write the parametric equations of the given surface as

$$x = a'\cos\theta,\ y = b'\sin\theta,\ z = k, \text{ where } a' = a\sqrt{1+\frac{k^2}{c^2}} \text{ and } b' = b\sqrt{1+\frac{k^2}{c^2}}.$$

As the domain of z is $-\infty < z < \infty$, on setting $z = c\sinh\alpha$, we get another set of parametric equations as

$$x = a\cosh\alpha\cos\theta,$$

$$y = b\cosh\alpha\sin\theta$$

$$z = c\sinh\alpha,$$

where $0 \le \theta \le 2\pi,\ -\infty < \alpha < \infty$.

Note that the surface given in Example 1 is called an **elliptic hyperboloid of one-sheet**.

***

You may now try the following exercise.

E1) i) Find the parametric equations of the surface

$$x^2 + y^2 = 2z,\ -\infty < z < \infty$$

ii) Find the Cartesian equation of the surface

$$x = u\cos v,\ y = u\sin v,\ z = u\cot v.$$

So far, we have considered only a surface in space. We next take up family of surfaces.

Family of Surfaces

A one-parameter family of surfaces in three-dimensional Euclidean space is given by the equation

$$f(x,\ y,\ z,\ c) = 0 \tag{11}$$

where the function f has continuous first order partial derivatives w.r.t. $x,\ y,\ z \in D,\ a$ domain in space, and c being a parameter. For different values of c we get different members of the family. For example, the tangent planes to a surface along a curve in the surface form such a family.
Let us consider a member of the family (11) for a prescribed value of c and also consider another member corresponding to the slightly different value $c + \delta c$, having the equation

$$f(x,\ y,\ z,\ c + \delta c) = 0 \tag{12}$$

The two surfaces (11) and (12) will intersect in a curve represented by the equations

$$f(x,\ y,\ z,\ c) = 0,\ f(x,\ y,\ z,\ c + \delta c) = 0$$

This curve may also be considered to be the intersection of the surfaces with equations

$$f(x,\ y,\ z,\ c) = 0 \text{ and } \frac{1}{\delta c}\{f(x,\ y,\ z,\ c + \delta c) - f(x,\ y,\ z,\ c)\} = 0 \tag{13}$$

As the parameter difference δc tends to zero, the curve of intersection tends to a limiting position given by the equations

$$f(x, y, z,\ c) = 0,\ \frac{\partial}{\partial c} f(x,\ y,\ z, c) = 0 \tag{14}$$

We call this limiting curve as the **characteristic curve** of the family and it lies on the surface (11). As the parameter c varies, the characteristic curve (14) traces out a surface, whose equation is obtained by eliminating the parameter c between the two Eqns. (14) in the form

$$g(x,\ y,\ z) = 0$$

We call this surface to be the **envelope of the one-parameter family** given by Eqn. (11).

We shall now show that the envelope of the family of spheres of unit radius with centres on the z-axis is the cylinder.

Example 2: Find the envelope of the family of spheres $x^2 + y^2 + (z - c)^2 = 1$, c being the parameter.

Solution: Let $f = x^2 + y^2 + (z - c)^2 - 1 = 0$

$$\therefore\ \frac{\partial f}{\partial c} = -2(z - c) = 0 \Rightarrow z = c$$

Eliminating c from $f = 0$ and $\dfrac{\partial f}{\partial c} = 0$, we get the envelope of the family of spheres as

$$x^2 + y^2 = 1$$

which is a cylinder with base in xy-plane, the centre $(0, 0)$ and radius 1 (see Fig. 5).

The characteristic curve is given by

$$z = c \text{ and } x^2 + y^2 + (z-c)^2 = 1.$$

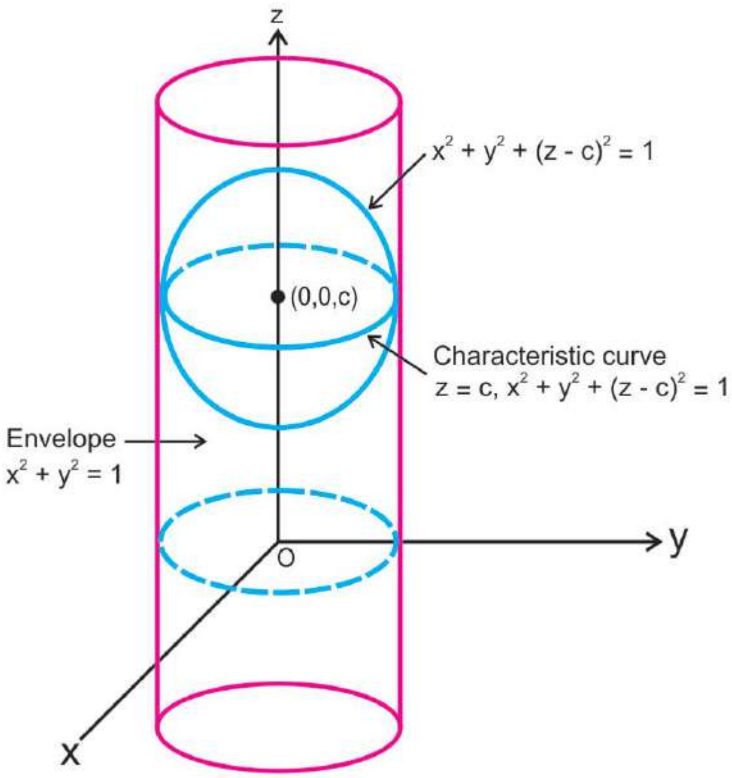

Fig. 5

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You may now try this exercise.

E2) Find the envelope of the family of spheres
$x^2 + y^2 + (z-c)^2 = c^2 \sin^2 \alpha$, c being the parameter..

Next, we consider a **two parameter family of surfaces** defined by the equation

$$f(x, y, z, a, b) = 0 \tag{15}$$

where f is a function having continuous first order partial derivatives w.r.t. $x, y, z \in D$, a domain in the 3-dimensional space and a and b are parameters. From Eqn. (15) we can obtain a one-parameter family of surfaces by taking b as some definite function of a, say

$$b = \phi(a) \tag{16}$$

We can then obtain the **envelope** of this one-parameter family by eliminating a and b from Eqns. (15) and (16) and the relation

$$\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\frac{db}{da} = 0 \tag{17}$$

The **characteristic curve** of this one-parameter family is then given by the Eqns. (15) and (17) in which $b = \phi(a)$.

You may **note** here that for every choice of $\phi(a)$, the characteristic curve of the one-parameter family passes through the point defined by the equations

$$f(x,\ y,\ z,\ a,\ b) = 0,\ f_a = 0,\ f_b = 0 \tag{18}$$

This point is the **characteristic point** of the two-parameter family (18) on the particular surface of the family. As we vary the parameters a and b the characteristic point generates a surface which we call the **envelope** of the two-parameter family of surfaces (15). Its equation is obtained by eliminating a and b from the three Eqns. (18).

We now take up an example to illustrate the theory outlined above.

Example 3: Find the envelope of the two-parameter family of planes $z = ax + by + a^2 + b^2$.

Solution: Let $f(x,\ y,\ z,\ a,b) = z - ax - by - a^2 - b^2 = 0$ (19)

$\therefore\ \ f_a = 0 \Rightarrow x + 2a = 0$

$f_b = 0 \Rightarrow y + 2b = 0$

Eliminating a and b from $f_a = 0,\ f_b = 0$ and Eqn. (19), we obtain the envelope as

$$4z = -(x^2 + y^2) \tag{20}$$

which is a paraboloid of revolution. The characteristic point is $(-2a,\ -2b,\ -(a^2 + b^2))$. Next, we take $a^2 + b^2 = 1$. Substituting in Eqn. (19) we obtain a one-parameter family of planes

$$z = ax \pm y\sqrt{1 - a^2} + 1 \tag{21}$$

whose envelope is the right circular cone

$$(z-1)^2 = x^2 + y^2 \tag{22}$$

It is easy to verify that the characteristic point for $a^2 + b^2 = 1$ is $\left(-2a,\ \pm 2\sqrt{1-a^2},\ -1\right)$ and this point lies on both the surfaces (21) and (22).

***

And now an exercise for you.

E3) Find the envelope of the two-parameter family of spheres $(x-a)^2 + (y-b)^2 + z^2 = 1$. Also obtain the equation of the characteristic curve of the one-parameter family, $(x-a)^2 + (y-a)^2 + z^2 = 1$.

With the above background of curves and surfaces in space we now move on to the study of simultaneous differential equations. Earlier you saw that the equation of a straight line in 3D-coordinate system can be represented as Eqn. (3) which are simultaneous differential equations. You can see that ordinary differential equations of order two and above can be expressed as a system of simultaneous differential equations. As a simple example consider the following second order equation

$$\frac{d^2x}{dt^2} - 7\frac{dx}{dt} + 9 = 0$$

or, $\dfrac{d^2x}{dt^2} = 7\dfrac{dx}{dt} - 9 = f\left(t,\, x,\, \dfrac{dx}{dt}\right)$, say.

The above equation can be expressed as a system of two first order differential equations, viz.,

$$\frac{dx}{dt} = y,\ \frac{dy}{dt} = f$$

or, $$\frac{dx}{y} = \frac{dy}{f} = dt \tag{23}$$

where y is a function of t.

Similarly, an n^{th} order differential equation of the form

$$\frac{d^n x}{dt^n} = f\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \cdots, \frac{d^{n-1}x}{dt^{n-1}}\right) \tag{24}$$

can be expressed as a system of n first-order differential equations, viz.,

$$\frac{dx}{dt} = y_1,\ \frac{dy_1}{dt} = y_2,\ \cdots,\ \frac{dy_{n-2}}{dt} = y_{n-1},$$

$$\frac{dy_{n-1}}{dt} = f(x, t, y_1, \ldots, y_{n-1}) \tag{25}$$

or,

$$\frac{dx}{y_1} = \frac{dy_1}{y_2} = \frac{dy_2}{y_3} = \cdots = \frac{dy_{n-1}}{f} = dt \tag{26}$$

Eqns. (23) and (26) are system of **simultaneous differential equations** of the first order. These equations arise frequently in mathematical physics. For instance, equations of the type (26) arise in the general theory of radioactive transformations as discovered by Rutherford and Soddy (1930). In analytical mechanics, the equations of motion of a dynamical system of n degree of freedom are

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},\ \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i},\ i = 1, 2, \ldots, n \tag{27}$$

where $H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, t)$ is the Hamiltonian function and $q_i,\ p_i (i = 1, 2, \ldots, n)$ are $2n$ unknown functions. Eqns. (27) are a system of $2n$ first order equations, the solution of which provides a description of the properties of the dynamical system at any time t.

In the succeeding units of this block, you will observe that quasi-linear and non-linear partial differential equations, also give rise to simultaneous differential equations. We shall now take up, in the next section, the formation of simultaneous differential equations.

14.3 FORMATION OF SIMULATNEOUS DIFFERENTIAL EQUATIONS

Let us consider two families of surfaces

$$u(x, y, z) = c_1,\ v(x, y, z) = c_2 \tag{28}$$

c_1 and c_2 being the parameters.

You know that these surfaces intersect in a two-parameter family of space curves. Also, along any curve of the family, $du = 0$ and $dv = 0$.

Now,

$$du = 0 \Rightarrow u_x dx + u_y dy + u_z dz = 0 \tag{29}$$

$$dv = 0 \Rightarrow v_x dx + v_y dy + v_z dz = 0 \tag{30}$$

Solving Eqns. (29) and (30) for dx, dy and dz, we obtain

$$\frac{dx}{u_y v_z - u_z v_y} = \frac{dy}{u_z v_x - u_x v_z} = \frac{dz}{u_x v_y - u_y v_x}$$

or,

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)}, \tag{31}$$

where P, Q, R are known functions of x, y, z.

Eqns. (31) are the simultaneous differential equations of the two-parameter family of space curves in which two families of surface $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ intersect.

We now illustrate the formation of simultaneous differential equations with the help of examples.

Example 4: Find the differential equations of the space curves in which the two families of surfaces

$$u = x^2 + y^2 + z^2 = c_1 \text{ and } v = x + z = c_2 \tag{32}$$

intersect.

Solution: Here the given families of surfaces are

$$u = x^2 + y^2 + z^2 = c_1,\ v = x + z = c_2$$

Along any curve of the family, we have $du = 0$ and $dv = 0$.

$$du = 0 \Rightarrow 2xdx + 2ydy + 2zdz = 0 \tag{33}$$

and $dv = 0 \Rightarrow dx + dz = 0$ (34)

Solving Eqns. (33) and (34), we get

$$\frac{dx}{2y - 0} = \frac{dy}{2z - 2x} = \frac{dz}{0 - 2y}$$

$$\Rightarrow \quad \frac{dx}{y} = \frac{dy}{z - x} = \frac{dz}{-y},$$

which are the required differential equations of the space curves.

***

Let us look at another example.

Example 5: Find the differential equation of the space curves in which the families of surfaces $u = xy = c_1$ and $v = x^4 - z^4 - 2xyz^2 = c_2$ intersect.

Solution: The given families of surfaces are

$$u = xy = c_1 \text{ and } v = x^4 - z^4 - 2xyz^2 = c_2$$

$$du = 0 \Rightarrow ydx + dy = 0$$

$$dv = 0 \Rightarrow 4x^3dx - 4z^3dz - 2yz^2dx - 2xz^2dy - 4xyzdz = 0$$

Solving the above two equations, we get

$$\frac{dx}{-4x(z^3+xyz)} = \frac{dy}{4zy(z^2+xy)} = \frac{dz}{-2xyz^2 - 4x^4 + 2xyz^2}$$

$$\Rightarrow \quad \frac{dx}{xz(z^2+xy)} = \frac{dy}{-zy(z^2+xy)} = \frac{dz}{x^4}$$

as the required differential equations of the space curves.

***

You may now try the following exercise.

E4) Find the differential equations of the space curves in which the following two families of surfaces $u = c_1$ and $v = c_2$ intersect.

i) $u = 3x + 4y + z, \ v = x + z$.

ii) $u = x^2 + y^2, \ v = 3x + 4z$.

iii) $u = xy, \ v = z(x+y) + x^2 + y^2$.

From the formation of simultaneous differential equations it is evident that the solution set of simultaneous equations of the type (31) is a two-parameter family of space curves obtained as an intersection of two one-parameter families of surfaces. But then how to find this solution? In the next section, we discuss the methods of finding the solution of the equations of form (31).

14.4 METHODS OF SOLUTION OF $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

As you have seen above, the curves of intersection of family of surfaces, given by Eqn. (28), namely,

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$

are defined by the system of simultaneous differential Eqns. (31) i.e.,

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)}$$

where

$P(x,y,z) = u_y v_z - u_z v_y, Q(x,y,z) = u_z v_x - u_x v_z$ and $R(x,y,z) = u_x v_y - u_y v_x$

Thus, in order to find the solution of Eqns. (31), we need to derive from it two relations of the form (28) involving two arbitrary constants c_1 and c_2. By varying these constants, we can then arrive at a two-parameter family of curves satisfying Eqns. (31).

We shall now discuss the methods of finding the surfaces of the type (28) starting with Eqns. (31) for which the functions P, Q, R are known. We start by considering some simple situations when the solution of Eqns. (31) can be obtained easily.

The simplest is the case when by equating two fractions out of three in Eqns. (31), it is possible to get an equation in only two variables. For example, consider equations of the form

$$\frac{x\,dx}{y^2z} = \frac{dy}{xz} = \frac{dz}{y^2} \tag{35}$$

By equating the first two fractions we get an equation in only two variables, viz.,

$$\frac{x\,dx}{y^2} = \frac{dy}{x} \text{ or, } x^2dx = y^2dy \tag{36}$$

Eqn. (36) can be solved to obtain the equation

$$x^2 - y^2 = c_1 \tag{37}$$

which gives one of the relations in the complete solution of Eqns. (35). In the same way, taking the first and the third fractions in Eqns. (35), we get

$$\frac{x\,dx}{z} = dz \text{ or } x\,dx = z\,dz$$

which on integration yields

$$x^2 - z^2 = c_2 \tag{38}$$

The two relations given by Eqns. (37) and (38) together constitute the complete solution of Eqns. (35).

Let us look at another example.

Example 6: Find the integral curves of the equations

$$\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}.$$

Solution: Taking the first two fractions of the above equations, we get

$$x^2dx = y^2dy$$

$$\Rightarrow \quad x^3 - y^3 = c_1$$

Again, taking the second and the third fraction, we obtain

$$dy = \frac{dz}{y^2z^2} \text{ or } y^2dy = \frac{1}{z^2}dz$$

$$\Rightarrow \quad \frac{y^3}{3} = -\frac{1}{z} + c$$

or, $y^3 + \frac{3}{z} = c_2$.

Thus, the required integral curves are given by the intersection of the family of surfaces

$$x^3 - y^3 = c_1$$

and $y^3 + \frac{3}{z} = c_2$.

***

Sometimes you may come across the situation when, for given equations, it is comparatively simple to derive one of the sets of surfaces of the solution, but not so easy to derive the second set of surfaces. In such cases you may find the second set of surfaces by using the first solution set of surfaces. To do this, we express one variable in terms of the other two and possibly try to get an equation in two variables which can then be solved to obtain the second set of surfaces. Let us try to understand the method through an example.

Example 7: Solve the simultaneous equations

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}.$$

Solution: Taking the first two fractions, we obtain

$$\frac{dx}{x} = \frac{dy}{-y} \text{ or } \frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating the above equation, we get

$$xy = c_1 \tag{39}$$

Using Eqn. (39), the first and third fractions give

$$x^4 dx = xz(z^2 + c_1)\, dz$$

or, $x^3 dx - (z^3 + c_1 z) dz = 0$

which on integration yields

$$\frac{x^4}{4} - \left(\frac{1}{4} z^4 + \frac{1}{2} c_1 z^2\right) = c$$

or, $x^4 - z^4 - 2c_1 z^2 = c_2$

or, $x^4 - z^4 - 2xyz^2 = c_2 (\because c_1 = xy)$ (40)

Eqns. (39) and (40) constitute the required complete solution.

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In practice, it may not always be the situation as illustrated in Examples 6 and 7. We need to look for other methods of solving Eqns. (31). But before we take up other methods of solving Eqns. (31) you may try the following exercise.

E5) Find the integral curves of the following system of equations.

i) $\dfrac{dx}{y^2} = \dfrac{dy}{y} = \dfrac{dz}{z}$.

ii) $\dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{zxy - 2x^2}$.

iii) $\dfrac{xdx}{y^2 z} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$.

We now discuss the method of multipliers for solving Eqns. (31).

14.4.1 Method of Multipliers

Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be the two one-parameter families of surfaces for the system of Eqns. (31), viz.,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Then along any curve of the family, we have

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

and

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0.$$

To find u and v we try to determine functions (P_1, Q_1, R_1) and (P_2, Q_2, R_2) with the properties

$$P_1 = \frac{\partial u}{\partial x},\ Q_1 = \frac{\partial u}{\partial y},\ R_1 = \frac{\partial u}{\partial z}$$

$$P_2 = \frac{\partial v}{\partial x},\ Q_2 = \frac{\partial v}{\partial y},\ R_2 = \frac{\partial v}{\partial z}$$

such that

$$PP_1 + QQ_1 + RR_1 = 0 \tag{41}$$

$$PP_2 + QQ_2 + RR_2 = 0 \tag{42}$$

From componendo-dividendo rule in algebra, we know that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1 dx + Q_1 dy + R_1 dz}{PP_1 + QQ_1 + RR_1} = \frac{P_2 dx + Q_2 dy + R_2 dz}{PP_2 + QQ_2 + RR_2} \tag{43}$$

Thus, in view of Eqns. (41) and (42), we get from Eqns. (43),

$$P_1 dx + Q_1 dy + R_1 dz = 0 \tag{44}$$

$$P_2 dx + Q_2 dy + R_2 dz = 0 \tag{45}$$

Now, if Eqns. (44) and (45) are exact, then

$$du = P_1 dx + Q_1 dy + R_1 dz = 0$$

$$dv = P_2 dx + Q_2 dy + R_2 dz = 0$$

On integrating these equations, we get the surfaces

$$u(x, y, z) = c_1$$

and

$$v(x, y, z) = c_2$$

The curves of intersection of these surfaces are the integral curves of the system of Eqns. (31).

For better understanding of the method the following examples are in order.

Example 8: Solve the equations

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}.$$

where l, m and n are constants.

Solution: Here $P = mz - ny,\ Q = nx - lz,\ R = ly - mx$

If we take $P_1 = 1,\ Q_1 = m,\ R_1 = n$

and $P_2 = x,\ Q_2 = y,\ R_2 = z$

then

$$PP_1 + QQ_1 + RR_1 = 1(mz - ny) + m(nx - lz) + n(ly - mx)$$

$$= lmz - \ln y + mnx - mlz + \ln y - mnx = 0$$

and

$$PP_2 + QQ_2 + RR_2 = x(mz - ny) + y(nx - lz) + z(ly - mx)$$

$$= mxz - nxy + nxy - lyz + lyz - mxz = 0$$

For the given system, we have

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} = \frac{ldx + mdy + ndz}{0} = \frac{xdx + ydy + zdz}{0}$$

Also,

$$l\,dx + m\,dy + n\,dz = d(lx + my + nz) = du \text{ (say)}$$

and,

$$xdx + ydy + zdz = d\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}\right) = dv \text{ (say)}$$

Therefore the integral curves are given by the intersection of family of surfaces

$$lx + my + nz = c_1$$

and $x^2 + y^2 + z^2 = c_2$.

with c_1, c_2 being arbitrary constants.

***

Example 9: Find the integral curves of the equations

$$\frac{dx}{y(x+y)+\alpha z} = \frac{dy}{x(x+y)-\alpha z} = \frac{dz}{z(x+y)}.$$

Solution: Here, we have

$$P = y(x+y)+\alpha z,\ Q = x(x+y)-\alpha z,\ R = z(x+y)$$

If we take

$$P_1 = x,\ Q_1 = -y,\ R_1 = -\alpha$$

and

$$P_2 = 1,\ Q_2 = 1, R_2 = -\frac{x+y}{z}$$

then

$$PP_1 + QQ_1 + RR_1 = x[y(x+y)+\alpha z] - y[x(x+y)-\alpha z] - \alpha[z(x+y)]$$
$$= xy(x+y) + \alpha xz - xy(x+y) + \alpha yz - \alpha xz - \alpha yz = 0$$

and

$$PP_2 + QQ_2 + RR_2 = y(x+y)+\alpha z + x(x+y) - \alpha z - \left(\frac{x+y}{z}\right)[z(x+y)]$$
$$= y(x+y) + x(x+y) - (x+y)\ (x+y) = 0$$

Also, for the given system of equations, we have

$$\frac{dx}{y(x+y)+\alpha z} = \frac{dy}{x(x+y)-\alpha z} = \frac{dz}{z(x+y)} = \frac{xdx - ydy - \alpha dz}{0} = \frac{dx + dy - \left(\frac{x+y}{z}\right)dz}{0} \tag{46}$$

Thus,

$$xdx - ydy - \alpha dz = 0 \Rightarrow d\left(\frac{x^2}{2} - \frac{y^2}{2} - \alpha z\right) = 0$$

and

$$dx + dy - \left(\frac{x+y}{z}\right) dz = 0$$

$$\Rightarrow\ \frac{d(x+y)}{x+y} - \frac{dz}{z} = 0$$

$$\Rightarrow\ d\left[\ln\left|\frac{(x+y)}{z}\right|\right] = 0$$

Therefore, the integral curves are given by the intersection of family of surfaces

$$\left.\begin{array}{l} x^2 - y^2 - 2\alpha z = c_1 \\ \text{and} \\ \ln\left|\dfrac{x+y}{z}\right| = c_2 \end{array}\right\} \tag{47}$$

***

In the above Examples, you must have realised that determining solution by this method requires a good deal of intuition in determining the forms of the functions (P_1, Q_1, R_1) and (P_2, Q_2, R_2). In actual practice, sometimes it is much simpler to write given equations in a form which suggests its solution. For example, in Example 9 if we add the numerators and denominators of the first two fractions, we obtain

$$\frac{dx+dy}{(x+y)^2} = \frac{dz}{z(x+y)},$$

which can be written in the form

$$\frac{d(x+y)}{(x+y)} = \frac{dz}{z}$$

and its general solution is

$$\ln\left|\frac{x+y}{z}\right| = \text{constant}. \tag{48}$$

Similarly, we have

$$\frac{xdx - ydy}{\alpha(x+y)z} = \frac{dz}{z(x+y)}$$

$$\Rightarrow \quad xdx - ydy - \alpha dz = 0$$

$$\Rightarrow \quad d\left(\frac{1}{2}x^2 - \frac{1}{2}y^2 - \alpha z\right) = 0$$

and, hence, we get the solution

$$x^2 - y^2 - 2\alpha z = \text{constant} \tag{49}$$

Eqns. (48) and (49) together give us the solution (47).

Let us take up another example.

Example 10: Solve $\dfrac{dx}{3x+y-z} = \dfrac{dy}{x+y-z} = \dfrac{dz}{2(x-y)}$.

Solution: For the given system of equations, we have each ratio

$$= \frac{dx - 3dy - dz}{3x + y - z - 3\,(x+y-z) - 2(z-y)} = \frac{dx - 3dy - dz}{0}$$

Hence,

$$dx - 3dy - dz = 0$$

$$\Rightarrow \quad d\,(x - 3y - z) = 0$$

Integrating, we get one integral surface as

$$u = x - 3y - z = c_1. \tag{50}$$

From Eqn. (50), we have

$$z = x - 3y - c_1$$

To find the second integral surface we substitute this value of z in the first two ratios of the given equations and obtain

$$\frac{dx}{3x + y - (x - 3y - c_1)} = \frac{dy}{x + y - (x - 3y - c_1)}$$

$$\Rightarrow \quad \frac{dx}{2x + 4y + c_1} = \frac{dy}{4y + c_1} \tag{51}$$

Eqn. (51) is an ordinary differential equation in x and y and you can solve it by the methods you have learnt in Block 2.
If we write $4y + c_1 = t$, then Eqn. (51) reduces to

$$\frac{dx}{dt} - \frac{x}{2t} = \frac{1}{4} \tag{52}$$

Eqn. (52) is a linear equation with $\text{I.F} = e^{-\int \frac{1}{2t} dt} = \frac{1}{\sqrt{t}}$

The solution of Eqn. (52) is

$$\frac{x}{\sqrt{t}} = \frac{1}{2}\sqrt{t} + \text{constant}$$

$$\Rightarrow \quad \frac{2x - t}{\sqrt{t}} = \text{constant}$$

$$\Rightarrow \quad \frac{x - y + z}{\sqrt{x + y - z}} = \text{constant} = c_2 \quad [\text{substituting } t = 4y + c_1 \text{ and } c_1 = x - 3y - z] \tag{53}$$

The integral curves of the given equation are thus obtained as intersection of surfaces (50) and (53).

***

In the above examples, you must have observed that for solving Eqns. (31), we tried to find a set of functions (P_1, Q_1, R_1) and (P_2, Q_2, R_2) satisfying relations (41) and (42) and transform expressions (44) and (45) as exact differentials. Sometimes it is not possible to do so. In such cases, try to find functions (P_1, Q_1, R_1) and (P_2, Q_2, R_2) such that either the expressions

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{PP_1 + QQ_1 + RR_1} \text{ and } \frac{P_2 dx + Q_2 dy + R_2 dz}{PP_2 + QQ_2 + RR_2} \tag{54}$$

are exact differentials, as we have done above, or these fractions when taken with two of the fractions of Eqns. (31) yield exact differentials, as we have illustrated in the following example.

Example 11: Find the integral curves of the following system of equations

$$\frac{dx}{-1} = \frac{dy}{3y + 4z} = \frac{dz}{2y + 5z}$$

Solution: Each of the fractions of the given system of equations is equal to

$$\frac{dy - dz}{y - z} \text{ and } \frac{dy + 2dz}{7(y + 2z)} \tag{55}$$

Thus, we have

$$\frac{dy - dz}{y - z} = \frac{dy + 2dz}{7(y + 2z)}$$

Integrating, we get one family of surfaces as

$$y - z = c_1 (y + 2z)^{1/7} \tag{56}$$

To find the second family of surfaces we take the first fraction in the given equations with the first fraction in relation (55) i.e.,

$$\frac{dx}{-1} = \frac{dy - dz}{y - z}$$

Integrating, we get

$$y - z = c_2 e^{-x} \tag{57}$$

The intersection of the two families of surfaces (56) and (57) gives the integral curves of the given system of equations.

***

You may now try to solve the following exercise.

E6) Find the integral curves of the following system of equations

i) $$\frac{dx}{y^3 x - 2x^4} = \frac{dy}{2y^4 - x^3 y} = \frac{dz}{9z(x^3 - y^3)}.$$

ii) $$\frac{xdx}{z^2 - 2yz - y^2} = \frac{dy}{y+z} = \frac{dz}{y-z}.$$

iii) $$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}.$$

iv) $$\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{5z + \tan(y-2x)}.$$

There may be a situation where one variable is absent from one of the equations of the set of Eqns. (31). In such cases, we can derive the integral curves by a simpler method. In the next sub-section, we shall take up this method.

14.4.2 One Variable Absent

We illustrate the method through an example.

Consider a system of simultaneous differential equations

$$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2} \tag{58}$$

You may notice that variable z is absent in the first two fractions of the system and we can write the equation

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

in the form $\dfrac{dy}{dx} = f(x, y) = y/x$.

We can easily integrate the above equation and obtain the first solution as

$$x = c_1 \; y \tag{59}$$

Considering the last two fractions in Eqns. (58) and substituting for x from Eqn. (59), we have

$$\frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$$

or, $$\frac{dy}{y^2} = \frac{dz}{c_1 y^2 z - 2c_1^2 y^2}$$

or, $$c_1 dy = \frac{dz}{z - 2c_1}$$

which can be integrated easily to obtain the second solution of Eqns. (58). Integrating the above equation, we get

$$c_1 y = \ln|(z-2c_1)| + c_2$$

or, $$x = \ln\left|\left(z - \frac{2x}{y}\right)\right| + c_2 \qquad [\because x = c_1 y]$$

or, $$x = \ln|(yz-2x)| - \ln|y| + c_2 \qquad (60)$$

Hence, the complete solution of Eqns. (58) is constituted by Eqns. (59) and (60). The integral curves of Eqns. (58) are the intersection of surfaces $x/y = c_1$ and $x - \ln(yz-2x) + \ln y = c_2$.

In general, suppose that the variable x does not occur in functions Q and R of the system of Eqns. (31). Then the equations to be solved are

$$\frac{dx}{P(x,\ y,\ z)} = \frac{dy}{Q(y,\ z)} = \frac{dz}{R(y,\ z)} \qquad (61)$$

From the last two fractions, in the above set, we obtain

$$\frac{dy}{dz} = \frac{Q(y,\ z)}{R(y,\ z)} = f(y,\ z) \qquad (62)$$

Eqn. (62) is a first order equation in $y,\ z$ and has a solution of the form

$$\phi(y,\ z,\ c_1) = 0, \qquad (63)$$

where c_1 is an arbitrary constant.

On solving the relation (63) for z and substituting in the first two fractions of Eqns. (61), we obtain

$$\frac{dy}{dx} = \frac{Q(y,\ z)}{P(x,\ y,\ z)} = g(x,\ y,\ c_1)$$

Above equation is again a first order equation in $x,\ y$ and its solution is of the form

$$\psi(x,\ y,\ c_1,\ c_2) = 0$$

where c_2 is another arbitrary constant.

Thus, the integral curves of the set of Eqns. (61) are the intersection of the surfaces

$\phi(y,\ z, c_1) = 0$ and $\psi(x,\ y,\ c_1,\ c_2) = 0$.

We illustrate this method with the help of an example.

Example 12: Find the integral curves of the equations

$$\frac{dx}{x+z} = \frac{dy}{y} = \frac{dz}{z+y^2}$$

Solution: From the last two ratios of the given set of equations, we have

$$\frac{dz}{dy} - \frac{z}{y} = y$$

It is a linear equation and its solution is

$$z - y^2 = c_1 y \qquad (64)$$

Substituting for z from Eqn. (64) in the first ratio of the given equations and taking the first two ratios, we get

$$\frac{dx}{dy} = \frac{x}{y} + y + c_1$$

It is again a linear equation in x and y and its solution is

$$x = c_1\, y \ln |\, y\,| + y^2 + c_2 y$$

Substituting the value of c_1 from Eqn. (64) in the above equation, we obtain

$$x = (z - y^2)\ \ln |\, y\,| + y^2 + c_2 y \tag{65}$$

The integral curves of the given equations are given by Eqns. (64) and (65).

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And now a few exercises for you.

E7) Find the integral curves of the following simultaneous equations

i) $\dfrac{dx}{1} = \dfrac{dy}{-2} = \dfrac{dz}{3x^2 \sin(y + 2x)}$

ii) $\dfrac{dx}{y^2} = \dfrac{dy}{x^2} = \dfrac{dz}{x^2 y^2 z^2}$

iii) $\dfrac{dx}{-\dfrac{y^2 + z^2}{x}} = \dfrac{dy}{y} = \dfrac{dz}{z}$

iv) $\dfrac{dx}{\cos(x + y)} = \dfrac{dy}{\sin(x + y)} = \dfrac{dz}{z + \dfrac{1}{z}}$

E8) Solve the following system of equations

i) $\dfrac{adx}{(b - c)yz} = \dfrac{bdy}{(c - a)zx} = \dfrac{cdz}{(a - b)xy}$

ii) $\dfrac{dx}{xz - y} = \dfrac{dy}{yz - x} = \dfrac{dz}{1 - z^2}$

iii) $\dfrac{dx}{x^2(y^3 - z^3)} = \dfrac{dy}{y^2(z^3 - x^3)} = \dfrac{dz}{z^2(x^3 - y^3)}$.

In Sec. 14.4 we have discussed various methods of solving the system of simultaneous differential equations. As already mentioned in Secs. 14.1 and 14.2 simultaneous equations arise frequently in geometry and mathematical physics. In the next section we shall take up some applications of simultaneous differential equations.

14.5 APPLICATIONS

Geometrically when all the curves of one family of curves $G(x,\ y,\ c_1) = 0$ intersect orthogonally all the curves of another family $H(x,\ y,\ c_2) = 0$, then we say that the families are **orthogonal trajectories** of each other. In other words, an orthogonal trajectory is any one curve that intersects every curve of another family at right angles.

The problem of finding the orthogonal trajectories of a system of curves on a given surface provides an interesting application of simultaneous differential equations. In the next sub-section we shall discuss this problem.

14.5.1 Orthogonal Trajectories of a System of Curves on a given Surface

You know that the intersection of a surface by a plane is a curve. When we take the intersection of the cone

$$x^2 + y^2 = z^2 \tan^2 \alpha$$

by the system of parallel planes

$$z = c$$

where c is a parameter, we obtain a system of circles (see Fig. 6). Geometrically, we can say that, in this case the orthogonal trajectories are the generators of the cone shown dotted in Fig. 6. We shall show this in Example 13.

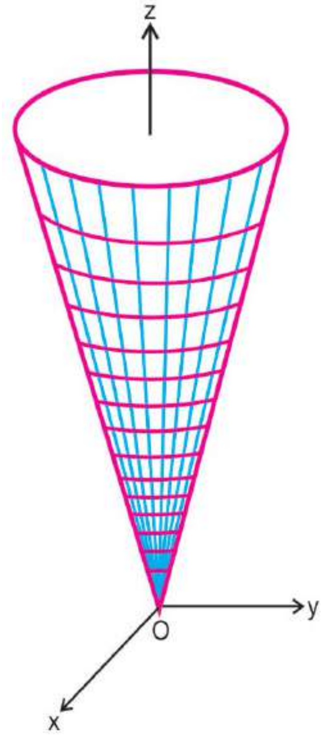

Fig. 6

You may recall that in Unit 6, Sec. 6.4, we discussed the problem of finding the orthogonal trajectories of a system of plane curves. In three dimensions the corresponding problem is, given a one-parameter family of surfaces

$$F(x, y, z) = c_1 \tag{66}$$

and a system of curves on it, to find another system of curves each of which lies on the surface (66) and cuts every curve of the given system at right angles.

Let the given system of curves be the intersection of a one-parameter family of surfaces

$$G(x, y, z) = c_2 \tag{67}$$

with the surface (66).

Now we want to find another system of curves on the surfaces (66) which intersect orthogonally the given system of curves.

If (dx, dy, dz) define the tangential direction of the given system of curves through a point say, $T(x, y, z)$ on the surface (66), then we have

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0 \tag{68}$$

and $$\frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial z}dz = 0 \tag{69}$$

On solving Eqns. (68) and (69), for dx, dy, dz, we get

$$\frac{dx}{F_y G_z - F_z G_y} = \frac{dy}{F_z G_x - F_x G_z} = \frac{dz}{F_x G_y - F_y G_x}$$

or $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{70}$$

with $P = F_y G_z - F_z G_y,\ Q = F_z G_x - F_x G_z,\ R = F_x G_y - F_y G_x$ (71)

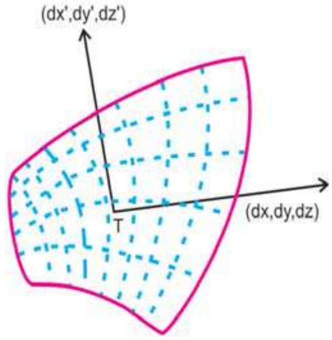

Fig. 7

Eqns. (70) are the simultaneous differential equations of the given system of curves on the surface (66).
Similarly on the curves of the orthogonal system through the point $T(x, y, z)$ on the surface (66) (see Fig. 7), we can write

$$\frac{\partial F}{\partial x}dx' + \frac{\partial F}{\partial y}dy' + \frac{\partial F}{\partial z}dz' = 0 \tag{72}$$

where (dx', dy', dz') define the tangential direction of the orthogonal system at $T(x, y, z)$ on the surface (66).
Also because of the orthogonality condition, we have from Eqn. (70)

$$Pdx' + Qdy' + Rdz' = 0 \tag{73}$$

On solving Eqns. (72) and (73) for dx', dy' and dz', we get the system of equations

$$\frac{dx'}{P'} = \frac{dy'}{Q'} = \frac{dz'}{R'} \tag{74}$$

where

$$\left.\begin{aligned} P' &= R\frac{\partial F}{\partial y} - Q\frac{\partial F}{\partial z} \\ Q' &= P\frac{\partial F}{\partial z} - R\frac{\partial F}{\partial x} \\ R' &= Q\frac{\partial F}{\partial x} - P\frac{\partial F}{\partial y} \end{aligned}\right\} \tag{75}$$

The solution of the Eqns. (75) with the relation (66) gives the required system of orthogonal trajectories.

We illustrate the method through the following examples.

Example 13: Find the orthogonal trajectories on the cone $x^2 + y^2 = z^2 \tan^2 \alpha$ of its intersections, with the family of planes parallel to $z = c$.

Solution: We are given

$$F(x, y, z) = x^2 + y^2 - z^2 \tan^2 \alpha = 0$$

and $G(x, y, z) = z - c = 0$

Now $F_x = 2x,\ F_y = 2y,\ F_z = -2z \tan^2 \alpha$,

$$G_x = 0,\ G_y = 0,\ G_z = 1$$

Therefore, from relations (71), we have

$$P = F_y G_z - F_z G_y = 2y$$
$$Q = F_z G_x - F_x G_z = -2x$$
$$R = F_x G_y - F_y G_x = 0$$

Also from relations (75), we have

$$P' = RF_y - QF_z = -4xz \tan^2 \alpha$$
$$Q' = PF_z - RF_x = -4yz \tan^2 \alpha$$
$$R' = QF_x - F_y G_x = -4x^2 - 4y^2$$

Hence, the orthogonal trajectories are given by

$$\frac{dx}{-4xz \tan^2 \alpha} = \frac{dy}{-4yz \tan^2 \alpha} = \frac{dz}{-4(x^2 + y^2)}$$

From the first two fractions, we obtain

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow |x| = c_1 |y|,\ c_1 \text{ is a parameter.}$$

Also, we have

$$\frac{xdx + ydy}{-4z \tan^2 \alpha (x^2 + y^2)} = \frac{dz}{-4(x^2 + y^2)}$$

$$\Rightarrow \quad xdx + ydy = z \tan^2 \alpha \; dz$$

Integrating, we obtain

$$x^2 + y^2 - z^2 \tan^2 \alpha = c_2$$

where c_2 is a parameter.

Hence the orthogonal trajectories are the generators of the cone formed by the intersection of its surface with the planes $x = c_1 y$ passing through the z-axis (see Fig. 6).

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Let us consider another example.

Example 14: Find the orthogonal trajectories on the sphere $x^2 + y^2 + z^2 = 1$ of its intersections with the family of planes $z = k,\ -1 \le k \le 1$.

Solution: Here

$$F(x,\ y,\ z) = x^2 + y^2 + z^2 - 1 = 0$$
$$G(x,\ y,\ z) = z - k = 0$$

Hence

$$F_x = 2x,\ F_y = 2y,\ F_z = 2z$$
$$G_x = 0,\ G_y = 0,\ G_z = 1$$

Thus the system of equations defining the given integral curves are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where

$$P = F_y G_z - F_z G_y = 2y$$
$$Q = F_z G_x - F_x G_z = -2x$$
$$R = F_x G_y - F_y G_x = 0$$

The orthogonal trajectories are given by the system of equations

$$\frac{dx}{P'} = \frac{dy}{Q'} = \frac{dz}{R'}$$

where

$$P' = RF_y - QF_z = 4xz\ ,$$
$$Q' = PF_z - RF_x = 4yz\ ,$$
$$R' = QF_x - PF_y = -4(x^2 + y^2)\ .$$

$$\Rightarrow\ \frac{dx}{4xz} = \frac{dy}{4yz} = \frac{dz}{-4(x^2 + y^2)}\ . \qquad (76)$$

Eqns. (76) have the solutions

$$|\,y\,| = c_1 |\,x\,|, \text{ and } x^2 + y^2 + z^2 = c_2$$

where $c_1,\ c_2$ are the parameters.

Hence the orthogonal trajectories are the curves of intersection of the planes $|\,y\,| = c_1 |\,x\,|$ with the surfaces $x^2 + y^2 + z^2 = c_2$.

***

You may now try the following exercises.

E9) Find the orthogonal trajectories on the hyperboloid $x^2 + y^2 - z^2 = 1$ of the conics in which it is cut by the family of planes $x + y = c$.

E10) Find the orthogonal trajectories on the conicoid $(x+y)z=1$ of the conics in which it is cut by the system of planes $x-y+z=k$ where k is a parameter.

We now take up an application from particle dynamics in phase-plane. Here the phase-plane is the xy-plane in which the particle is moving. The problem consists of finding a curve in the xy-plane which passes through a given point, say (x_0, y_0), along which a particle is moving in simple harmonic motion.

14.5.2 Particle Motion in Phase-Plane

You know (ref. Sec. 13.6, Unit 13) that the equation governing the simple harmonic motion of a particle is

$$\frac{d^2x}{dt^2}+w^2x=0,\ x(0)=x_0,\ \frac{dx}{dt}(0)=y_0 \tag{77}$$

Let $$\frac{dx}{dt}=y \tag{78}$$

From Eqns. (77) and (78), we get

$$\frac{dy}{dt}=-w^2x$$

or,

$$\frac{dx}{y}=\frac{dy}{-w^2x}=dt,\ y(x_0)=y_0 \tag{79}$$

The time variable does not appear explicitly in the system of Eqns. (79). These equations are called **autonomous**.

Eqns. (79) are the simultaneous differential equations defining the particle motion in phase-plane.
Integrating Eqns. (79), we get

$$w^2x^2+y^2=w^2x_0^2+y_0^2. \tag{80}$$

The integral curve (trajectory) is shown in Fig. 8.

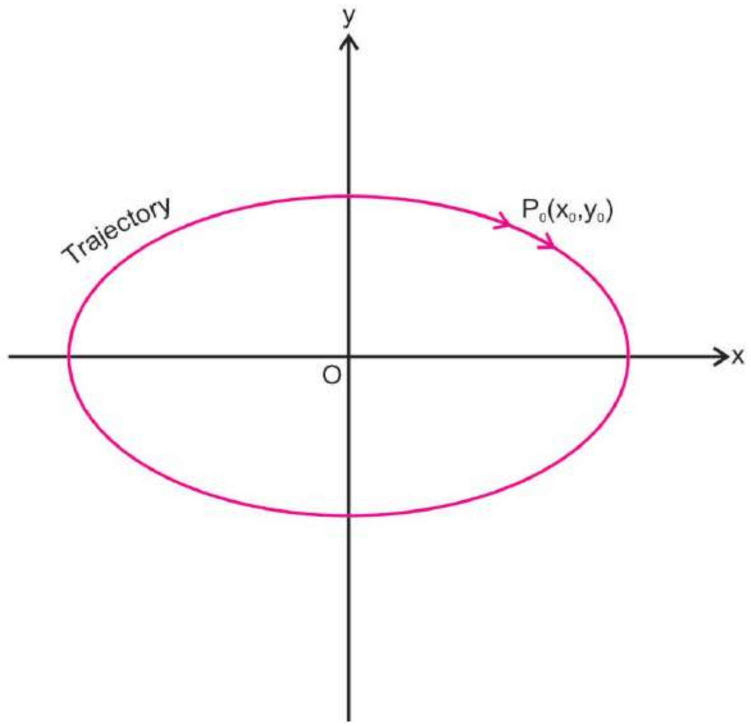

Fig. 8

The arrows on the trajectory tells about the evolution of the dynamical system with time.

You may now try the following exercise.

E11) Determine the system of simultaneous differential equations governing the equilibrium of a heavy string hanging from two points of support, where H is the horizontal tension at the lowest point L of the string, T is the tension in the string at the point P and W is the weight borne by the portion $\widehat{LP}$ of the string.

We now take up another application of simultaneous equations in which the problem of an electric circuit is reduced to a system of simultaneous differential equations.

14.5.3 Electric Circuits

You may recall that in Sec. 13.6 of Unit 13, Block 3, we obtained the differential equation of an electric circuit containing an inductance L, a resistance R, a conductor of capacitance C and an electromagnetic force $E(t)$ in the form

$$L\frac{di}{dt}+Ri+\frac{q}{C}=E(t) \tag{81}$$

where i is the current and q is the charge (ref. Eqn. (127), Unit 13).

Also, we know that the current i is just the instantaneous rate of change in charge q and we have

$$i=\frac{dq}{dt} \tag{82}$$

From Eqns. (81) and (82), we obtain

$$\frac{di}{dt}=\frac{E(t)-iR-\frac{q}{C}}{L} \text{ and } \frac{dq}{dt}=i$$

$$\Rightarrow \quad \frac{di}{\frac{\left(E(t)-iR-\frac{q}{C}\right)}{L}}=\frac{dq}{i}=\frac{dt}{1} \tag{83}$$

Eqns. (83) are the system of simultaneous differential equations defining the vibrations in an electric circuit. The solution of Eqn. (83) yields the charge and the current at any time t.

Let us consider the following example.

Example 15: An electric circuit consists of an inductance of 0.1 henry, a resistance of 20 ohms and a condenser of capacitance 25 microfarads. Find the system of equations governing the charge and the current at time t.

Solution: Here $L=0.1,\ R=20,\ C=25$ microfarads $=25\times10^{-6}$ farads, $E(t)=0$.

$\therefore$ Simultaneous Eqns. (83) governing the vibrations in the electric circuit in this case reduce to

$$\frac{di}{\frac{\left(E(t)-Ri-\frac{q}{c}\right)}{L}}=\frac{dq}{i}=\frac{dt}{1}$$

$$\Rightarrow \quad \frac{di}{-200i-400{,}000q}=\frac{dq}{i}=\frac{dt}{1}$$

where q is the charge and i is the current at time t.

***

You may now try the following exercise.

E12) In Example 15 above, what will be the form of the governing equation if there is a variable electromagnetic force of $100\cos 200t$ volts.

We now conclude this unit by giving a summary of what we have covered in it.

14.6 SUMMARY

In this unit, we have covered the following:

1. i) The parametric equation of a space curve $f(x, y, z)=0$ are

$$x=\phi(t),\ y=\psi(t),\ z=\theta(t),\ t_1 \le t \le t_2$$

provided $f(\phi(t), \psi(t), \theta(t))=0$, for $t_1 \le t \le t_2$.

ii) The equation of a surface is $f(x, y, z)=0$ or $z=F(x, y)$.

iii) The equation $f(x, y, z, c)=0$ represents one-parameter family of surfaces with c as a parameter.

iv) The envelope of one-parameter family of surfaces is a surface obtained by eliminating c between $f(x, y, z, c)=0$ and $\frac{\partial f}{\partial c}=0$.

v) The characteristic curve is the intersection of surfaces given by $f(x, y, z, c)=0$ and $\frac{\partial f}{\partial c}=0$.

vi) The equation $f(x, y, z, a, b)=0$ represents two-parameter family of surfaces, with a and b as two parameters. If $b=\phi(a)$, then characteristic curve for the surface $f(x, y, z, a, b)=0$ is the intersection of the surface given by $f(x, y, z, a, \phi(a))=0$ and $\frac{\partial f}{\partial a}+\frac{\partial f}{\partial b}\frac{db}{da}=0$.

vii) The equation $f(x, y, z, a, b)=0,\ \frac{\partial f}{\partial a}=0$ and $\frac{\partial f}{\partial b}=0$ represent a point known as characteristic point for two parameter family of surfaces.

viii) The envelope of two-parameter family of surfaces is generated by its characteristic point and is obtained by eliminating a and b from the equations $f(x, y, z, a, b)=0,\ \frac{\partial f}{\partial a}=0,\ \frac{\partial f}{\partial b}=0$.

2. The simultaneous differential equations represent
   - i) a space curve as the intersection of two surfaces
   - ii) the equations of motions of a dynamical system
   - iii) the equations governing the theory of radioactive transformations..

3. The system of simultaneous differential equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

can be solved by the following methods:

i) **Method of multipliers:** In this method we find multipliers P_1, Q_1, R_1 and P_2, Q_2, R_2 (constants or functions of x, y, z) such that $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1} = \frac{P_2dx + Q_2dy + R_2dz}{PP_2 + QQ_2 + RR_2}$.
with

$$PP_1 + QQ_1 + RR_1 = 0$$

$$PP_2 + QQ_2 + RR_2 = 0$$

Also $P_1dx + Q_1dy + R_1dz = 0$ and $P_2dx + Q_2dy + R_2dz = 0$ are either exact differentials or reducible to exact differentials and their integration yields the family of surfaces, whose intersection is the integral curves of the given equations.

ii) **One variable absent:** In this case we find one solution surface and use it to find the second solution surface from the given system.

4. As an application of simultaneous differential equations, we find that

i) Orthogonal trajectories of space curves obtained as the intersection of surfaces $F(x, y, z) = C_1$ and $G(x, y, z) = C_2$ are the solutions of the following system of simultaneous differential equations.

$$\frac{dx}{F_y(F_xG_y - F_yG_x) - F_z(F_zG_x - F_xG_z)}$$

$$= \frac{dy}{F_z(F_yG_z - F_zG_y) - F_x(F_xG_y - F_yG_x)}$$

$$= \frac{dz}{F_x(F_zG_x - F_xG_z) - F_y(F_yG_z - F_zG_y)}.$$

ii) Particle motion in phase-space is governed by the system of equations.

$$\frac{dx}{y} = \frac{dy}{-w^2x} = dt, \; y(x_0) = y_0$$

ii) The vibration in an electric circuit can be represented by the system of equations

$$\frac{di}{\frac{\left(E(t) - iR - \frac{q}{C}\right)}{L}} = \frac{dq}{i} = \frac{dt}{1}.$$

14.7 SOLUTIONS/ANSWERS

E1) i) The given equation of the surface is

$x^2 + y^2 = 2z, -\infty < z, < \infty$

The intersection of this surface with plane $z = k$ is the circle $x^2 + y^2 = 2k$. This equation is satisfied if we take $x = \sqrt{2k}\cos\theta,\ y = \sqrt{2k}\sin\theta$. Hence the parametric equations of the given surface are $x = \sqrt{2k}\cos\theta,\ y = \sqrt{2k}\sin\theta,\ z = k$.
As the domain of z is $-\infty < z < \infty$, we set

$z = \sinh\alpha$

and then parametric equations reduce to

$x = \sqrt{2\sinh\alpha}\cos\theta,\ y = \sqrt{2\sinh\alpha}\sin\theta$

where $0 \le \theta \le 2\pi, -\infty < \alpha < \infty$.

ii) Given equation of the surface is

$$\left.\begin{aligned} x &= u\cos v \\ y &= u\sin v \\ z &= u\cot v \end{aligned}\right\} \Rightarrow \begin{aligned} \frac{x}{u} &= \cos v \\ \frac{y}{u} &= \sin v \\ \textit{and}\ z &= u\cot v \end{aligned}$$

$$\therefore\ \frac{x^2}{u^2} + \frac{y^2}{u^2} = \cos^2 v + \sin^2 v = 1$$

i.e, $x^2 + y^2 = u^2 = \left(\frac{z}{\cot v}\right)^2 = z^2\tan^2 v$

$\Rightarrow\ x^2 + y^2 = z^2\tan^2 v$, which is the required Cartesian equation of the surface.

E2) We obtain the envelope by eliminating the parameter c between the equations

$$f = x^2 + y^2 + (z-c)^2 - c^2\sin^2\alpha = 0$$

and

$$\frac{\partial f}{\partial c} = -2(z-c) - 2c\sin^2\alpha = 0 \Rightarrow z = c\ \cos^2\alpha$$

On eliminating c, we find that the envelope is a cone

$$x^2 + y^2 = z^2\tan^2\alpha$$

and the characteristic curve is given by

$$x^2 + y^2 = c^2\sin^2\alpha\cos^2\alpha$$

$$z = c\ \cos^2\alpha$$

which is a circle of radius $c\sin\alpha\cos\alpha$ in the plane $z = c\ \cos^2\alpha$.

E3) The equation of two-parameter family of sphere is

$$f = (x-a)^2 + (y-b)^2 + z^2 - 1 = 0$$

Here $f_a = 0 \Rightarrow -2(x-a) = 0 \Rightarrow (x-a) = 0$

$f_b = 0 \Rightarrow -2(y-b) = 0 \Rightarrow (y-b) = 0$

Eliminating a and b between $f = 0,\ f_a = 0$ and $f_b = 0$, the envelope of the given family of sphere is

$z^2 - 1 = 0$

$\Rightarrow \quad z = 1$ and $z = -1$

Now one-parameter family is

$$g = (x-a)^2 + (y-a)^2 + z^2 - 1 = 0$$

The characteristic curve is given by

$$g = 0 \text{ and } \frac{\partial g}{\partial a} = 0$$

$\Rightarrow \quad (x-a)^2 + (y+a)^2 + z^2 - 1 = 0$ and $2(x-a) + 2(y-a) = 0$

$\Rightarrow \quad (x-a)^2 + (y-a)^2 + z^2 - 1 = 0$ and $(x+y) = 2a$

Hence characteristic curve is (eliminating a from $g = 0$ and $\frac{\partial g}{\partial a} = 0$)

$$\left(x - \frac{x+y}{2}\right)^2 + \left(y - \frac{x+y}{2}\right)^2 + z^2 - 1 = 0$$

$$\Rightarrow \quad \frac{1}{4}(x-y)^2 + \frac{1}{4}(y-x)^2 + z^2 - 1 = 0$$

$$\Rightarrow \quad (x-y)^2 = 2(1-z^2)$$

E4) i) The given families of surfaces are

$$u = 3x + 4y + z - c_1 = 0 \text{ and } v = x + z - c_2 = 0$$

Along any curve of the families

$du = 0$ and $dv = 0$

Now $du = 0 \Rightarrow 3dx + 4dy + dz = 0$

and $dv = 0 \Rightarrow dx + dz = 0$

Solving for $dx,\ dy,\ dz$, we get

$$\frac{dx}{4} = \frac{dy}{1-3} = \frac{dz}{-4}$$

$$\Rightarrow \quad \frac{dx}{2} = \frac{dy}{-1} = \frac{dz}{-2}$$

which are the required simultaneous differential equations of the space curves.

ii) The given families of surfaces are

$u = x^2 + y^2 - c_1$ and $v = 3x + 4z - c_2 = 0$

$du = 0 \Rightarrow 2x\,dx + 2y\ dy = 0$

$dv = 0 \Rightarrow 3dx + 4dz = 0$

Solving for $dx,\ dy,\ dz$, we get

$$\frac{dx}{8y} = \frac{dy}{-8x} = \frac{dz}{-6y}$$

$$\Rightarrow \quad \frac{dx}{4y} = \frac{dy}{-4x} = \frac{dz}{-3y}$$

which are the required simultaneous differential equations of the space curves.

iii) $du = ydx + xdy = 0$

$dv = (z+2x)dx + (3+2y)dy + (x+y)dz = 0$

Solving for dx, dy, dz, we get

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{(y-x)(z+2y+2x)}.$$

E5) i) The given equations are

$$\frac{dx}{y^2} = \frac{dy}{y} = \frac{dz}{z}.$$

Taking the second and the third fractions, we get

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow \quad \frac{y}{z} = c_1$$

From the first two fractions we obtain

$$\frac{dx}{y^2} = \frac{dy}{y} \text{ or } dx = y\ dy$$

$$\Rightarrow \quad 2x - y^2 = c_2$$

The integral curves are given by the intersection of families of surfaces

$$y = zc_1 \text{ and } 2x - y^2 = c_2.$$

ii) Taking the first two fractions of the given system, we get

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow x/y = c_1 \text{ or } x = c_1 y \tag{84}$$

Taking the second and third fractions, we obtain

$$\frac{dy}{y^2} = \frac{dz}{c_1 z\ y^2 - 2c_1^2 y^2} \text{ or } c_1 dy = \frac{dz}{z - 2c_1^2} \quad (\because x = c_1 y)$$

which on integration yields,

$$c_1 y - \ln|(z - 2c_1^2)| = c_2 \text{ or } x - \ln|(z - 2x^2/y^2)| = c_2 \tag{85}$$

Required integral curves are given by the intersection of families of surfaces (84) and (85).

iii) The required integral curves are the intersection of the families of surfaces

$$x^3 - y^3 = c_1 \text{ and } x^2 - z^2 = c_2.$$

E6) i) The given equations are

$$\frac{dx}{y^3 x - 2x^4} = \frac{dy}{2y^4 - x^3 y} = \frac{dz}{9z(x^3 - y^3)}$$

Here $P = y^3 x - 2x^4,\ Q = 2y^4 - x^3 y,\ R = 9z(x^3 - y^3)$

If we take $P_1 = \frac{1}{x},\ Q_1 = \frac{1}{y},\ R_1 = \frac{1}{3z}$, then

$$PP_1 + QQ_1 + RR_1 = \frac{1}{x}(y^3 x - 2x^4) + \frac{1}{y}(2y^4 - x^3 y) + \frac{1}{3z}(x^3 - y^3) = 0$$

Thus for the given system of equations, we have

$$\frac{dx}{y^3 x - 2x^4} = \frac{dy}{2y^4 - x^3 y} = \frac{dz}{9z(x^3 - y^3)} = \frac{dx/x + dy/y + dz/(3z)}{0}$$

From the last fraction, we get

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{3z}=0$$

$$\Rightarrow \quad d\left(\ln|x|+\ln\,|y|+\frac{1}{3}\ln|z|\right)=0$$

Integrating, we get $\ln|x||y||z|^{1/3}=$ constant

$$\Rightarrow \quad |x|.|y|.|z|^{1/3}=c_1$$

From 1st and 2nd fraction, we get

$$(2y^4-x^3y)\,dx-(y^3x-2x^4)\,dy=0$$

Dividing, by x^3y^3, we get

$$\frac{2y}{x^3}dx-\frac{1}{y^2}dx-\frac{1}{x^2}dy+\frac{2x}{y^3}dy=0$$

$$\Rightarrow \quad -\left(-\frac{2y}{x^3}dx+\frac{dy}{x^2}\right)-\left(\frac{dx}{y^2}-\frac{2x}{y^3}dy\right)=0$$

$$\Rightarrow \quad d\left(\frac{y}{x^2}+\frac{x}{y^2}\right)=0$$

Integrating, we get

$$\frac{y}{x^2}+\frac{x}{y^2}=c_2$$

Thus the integral curves are given by the intersection of families of surfaces

$$|x|.|y|.|z|^{1/3}=c_1 \text{ and } \frac{y}{x^2}+\frac{x}{y^2}=c_2$$

ii) The given equations are

$$\frac{x\,dx}{z^2-2yz-y^2}=\frac{dy}{y+z}=\frac{dz}{y-z} \tag{86}$$

From 2nd and 3rd fractions, we get

$$\frac{dy}{dz}=\frac{y+z}{y-z}$$

It is a homogeneous equation in y and z. To solve it, we put $y=vz$ and obtain the solution as

$$z^2(-v^2+2v+1)=\text{constant}$$

$$\Rightarrow \quad -y^2+2yz+z^2=c_1 \text{ say (substituting } v=y/z) \tag{87}$$

Also, each fraction in Eqns. (86) $=\dfrac{xdx+ydy+zdz}{0}$

$$\therefore\ x\,dx+y\,dy+z\,dz=0$$

$$\Rightarrow \quad d(x^2+y^2+z^2)=0$$

Integrating, we get

$$x^2+y^2+z^2=c_2 \tag{88}$$

From Eqns. (87) and (88) the integral curves are given by the intersection of families of surfaces

$$z^2 + 2yz - y^2 = c_1 \text{ and } x^2 + y^2 + z^2 = c_2.$$

iii) The given equations are

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} \tag{89}$$

Each fraction in Eqns. (89) $= \dfrac{d(x+y)}{\cos(x+y)+\sin(x+y)}$

Thus, $\dfrac{dz}{z} = \dfrac{d(x+y)}{\cos(x+y)+\sin(x+y)}$

$\Rightarrow \quad \dfrac{dz}{z} = \dfrac{dU}{\cos U + \sin U}$, where $U = x+y$

Integrating, we get

$$\begin{aligned}
\ln|z| &= \int \frac{1}{\sin U + \cos U} dU + \text{constant} \\
&= \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(U + \frac{\pi}{4}\right) dU + \text{constant} \\
&= \frac{1}{\sqrt{2}} \ln \left| \tan\left(\frac{\pi}{8} + \frac{U}{2}\right) \right| + \text{constant}
\end{aligned}$$

$$\Rightarrow \quad |z|^{\sqrt{2}} \left| \cot\left(\frac{\pi}{8} + \frac{x+y}{2}\right) \right| = c_1 \text{ (substituting } U = x+y) \tag{90}$$

Again taking first two fractions in Eqns. (89), we have

$$\frac{dy}{dx} = \tan(x+y) \tag{91}$$

On putting $x+y=V$, Eqn. (91) reduces to

$$\frac{dV}{1+\tan V} = dx$$

Integrating this equation, we get

$$\begin{aligned}
x + \text{constant} &= \frac{1}{2}[V + \ln|\cos V + \sin V|] \\
&= \frac{1}{2}[(y+x) + \ln|\cos(y+x) + \sin(y+x)|]
\end{aligned}$$

$$\Rightarrow \quad 2x + \text{constant} = (y+x) + \ln|\cos(y+x) + \sin(y+x)|$$

$$\Rightarrow \quad c_2 = e^{y-x} |\cos(y+x) + \sin(y+x)| \tag{92}$$

Hence integral curves are the intersection of families of surfaces (90) and (92).

iv) The given equations are

$$\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{5z + \tan(y-2x)} \tag{93}$$

From 1st and 2nd fractions, we get one family of surfaces as

$$y - 2x = c_1 \quad (94)$$

From 1st and 3rd fractions of Eqns. (93), and using relation (94), we get the second family of surfaces as

$$5z + \tan(y - 2x) = c_2 e^{5x} \quad (95)$$

The integral curves are the intersection of families of surfaces (94) and (95).

E7) i) $y + 2x = c_1$ and

$z - x^3 \sin(y + 2x) = c_2$

ii) $x^3 - y^3 = c_1$

and $y^3 + \dfrac{3}{z} = c_2$

iii) $\left| \dfrac{y}{z} \right| = c_1$ and $x^2 + y^2 + z^2 = c_2$

Hint: Here $\dfrac{dx}{-\dfrac{y^2 + z^2}{x}} = \dfrac{dy}{y} = \dfrac{dz}{z} = \dfrac{xdx + ydy + zdz}{0}$

iv) $e^{y-x}\{\cos(x + y) + \sin(x + y)\} = c_1$

and $(z^2 + 1)^{1/\sqrt{2}} \cot\left(\dfrac{\pi}{8} + \dfrac{x + y}{2}\right) = c_2$

Hint: Proceed as in E6) iii).

E8) i) The given equations are

$$\frac{adx}{(b-c)yz} = \frac{bdy}{(c-a)zx} = \frac{cdz}{(a-b)xy}$$

Here $P = \dfrac{b-c}{a} yz,\ Q = \dfrac{c-a}{b} zx,\ R = \dfrac{a-b}{c} xy$

Let $P_1 = ax,\ Q_1 = by, R_1 = cz$, then

$$PP_1 + QQ_1 + RR_1 = 0$$

Also if we take

$$P_2 = a^2 x,\ Q_2 = b^2 y,\ R_2 = c^2 z$$

then $PP_2 + QQ_2 + RR_2 = 0$

Thus each fraction in the given equation is equal to

$$\frac{ax\ dx + by\ dy + cz\ dz}{0} \text{ and } \frac{a^2 x\ dx + b^2 y\ dy + c^2 z\ dz}{0}$$

Using these fractions with any one of the fractions of the given equation, we get the two families of surface as

$$ax^2 + by^2 + cz^2 = c_1$$

and

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = c_2.$$

ii) The given equations yield

$$\frac{dx}{xz - y} = \frac{dy}{yz - x} = \frac{dz}{1 - z^2} = \frac{dx + dy}{(x + y)\,(z - 1)} = \frac{dx - dy}{(x - y)(z + 1)}$$

From which we get the two families of surfaces as

$$(x+y)(z+1)=c_1$$

and

$$(x-y)(z-1)=c_2$$

iii) Each fraction in the given equations is equal to

$$\frac{\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}}{0} \text{ and } \frac{xdx+ydy+zdz}{0}$$

Thus, the required families of surfaces are

$$x^2+y^2+z^2=c_1$$

and

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_2\,.$$

E9) Here $F=x^2+y^2-z^2-1=0$
and $G=x+y-c=0$
Hence

$$F_x=2x,\ F_y=2y,\ F_z=-2z$$

and

$$G_x=1,\ G_y=1,\ G_z=0$$

Thus system of equations defining the given integral curves are

$$\frac{dx}{(F_yG_z-F_zG_y)}=\frac{dy}{(F_zG_x-F_xG_z)}=\frac{dz}{(F_xG_y-F_yG_x)}$$

$$\Rightarrow\quad \frac{dx}{2z}=\frac{dy}{-2z}=\frac{dz}{2x-2y}$$

$$\Rightarrow\quad \frac{dx}{z}=\frac{dy}{-z}=\frac{dz}{x-y}$$

$$\Rightarrow\quad \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \text{ where } P=z,\ Q=-z,\ R=x-y$$

The orthogonal trajections are given by the system of equations

$$\frac{dx}{P'}=\frac{dy}{Q'}=\frac{dz}{R'}$$

where

$$P'=RF_y-QF_z=2[y(x-y)-z^2]$$

$$Q'=PF_z-RF_x=-2\,[x(x-y)+z^2]$$

$$R'=QF_x-PF_y=-2z\,(x+y)$$

$$\Rightarrow\quad \frac{dx}{y(x-y)-z^2}=\frac{dy}{-[x(x-y)+z^2]}=\frac{dz}{-z(x+y)}$$

$$=\frac{xdx+ydy-zdz}{0}=\frac{dx-dy}{x^2-y^2}$$

The 4[th] fraction yields

$$xdx+ydy-zdz=0$$

Integrating, we get

$$x^2 + y^2 - z^2 = c_1$$

The 3rd and 5th fractions yield

$$\frac{d(x-y)}{(x-y)} = \frac{dz}{-z}$$

Integrating, we get

$$|x-y|.|z| = c_2$$

Hence the orthogonal trajectories are given by the intersection of the surfaces

$$x^2 + y^2 - z^2 = c_1 = 1 \text{ and } |x-y|.|z| = c_2$$

where c_1 and c_2 are the parameters.

E10) Here $F = (x+y)z - 1 = 0$

$$G = x - y + z - k = 0$$

The system of equations defining the given integral curves are

$$\frac{dx}{z+x+y} = \frac{dy}{x+y-z} = \frac{dz}{-2z}$$

$$\Rightarrow \quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where $P = z + x + y,\ Q = x + y - z,\ R = -2z$

The orthogonal trajectories are given by the equations

$$\frac{dx}{P'} = \frac{dy}{Q'} = \frac{dz}{R'}$$

where

$$P' = -2z^2 - (x+y-z)(x+y),\ Q' = (x+y)\ (x+y+z) + 2z^2,\ R' = -2z^2$$

$$\Rightarrow \frac{dx}{-2z^2 - (x+y-z)(x+y)} = \frac{dy}{(x+y)\ (x+y+z) + 2z^2}$$

$$= \frac{dz}{-2z^2} = \frac{d(x+y)}{2z(x+y)}$$

From 3rd and 4th fractions, we get on integration $(x+y)z = 1$.

Using $x + y = \frac{1}{z}$ in 1st and 3rd fractions, and integrating we get

$$x + c_1 = z + \frac{1}{2z} - \frac{6}{6z^3}.$$

The orthogonal trajectories are the curves

$$x + c_1 = z + \frac{1}{2z} - \frac{1}{6z^3},\ (x+y)z = 1.$$

E11) Let the two points of support of a string be A and B (see Fig. 9). The portion $\widehat{LP}$ of the string is in equilibrium under the action of tensions T at P, H at L and weight W. Let the are $\widehat{LP} = s$.

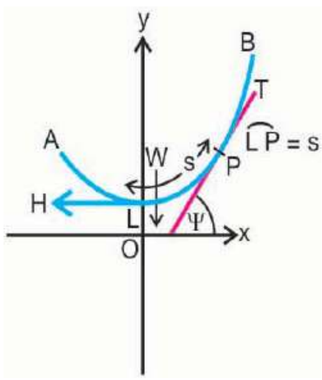

Fig. 9

If ψ is the inclination of tangent at P to the horizontal, then we have

$$T\cos\psi = H \tag{96}$$

and $T\sin\psi = W$ (97)

If w be the weight of the string per unit length, then $W = ws$ and hence from Eqns. (96) and (97), we get

$$\tan\psi = \frac{ws}{H} = \frac{s}{c}, \text{ (say)} \tag{98}$$

where $c = H/w$.

If we take axes of x and y as horizontal and vertical axes respectively, then

$$\frac{dy}{dx} = \tan\psi = \frac{s}{c} \quad \text{(using Eqn. (98))}$$

Differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{c}\frac{ds}{dx} = \frac{1}{c}\sqrt{1+\left(\frac{dy}{dx}\right)^2} \tag{99}$$

Let $\frac{dy}{dx} = v$, (100)

then Eqn. (99) becomes

$$\frac{dv}{dx} = \frac{1}{c}\sqrt{1+v^2} \tag{101}$$

From Eqns. (100) and (101), the required simultaneous equations governing the equilibrium of the string, under given conditions, are

$$\frac{c\,dv}{\sqrt{1+v^2}} = \frac{dy}{v} = dx.$$

E12) Here $L = 0.1,\ R = 20,\ c = 25\times10^{-6}$ and $E(t) = 100\cos 200t$.

Thus equations governing the electric circuit are

$$\left.\begin{aligned} &\frac{di}{dt} + 200i + 400{,}000q = 100\cos 200t \\ &and\ \ \frac{dq}{dt} = i \end{aligned}\right\} \tag{102}$$

Eqns. (102) yield the required system of simultaneous equations governing the motion of electric circuit as

$$\frac{di}{100\cos 200t - (200i + 400{,}000q)} = \frac{dq}{i} = dt$$

– x –

UNIT 15

TOTAL DIFFERENTIAL EQUATIONS

Structure **Page No.**

15.1 INTRODUCTION

In Block 2, Unit 6, we introduced the concepts of total differentials and total differential equations. These equations contain more than one dependent and one independent variable. In this unit we shall discuss such equations in detail. General form of total differential equation of first order in n variables is given by

$$F_1(x_1, x_2, \ldots, x_n)dx_1 + F_2(x_1, x_2, \ldots, x_n)dx_2 + \cdots + F_n(x_1, x_2, \ldots, x_n)dx_n = 0 \qquad (1)$$

where the F_i's $(i = 1, 2, \ldots, n)$ are continuous functions of some or all of the n independent variables $x_1, x_2, \ldots, x_n$.

The first important contribution to the solution of Eqn. (1) also called Pfaffian differential equation was made in 1814 in a memoir by the German mathematician Johann Friedrich Pfaff (1765-1825). He was described as one of Germany's most eminent mathematician during the 19th century. He studied integral calculus and is noted for his work on Pfaffian differential equations of the first order. It was later in 1909, the Greek mathematician Caratheodory who used these equations in the mathematical formulation of various principles in physics, for instance, in the formulation of first and second laws of thermodynamics. Legendre's transformation (a special type of transformation for changing a variable, say x, to another variable X) is generated by a total differential equation in two variable, x and X. However, we shall not be discussing these applications in this unit as they require the understanding of various concepts which are beyond the scope of this course.

We shall in this unit concentrate mainly on total differential equations in three variables, which can, under certain conditions, represent a one-parameter family of surfaces. We shall start by considering the formation of total differential equations in Sec. 15.2 and determine the conditions of their integrability in Sec. 15.3. In Sec. 15.4 we shall discuss various methods of solving total differential equations.

Objectives

After studying this unit, you should be able to:

- identify a total differential equation;
- show that a one-parameter family of surfaces in 3-dimensions leads to a total differential equation in three variables;
- show that a total differential equation in three variables can be integrated to obtain a one-parameter family of surfaces in 3-dimensions if and only if, certain integrability condition is satisfied; and
- use various methods of solving a total differential equation in three variables.

15.2 FORMATION OF TOTAL DIFFERENTIAL EQUATIONS

We start by considering a simple equation

$$xy + z^2 = c \tag{2}$$

in three variables x, y, z with c as a parameter.
You know that Eqn. (2) represents one-parameter family of surfaces.
The total derivative of Eqn. (2) gives

$$d(xy + z^2) = 0$$

$$\Rightarrow \quad y\,dx + x\,dy + 2z\,dz = 0 \tag{3}$$

which is of the form (1) and is a total differential equation. Thus the total differential Eqn. (3) corresponds to the one-parameter family of surfaces given by Eqn. (2).

In general, consider the equation of a one-parameter family of surfaces in 3-dimensional space given by

$$f(x, y, z) = c, \tag{4}$$

where c is the defining parameter of the surface. Differentiating Eqn. (4), we obtain

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \tag{5}$$

If f_x, f_y, f_z have a common factor say $\mu(x, y, z)$, then we may write

$$f_x = \mu P,\ f_y = \mu Q,\ f_z = \mu R,$$

where P, Q and R are functions of (x, y, z).

Eqn. (5) then reduces to

$$Pdx + Qdy + Rdz = 0 \qquad (6)$$

which is **total differential equation in three variables**. Thus, a one-parameter family of surfaces in 3-dimensions leads to a total differential equation in three variables. We can then integrate Eqn. (6), after multiplying it by the function $\mu(x, y, z)$ to obtain the family of surfaces (4).

We illustrate the method discussed above through the following examples.

Example 1: Find the total differential equation corresponding to the family of surfaces.

$$xy = c(a-z),$$

where c is a parameter.

Solution: The given family of surfaces is a one-parameter family of surfaces and can be expressed as

$$\frac{xy}{a-z} = c$$

The total derivative of the above relation gives

$$d\left(\frac{xy}{a-z}\right) = 0$$

$$\Rightarrow \quad \frac{(a-z)\, d(xy) - xy\, d(a-z)}{(a-z)^2} = 0$$

$$\Rightarrow \quad (a-z)\,[x\, dy + y\, dx] + xy\, dz = 0$$

which is the required differential equation corresponding to the given family of surfaces.

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Example 2: Find the total differential equation corresponding to the family of surfaces

$$x^2 + y^2 + z^2 = xc$$

with c as a parameter.

Solution: The given equation can be written as

$$\frac{x^2 + y^2 + z^2}{x} = c$$

The total derivative of the above equation gives

$$d\left(\frac{x^2 + y^2 + z^2}{x}\right) = 0$$

$$\Rightarrow \quad d\left(\frac{x^2}{x}\right) + d\left(\frac{y^2}{x}\right) + d\left(\frac{z^2}{x}\right) = 0$$

$$\Rightarrow \quad dx + \frac{2xydy - y^2dx}{x^2} + \frac{2xzdz - z^2dx}{x^2} = 0$$

$$\Rightarrow \quad x^2dx + 2xydy - y^2dx + 2xzdz - x^2dx = 0$$

$$\Rightarrow (x^2 - y^2 - z^2)dx + 2xydy + 2xzdz = 0$$

as the required total differential equation.

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You may now try the following exercise.

E1) Determine the total differential equation for the following family of surfaces, c being the defining parameter in each case.

i) $x^3z + x^2y = c$

ii) $x^2 + y^2 + (z+c)^2 = a^2$

iii) $xyz + x^2 - 2yz = c$

As you have seen above, starting with the relation (4) the Eqn. (6) can be formed. But does the converse also holds true? That is, does an equation of the form (6) always lead to the relation of the form (4)? This may not necessarily be possible always as the existence of the relation (4) implies that the three functions P, Q and R are proportional to the differential coefficient of one common function and this requirement may not be satisfied, in general, for any P, Q and R.

In the next section we shall seek the conditions on P, Q and R under which an equation of the form (6) always lead to the relation of the form (4). In other words, we shall obtain the conditions under which Eqn. (6) is integrable.

15.3 INTEGRABILITY OF TOTAL DIFFERENTIAL EQUATIONS

Let us consider the following differential equations

$$3x^2(y+z)dx + (z^2 + x^3)dy + (2yz + x^3)dz = 0 \tag{7}$$

$$(3xz + 2y)dx + xdy + x^2dz = 0 \tag{8}$$

$$ydx + (z - y)dy + xdz = 0 \tag{9}$$

They are all being of the form of Eqn. (1) are total differential equations.

Eqn. (7) is an exact differential of the function

$$f(x, y, z) = x^3y + x^3z + z^2y = c$$

where c is an arbitrary constant.
You can easily check that

$$d[x^3y + x^3z + z^3y] = 0$$

$$\Rightarrow 3x^2ydx + x^3dy + 3x^2zdx + x^3dz + z^2dy + 2zydz = 0$$

$$\Rightarrow 3x^2(y+z)dx + (x^3 + z^2)dy + (x^3 + 2zy)dz = 0$$

which is our Eqn. (7). Such an equation is called an **exact equation**. Thus Eqn. (7) is an exact equation.
Eqn. (8) is not an exact differential, but the use of x as an integrating factor yields

$$(3x^2z + 2xy)dx + x^2dy + x^3dz = 0$$

which is the exact differential of the function

$$f(x, y, z) = x^3z + x^2y = c, \ c \text{ being a constant.}$$

Eqn. (7) and (8) are called **integrable equations**.
Further, you can see that Eqn. (9) is not integrable as no function

$f(x, y, z) = c$

can be found for it whose exact differential leads to Eqn. (9)
We shall now state a theorem which gives the condition for the integrability of Eqn. (6).

Theorem 1: A **necessary** and **sufficient** condition that the total differential equation

$Pdx + Qdy + Rdz = 0$

is integrable is that

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad (10)$$

We shall not be proving the theorem here as it is beyond the scope of the present course. However, we shall illustrate it through examples.
You can see that in the case of Eqn. (8)

$P = 3xz + 2y, Q = x$ and $R = x^2$

$$\therefore \frac{\partial P}{\partial y} = 2, \frac{\partial P}{\partial z} = 3x, \frac{\partial Q}{\partial x} = 1, \frac{\partial Q}{\partial z} = 0, \frac{\partial R}{\partial x} = 2x, \frac{\partial R}{\partial y} = 0$$

and l.h.s. of Eqn. (10) becomes

$(3xz + 2y)\ (0 - 0) + x(2x - 3x) + x^2(2 - 1)$

$= 0 - x^2 + x^2 = 0 =$ r.h.s. of Eqn. (10)

Eqn. (8) is thus integrable.
Similarly, you can check that for Eqn. (9), l.h.s. of Eqn. (10) reduces to

$y(1-0) + (z - y)\ (1 - 0) + x(1 - 0)$

$= y + z - y + x = z + x \neq 0$

Eqn. (10) is not satisfied and hence Eqn. (9) is not integrable.

You may **note** that the condition (10) can be easily remembered because P, Q, R, x, y, z appear in it in a cyclic order. Another way to remember it is that condition (10) can be obtained by expanding the determinant given below in terms of the elements of its first row.

$$\begin{vmatrix} P & Q & R \\ P & Q & R \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{vmatrix} = 0$$

Further, the conditions for the equation $Pdx + Qdy + Rdz = 0$ to be exact are

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \qquad (11)$$

When conditions (11) are satisfied the condition (10) for integrability, namely,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

is also satisfied, for each term in brackets vanishes identically. Thus **an exact differential equation is always integrable**.
In the case of Eqn. (7) you will find that

$$P = 3x^2(y + z), \frac{\partial P}{\partial y} = 3x^2, \frac{\partial P}{\partial z} = 3x^2$$

$$Q = z^2 + x^3, \frac{\partial Q}{\partial x} = 3x^2, \frac{\partial Q}{\partial z} = 2z$$

$$R = 2yz + x^3, \ \frac{\partial R}{\partial x} = 3x^2, \ \frac{\partial R}{\partial y} = 2z$$

and condtions (11) are satisfied. Eqn. (7) is an exact equation.

Remember that a total differential equation may be integrable but may not necessarily be exact as is the case with Eqn. (8). Eqn. (8) when multiplied with x satisfies condition (10) but it does not satisfy condition (11).

When the given total differential equation is exact, then the equation can be integrated after regrouping its terms so that each new term is an exact differential.

We illustrate it through an example.

Example 3: Verify that the following differential equation is exact and find its solution

$$(yz + 2x)dx + (zx + 2y)dy + (xy + 2z)dz = 0.$$

Solution: For the given equation

$$P = (yz + 2x), Q = (zx + 2y), R = (xy + 2z)$$

Now, $\dfrac{\partial P}{\partial y} = z = \dfrac{\partial Q}{\partial x}$

$$\frac{\partial Q}{\partial z} = x = \frac{\partial R}{\partial y}$$

$$\frac{\partial R}{\partial x} = y = \frac{\partial P}{\partial z}.$$

Thus conditions (11) are satisfied. The given equation is exact.
Further, re-writing the given equation, we get

$$(yzdx + zxdy + xydz) + (2xdx + 2ydy + 2zdz) = 0$$

or, $\quad d(xyz) + d(x^2 + y^2 + z^2) = 0$

Integrating the above equation, we get

$$xyz + x^2 + y^2 + z^2 = c$$

as the required solution.

***

Before we take up various methods of solving equations of the form (6), you may try the following exercises to check your understanding of what we have discussed above.

E2) Verify that the following total differential equations are integrable but not exact

i) $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0$

ii) $y^2dx - zdy + ydz = 0$

iii) $(2x^3y + 1)dx + x^4dy + x^2 \tan z dz = 0.$

E3) Verify that the following total differential equations are exact and find their solutions

i) $(x - y)dx - xdy + zdz = 0$

ii) $(y - z)(y + z - 2x)dx + (z - x)(z + x - 2y)dy$
$\quad + (x - y)(x + y - 2z)dz = 0$

iii) $(3x^2y^2 - e^x z)dx + (2x^3 y + \sin z)dy + (y\cos z - e^x)dx = 0$

We now discuss various methods of finding solutions of total differential equations.

15.4 METHODS OF INTEGRATION

Consider the total differential Eqn. (6), namely,

$$P\ dx + Q\ dy + R\ dz = 0$$

We now discuss the various methods of solving it when it is integrable i.e., when the integrability condition (10) is satisfied by it.

15.4.1 By Inspection

Sometimes rearranging the terms of the given equation and/or dividing the terms by a suitable function of x, y, z, we may get an equation whose terms can be combined to get exact differentials. These terms can then be integrated to obtain the desired solution as illustrated through the following examples:

Example 4: Verify that the given equation

$$yz\ dx + zx\ dy + xy\ dz = 0$$

is integrable, and find its integral surfaces.

Solution: Here $P = yz, Q = zx, R = xy$.

$$\frac{\partial P}{\partial y} = z,\ \frac{\partial P}{\partial z} = y,\ \frac{\partial Q}{\partial x} = z,\ \frac{\partial Q}{\partial z} = x,\ \frac{\partial R}{\partial x} = y,\ \frac{\partial R}{\partial y} = x$$

Applying condition (10) of integrability, we get

l.h.s. $= yz(x-x) + zx(y-y) + xy(z-z) = 0 =$ r.h.s.

and hence the given equation is integrable.
By looking at the form of the equation we can immediately write it as

$$d\ (xyz) = 0$$

so that the required solution of the given equation is $xyz = c$, where c is a constant.

***

Let us consider another example.

Example 5: Verify that the equation

$$(x^2 z - y^3)\ dx + 3xy^2 dy + x^3 dz = 0$$

is integrable, and find its integral surfaces.

Solution: Here $P = x^2 z - y^3,\ Q = 3xy^2,\ R = x^3$.
Applying condition (10), we get
l.h.s. $= (x^2 z - y^3)\ (0-0) + 3xy^2(3x^2 - x^2) + x^3(-3y^2 - 3y^2)$

$= 0 + 6x^3y^2 - 6x^3y^2$

$= 0 =$ r.h.s

Thus, the given equation is integrable and may be re-written as

$$x^2(zdx + xdz) - y^3dx + 3xy^2dy = 0$$

$$\Rightarrow \ (zdx + xdz) - \frac{y^3}{x^2}dx + \frac{3y^2}{x}dy = 0$$

$$\Rightarrow \ d(xz) + d\left(\frac{y^3}{x}\right) = 0$$

Thus, the integral surfaces of the given equation are

$$xz + \left(\frac{y^3}{x}\right) = c,$$

where c is a constant.

***

Note that for a given problem unless mentioned, you need not verify the integrability condition and directly proceed to find its solution.

Example 6: Find the integral surfaces of the following total differential equation

$$z(1 - z^2)dx + zdy - (x + y + xy^2)dz = 0.$$

Solution: The given equation can be written as

$$z(dx + dy) - z^2(zdx + xdz) - (x + y)dz = 0$$

or, $$zd(x + y) - z^2d(xz) - (x + y)dz = 0.$$

Dividing the above equation by z^2, we obtain

$$\frac{zd(x + y) - (x + y)dz}{z^2} - d(xz) = 0$$

or, $$d\left(\frac{x + y}{z}\right) - d(xz) = 0$$

Integrating, we get the required integral surfaces as

$$\frac{x + y}{z} - xz = c$$

where c is a constant.

***

You may now try the following exercises.

E4) Verify that the following equations are integrable and find their integral surfaces.

i) $yz\,dx + zx\,dy - y^2dz = 0$

ii) $(y^2 + z^2)\,dx + xy\,dy + xz\,dz = 0$

iii) $(y + z)\,dx + dy + dz = 0$.

E5) Find $f(z)$ such that $[(y^2 + z^2 - x^2)/2x]dx - ydy + f(z)dz = 0,\ x > 0$, is integrable. Hence solve it.

You may recall that in Unit 7 of Block 2, we discussed differential equations in two variables for which variables were separable. In the next sub-section, we take up total differential equations in three variables for which variables are separable.

15.4.2 Variables Separable

Let us look at Example 4 once again. The given differential equation to be solved is

$$yz\,dx + zx\,dy + xy\,dz = 0$$

If we divide the equation by xyz, we obtain

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \qquad (12)$$

You may notice that Eqn. (12) is in variable separable form as each of the terms is a function of one variable only. Eqn. (12) can be easily integrated to obtain the required solution as

$$\ln|x| + \ln|y| + \ln|z| = \ln c$$

or, $|xyz| = c$, where c is a positive constant.

Similarly, in certain cases, it is possible to write Eqn. (6) in the form

$$P(x)dx + Q(y)dy + R(z)dz = 0 \qquad (13)$$

where variables are separable and we can obtain the integral surfaces by integrating Eqn. (13) as

$$\int P(x)\,dx + \int Q(y)\,dy + \int R(z)\,dz = c,$$

where c is an arbitrary constant.

We consider some more examples to illustrate the method above.

Example 7: Solve the differential equation

$$a^2y^2z^2dx + b^2z^2x^2dy + c^2x^2y^2dz = 0.$$

Solution: On dividing both the sides of this equation by $x^2y^2z^2$, we have

$$\frac{a^2}{x^2}dx + \frac{b^2}{y^2}dy + \frac{c^2}{z^2}dz = 0,$$

which is in variable separable form and its integral is given by

$$\int \frac{a^2}{x^2}dx + \int \frac{b^2}{y^2}dy + \int \frac{c^2}{z^2}dz = 0$$

$$\Rightarrow \quad \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = k,$$

where k is a constant.

***

Example 8: Solve the differential equation

$$2yz\,dx - 2xz\,dy - xyz(z-1)dz = 0.$$

Solution: Dividing the given equation by xyz, we get

$$\frac{2}{x}dx - \frac{2}{y}dy - (z-1)dz = 0.$$

Integrating, we obtain the required solution as

$$2\ln|x| - 2\ln|y| - \left(\frac{z^2}{2} - z\right) = c$$

or, $\dfrac{x^2}{y^2} = c_1 e^{z\left(\frac{z}{2}-1\right)}$

where c and c_1 are arbitrary constants.

***

And now a few exercise for you.

E6) Solve the differential equation

$$yz \ln z \, dx - zx \ln z \, dy + xy \, dz = 0, \; z > 0$$

E7) Express the differential equation

$$f(z)\,(ydx + xdy) + f'\,(z)\; xy \; dz = 0$$

where $f'(z) = \frac{df}{dz}$, in the variable separable form and hence find its integral surfaces.

Sometimes it may happen that in a given total differential equation of the form (6) not all the variables are separable but only one variable is separable. We now take up such differential equations.

15.4.3 One Variable Separable

Let us start by considering the following example:

Example 9: Find the integral surfaces of the equation

$$(x^2 - y^2)\, dx - 2xydy + e^{-z}dz = 0.$$

Solution: You may note that the given equation is separable in z.
Consider the first two terms of the equation viz.,

$$(x^2 - y^2)dx - 2xydy.$$

Here $P = x^2 - y^2$ and $Q = -2xy$.

These terms are exact differential if P and Q satisfy the exactness condition for $Pdx + Qdy = 0$ to be exact, i.e.,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \text{ (ref. Sec. 7.4, Unit 7)}$$

Here $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2 - y^2) = -2y + 2y = 0$

Hence the condition is satisfied. The first two terms are exact differential and can be written as

$$x^2dx + (-y^2dx - 2xy\; dy)$$

or, $$d\left(\frac{x^3}{3}\right) + d(-xy^2).$$

Combining this with the third term of the given equation, we get

$$d\left(\frac{x^3}{3}\right) + d\;(-xy^2) + d\,(-e^{-z}) = 0$$

Integrating the above equation the required integral surfaces are obtained as

$$\frac{x^3}{3} - xy^2 - e^{-z} = c,$$

where c is an arbitrary constant.

***

In general, let us consider Eqn. (6) and assume that the variable z is separable. Then it can be written in the form

$$P(x, y)\,dx + Q(x, y)\,dy + R(z)\,dz = 0 \tag{14}$$

The condition (10) of inegrability of Eqn. (6) in this case reduces to

$$P(x, y)[0-0] + Q(x, y)[0-0] + R(z)\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = 0$$

or, $\quad R(z)\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] = 0$

Now, since $R(z) \neq 0$, then $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$. (15)

Did you **notice** here that Eqn. (15) gives the condition for an equation $P\,dx + Q\,dy = 0$ to be an exact equation? We can therefore remark that the expression $Pdx + Qdy$ is an exact differential, say du, and Eqn. (14) thus reduces to

$$du + R(z)\,dz = 0$$

Integrating the above equation, we get

$$u + \int R(z)\,dz = c$$

which is the required solution.

Let us take up another example to illustrate the method above.

Example 10: Solve the following differential equation

$$(2x^3y+1)dx + x^4dy + x^2\tan z\,dz = 0.$$

Solution: Dividing the given equation throughout by x^2, we obtain

$$\frac{(2x^3y+1)dx}{x^2} + x^2dy + \tan z\,dz = 0 \tag{16}$$

Eqn. (16) is separable in z.

Here $P = \dfrac{2x^3y+1}{x^2}$, $Q = x^2$.

Also, $\dfrac{\partial P}{\partial y} = \dfrac{2x^3}{x^2} = 2x$ and $\dfrac{\partial Q}{\partial x} = 2x$.

Thus P and Q satisfy the condition $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$.

Hence the expression $Pdx + Qdy$ is an exact differential. We can write $Pdx + Qdy$ in Eqn. (16) as

$$\left(2xy + \frac{1}{x^2}\right)dx + x^2dy = d(x^2y) + d\left(-\frac{1}{x}\right)$$

The given equation can then be written as

$$d\left(x^2y - \frac{1}{x}\right) + \tan z\,dz = 0$$

Integrating we obtain

$$x^2y - \frac{1}{x} + \ln|\sec z| = c$$

as the required solution where c is a constant.

***

You may, now, try the following exercise.

E8) Verify that the following differential equations are integrable and solve them:

i) $x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0$

ii) $2yz\,dx - 2xz\,dy - (x^2 - y^2)\,(z-1)\,dz = 0$

You may recall that in Sec. 7.3 of Unit 7, Block 2, we defined homogeneous functions of two variables and discussed the methods of solving homogeneous differential equations in two variables. We now extend the definition of homogeneous functions to three variables and also discuss the method of solving homogeneous differential equations in three variables.

Definition: A real-valued function $P(x, y, z)$ of three variables x, y, z is called a **homogeneous function** of degree n, where n is a real number, if we have

$$P(x, y, z) = x^n f_1(y/x, z/x) = x^n f_1(u, v)$$

where $u = y/x$ and $v = z/x$.

Equivalently, we can have

$$P(x, y, z) = y^n f_2\left(\frac{x}{y}, \frac{z}{y}\right)$$

or, $$P(x, y, z) = z^n f_3\left(\frac{x}{z}, \frac{y}{z}\right).$$

For example, the function

$$f(x, y, z) = x^3 + y^3 + 3xz^2 = x^3 f_1\left[1 + \left(\frac{y}{x}\right)^3 + 3\left(\frac{z}{x}\right)^2\right] = x^3 f_1\left(\frac{y}{x}, \frac{z}{x}\right)$$

or, $$f(x, y, z) = x^3 + y^3 + 3xz^2 = y^3 f_3\left[\left(\frac{x}{y}\right)^3 + 1 + 3\left(\frac{x}{y}\right)\left(\frac{z}{y}\right)^2\right] = y^3 f_2\left(\frac{x}{y}, \frac{z}{y}\right)$$

or, $$f(x, y, z) = z^3 f_3\left[\left(\frac{x}{z}\right)^3 + \left(\frac{y}{z}\right)^3 + 3\left(\frac{x}{z}\right)\right] = z^3 f_3\left(\frac{x}{z}, \frac{y}{z}\right)$$

is a homogeneous function in (x, y, z) of degree 3.

A total differential equation of the form (6), namely,

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0$$

in which $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ are homogeneous functions of the same degree, is called the **homogeneous total differential equation**.

For instance, consider the total differential equation

$$(y^2 + yz + z^2)dx + (z^2 + xz + x^2)dy + (x^2 + xy + y^2)dz = 0 \tag{17}$$

Here, $P = y^2 + yz + z^2 = x^2(u^2 + uv + v^2)$

$Q = z^2 + xz + x^2 = x^2(v^2 + v + 1)$

$R = x^2 + xy + y^2 = x^2(1 + u + u^2)$

where $u = y/x$ and $v = z/x$.

Thus P, Q, R are all homogeneous functions of degree 2 and Eqn. (17) is a homogeneous equation.

In the next sub-section, we take up the method of solving such homogeneous equations.

15.4.4 Homogeneous Total Differential Equations

Let us start by considering a simple example.

Example 11: Solve the following differential equation

$$2(y+z)dx-(x+z)dy+(2y-x+z)dz=0 \tag{18}$$

Solution: In Eqn. (18), we have, $P=y+z=x(u+v)$, $Q=-(x+z)=-x(1+v)$ and $R=(2y-x+z)=x(2u-1+v)$, where $u=y/x$, $v=z/x$. P, Q and R are all homogeneous functions of degree one and hence Eqn. (18) is a homogeneous equation.

You may recall here the method you learnt in Sec.7.3 of Unit 7 for solving homogeneous differential equations in two variables. Extending the method to differential equations in three variables, let us make use of the substitution

$$y=xu \text{ and } z=xv \tag{19}$$

Putting the values from Eqn. (19) in Eqn. (18), we get

$$2(xu+xv)dx-(x+xv)(xdu+udx)+(2xu-x+xv)(xdv+vdx)=0$$

or, $(u+v)(1+v)dx-x(1+v)du+x(2u-1+v)dv=0$

Dividing the above equation throughout by $x(u+v)(v+1)$, we get

$$\frac{dx}{x}-\frac{1}{u+v}du+\frac{(2u-1+v)}{(u+v)\,(1+v)}\,dv=0 \tag{20}$$

You may note here that in Eqn. (20) one variable i.e., x is separable.

Using the method discussed in Sub-sec. (15.4.3) we find the expression

$$\frac{-1}{u+v}du+\frac{(2u-1+v)}{(u+v)\,(1+v)}\,dv \tag{21}$$

to be an exact differential.

By combining and re-arranging the terms of expresion (21), we can write Eqn. (20) in the following form

$$\frac{dx}{x}-\frac{1}{u+v}du+\frac{2dv}{1+v}-\frac{1}{u+v}dv=0$$

$$\Rightarrow\ \frac{dx}{x}-\left(\frac{du+dv}{u+v}\right)+\frac{2}{1+v}dv=0$$

Integrating the above equation, we get

$$\ln|x|-\ln|(u+v)|+2\ln|(1+v)|=\ln|c|$$

or, $\left|\dfrac{x(1+v)^2}{u+v}\right|=c$

substituting the values of u and v from Eqn. (19) in the above equation, we get the desired solution as

$$(x+z)^2=c\,|(y+z)|.$$

***

The method illustrated above can be used in general for the total differential Eqn. (6) in which the functions P, Q and R are homogeneous in x, y, z of the same degree, say n.

In order to find the solution, we use the substitution

$$y=u\ x \text{ and } z=v\ x \tag{22}$$

in Eqn. (6). On cancelling out factor x^n throughout, the equation reduces to

$$P(u,v)dx+Q(u,v)\,(u\ dx+x\ du)+R(u,v)\,(v\ dx+x\ dv)=0$$

$$\Rightarrow\ [P(u,v)+uQ(u,v)+vR(u,v)]\,dx+xQ(u,v)du+xR(u,v)\,dv=0$$

$$\Rightarrow \quad \frac{dx}{x} + A(u, v)\,du + B(u, v)\,dv = 0, \tag{23}$$

where $A(u, v) = \dfrac{Q(u, v)}{P(u, v) + uQ(u, v) + vR(u, v)}$, (24)

and $B(u, v) = \dfrac{R(u, v)}{P(u, v) + uQ(u, v) + vR(u, v)}$. (25)

Note that Eqn. (23) is of the same type as Eqn. (14), i.e., one variable separable. Condition (15), in this case, reduces to

$$\frac{\partial A}{\partial v} = \frac{\partial B}{\partial u} \tag{26}$$

Once the condition (26) is satisfied, we can obtain the solution of Eqn. (23) in the form

$$f\ (u, v) + \ln | x | = c \tag{27}$$

where c is an arbitrary constant.

We can then use Eqn. (22) and replace u, v by their values in terms of x, y and z to obtain the solution of Eqn. (6) in the form

$$f\,(y/x, z/x) + \ln | x | = c.$$

Note that in Eqn. (22) we have used the transformation $y = ux$ and $z = vx$ and obtained the equation with variable x as separable. Instead, we could also use other transformations like $x = uz$ and $y = vz$ or, $x = uy$ and $z = vy$ and accordingly obtain the equations separable in variables z and y, respectively.

For instance, in Example 11, if we use the substitution

$$x = uz \text{ and } y = vz$$

then Eqn. (18) reduces to the following form

$$z[2(v+1)du - (u+1)dv] + (u+1)\,(v+1)dz = 0$$

or, $$2\frac{du}{u+1} - \frac{dv}{v+1} + \frac{dz}{z} = 0.$$

The above equation is with variable z separable and it can be integrated to obtain the desired solution.

We now illustrate this method with the help of a few examples.

Example 12: Verify that the integrability condition for the equation

$$y(y+z)dx + z(z+x)dy + y(y-x)dz = 0$$

is satisfied and find its integral surfaces.

Solution: Here $P = y(y+z), Q = z(z+x)$ and $R = y(y-x)$, since P, Q, R are homogeneous functions of x, y, z of degree 2 therefore the given equation is homogeneous. Further, for the case under consideration, the integrability condition (10), yields

$$\text{l.h.s.} = (y^2 + yz)(2z + x - 2y + x) + (z^2 + zx)(-y - y) + (y^2 - yx)(2y + z - z)$$

$$= 2(y^2 + yz)(x + z - y) - 2y(xz + z^2) + 2y(y^2 - xy) = 0 = \text{r.h.s.}$$

Hence the integrability condition is satisfied.

Putting $y = x\,u$ and $z = x\,v$, the given equation reduces to

$$xu(xu + xv)dx + xv(xv + x)\,(xdu + udx) + xu(xu - x)\,(xdv + vdx) = 0,$$

$\Rightarrow \quad x^2u(v+1)\,(u+v)\,dx + x^3v(v+1)\,du + x^3u(u-1)\,dv = 0$,

$$\Rightarrow \quad \frac{dx}{x} + \frac{v}{u(u+v)}du + \frac{u-1}{(v+1)\,(u+v)}\,dv = 0,$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{du}{u} + \frac{dv}{v+1} - \frac{du+dv}{u+v} = 0.$$

Integrating, we get

$\ln|x| + \ln|u| + \ln|(v+1)| - \ln|(u+v)| = \ln|c|$, say

$$\Rightarrow \quad \left|\frac{xu(v+1)}{u+v}\right| = c$$

Substituting back the values of u and v in terms of $x,\ y$ and z, we get the required integral surfaces in the form

$|y(x+z)| = c\,|(y+z)|$.

***

Remark: By looking at the forms of $A(u,\ v)$ and $B(u,\ v)$ in Eqns. (24) and (25) respectively, one can remark that in Eqn. (6) if the condition of integrability is satisfied and $P,\ Q,\ R$ are homogeneous function of $x,\ y,\ z$ of the same degree and also $xP + yQ + zR$ does not vanish identically then $\frac{1}{xP + yQ + zR}$ is the integrating factor of the given equation.

Let us take up an example to examine the above claim.

Example 13: Verify that the equation

$$(x^2 + xy + yz)\,dx - x(x+z)\,dy + x^2dz = 0. \qquad (28)$$

is integrable and determine its solution.

Solution: Here $P = x^2 + xy + yz,\ Q = -x(x+z),\ R = x^2$.

The given equation is homogeneous. You may also check that the given equation is integrable (see E9)).

Further, $xP + yQ + Rz = x(x^2 + xy + yz) - xy(x+z) + x^2z$

$$= x^2(x+z)$$

$$\neq 0$$

Hence $\frac{1}{x^2(x+z)}$ is an I.F. for the given equation.

Multiplying the given equation by $\text{I.F} = \frac{1}{x^2(x+z)}$, we obtain

$$\frac{x^2 + (x+z)y}{x^2(x+z)}dx - \frac{x(x+z)}{x^2(x+z)}dy + \frac{x^2}{x^2(x+z)}dz = 0,$$

combining various terms in the equation above, we get

$$\frac{1}{x+z}dx + \frac{y}{x^2}dx - \frac{dy}{x} + \frac{1}{x+z}dz = 0,$$

or, $$\frac{d(x+z)}{x+z} + \frac{d}{dx}\left(-\frac{y}{x}\right) = 0.$$

which is an exact differential.
Integrating, we get the required solution as

$$\ln|(x+z)| - \frac{y}{x} = c\,,$$

where c is an arbitrary constant.

***

You may now try the following exercises.

E9) Verify that Eqn. (28) is integrable.

E10) Verify that the following equations are integrable and determine their solutions.

i) $yz^2(x^2 - yz)dx + zx^2(y^2 - zx)dy + xy^2(z^2 - xy)dz = 0$

ii) $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0$

iii) $(y^2 + z^2)dx + xy\,dy + xz\,dz = 0$

iv) $yz(y+z)dx + xz(x+z)dy + xy(x+y)dz = 0$

We now conclude this unit by giving a summary of what we have covered in it.

15.5 SUMMARY

In this unit, we have covered the following:

1. An equation of the form

$$\sum_{i=1}^{n} F_i(x_1, x_2, \ldots, x_n)\, dx_i = 0$$

where $F_i (i = 1, 2, \ldots, n)$ are continuous functions of some or all of the n independent variables $x_1, x_2, \ldots, x_n$ is called a total differential equation.

2. One-parameter family of surfaces in 3-dimensional space gives rise to a total differential equation in three variables of the form

$$Pdx + Qdy + Rdz = 0.$$

3. Total differential equation $P\,dx + Q\,dy + R\,dz = 0$ is integrable, if and only if

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

If integrable, then its integral curve is a one-parameter family of surfaces in 3-dimensional space.

4. Total differential Eqn. (6), viz.,

$$P\,dx + Q\,dy + R\,dz = 0$$

can be solved if

i) it is exact and then solution is evident after, at most, regrouping of terms **by inspection**.

ii) it is integrable and can be written in variable separable form

$$X(x)dx + Y(y)dy + Z(z)dz = 0$$

Its integral then yields the solution by **variable separable method**.

iii) it is integrable and can be expressed in the form

$$P(x,\ y)dx + Q(x,\ y)dy + R(z)dz = 0$$

with **one variable separable** with the expression

$$P(x,\ y)dx + Q(x,\ y)\ dy$$

as exact function say, du of x and y. The solution can then be written as

$$u(x,\ y) + \int R(z)dz = \text{constant}.$$

iv) it is homogeneous and integrable by separating one variable, say z, using the transformation $x = uz,\ y = vz$. The equation can then be integrated by the method given in 4. iii) above.

Further, in case, if $(xP + yQ + zR) \neq 0$, then $\dfrac{1}{xP + yQ + zR}$ is an integrating factor of homogeneous integrable equation of the form (6).

15.6 SOLUTIONS/ANSWERS

E1) i) The given family of surfaces is

$$x^3 z + x^2 y = c$$

It is a one-parameter family of surfaces.
The total derivative of the above relation gives

$$d(x^3 z + x^2 y) = 0$$

$$\Rightarrow\ x^3 dz + 3x^2 dx.z + x^2 dy + 2x\,dx.\,y = 0$$

$$\Rightarrow\ (3x^2 z + 2xy)\ dx + x^2 dy + x^3 dz = 0$$

$$\Rightarrow\ (3xz + 2y)dx + xdy + x^2 dz = 0 \text{ (dividing by } x\text{, as } x \neq 0\text{)}$$

which is the required total differential equation.

ii) The given family of surfaces can be written as

$$(z + c)^2 = a^2 - x^2 - y^2$$

$$\Rightarrow\ c = \sqrt{a^2 - x^2 - y^2} - z$$

Taking total derivative of the above equation, we get

$$d\left(\sqrt{a^2 - x^2 - y^2}\right) - dz = 0$$

$$\Rightarrow\ \frac{1}{2}\frac{1}{\sqrt{a^2 - x^2 - y^2}}(-2xdx - 2ydy) - dz = 0$$

$$\Rightarrow\ xdx + ydy + \sqrt{a^2 - x^2 - y^2}\ dz = 0$$

which is the required total differential equation.

iii) $(yz + 2x)dx + (xz - 2z)dy + (xy - 2y)dz = 0$.

E2) i) Here $P = y^2 + z^2 - x^2,\ Q = -2xy,\ R = -2xz$

The integrability condition (10) assume the following form

l.h.s. $= (y^2+z^2-x^2)(0-0) - 2xy(-2z-2z) - 2xz(2y+2y)$

$$= 8xyz - 8xyz = 0 = \text{r.h.s.}$$

which indicates that condition (10) is satisfied.

Also, $\dfrac{\partial R}{\partial x} = -2z \neq 2z = \dfrac{\partial P}{\partial z}$ and $\dfrac{\partial P}{\partial y} = 2y \neq -2y = \dfrac{\partial Q}{\partial x}$

$\therefore$ Given equation is not exact.

ii) $P = y^2,\ Q = -z,\ R = y$

Integrability condition (10) is satisfied. But $\dfrac{\partial Q}{\partial z} \neq \dfrac{\partial R}{\partial y}$ and $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$. The given equation is not exact.

iii) $P = 2x^3y + 1,\ Q = x^4,\ R = x^2 \tan z$

Integrability condition (10) is satisfied. But $\dfrac{\partial R}{\partial x} \neq \dfrac{\partial P}{\partial z}$ and $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$. The given equation is not exact.

E3) i) $P = x - y,\ Q = -x,\ R = z$.

Here $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y} = 0,\ \dfrac{\partial R}{\partial x} = \dfrac{\partial P}{\partial z} = 0$ and $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} = -1$

$\therefore$ The given equation is exact.

Also, $(x-y)dx - xdy + zdz = 0$

$$\Rightarrow\ xdx - (ydx + xdy) + zdz = 0$$

$$\Rightarrow\ x^2 - 2xy + z^2 = c$$

which is the required solution.

ii) Here $P = (y-z)(y+z-2x),\ Q = (z-x)(z+x-2y)$,
$R = (x-y)(x+y-2z)$

The given equation is exact since

$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = -2y + 2z,\ \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} = 2x - 2z \text{ and } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2y - 2x$$

We can re-write the equation in the form

$$(y^2dx + 2xydy) - (z^2dx + 2zxdz) + (z^2dy + 2yzdz) - (x^2dy + 2xydy)$$
$$+ (x^2dz + 2xzdx) - (y^2dz + 2yzdy) = 0$$
$$\Rightarrow\ d(y^2x) - d(z^2x) + d(z^2y) - d(x^2y) + d(x^2z) - d(y^2z) = 0$$

Integrating, we get the required solution as

$$y^2x - z^2x + z^2y - x^2y + x^2z - y^2z = c.$$

iii) Here $P = (3x^2y^2 - e^xz),\ Q = (2x^3y + \sin z),\ R = (y\cos z - e^x)$

Since $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y} = \cos z,\ \dfrac{\partial R}{\partial x} = \dfrac{\partial P}{\partial z} = -e^x$ and $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} = 6x^2y$

the given equation is exact.
Re-arranging the terms of the given equation it can be written as

$$(3x^2y^2dx + 2x^3ydy) - (e^x zdx + dz) + (\sin zdy + y\cos zdz) = 0$$

$$\Rightarrow \ d(x^3y^2) - d(e^x z) + d(y\sin z) = 0$$

Integrating, the required solution is

$$x^3y^2 - e^x z + y\sin z = c.$$

E4) i) Integrable; $\frac{x}{y} - \ln|z| = c$

Hint: Dividing by $y^2 z$, the given equation can be written as

$$\frac{dx}{y} - \frac{x}{y^2}dy - \frac{dz}{z} = 0$$

$$\Rightarrow \ d\left(\frac{x}{y}\right) - d(\ln|z|) = 0.$$

ii) Integrable; $|x|.\sqrt{y^2+z^2} = c$

Hint: Given equation can be written as

$$\frac{dx}{x} + \frac{y\,dy + z\,dz}{y^2+z^2} = 0$$

$$\Rightarrow \ d\left(\ln|x| + \frac{1}{2}\frac{d(y^2+z^2)}{y^2+z^2}\right) = 0.$$

iii) Integrable; $x + \ln|y+z| = c$

Hint: Given equation can be written as

$$dx + \frac{dy+dz}{y+z} = 0.$$

E5) Multiplying throughout by $2x$, the given equation reduces to

$$(y^2+z^2-x^2)dx - 2xydy + 2xf(z)dz = 0 \qquad (29)$$

Here $P = y^2+z^2-x^2,\ Q = -2xy,\ R = 2xf(z).$ (30)

If Eqn. (29) is integrable then it must satisfy the integrability condition

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad (31)$$

Substituting in Eqn. (31) from Eqn. (30), we get

$$(y^2+z^2-x^2)\,(0-0) - 2xy(2f(z) - 2z) + 2xf(z)(2y+2y) = 0$$

$$\Rightarrow \ -4xy(f(z) - z) + 8xy\ f(z) = 0$$

$$\Rightarrow \ f(z) = -z$$

Putting the above value of $f(z)$ in Eqn. (29), we get

$$(y^2+z^2-x^2)dx - 2x\ ydy - 2x\,zdz = 0$$

or, $(y^2+z^2+x^2)dx - 2x^2dx - 2xy\ dy - 2xz\,dz = 0$

or, $(y^2+z^2+x^2)dx - 2x(x\,dx + y\ dy + z\ dz) = 0$

or, $\dfrac{dx}{x} = \dfrac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$

Integrating the above equation we get the required solution as

$$\ln|x| + \ln|c| = \ln|(x^2+y^2+z^2)|$$

or, $$|xc| = (x^2+y^2+z^2).$$

E6) The given equation is

$$yz\ \ln z\ dx - zx\ \ln z\ dy + xy\ dz = 0,\ z > 0$$

Dividing throughout by $xyz\ln z$, we get

$$\frac{dx}{x} - \frac{dy}{y} + \frac{dz}{z\ln z} = 0$$

It is in variable separable form. Integrating, we have

$$\ln|x| - \ln|y| + \ln(\ln z) = \ln c$$

$$\Rightarrow\ \ln\left(\frac{|x|}{|y|}.\ln z\right) = \ln c$$

$$\Rightarrow\ c|y| = |x|\ln z.$$

E7) The given equation is

$$f(z)\ (y\ dx + x\ dy) + f'(z)\ xy\,dz = 0$$

$$\Rightarrow\ \frac{ydx + xdy}{xy} + \frac{f'(z)}{f(z)}dz = 0$$

$$\Rightarrow\ \frac{dx}{x} + \frac{dy}{y} + \frac{f'(z)}{f(z)}dz = 0$$

It is in variable separable form. Integrating, we get

$$\ln|x| + \ln|y| + \ln|f(z)| = \ln c$$

$$\Rightarrow\ |x|.|y|.|f(z)| = c.$$

E8) i) The given equation is

$$x(y^2-a^2)dx + y(x^2-z^2)dy - z(y^2-a^2)dz = 0$$

Dividing throughout by $(y^2-a^2)\ (x^2-z^2)$, we get

$$\frac{xdx - zdz}{x^2-z^2} + \frac{ydy}{y^2-a^2} = 0 \qquad (32)$$

Thus given equation is separable in y.
It is, therefore, integrable if

$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$

where $P = \dfrac{x}{x^2-z^2},\ R = \dfrac{-z}{x^2-z^2}$

Here $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$ and the given equation is integrable.

Eqn. (32) can be written as

$$\frac{1}{2} d \frac{(x^2 - z^2)}{x^2 - z^2} + \frac{1}{2} \frac{d(y^2 - a^2)}{y^2 - a^2} = 0$$

Integrating, we get

$$\frac{1}{2} \ln | x^2 - z^2 | + \frac{1}{2} \ln | y^2 - a^2 | = \frac{1}{2} \ln c \Rightarrow | x^2 - z^2 | . | y^2 - a^2 | = c$$

where c is an arbitrary constant.

ii) Integrable; $\ln \left| \frac{x-y}{x+y} \right| - z + \ln | z | = c$

Hint: Given equation can be written as

$$\frac{2y dx - 2x dy}{x^2 - y^2} - \frac{z-1}{z} dz = 0 .$$

E9) Integrable.

E10) i) The given equation is

$$yz^2 (x^2 - yz) dx + zx^2 (y^2 - zx) dy + xy^2 (z^2 - xy) dz = 0$$

Check that it satisfies the integrability condition (10).
The given equation is a homogeneous total differential equation.
Substituting $x = uz$ and $y = vz$, it is reduced to

$$v(u^2 - v)\, du + u^2 (v^2 - u)\, dv = 0$$

Dividing the above equation by $u^2 v^2$, we get

$$\left(\frac{1}{v} - \frac{1}{u^2} \right) du + \left(1 - \frac{u}{v^2} \right) dv = 0$$

$$\Rightarrow \left(\frac{du}{v} - \frac{u}{v^2} dv \right) + dv - \frac{1}{u^2} du = 0$$

$$\Rightarrow d \left(\frac{u}{v} + v + \frac{1}{u} \right) = 0$$

Integrating, we get

$$\frac{u}{v} + v + \frac{1}{u} = \text{constant}$$

$\Rightarrow \frac{x}{y} + \frac{z}{x} + \frac{y}{z} = c$, (substituting for u and v in terms of x, y, z)

where c is an arbitrary constant.

ii) Integrable; $\frac{| x + z | | y |}{| y + z |} = c$.

Hint: Substituting $x = uz,\ y = vz$ and simplifying, we get

$$\frac{(v^2 + v) du + (u+1) dv}{(u+1)\,(v^2 + v)} + \frac{dz}{z}$$

or, $\frac{du}{u+1} + \frac{dv}{v^2 + v} + \frac{dz}{z} = 0$

iii) Integrable; $|x||y^2+z^2|^{1/2}=c$

Hint: Substituting $y = xu,\ z = xv$ and simplifying, we get

$$\frac{2dx}{x}+\frac{udu+v\,dv}{u^2+v^2}=0$$

iv) Integrable; $|xyz| = c\,|(x+y+z)|$

– x –

UNIT 16

LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Structure **Page No.**

16.1 INTRODUCTION

In Units 14 and 15 we have presented the necessary tools required for the study of partial differential equations (PDEs). Many problems in geometry, physics and other areas of science and engineering, when formulated mathematically give rise to PDEs. Such equations arise when the number of independent variables in the problem under discussion is two or more. Any dependent variable is then likely to be a function of more than one variable and possesses not the ordinary derivatives with respect to a single variables, but partial derivatives with respect to several variables. For instant, if u is a function of (x, y, t), then

$$u_t + u_x = u \qquad (1)$$

is a partial differential equation. Other well known examples of PDEs in two dimensions are

$$u_{xx} + u_{yy} = 0 \qquad \text{(Laplace equation)} \qquad (2)$$

$$u_{xx} + u_{yy} = \frac{1}{k} u_t \qquad \text{(Heat equation)} \qquad (3)$$

$$u_{xx} + u_{yy} = c^2 u_{tt} \qquad \text{(Wave equation)} \qquad (4)$$

In general, a PDE may be written in the form

$$f(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots) = 0 \qquad (5)$$

which involves several independent variables $x, y, \ldots,$ an unknown function u of these variables, and partial derivatives $u_x, u_y, \ldots u_{xx}, u_{yy} \ldots$ of the function. Eqn. (5) is defined in a suitable domain D of the n-dimensional space $\mathbb{R}^n$ in n independent variables $x, y, \ldots$. As in the case of ordinary differential equations, the order of a PDE is the order of the highest order partial derivative occurring in the equation. Thus, Eqn. (1) is a PDE of first order whereas, Eqns. (2)-(4) are second order PDEs. In this unit we shall concentrate on the first order partial differential equations. In Sec. 16.2, we shall begin with the origin of partial differential equations, restricting ourselves to one dependent variable z and two independent variables (x and y).

Unlike ordinary differential equations, where equations are either linear or non-linear (refer Unit 6 of Block 2), partial differential equations have further classification of linear equations. In Sec. 16.3 we have taken up this classification for first order partial differential equations. Also, the classification of integrals/solutions of partial differential equations of first order, as made by Lagrange (1736-1813), an Italian mathematician, in 1769 has been discussed in Sec. 16.4 of the unit.

Objectives

After studying this unit, you should be able to:

- describe the origin of the first order partial differential equations;
- identify linear, semi-linear, quasi-linear and non-linear PDEs of the first order; and
- distinguish the integrals of first order PDEs into the complete integral, the general integral, the singular integral and the special integral.

16.2 ORIGIN OF THE FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

First order partial differential equations can arise in geometry in a variety of ways. We begin by examining the interesting question of how they arise. We consider the various situations one by one.

Let us start by taking a simple example. Consider an equation of a family of spheres with centres lying along the z-axis.

$$x^2 + y^2 + (z-c)^2 = r^2 \qquad (6)$$

and try to eliminate the arbitrary constant c from it. On differentiating Eqn. (6) with respect to x and y, we get

$$x + (z-c)\frac{\partial z}{\partial x} = 0 \text{ and } y + (z-c)\frac{\partial z}{\partial y} = 0.$$

Eliminating c from the above equations, we obtain

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0 \qquad (7)$$

which is a first order partial differential equation. Eqn. (7) is of first order as it involves only the first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Thus, a family of spheres given by Eqn. (6) satisfies the first order partial differential Eqn. (7). This holds, in general, for all surfaces of revolution with the z-axis as the axis of symmetry.

Surfaces of revolution

Let us consider the equation

$$z = f(r),\ r = (x^2 + y^2)^{1/2} \tag{8}$$

Where f is an arbitrary function on some domain D having continuous partial derivatives. Eqn. (8) represents surfaces of revolution with z-axis as the axis of revolution; for example, sphere, cone, etc.

On differentiating Eqn. (8), with respect to x and y respectively, we obtain

$$\frac{\partial z}{\partial x} = p = f'(r)\frac{\partial r}{\partial x} \text{ and } \frac{\partial z}{\partial y} = q = f'(r)\frac{\partial r}{\partial y}$$

where

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}$$

and

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\Rightarrow\ p = \frac{x}{r} f'(r) \text{ and } q = \frac{y}{r} f'(r) \tag{9}$$

Eliminating the function $\mathrm{f(r)}$ from Eqns. (9), we get

$$yp - xq = 0, \tag{10}$$

or, $$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$$

which is a partial differential equation of the first order.

Note that throughout the discussion of partial differential equations with z as dependent variable and x and y as independent variables, we shall be denoting the partial derivatives of z with respect to x and y by p and q, respectively.

We next consider the two parameter family of surfaces.

Two-parameter family of surfaces

Let us take up an example of two parameter family of spheres.

Example 1: Obtain the PDE corresponding to the equation

$$(x-a)^2 + (y-b)^2 + z^2 = 1.$$

Solution: The given equation can be written as

$$z^2 = 1 - (x-a)^2 - (y-b)^2 \tag{11}$$

Differentiating Eqn. (11) partially w.r.t. x and y, we get

$$2z\frac{\partial z}{\partial x} = -2(x-a) \tag{12}$$

and

$$2z\frac{\partial z}{\partial y} = -2(y-b) \tag{13}$$

From Eqns. (12) and (13), we obtain

$$(x-a) = -z\frac{\partial z}{\partial x} \text{ and } (y-b) = -z\frac{\partial z}{\partial y}.$$

Substituting these values of $(x-a)$ and $(y-b)$ in Eqn. (11), we get

$$z^2 = 1 - \left(z\frac{\partial z}{\partial x}\right)^2 - \left(z\frac{\partial z}{\partial y}\right)^2$$

or, $\quad z^2(1+p^2+q^2) = 1$

which is again a first order PDE.
Thus, the two parameter family of spheres satisfies the first order partial differential equation.

***

The result holds true, in general, for any two parameter family of surfaces

$$z = F(x,\ y,\ a,\ b) \tag{14}$$

where a and b are two parameters. If we differentiate Eqn. (14) with respect to x and y, respectively, we get

$$p = F_x(x,\ y,\ a,\ b) \tag{15}$$

and

$$q = F_y(x,\ y,\ a,\ b) \tag{16}$$

If we take Eqns. (14) and (15), we can solve them for a and b provided,

$$\begin{vmatrix} F_a & F_b \\ F_{xa} & F_{xb} \end{vmatrix} = F_a F_{xb} - F_b F_{xa} \neq 0.$$

Any two of the three Eqns. (14), (15) and (16) can be solved to find a and b in terms of $x,\ y,\ p,\ q$, provided, the following holds:

$$\left.\begin{array}{ll} \text{i)} & F_a F_{xb} - F_b F_{xa} \neq 0 \text{ (if Eqns. (14) and (15) are chosen)} \\ \text{ii)} & F_{xa} F_{yb} - F_{xb} F_{ya} \neq 0 \text{ (if Eqns. (15) and (16) are chosen)} \\ \text{iii)} & F_a F_{yb} - F_b F_{ya} \neq 0 \text{ (if Eqns. (14) and (16) are chosen)} \end{array}\right\} \tag{17}$$

Thus, to solve for a and b, we require either (17i) or (17ii) or (17iii).

Substituting the values of a and b so obtained from any two equations of (17) into the third equation, we will get a relation of the form

$$f(x,\ y,\ z,\ p,\ q) \tag{18}$$

which is a **general partial differential equation of the first order** in two independent variables x and y.

Let us consider the following examples.

Example 2: Obtain the PDE corresponding to the equation

$$z^2(1+a^3) = 8(x+ay+b)^3.$$

Solution: The given two-parameter family of surface is

$$z^2(1+a^3) = 8(x+ay+b)^3 \tag{19}$$

Differentiating Eqn. (19) w.r.t. x and y, respectively, we obtain

$$zp(1+a^3) = 12(x+ay+b)^2 \tag{20}$$

and $\quad zq(1+a^3)=12a(x+ay+b)^2 \qquad (21)$

In order to obtain the PDE we have to eliminate a and b from Eqns. (20) and (21). Taking the cubes of Eqns. (20) and (21) and adding, we get

$$z^3(1+a^3)^3(p^3+q^3)=12^3(x+ay+b)^6(1+a^3)$$

$$=\frac{12^3}{8^2}z^4(1+a^3)^3 \qquad \text{(using Eqn. (19))}$$

$$\Rightarrow \quad (p^3+q^3)=27z, \qquad (22)$$

which is the required PDE.

***

Example 3: Obtain the PDE corresponding to the equation

$$2z=(ax+y)^2+b \qquad (23)$$

Solution: Differentiating Eqn. (23) partially w.r.t. x, we get

$$2\frac{\partial z}{\partial x}=2(ax+y)a$$

or,

$$\frac{1}{a}\frac{\partial z}{\partial x}=ax+y,\ a\neq 0 \qquad (24)$$

Differentiating Eqn. (23) partially w.r.t. y, we get

$$\frac{\partial z}{\partial y}=ax+y \qquad (25)$$

From Eqns. (24) and (25), we can write

$$\frac{\partial z}{\partial x}=a\frac{\partial z}{\partial y}$$

Substituting in the above equation value of a from Eqn. (25), we get

$$\frac{\partial z}{\partial x}=\frac{1}{x}\left(\frac{\partial z}{\partial y}-y\right)\frac{\partial z}{\partial y}$$

or $\quad x\dfrac{\partial z}{\partial x}=\left(\dfrac{\partial z}{\partial y}\right)^2-y\dfrac{\partial z}{\partial y}$

or $\quad px+qy=q^2$

which is the required first order PDE.

***

Sometimes instead of eliminating arbitrary constants or parameters we need to eliminate an arbitrary function from the given equation in order to obtain a PDE corresponding to it. We are illustrating this situation in our next example.

Example 4: Obtain a PDE corresponding to the equation

$$z=xy+f(x^2+y^2) \qquad (26)$$

where f is an arbitrary function.

Solution: Differentiating Eqn. (26) w.r.t. x, we get

$$\frac{\partial z}{\partial x}=y+f'(x^2+y^2).2x$$

$$\Rightarrow \quad \frac{1}{x}\left(\frac{\partial z}{\partial x}-y\right)=2\,f'(x^2+y^2) \qquad (27)$$

Again differentiating Eqn. (26) w.r.t. y, we get

$$\frac{\partial z}{\partial y} = x + f'(x^2+y^2).2y$$

$$\Rightarrow \frac{1}{y}\left(\frac{\partial z}{\partial y} - x\right) = 2\, f'(x^2+y^2) \qquad (28)$$

From Eqns. (27) and (28), we obtain

$$\frac{1}{x}\left(\frac{\partial z}{\partial x} - y\right) = \frac{1}{y}\left(\frac{\partial z}{\partial y} - x\right)$$

or, $yp - y^2 = xq - x^2$

which is the required PDE of the first order.

***

We shall now take up an example to illustrate that if the given equation involves more number of arbitrary constants than the number of independent variables, then the above procedure of elimination would yield partial differential equations of higher order than the first.

Example 5: Find a PDE corresponding to the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (29)$$

where a, b and c are arbitrary constants.

Solution: Differentiating Eqn. (29) w.r.t. x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \text{ or } c^2x + a^2z\frac{\partial z}{\partial x} = 0 \qquad (30)$$

and $$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} \text{ or } c^2y + b^2z\frac{\partial z}{\partial y} = 0 \qquad (31)$$

once again differentiating Eqn. (30) w.r.t. x and Eqn. (31) w.r.t. y, we obtain

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0 \qquad (32)$$

and $$c^2 + b^2\left(\frac{\partial z}{\partial y}\right)^2 + b^2z\frac{\partial^2 z}{\partial y^2} = 0 \qquad (33)$$

From Eqn. (30), we have

$$c^2 = \frac{-a^2z}{x}\frac{\partial z}{\partial x}$$

Putting this value of c^2 in Eqn. (32) and simplifying, we get

$$zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0 \qquad (34)$$

Similarly putting the value of c^2 from Eqn. (31) in Eqn. (33), we get

$$zy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0 \qquad (35)$$

Thus, Eqns. (34) and (35) give the required PDEs. **Note** that both the PDEs are of order two.

***

We now move on to the next situation

Surfaces of the form $F(u, v) = 0$

We start by considering the following example.

Example 6: Eliminate the arbitrary function F from the equation $F(x+y+z,\, x^2+y^2-z^2)=0$ and obtain the corresponding PDE.

Solution: Let $u = x+y+z$ and $v = x^2+y^2-z^2$ (36)

then the given equation becomes

$$F(u,\, v)=0 \tag{37}$$

Differentiating Eqn. (37) partially w.r.t. x, we obtain

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\,\frac{\partial z}{\partial x}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\,\frac{\partial z}{\partial x}\right)=0 \tag{38}$$

Substituting the values of u_x, u_z, v_x, v_z from Eqn. (36) in Eqn. (38), we get

$$\frac{\partial F}{\partial u}\left(1+\frac{\partial z}{\partial x}\right)+2\frac{\partial F}{\partial v}\left(x-z\frac{\partial z}{\partial x}\right)=0$$

or, $$\frac{\partial F}{\partial u}\Big/\frac{\partial F}{\partial v}=\frac{-2(x-pz)}{1+p} \tag{39}$$

Similarly, differentiating Eqn. (37) partially w.r.t. y, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\,\frac{\partial z}{\partial y}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\,\frac{\partial z}{\partial y}\right)=0$$

Substituting the values of u_y, u_z, v_y, v_z from Eqn. (36) in the above equation, we get

$$\frac{\partial F}{\partial u}\Big/\frac{\partial F}{\partial v}=\frac{-2(y-qz)}{1+q} \tag{40}$$

Eliminating F from Eqns. (39) and (40), we obtain

$$\frac{x-pz}{1+p}=\frac{y-qz}{1+q}$$

or, $$(1+q)\,(x-pz)=(1+p)\,(y-qz)$$

or, $$(y+z)\,p-(x+z)\,q=x-y$$

which is the desired PDE of the first order.

***

The method above can be applied is general, for surfaces of the form

$$F(u,\, v)=0 \tag{41}$$

where $u=u(x,y,z),\, v=v(x,\,y,\,z)$ are known functions of $x,\ y$ and z, and F is an **arbitrary function** of u and v. If we differentiate relation (41) with respect to x and y, respectively, we obtain

$$\frac{\partial F}{\partial u}\,(u_x+pu_z)+\frac{\partial F}{\partial v}\,(v_x+pv_z)=0 \tag{42}$$

$$\frac{\partial F}{\partial u}\,(u_y+qu_z)+\frac{\partial F}{\partial v}\,(v_y+qv_z)=0 \tag{43}$$

To eliminate $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from Eqns. (42) and (43), we calculate $\frac{\partial F/\partial u}{\partial F/\partial v}$ from Eqns. (42) and (43) and equate them to obtain

$$\frac{(v_x+pv_z)}{(u_x+pu_z)}=\frac{(v_y+qv_z)}{(u_y+qu_z)}$$

$$\Rightarrow \quad p(v_z u_y - u_z v_y) + q(u_z v_x - u_x v_z) = u_x v_y - v_x u_y$$

$$\Rightarrow \quad p\frac{\partial(u, v)}{\partial(y, z)} + q\frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial(u,v)}{\partial(x, y)}, \tag{44}$$

which is a partial differential equation of the first order.

We now illustrate the above method through an example.

Example 7: Eliminate the arbitrary function F from the equation $F(z-x,\ xy)=0$ and obtain the corresponding PDE.

Solution: Let $u = z - x$ and $v = xy$, then

$$F(z-x,\ xy) = F(u,\ v) = 0$$

From Eqn. (44), we have

$$p\frac{\partial(u,v)}{\partial(y, z)} + q\frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial(u, v)}{\partial(x, y)}$$

$\Rightarrow x\,p - yq - x = 0$

which is the required PDE of the first order.

***

There are many other situations where you would have come across with PDE/(s) of the first order. We recall some of them here.

Integrating Factor

You may recall the method you learnt in Sub-sec.15.4.3 of Unit 15.
There the problem of finding an integrating factor for a particular form of the total differential equation

$$P(x,\ y)dx + Q(x,\ y)dy = 0$$

consists in determining a function $\mu(x,\ y)$ for which $(\mu\ Pdx + \mu\ Qdy)$ is an exact differential. This leads to

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q) \tag{45}$$

which is a partial differential equation of first order for the function $\mu(x,\ y)$ and for known P and Q.

Euler's Equation for a Homogeneous Function

In Unit 7 of Block 2, we defined a function f to be a homogeneous function of x and y of degree n, where n is a real number, if it satisfies

$$f(\lambda x,\ \lambda y) = \lambda^n f(x,\ y)$$

for all $x,\ y$ and any constant $\lambda > 0$.
Let $z = f(x,\ y)$ be a homogeneous function of x and y of degree n. Then by Euler's theorem, the function f satisfies the first order PDE

$$xf_x + yf_y = nf\ .$$

Hamiltonian Function

In Unit 14 we mentioned that the equation of motion of a dynamical system of n degrees of freedom (ref. Eqn. (27), Unit 14), are given by

$$\frac{dp_i}{dt} = \frac{-\partial H}{\partial q_i}, \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, i = 1, 2, \ldots, n \tag{46}$$

where $H(q_i, q_2, \ldots, q_n, p_1, p_2 \ldots, p_n, t)$ is the **Hamiltonian function** which is equal to the total energy of the system. In terms of time t, the $q_i{}'s(i = 1, 2, \ldots, n)$ are the generalized coordinates and $p_i{}'s(i = 1, 2, \ldots, n)$ are the generalized momenta, which characterize the state of a dynamical system. Eqns. (46) are a system of $2n$ first order partial differential equations called **Hamilton canonical equations of motion**. The solution of these equations provide a description of the properties of the dynamical system at any time t. The equations are named after W. R. Hamilton (1805-1865), an Irish physicist, astronomer and mathematician. He made important contributions to classical mechanics, optics and algebra. His best known contribution to mathematical physics is the reformulation of Newtonian mechanics now called Hamiltonian mechanics.

We can thus say that there are many situations which lead to partial differential equations of the first order. You may now yourself obtain a few PDEs while doing the following exercises.

E1) Show that the family of right circular cones

$$x^2 + y^2 = (z - c)^2 \tan^2 \alpha$$

whose axes coincide with the z-axis satisfies a first order partial differential equation.

E2) Eliminate the arbitrary constants a and b from the following equations and obtain the corresponding partial differential equations. Also specify the order of these PDEs.

i) $z = ax + by + a^2 + b^2$

ii) $z = (x - a)^2 + (y - b)^2$

iii) $z = x + ax^2y^2 + b$

iv) $az + b = a^2x + y$

v) $z = ae^{bt} \sin bx$

vi) $z = ae^{-b^2t} \cos bx$

vii) $z = ax + by + a + b - ab$.

E3) Find the partial differential equation arising from each of the following surfaces:

i) $z = f\left(\frac{xy}{z}\right)$

ii) $z = xy + f(x^2 + y^2)$

iii) $F(x^2 + y^2 + z^2, z^2 - 2xy) = 0$

iv) $2z = (\alpha x + y)^2 + \beta$

You may recall that in Sec. 6.2 of Unit 6, Block 2, we classified the ordinary differential equations depending upon the degree of dependent variables and its derivatives into two classes, namely, linear and non-linear. We term the ODE which is not linear as the non-linear one. But, in the case of PDEs, as we have already mentioned in the introduction of this unit, these equations have further classifications. If a partial differential equation is not linear, it can be quasi-linear, semi-linear or non-linear. We now take up this classification for the first order partial differential equations.

16.3 CLASSIFICATION OF THE FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Consider the general form of the first order PDE in two independent variables x and y given by Eqn. (18), viz.,

$$f(x, y, z, p, q) = 0,$$

with $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$.

Eqn. (18) can be classified into the following types.

1) Eqn. (18) is said to be **linear** if f is linear in each of the variables z, p and q and the coefficients of these variables are functions only of the independent variables x and y. For instance, the equation

$$x^2 p + y^2 q = (x + y)\, z$$

is a linear equation. The most general linear, first order PDE has the form

$$P(x, y)p + Q(x, y)q + R(x, y)z = H(x, y) \tag{47}$$

where P, Q, R and H are functions only of the independent variables x and y. Eqn. (47) is called **homogeneous** if $H(x, y) = 0$ and **non-homogeneous** if $H(x, y) \neq 0$. Examples of some more linear PDEs of the first order are

$$yp + xq = xy \tag{48}$$

$$np + (x + y)q - u = e^x \tag{49}$$

$$xp - yq - nu = 0 \tag{50}$$

Here Eqns. (48) and (49) are non-homogeneous whereas, Eqn. (50) is a homogeneous equation.

2) If we can write Eqn. (18) in the form

$$P(x, y)p + Q(x, y)q = R(x, y, z) \tag{51}$$

it is called a **semi-linear** PDE of first order. Here the coefficients P and Q are independent of z i.e., they are functions of only independent variables whereas, R is an arbitrary function of both dependent and independent variables.
Equations

$$px(x + y) = qy(x + y) - (x - y)(2x + 2y + z^2) \tag{52}$$

$$xp + yq = z^2 + x^2 \tag{53}$$

$$(x+1)^2 p + (y-1)^2 q = (x + y)z^2 \tag{54}$$

are all examples of semi-linear equations.

3) If it is possible to express Eqn. (18) in the form

$$P(x\ y,\ z)p + Q(x,\ y,\ z)q = R(x,\ y,\ z) \tag{55}$$

then it is called a **quasi-linear** PDE of the first order where the coefficients. P, Q and R are functions of x, y and z. Such equations are linear in their highest order derivatives.
Eqn. (55), being of first order is linear in its first order partial derivatives of the dependent variable $z(x,\ y)$ i.e., in p and q.
Equations

$$z(xp - yq) = y^2 - x^2, \tag{56}$$

$$(y + zx)p - (x + yz)q = (x^2 - y^2)z \tag{57}$$

and

$$x(y^2 + z)p - y(x^2 + z)q + z^2 = 0 \tag{58}$$

are all first order quasi-linear equations. In Eqns. (56)-(58), the highest derivative is of order one and its power throughout is one.

You might have also **noticed** here that the linear and the semi-linear equations are the special cases of the quasi-linear equations.

If Eqn. (18) is none of the types 1), 2) and 3) mentioned above, we call it a **non-linear** PDE of the first order.

For instance, we cannot put equations

$$z^2(1 + p^2 + q^2) = 1 \tag{59}$$

and $$2(y + zp) = q(xp + yq) \tag{60}$$

in any of the forms 1) to 3) discussed above. Eqns. (59) and (60) are non-linear equations.

Note that unlike ODEs, linearity in PDEs (semi-linear and quasi-linear) does not depend on the degree of the dependent variable z as you can see in Eqns. (52)-(54) and (58) above. Here the linearity of the derivatives of the dependent variable is considered.

You may now try the following exercise.

E4) Classify the following equations into linear, semi-linear, quasi-linear and non-linear equations.

i) $xp + yq = zx + x^2 + y^2$

ii) $9x^2y^2z = 3px^3y^2 + 3qx^2y^3 + pq$

iii) $xp + 2yq = x^2 / y$

iv) $py - qx = x^2 - y^2$

v) $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

vi) $(xz + y^2)p + (yz - 2x^2)q + 2xy + z^2 = 0$

vii) $\cos(x + y)p + \sin(x + y)q = z + \dfrac{1}{z}$

viii) $p^2 + q^2 + 2px + 2qy + zp = 0$.

Having discussed the classification of PDEs of the first order, you can now classify any given PDE of the first order into linear, semi-linear, quasi-linear and non-linear equation. Next, your natural curiosity may lead you to enquire about is solution. Since partial derivatives of multivariable functions are ordinary derivatives with respect to one variable (the others being held constant), it might occur to you that the study of partial differential equations should be an easy extension of the theory for ordinary differential equations. But such is not the case. Partial differential equations and ordinary differential equations are approached in different ways. To understand why, you may recall that in the case of ODEs, a general solution of the second-order linear ODE

$$p\frac{d^2y}{dt^2}+q\frac{dy}{dt}+ry=0 \tag{61}$$

is $y(t)=Ay_1(t)+By_2(t)$, where A and B are arbitrary constants and $y_1(t)$ and $y_2(t)$ are any two linearly independent solutions of the equation. Once $y_1(t)$ and $y_2(t)$ are known, every solution of the equation is of the form $Ay_1(t)+By_2(t)$ for some A and B. Solution of Eqn. (61) for particular values of A and B determined under given conditions is a particular solution of the equation. But PDEs are approached in different ways because arbitrary constants are replaced by arbitrary functions and determination of these arbitrary functions using subsidiary conditions is usually difficult or impossible. We shall, therefore, in the next section first define what we mean by a solution/integral of a PDE of the first order and then classify different types of integrals that might arise and give the relation between these different integrals.

16.4 SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Consider a general PDE of the first order as given by Eqn. (18), viz.,

$$f(x, y, z, p, q)=0$$

By a solution of this equation we mean a continuously differentiable function say, ϕ defined for all x, y on some domain $D\subset\mathbb{R}\times\mathbb{R}$ such that

$$f(x, y, \phi(x, y), \phi_x(x, y), \phi_y(x, y))=0 \tag{62}$$

If we write $z=\phi(x, y)$, then this solution represents a family of surfaces in the xyz-space. For example, you may refer to Eqn. (23) of Example 3, viz;

$$2z=(ax+y)^2+b$$

where a and b are real numbers, is a two parameter family of surface and it represents the PDE

$$px+qy=q^2, \tag{63}$$

which is of the first order. Eqn. (23) is the solution/integral of the PDE (63).

In Examples 1 to 3 considered in Sec. 16.2, you have seen that the two-parameter family of surfaces give rise to linear or non-linear first order partial differential equations. We shall, therefore, assume that a general first order partial differential Eqn. (18), viz.,

$$f(x, y, z, p, q)=0$$

can have a solution

$$z = F(x, y, a, b), \tag{64}$$

which depends on two parameters a, b.

Depending on the number of parameters, we now classify the solutions of first order PDE as follows:

Classification of Integrals

1) The Complete Integral

A two-parameter family of solution (64), i.e., $z = F(x, y, a, b)$ is called a **complete integral** of Eqn. (18) if, in the region considered, F satisfies any of the Eqns. 17 (i, ii, iii). In case of Example 3, Eqn. (23) is a complete integral of the PDE (63).

2) The General Integral

In Eqn. (64) if we take $b = b(a)$, we obtain

$$z = F(x, y, a, b(a)), \tag{65}$$

which is a one-parameter family of solutions of Eqn. (18) and is a subsystem of the two-parameter family given by Eqn. (64). We can obtain the envelope of Eqn. (65) by eliminating a between relation (65) and relation

$$F_a + F_b b'(a) = 0$$

In fact we can solve the above relation for a, then

$$a = a(x, y)$$

and substituting this value of a in relation (65), we obtain the **general integral** of Eqn. (18) as

$$z = F(x, y, a(x, y), b(a(x, y))) \tag{66}$$

The surface given by Eqn. (66), being the envelope of the one-parameter family of surfaces (65), touches every member of the family along the characteristic curve and has the same values of p, q all along the curve as Eqn. (65). Hence, it is a set of solutions of Eqn. (18) depending on an arbitrary function. In case of Example 3, one parameter family of solution is written by taking $b = b(a)$ in Eqn. (23) as follows:

$$2z = (ax + y)^2 + b(a) \tag{67}$$

The envelope of Eqn. (67) is obtained by solving the equation

$$(ax + y)x + \frac{1}{2}b'(a) = 0$$

for $a = a(x, y)$ and then substituting the value of $a(x, y)$ in Eqn. (67). Thus, the equation

$$2z = (ax + y)^2 + b(a(x, y)) \tag{68}$$

where b is an arbitrary function gives the general solution of Eqn. (63) If in Eqn. (66) a particular function $b(a)$ is used, we obtain a **particular solution** of the PDE. Different choices of b may give different particular solution of the PDE (18). For instance, we may take $b = a$ in Eqn. (67) and obtain a particular solution $2z = (ax + y) + a$ of Eqn. (63).

3) Singular Integral

In addition to the general integral, we can sometimes obtain still another solution by finding the envelope of the two-parameter family (64). This is obtained by eliminating a and b from the equations

$$\left.\begin{aligned} z &= F(x,\ y,\ a,\ b) \\ 0 &= F_a \\ 0 &= F_b \end{aligned}\right\} \tag{69}$$

and is called the **singular integral** of Eqn. (18). For instance, you know from E2)vii) that the two parameter family of planes

$$z = ax + by + a + b - ab = F(x,\ y,\ a,\ b)$$

is the complete integral of the PDE

$$z = px + qy + p + q - pq \tag{70}$$

The envelope of this two parameter family of planes is obtained from

$$z = F = ax + by + a + b - ab$$

$$F_a = 0 \Rightarrow x + 1 - b = 0 \Rightarrow x + 1 = b$$

$$F_b = 0 \Rightarrow y + 1 - a = 0 \Rightarrow y + 1 = a$$

Thus, we get the envelope as

$$\begin{aligned} z &= x(y+1) + y(x+1) + y + 1 + x + 1 - (x+1)\ (y+1) \\ &= xy + x + y + 1 \end{aligned}$$

which is the singular solution of the PDE (70).

You may also note that not all equations have singular solution. For instance, consider the two parameter family of solution

$$2z = (ax + y)^2 + b = F(x,\ y,\ a,\ b) \tag{71}$$

of the PDE $px + qy = q^2$ in Example 3. Here we have

$$F_a = 0 \Rightarrow (ax + y)x = 0$$

$F_b = 0 \Rightarrow 1 = 0$, which is ambiguous.

Thus, the envelope of Eqn. (71) and hence the singular integral of equation $px + qy = q^2$ does not exist.

However, the singular solution can also be obtained **directly** from the PDE (18) without going through the process described above. This can be done by eliminating p and q from the equations

$$\left.\begin{aligned} f(x,\ y,\ z,\ p,\ q) &= 0 \\ f_p = 0,\ f_q &= 0 \end{aligned}\right\} \tag{72}$$

In fact, the two processes are equivalent is evident from the following discussion.

Since $z = F(x,\ y,\ a,\ b)$ is a two-parameter family of solutions of Eqn. (18), the equation

$$f(x,\ y,\ F(x,\ y,\ a,\ b),\ F_x(x,\ y,\ a,\ b),\ F_y(x,\ y,\ a,\ b)) = 0 \tag{73}$$

holds identically in a and b. We differentiate Eqn. (73) with respect to a and b, and obtain

$$\left.\begin{aligned} f_z F_a + f_p F_{xa} + f_q F_{ya} = 0, \\ f_z F_b + f_p F_{xb} + f_q F_{yb} = 0, \end{aligned}\right\} \tag{74}$$

But we know from Eqn. (69) that on the singular integral,

$$F_a = 0,\ F_b = 0.$$

Therefore, relation (74) reduces to

$$\left.\begin{aligned} f_p F_{xa} + f_q F_{ya} = 0 \\ f_p F_{xb} + f_q F_{yb} = 0 \end{aligned}\right\} \tag{75}$$

In order to solve the system of Eqns. (75) for f_p and f_q, consider

$$\begin{vmatrix} F_{xa} & F_{ya} \\ F_{xb} & F_{yb} \end{vmatrix} = F_{xa}F_{yb} - F_{xb}F_{ya} \tag{76}$$

But you know from Eqns. (17) that F satisfies $F_{xa}F_{yb} - F_{xb}F_{ya} \neq 0$ (Eqn. 17 ii) and hence the only solution of Eqn. (75) is

$f_p = 0,\ f_q = 0$ (ref. appendix Unit 10)

Therefore, we can find the equation of the singular integral from Eqns. (72) by eliminating p and q. Eqns. (72) provide an **alternative characterisation** of the singular integral in terms of the given PDE whenever such an integral exists.

4) Special Integral

Usually (but not always), the three classes 1), 2) and 3) discussed above include all the integrals of the first order partial differential equation. Exceptions may arise in special cases for equations of particular forms. These equations have solutions which we call **special integrals** and these cannot be obtained from 1) or 2) or 3) above.
For example, if we eliminate the function F from the equation $F(x+y,\ y-\sqrt{z}) = 0$ we obtain the first order partial differential equation $p - q = 2\sqrt{z}$. Therefore, $F(x+y,\ x-\sqrt{z})$ involving one arbitrary function, is the general integral of the equation $p - q = 2\sqrt{z}$. But $z = 0$ also satisfies this equation and it cannot be obtained from the general integral. It is, therefore, the special integral of the equation.

We illustrate the ideas presented in 1) to 4) above, about the different types of integrals of a first order partial differential equation, through the following examples. We shall start with a two-parameter family of surfaces, construct the corresponding partial differential equation and then derive the general integral and the singular integral from the complete integral.

Example 8: Eliminating a and b from the family of planes

$$z = ax + by + a^2 + b^2,$$

determine the partial differential equation of this family of planes. State the complete integral of the equation and find its singular integral and the general integral.
Solution: The family of planes is

$$f = z - (ax + by + a^2 + b^2) = 0 \tag{77}$$

Differentiating Eqn. (77) partially w.r. to x and y, we get

$$\left.\begin{aligned} p - a = 0 \\ q - b = 0 \end{aligned}\right\} \tag{78}$$

Eliminating a and b from Eqns. (77) and (78), we get

$$z = px + qy + p^2 + q^2 \tag{79}$$

which is the partial differential equation for the given family of planes. Eqn. (79) is linear first order PDE.

Since Eqn. (77) is a two-parameter family of planes, it represents the complete integral of the PDE (79).

Using relation (69), the singular integral is obtained by eliminating a and b from Eqn. (77) and equations

$$\frac{\partial f}{\partial a} = 0 \Rightarrow x + 2a = 0$$

$$\frac{\partial f}{\partial b} = 0 \Rightarrow y + 2b = 0$$

Substituting the values of a and b from above in Eqn. (77), we get the singular integral as

$$z = x.\left(-\frac{x}{2}\right) + y\left(-\frac{y}{2}\right) + \left(-\frac{x}{2}\right)^2 + \left(-\frac{y}{2}\right)^2$$

$$\Rightarrow \quad 4z = -(x^2 + y^2), \tag{80}$$

which is **paraboloid of revolution**.
If we set $b = b(a)$, then we get the one-parameter family as

$$z - (ax + b(a)y + a^2 + b^2(a)) = 0 \tag{81}$$

The envelope of this one-parameter family is obtained by eliminating a from Eqn. (81) and its derivative w.r.t a, viz.,

$$x + b'(a)y + 2a + 2b(a)b'(a) = 0, \tag{82}$$

which will be the general integral of the given equation.

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Example 9: Find the partial differential equation for the twp parameter family of spheres of radius 1 in the xyz-space whose centres $(a, b, 0)$ lie on the xy-plane. State their complete integral and find the general integral and the singular integral.

Solution: The two-parameter family of spheres of radius 1 in the xyz-space whose centres $(a, b, 0)$ lie on the xy-plane is

$$(x-a)^2 + (y-b)^2 + z^2 = 1 \tag{83}$$

You know from Example 1, the first order PDE satisfied by two parameter family of surfaces (83) is given by

$$z^2(1 + p^2 + q^2) = 1 \tag{84}$$

Relation (83) is the **complete integral** of the PDE (84) since it involves two arbitrary constants. If we set $b = b(a)$ in Eqn. (83), we get the one-parameter family of spheres whose centres are $(a, b(a), 0)$ and which lie on the curve $y = b(x)$ in the xy-plane. The envelope of this family is then obtained by eliminating a from the equations

$$(x-a)^2 + (y - b(a))^2 + z^2 = 1 \tag{85}$$

and its derivative w.r.t. a, namely,

$$x - a + b'(a)\,(y - b(a)) = 0 \tag{86}$$

Eqns. (85) and (86) determine a surface whose axis is $y = b(x)$ and which is the **general integral** of Eqn. (84). If a particular value $b = 2a$ is used in Eqn. (86) then we get $a = \frac{x+2y}{5}$. This value of a when substitutded in Eqn. (85), yields

$$\frac{4(2x-y)^2}{25} + \frac{(y-2x)^2}{25} + z^2 = 1$$

or $\quad (y-2x)^2 + 5z^2 = 5 \qquad (87)$

which is a **particular integral** of the PDE (84).

The two-parameter family (83), gives yet another envelope which we can obtain by eliminating a and b from equations

$$(x-a)^2 + (y-b)^2 + z^2 = 1$$
$$x - a = 0,$$
$$y - b = 0$$

$\Rightarrow \quad z = 1$ and $z = -1 \qquad (88)$

Thus the envelope of two-parameter family is the pair of planes $z = \pm 1$. Eqn. (88) gives us the **singular integral** of the PDE (84). Alternatively, we can obtain the singular integral from the PDE directly by using the relation (72). Here

$$\left.\begin{aligned} f &= z^2(1+p^2+q^2) - 1 = 0 \\ f_p &= 2z^2 p = 0 \Rightarrow p = 0 \\ f_q &= 2z^2 q = 0 \Rightarrow q = 0 \end{aligned}\right\} \qquad (89)$$

Eliminating p, q from Eqns. (89), we obtain $z = 1$ and $z = -1$ which is same as given by Eqn. (88).

***

It is easy to verify that a particular solution (87) and the singular solutions (88) satisfy the PDE (84). Also, the singular solution (88) touches the solution (87) along $y - 2x = 0,\ z = 1$ and $y - 2x = 0,\ z = -1$. We leave it for you to verify it yourself.

E5) Verify that

i) a particular solution (87) and the singular solution (88) satisfy the PDE (84).

ii) the singular solutions (88) touches the solution (87) along, $y - 2x = 0,\ z = 1$ and $y - 2x = 0,\ z = -1$.

Let us take up another example.

Example 10: Given that the two parameter family of planes

$$z = ax + by + a^2 + b^2, \qquad (90)$$

is the complete integral of the PDE.

$$z = px + qy + p^2 + q^2,$$

determine its particular integral.

Solution: We can find any number of particular integrals starting with the complete integral. Here, we give two particular integrals. Take $b = \sqrt{1-a^2}$ in the complete integral (90). This means we have to take only a subsystem of planes from relation (90). The equation of this one-parameter family of planes is

$$F = z - ax - \sqrt{1-a^2}\; y - 1 = 0 \tag{91}$$

Here $\dfrac{\partial F}{\partial a} = 0 \Rightarrow -x + \dfrac{ay}{\sqrt{1-a^2}} = 0$ (92)

Now we eliminate a from Eqns. (91) and (92). Solving Eqn. (92) for a, we get

$$a = \frac{x}{\sqrt{x^2 + y^2}}$$

Substituting the above value of a in Eqn. (91) we get the envelope of the family of planes (91) as the **right circular cone** whose equation is

$$(z-1)^2 = x^2 + y^2 \tag{93}$$

Eqn. (93) is a particular integral of the PDE (90). In order to find one more particular integral of Eqn. (90) we make another choice of $b(a)$, say,

$$b = b(a) = a\,.$$

Then

$$F = z - ax - ay - 2a^2 \tag{94}$$

and

$$\frac{\partial F}{\partial a} = 0 \Rightarrow x + y + 4a = 0 \tag{95}$$

By eliminating a between Eqns. (94) and (95), we obtain the envelope of Eqn. (94) as

$$8z = -(x+y)^2 \tag{96}$$

which is a **parabolic cylinder** and constitutes another particular integral of the given PDE.

***

You may now try the following exercises.

E6) In Example 10 obtain the singular integral of the given PDE, it it exists.

E7) Given that $z = ax + by + ab$ is the complete integral of the PDE.

$$z = px + qy + pq$$

obtain its general integral and the singular integral.

We now end this unit by giving a summary of what we have covered in it.

16.5 SUMMARY

In this unit, we have covered the following:

1. PDEs can arise in many ways in geometry, physics and mathematics. For instance,

   i) on elimination of arbitrary function defining the surfaces of revolution.

ii) on elimination of two constants, defining two parameter family of surfaces, between the equation defining the family of surfaces and its partial derivatives w.r. to independent variables.

iii) on elimination of the function F defining the surfaces of the form $F(u, v) = 0$ where $u = u(x, y, z)$ and $v = v(x, y, z)$ are known functions of x, y and z.

iv) while satisfying the conditions of an equation to be exact.

v) while writing the Hamilton canonical equations of motion of a dynamical system.

vi) while obtaining the Euler's equation for a homogeneous function.

2. The general form of the first order PDE is $f(x, y, z, p, q) = 0$.

3. A first order PDE is classified as

i) **linear**, (non-homogeneous) if it can be expressed as

$$P(x, y)p + Q(x, y)q + R(x, y)z = H(x, y)$$

It is homogeneous if $H(x, y) = 0$.

ii) **semi-linear**, if it can be expressed in the form

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

iii) **quasi-linear**, if it can be expressed in the form

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$$

iv) **non-linear** if it cannot be expressed in any of the forms given in i), ii) and iii) above.

4. The solutions of the first-order PDEs are classified as

i) **complete integral**, which is a relation between the variables involving as many constants as there are independent variables.

ii) **general integral**, which is obtained by eliminating 'a' between the complete integral

$$\mathrm{f}(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{a}, \mathrm{b}) = 0$$

and the equations

$$b = b(a)$$

and $\dfrac{\partial f}{\partial a} + \dfrac{\partial f}{\partial b} b'(a) = 0$,

where b is an arbitrary function.
The general integral represents the envelope of a one-parameter family of surfaces.

iii) **singular integral**, which is obtained on eliminating a and b between the complete integral $f(x, y, z, a, b) = 0$

$$\left.\begin{aligned} &\frac{\partial f}{\partial a} = 0 \\ \text{and } &\frac{\partial f}{\partial b} = 0 \end{aligned}\right\}$$

The singular integral represents the envelope of the two-parameter family of surfaces.

Alternatively, the singular integral can as well be obtained by eliminating p and q from the PDE

$$F(x, y, z, p, q) = 0$$

and $\frac{\partial F}{\partial p} = 0, \frac{\partial F}{\partial q} = 0$

iv) In the exceptional cases, if there are integrals of the given PDE which are not included in the complete integral, the general integral or the singular integral, these integrals are called the **special integrals.**

16.6 SOLUTIONS/ANSWERS

E1) Given equation is

$$x^2 + y^2 = (z-c)^2 \tan^2 \alpha$$

Differentiating the above equation partially w.r.t. x and y, we get

$$2x = 2(z-c)\tan^2 \alpha\ p$$

$$2y = 2(z-c)\tan^2 \alpha q$$

From the above two equations, we obtain

$$yp - xq = 0$$

which is the required first order PDE

E2) i) The given equation is

$$z = ax + by + a^2 + b^2 \quad (97)$$

Differentiating partially with respect to x and y, we get

$$p = a \quad (98)$$

and $q = b$ (99)

Eliminating a and b from Eqn. (97) by using Eqns. (98) and (99), we get

$$z = px + qy + p^2 + q^2$$

which is the required PDE of the first order.

ii) $4z = p^2 + q^2$, First-order.

iii) $xp - yq - x = 0$, First-order

iv) Given equation is

$$az + b = a^2 x + y$$

Differentiating w.r.t. x and y, we get

$$ap = a^2$$

and $aq = 1 \Rightarrow a = \frac{1}{q}$

and hence $pq = 1$ is the required first order PDE.

v) $z = ae^{bt} \sin bx$

Differentiating above equation w.r.t. x and t, we get

$$\frac{\partial z}{\partial x} = abe^{bt} \cos bx \quad (100)$$

$$\frac{\partial z}{\partial t} = abe^{bt} \sin bx \qquad (101)$$

Differentiating Eqns. (100) and (101) once again w.r.t. x and t, respectively we get

$$\frac{\partial^2 z}{\partial x^2} = -ab^2 e^{bt} \sin bx$$

$$\frac{\partial^2 z}{\partial t^2} = ab^2 e^{bt} \sin bx$$

Adding the above two equations, we obtain

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

which is the required PDE of the second order.

vi) $z = ae^{-b^2 t} \cos bx$

$$\therefore \frac{\partial z}{\partial x} = -bae^{-b^2 t} \sin bx \text{ and } \frac{\partial^2 z}{\partial x^2} = -ab^2 e^{-b^2 t} \cos bx \qquad (102)$$

and $$\frac{\partial z}{\partial t} = -ab^2 e^{-b^2 t} \cos bx \qquad (103)$$

From Eqns. (102) and (103), we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}$$

which is a second order PDE.

vii) $z = ax + by + a + b - ab$

Differentiating the given equation w.r.t. x and y, we get

$$p = z_x = a,\ q = z_y = b$$

Substituting the above values of a and b in the given equation, we obtain

$$z = px + qy + p + q - pq$$

which is the required first order PDE.

E3) i) The given equation is

$$z = f\left(\frac{xy}{z}\right) \qquad (104)$$

Differentiating Eqn. (104) partially w.r.to x and y respectively, we get

$$p = f'\left(\frac{y}{z} - \frac{xy}{z^2} p\right) \qquad (105)$$

and $$q = f'\left(\frac{x}{z} - \frac{xy}{z^2} q\right) \qquad (106)$$

where f' is derivative of f w.r.to $\left(\frac{xy}{z}\right)$

Eliminating f' from Eqns. (105) and (106), we get

$$\frac{p}{\frac{y}{z} - \frac{xy}{z^2} p} = \frac{q}{\frac{x}{z} - \frac{xy}{z^2} q}$$

$$\Rightarrow \quad p(xz - xyq) = q(yz - xyp)$$

$\Rightarrow \quad z(px - qy) = 0$,

which is the required PDE.

ii) $z = xy + f(x^2 + y^2)$

Differentiating the above equation partially, w.r.t. x and y, we get

$$p = y + 2xf' \Rightarrow \frac{p-y}{x} = 2f'$$

$$q = x + 2yf' \Rightarrow \frac{q-x}{y} = 2f'$$

Equating the above two equations, we obtain

$$py - qx = y^2 - x^2$$

which is the required PDE.

iii) The given equation is

$$F(x^2 + y^2 + z^2, z^2 - 2xy) = 0$$

Let $x^2 + y^2 + z^2 = u,\ z^2 - 2xy = v$ (107)

Then given equation reduces to

$$F(u, v) = 0$$

Differentiating the above equation w.r.to x and y partially and eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$, we obtain (following Eqn. (44)).

$$p\frac{\partial(u, v)}{\partial(y, z)} + q\frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial(u, v)}{\partial(x, y)} \tag{108}$$

Here
$$\left.\begin{aligned}
\frac{\partial(u, v)}{\partial(y, z)} &= u_y v_z - u_z v_y = 2y.2z - 2z(-2x) = 4(yz + xz)\\
\frac{\partial(u, v)}{\partial(z, x)} &= u_z v_x - u_x v_z = 2z(-2y) - 2x.2z = -4(yz + xz)\\
\frac{\partial(u, v)}{\partial(x, y)} &= u_x v_y - u_y v_x = 2x(-2x) - (2y)(-2y) = 4(y^2 - x^2)
\end{aligned}\right\} \tag{109}$$

Substituting from Eqn. (109) into Eqn. (44), we get

$$(yz + zx)\ (p - q) = (y^2 - x^2)$$

which is the required PDE.

iv) $2z = (\alpha x + y)^2 + \beta$

Differentiating the given equation w.r.t. x and y, we get

$$2p = 2(\alpha x + y)\ \alpha \tag{110}$$

$$2q = 2(\alpha x + y) \tag{111}$$

Solving Eqn. (111) for α and substituting the value of α obtained in Eqn. (110), we get

$$p = (q - y + y)\left(\frac{q-y}{x}\right)$$

$\Rightarrow \quad xp + yq = q^2$

which is the required PDE.

E4) i) Semi-linear

ii) non-linear

iii) linear

iv) linear

v) quasi-linear

vi) quasi-linear

vii) semi-linear

viii) non-linear.

E5) i) Particular solution (87) is

$$(y-2x)^2+5z^2=5$$

$$\Rightarrow \quad z^2=1-\frac{1}{5}(y-2x)^2$$

Differentiating w.r.t. x and y, we get

$$zp=\frac{2}{5}(y-2x),\ zq=\frac{-1}{5}(y-2x)$$

Substituting the values of z^2, z^2p^2 and z^2q^2 from above in Eqn. (84), we get

$$\text{LH.S.}=1-\frac{(y-2x)^2}{5}+\frac{4}{25}(y-2x)^2+\frac{(y-2x)^2}{25}=1=\text{R.H.S}$$

ii) Substituting $z=1$ and $z=-1$ and their derivatives w.r.t. x and y in Eqn. (84), we obtain $(y-2x)=0$.

E6) From Eqn. (90), we have

$$z=F(x,\ y,\ a,\ b)=ax+by+a^2+b^2$$

$Fa=0$ and $Fb=0$ gives

$$x+2a=0 \tag{112}$$

and $$y+2b=0 \tag{113}$$

Eliminating a and b from Eqns. (90), (112) and (113) we obtain the singular solution

$$4z=-(x^2+y^2)$$

which is a paraboloid of revolution.

E7) The complete integral is

$$z=ax+by+ab$$

Let $b=b(a)$

Then the complete integral yields

$$z=ax+b(a)+ab(a) \tag{114}$$

Differentiating Eqn. (114) partially w.r.t. a, we get

$$0=x+b'(a)y+b(a)+ab'(a) \tag{115}$$

Elimination of a between Eqns. (114) and (115) gives us the general integral. Singular integral is obtain by eliminating a and b between the equations

$$\left.\begin{aligned} z &= ax + by + ab \\ 0 &= x + b \\ 0 &= y + a \end{aligned}\right\}$$

in the form $z = -xy$.

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UNIT 17

FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Structure **Page No.**

17.1 INTRODUCTION

In Unit 16, we observed that unlike ordinary differential equations, where equations are either linear or non-linear, partial differential equations of the first order have further classifications of linear equations into quasi-linear, semi-linear and linear equtions. We also discussed there various types of solutions/integrals of the first order PDEs. In this unit we shall discuss the methods of finding the solutions of both linear and non-linear first order PDEs. You would observe that the construction of an integral of the linear PDE of first order is a multistage process and in this respect it differs from the usual construction of an integral of an ordinary differential equation. The problem of finding the integral of a non-linear PDE of the first order is more involved than that for the corresponding linear equation although there are some striking similarities.

We shall start by discussing in Sec. 17.2 the Lagrange's method of finding the general solution of linear first order PDE which is due to Lagrange (1736-1813), an Italian mathematician. The method of solving non-linear PDEs of the first order is partly due to Lagrange. But later on, it was the French mathematician Charpit who perfected it and presented it in a memoir in 1784 to Paris Academy of Sciences. The method is known as Charpit's method and it gives the complete integral of the first order non-linear PDE. We shall discuss the method in Sec. 17.3. Since the method is based on the consideration of compatible system of first order equations, we have, started Sec. 17.3 by first defining the compatible systems of equations and obtained the conditions for systems to be compatible. We shall also take up in this section some special types, called the standard forms of the first order non-linear PDEs, for which the application of Charpit's method become shorter and the complete integrals can be obtained easily.

Objectives

After studying this unit, you should be able to:

- use Lagrange's method for solving the first order linear PDEs;
- define compatible systems of first order PDEs;
- obtain the conditions for systems of two first order non-linear PDEs to be compatible;
- use Charpit's method for finding the complete integral of a non-linear PDE of first order; and
- identify standard forms of non-linear first order PDEs and obtain their complete integrals, using shorter method.

17.2 LINEAR EQUATIONS OF THE FIRST ORDER

Consider the quasi-linear equation

$$P(x,\ y,\ z)p + Q(x,\ y,\ z)q = R(x,\ y,z) \tag{1}$$

where P, Q and R are given functions of x, y, z not involving p or q and having continuous partial derivatives w.r.t. x, y, z on some domain containing x, y, z. In order to obtain the general solution of Eqn. (1) we need to find a relation between x, y and z involving an arbitrary function. The first systematic approach to solve equations of this type was given by Lagrange. For this reason Eqn. (1) is called the **Lagrange's equation**. The method of solution of this equation is based on the following theorem which gives its **general solution**.

Lagrange (1736-1813)

Theorem 1: The general solution of the quasi-linear Eqn. (1) (or Lagrange's equation)

$$P(x,\ y,\ z)p + Q(x,\ y,\ z)q = R(x,\ y,\ z)$$

is

$$F(u,\ v) = 0 \tag{2}$$

where F is an arbitrary function of u and v, and $u(x,y,z) = c_1$, $v(x,y,z) = c_2$ are the solutions of the system of simultaneous equations

$$\frac{dx}{P(x,\ y,\ z)} = \frac{dy}{Q(x,\ y,\ z)} = \frac{dz}{R(x,\ y,\ z)}. \tag{3}$$

Let us now prove the theorem.

Proof: Consider the two families of surfaces

$$u(x,\ y,z) = c_1 \text{ and } v(x,\ y,z) = c_2 \tag{4}$$

If they form a solution of the system of Eqns. (3), viz.,

$$\frac{dx}{P(x,\ y,\ z)} = \frac{dy}{Q(x,\ y,\ z)} = \frac{dz}{R(x,\ y,\ z)}$$

then along any curve, given by Eqns. (3) we have

$$\left.\begin{array}{l} u_x dx + u_y dy + u_z dz = 0 \\ v_x dx + v_y dy + v_z dz = 0 \end{array}\right\} \tag{5}$$

Solving Eqns. (5) for $dx,\ dy,\ dz$, we get

$$\frac{dx}{u_y v_z - u_z v_y} = \frac{dy}{u_z v_x - u_x v_z} = \frac{dz}{u_x v_y - u_y v_x}, \tag{6}$$

which are the differential equations of the surfaces given by Eqn. (4). Hence the system of Eqns. (3) and (6) should both represent the same integral curves. Comparing Eqns. (3) and (6), we get

$$P = K(u_y v_z - u_z v_y),\ Q = K(u_z v_x - u_x v_z), R = K(u_x v_y - u_y v_x), \tag{7}$$

where K is a non-zero function of $x,\ y,\ z$.

Now, consider Eqn. (2), viz.,

$$F(u,\ v) = 0$$

where u and v are known functions of $x,\ y,\ z$ and F is an arbitrary function of u and v.

Differentiating relation (2) with respect to x and y, respectively, we obtain

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right] = 0 \tag{8}$$

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right] = 0 \tag{9}$$

For non-zero solution of $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$, Eqns. (8) and (9), yield

$$\begin{vmatrix} u_x + pu_z & v_x + pv_z \\ u_y + qu_z & v_y + qv_z \end{vmatrix} = 0$$

$$\Rightarrow\ (u_x + pu_z)\ (v_y + qv_z) - (v_x + pv_z)\ (u_y + qu_z) = 0$$

$$\Rightarrow\ p(u_y v_z - u_z v_y) + q(u_z v_x - u_x v_z) = (u_x v_y - u_y v_x), \tag{10}$$

which is a partial differential equation of the type (1).

Substituting from Eqn. (7) in Eqn. (1) and using Eqn. (10), we find that Eqn. (1) is satisfied identically. Thus $F(u,v) = 0$ is a solution of Eqns. (3), which is the intersection of the two surfaces u and v given by Eqn. (4), and is called its **general solution**.

This complete the proof of Theorem 1.

– ■ –

The system of Eqns. (3) are called the **Lagrange's auxiliary or subsidiary equations**.

The integral curves, given by the intersection of surfaces

$u(x,\ y,\ z) = c_1$ and $v(x,\ y,\ z) = c_2$,

are called the **characteristic curves** or **characteristics** of Eqn. (1) and are the solutions of Eqns. (3).

Before we illustrate the method, let us summarise the **steps** involved in it.

1) Write the Langrange's auxiliary equations for Eqn. (1). Viz.,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

2) Solve these simultaneous equations by the methods you have learnt in Sec. 14.4 of Unit-14 and obtain the two independent solutions

$$u(x,\ y,\ z) = c_1 \text{ and } v(x,\ y,\ z) = c_2$$

3) The general solution/integral of Eqn. (1) which is the intersection of two surfaces $u(x,\ y,\ z) = c_1$ and $v(x,\ y,\ z) = c_2$ can then be written in any one of the following three equivalent forms

$$F(u,\ v) = 0,\ u = \phi(v) \text{ or } v = \psi(u)$$

where F, ϕ and ψ are arbitrary functions.

We shall now illustrate the method through various examples. You may **notice** that in these examples while performing step 2) mentioned above, we have used different methods, that you learnt in Sec. 14.4 of Unit 14, for finding the two independent solutions $u = c_1$ and $v = c_2$ of the given problem.

Example 1: Find the general integral of the partial differential equation

$$z_t + zz_x = 0.$$

Solution: The auxiliary equations are

$$\frac{dt}{1} = \frac{dx}{z} = \frac{dz}{0} \tag{11}$$

The two integrals of Eqns. (11) are

$$u = z = c_1,\ v = x - c_1 t = c_2 \text{ or } v = x - zt = c_2$$

The general integral of the given equation is then

$$F(z,\ x - zt) = 0$$

or

$$z(x,\ t) = \phi(x - zt). \tag{12}$$

You may also check that the solution obtained above satisfies the given PDE. This can be done by differentiating Eqn. (12) w.r.t. to t and x and then substituting it in the given equation. In this case we have

$$z_t = \phi'(x - zt)\ (-z - tz_t)$$

$$= -z\phi'(x - zt) - t\phi'(x - zt)\ z_t$$

$$\Rightarrow \quad z_t = \frac{-z\phi'}{1 + t\phi'} \tag{13}$$

Similarly,

$$z_x = \frac{\phi'}{1 + t\phi'},. \tag{14}$$

Substituting from Eqns. (13) and (14) in the given equation, we obtain

$$z_t + zz_x = \frac{-z\phi'}{1 + t\phi'} + z\left(\frac{\phi'}{1 + t\phi'}\right) = 0.$$

Thus the solution given by Eqn. (12) satisfies the given equation.

***

Let us consider another example.

Example 2: Find the general integral of the PDE

$$x^2 p + y^2 q = (x + y) z \tag{15}$$

Solution: The auxiliary equations of Eqn. (15) are

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{z(x+y)}=\frac{x^{-1}dx+y^{-1}dy-z^{-1}dz}{0} \tag{16}$$

Considering the first two fractions of Eqns. (16), we obtain

$$\frac{dx}{x^2}=\frac{dy}{y^2}$$

$$\Rightarrow \quad \frac{-1}{x}=-\frac{1}{y}+c_1$$

$$\Rightarrow \quad u(x,\ y,\ z)=\frac{1}{y}-\frac{1}{x}=c_1$$

Considering the last fraction of Eqns. (16), we get

$$\frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}=0$$

$$\Rightarrow \quad \ln x+\ln y-\ln z= \text{ constant } \Rightarrow v(x,\ y,\ z)=\frac{xy}{z}=c_2$$

Thus the general integral of Eqn. (15) is

$$F\left(\frac{1}{y}-\frac{1}{x},\ \frac{xy}{z}\right)=0$$

where F is an arbitrary function.
The general integral can also be written as

$$z=xy\ F_1\left(\frac{1}{y}-\frac{1}{x}\right),$$

where F_1 is an arbitrary function.

***

Example 3: Find the general integral of the following PDE

$$p+q=x+y+z.$$

Solution: Lagrange's auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{x+y+z} \tag{17}$$

Taking the first two fractions in Eqns. (17) and integrating, we get

$$x-y=c\text{, or } x=c_1+y \tag{18}$$

Taking the last two fractions in Eqn. (17) and using Eqn. (18), we get

$$dy=\frac{dz}{c_1+2y+z} \text{ or } \frac{dz}{dy}=c_1+2y+z$$

$$\Rightarrow \quad \frac{dz}{dy}-z=c_1+2y \tag{19}$$

Eqn. (19) is a linear equation in z and y and its I.F.$=e^{\int -dy}=e^{-y}$. The solution of Eqn. (19) is obtained as

$$\begin{aligned} ze^{-y} &= \int (c_1+2y)e^{-y}dy+c_2 \\ &=-(c_1+2y)e^{-y}+\int e^{-y}2dy+c_2 \\ &=-(c_1+2y)e^{-y}-2e^{-y}+c_2 \\ &=-e^{-y}(c_1+2y+2)+c_2 \end{aligned}$$

$$\therefore \quad z=c_2e^{y}-(c_1+2y+2)$$

$$= c_2 e^y - (x + y + 2)$$

or $\quad (z + x + y + 2)e^{-y} = c_2 \qquad (20)$

From Eqns. (18) and (20), the required general solution is

$$F[x - y, e^{-y}(z + x + y + 2)] = 0.$$

***

Example 4: Find the general integral of the following PDE

$$(x - y)p + (x + y)q = 2xz.$$

Solution: Lagrange's auxiliary equations are

$$\frac{dx}{x - y} = \frac{dy}{x + y} = \frac{dz}{2xz} \qquad (21)$$

Taking the first two fractions in Eqn. (21), we get the homogeneous equations

$$\frac{dy}{dx} = \frac{x + y}{x - y} = \frac{1 + y/x}{1 - y/x} \qquad (22)$$

Let $y/x = v$ or $y = xv$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Substituting these values in Eqn. (22), we obtain

$$v + x\frac{dv}{dx} = \frac{1 + v}{1 - v}$$

or $\quad x\frac{dv}{dx} = \frac{1 + v^2}{1 - v}$

or $\quad \frac{2dx}{x} = \frac{2(1 - v)}{1 + v^2}dv = \left[\frac{2dv}{1 + v^2} - \frac{2vdv}{1 + v^2}\right]$

Integrating the above equation, we get

$$2\ln x = 2\tan^{-1} v - \ln(1 + v^2) + \ln c_1$$

or $\quad \ln x^2 = 2\tan^{-1}(y/x) - \ln(1 + y^2/x^2) + \ln c_1$

or $\quad \ln\frac{(x^2 + y^2)}{c_1} = 2\tan^{-1}(y/x)$

or $\quad (x^2 + y^2) = c_1 e^{2\tan^{-1} y/x}$

or $\quad (x^2 + y^2)e^{-2\tan^{-1} y/x} = c_1 \qquad (23)$

Now choosing $1, 1, -\frac{1}{z}$ as the multipliers, each fraction of Eqns. (21)

$$= \frac{dx + dy - dz/z}{(x - y) + (x + y) - 2x} = \frac{dx + dy - dz/z}{0}$$

$\therefore\ dx + dy - \frac{dz}{z} = 0 \Rightarrow x + y - \ln z = c_2 \qquad (24)$

Hence, the required general integral can be written from Eqns. (23) and (24) in the form

$$F[(x^2 + y^2)e^{-2\tan^{-1} y/x},\ x + y - \ln z] = 0.$$

***

You may now try the following exercise.

E1) Find the general integrals of the following differential equations:

i) $\quad z(xp - yq) = y^2 - x^2$

ii) $\quad y^2 p - xyq = x(z - 2y)$

iii) $(z^2 - 2yz - y^2)p + x(y+z)q = x(y-z)$

iv) $\dfrac{y-z}{yz}p + \dfrac{z-x}{zx}q = \dfrac{x-y}{xy}$

v) $(x^3 + 3xy^2)p + (y^3 + 3x^2 y)q = 2(x^2 + y^2)z$

You may recall that in Unit 6. Sec. 6.3, we defined an initial value problem for ordinary differential equation of the first order. Therein, we stated the existence theorem for the solution of an ordinary differential equation

$$\frac{dy}{dx} = f(x,\ y)\,,$$

where $y = y_0$ at $x = x_0$.

We shall now take up the initial-value problem for a quasi-linear partial differential equation.

As you have seen above, the solution of a PDE of the first order is the intersection of two surfaces. Thus the initial conditions in this case will not be at a point, but the solution surfaces will pass through a curve say Γ_0, called the **initial data curve**. We shall not be proving the existence of a solution of a PDE of the first order here in this course. However, we shall give you the method which shows how a general solution of Eqn. (1), viz.,

$$Pp + Qq = R$$

may be used to determine the integral surface which passes through a given initial data curve.

We shall illustrate the method through the following examples.

Example 5: Find the general solution of the equation

$$(2xy - 1)\ p + (z - 2x^2)\ q = 2(x - yz) \qquad (25)$$

and the solution surface which passes through the line $x = 1,\ y = 0$.

Solution: The auxiliary equations corresponding to Eqn. (25) are

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)} \qquad (26)$$

Each of these fractions is equal to

$$\frac{zdx + dy + xdz}{0} = \frac{xdx + ydy + (dz/2)}{0} \qquad (27)$$

Integrating Eqns. (27), we obtain

$$u = y + xz = c_1,\ v = x^2 + y^2 + z = c_2$$

where c_1 and c_2 are arbitrary constants.
The general solution of the given equation can be written as

$$x^2 + y^2 + z = F(y + xz) \qquad (28)$$

where F is an arbitrary function. We shall determine the form of F using the initial data:

$$x = 1,\ y = 0$$

Substituting this data in Eqn. (28), we obtain

$$F(z) = 1 + z$$

and therefore,

$$F(y+xz)=1+y+xz \tag{29}$$

Substituting from Eqn. (29) into Eqn. (28), we obtain the required solution surface as

$$x^2+y^2-xz-y+z=1$$

***

Let us consider another example.

Example 6: Find the equation of the integral surface of the equation

$$x^3 p+y(3x^2+y)q=z(2x^2+y) \tag{30}$$

which passes through the curve

$$\Gamma_0 : x_0=1,\ y_0=s, z=s(1+s) \tag{31}$$

where s is the defining parameter of the curve.

Solution: The auxiliary equations corresponding to Eqn. (30) are

$$\frac{dx}{x^3}=\frac{dy}{y(3x^2+y)}=\frac{dz}{z(2x^2+y)} \tag{32}$$

Each of the fractions of Eqns. (32) is equal to

$$\frac{-x^{-1}dx+y^{-1}dy-z^{-1}dz}{0} \Rightarrow x^{-1}dx-y^{-1}dy+z^{-1}dz=0$$

Therefore, integrating the above equation, we get

$$y=c_1 xz \tag{33}$$

On solving the pair formed by the first and third fractions of Eqns. (32) and using Eqn. (33), we obtain

$$\frac{2x}{z}-\frac{x^2}{z^2}\frac{dz}{dx}=c_1,$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{x^2}{z}\right)=c_1,$$

$$\Rightarrow \quad \frac{x^2}{z}=c_1 x+c_2,$$

or

$$x^2=y+c_2 z. \tag{34}$$

Substituting the initial data in Eqns. (33) and (34), we get

$$1=c_1(1+s),\ 1-s=c_2 s(1+s)$$

On eliminating s from above equations, we get a relation between c_1 and c_2 as

$$c_1(2c_1-1)=c_2(1-c_1). \tag{35}$$

Substituting for c_1 and c_2 from Eqns. (33) and (34) in Eqn. (35), we obtain the required solution surface as

$$2y^2-xyz=x(x^2-y)\,(xz-y).$$

***

You may now try the following exercises.

E2) Find the integral surface of the equation

$$x(y^2+z)p-y(x^2+z)q=(x^2-y^2)z$$

Passing through the curve

$$x + y = 0,\ z = 1.$$

E3) Find the integral surface of the equation $yp + xq - z = 0$ which passes through the curve $z = x^3,\ y = 0$.

E4) Find the integral surface of the equation

$$(y-z)p + (z-x)q = x - y$$

which passes through the curve $z = 0,\ xy = 1$.

We shall now consider the method of solving non-linear equations of the first order.

17.3 NON-LINEAR EQUATIONS OF THE FIRST ORDER

Consider the general first order non-linear PDE

$$f(x,\ y,\ z,\ p,\ q) = 0$$

As we have already mentioned in Sec. 17.1, the method of finding the complete integral of a first order non-linear PDE has been developed by Charpit, a French mathematician. This method consists in finding another first order PDE which is compatible with the given Eqn. (37) and which involves an arbitrary parameter. Thus, before taking up the Charpit's method, we shall define when a system of first order PDEs are compatible and what are the conditions satisfied by such systems.

17.3.1 Compatible Systems of First Order Equations

Consider the first order PDEs

$$f(x, y, z,\ p, q) = 0 \qquad (36)$$

and $g(x,\ y, z,\ p,\ q) = 0$ (37)

Eqns. (36) and (37) are said to be **compatible**, if and only if,

i) $$J = \frac{\partial(f,\ g)}{\partial(p,\ q)} = \begin{vmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial q} \end{vmatrix} \neq 0 \qquad (38)$$

ii) $p = \phi(x,\ y,\ z),\ q = \psi(x,\ y,\ z)$ (39)

obtained by solving Eqns. (36) and (37), makes the equation

$$dz = \phi(x,\ y,\ z)dx + \psi(x,\ y,\ z)dy \qquad (40)$$

integrable.

For more clarity, let us consider the following example.

Example 7: Show that the equations

$$xp = yq,\ z(xp + yq) = 2xy$$

are compatible and solve them.

Solution: Let $f = xp - yq = 0$ (41)

and $g = z(xp + yq) - 2xy = 0$ (42)

Then $\dfrac{\partial(f,\, g)}{\partial(p,\, q)} = \begin{vmatrix} x & -y \\ xz & yz \end{vmatrix} = 2xyz \neq 0$

Thus, f and g satisfy condition (38). In order to check condition (40) we solve Eqns. (41) and (42) for p and q and obtain

$$p = \frac{y}{z} \text{ and } q = \frac{x}{z}.$$

Substituting the values of p and q in Eqn. (40), we get

$$z\ dz = y\ dx + x\ dy$$

which on integration gives $z^2 = 2xy + c$.

Thus, Eqns. (41) and (42) are compatiable with $z^2 = 2xy + c$ as a one-parameter family of common solution.

***

Let us look at Eqn. (40) viz.,

$$\phi dx + \psi dy - dz = 0$$

once again. You know that it is a total differential equation and the necessary and sufficient condition for integrability (ref. Theorem 1, Unit-15) is

$$\phi \frac{\partial \psi}{\partial z} - \psi \frac{\partial \phi}{\partial z} - \left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \right) = 0$$

$$\Rightarrow \quad \phi \psi_z + \psi_x = \phi_y + \psi \phi_z \tag{43}$$

Thus we can say that Eqns. (36) and (37) are compatible if Eqn. (43) is satisfied where ϕ and ψ are given by Eqn. (39). In order to obtain the compatibility condition in terms of f and g, we substitute in Eqns. (36) and (37) the values of p and q from Eqn. (39) and then differentiate the resulting equations with respect to x and y, and obtain

$$f_x + f_z \phi + f_p(\phi_x + \phi_z \phi) + f_q(\psi_x + \psi_z \phi) = 0 \tag{44}$$

$$g_x + g_z \phi + g_p(\phi_x + \phi_z \phi) + g_q(\psi_x + \psi_z \phi) = 0 \tag{45}$$

$$f_y + f_z \psi + f_p(\phi_y + \phi_z \psi) + f_q(\psi_y + \psi_z \psi) = 0 \tag{46}$$

$$g_y + g_z \psi + g_p(\phi_y + \phi_z \psi) + g_q(\psi_y + \psi_z \psi) = 0 \tag{47}$$

Multiplying Eqns. (44) and (45) by g_p and f_p, respectively and subtracting the resulting equations, we get

$$(f_x g_p - g_x f_p) + \phi(f_z g_p - g_z f_p) + (\phi_x + \phi_z \phi)\ (f_p g_p - g_p f_p)$$
$$+ (\psi_x + \psi_z \phi)\ (f_q g_p - g_q f_p) = 0$$

$$\Rightarrow \quad \frac{\partial(f,\, g)}{\partial(x,\, p)} + p \frac{\partial(f,\, g)}{\partial(z,\, p)} + (\psi_x + \psi_z \phi)\ \frac{\partial(f,\, g)}{\partial(q,\, p)} = 0 \tag{48}$$

where using relation (39) we have replaced ϕ by p in the second term of Eqn. (48). Similarly, multiplying Eqns. (46) and (47) by g_p and f_q, respectively and subtracting the resulting equations, we obtain

$$\frac{\partial(\mathrm{f},\, \mathrm{g})}{\partial(\mathrm{y},\, \mathrm{q})} + \mathrm{q} \frac{\partial(\mathrm{f},\, \mathrm{g})}{\partial(\mathrm{z},\, \mathrm{q})} - (\phi_y + \phi_z\, \psi)\ \frac{\partial(\mathrm{f},\, \mathrm{g})}{\partial(\mathrm{q},\, \mathrm{p})} = 0 \tag{49}$$

Adding Eqns. (48) and (49) and using relation (43) we obtain the condition for the compatibility of Eqns. (36) and (37) as

$$\frac{\partial(\mathrm{f},\,\mathrm{g})}{\partial(\mathrm{x},\,\mathrm{p})}+\frac{\partial(\mathrm{f},\,\mathrm{g})}{\partial(\mathrm{y},\,\mathrm{q})}+\mathrm{p}\frac{\partial(\mathrm{f},\,\mathrm{g})}{\partial(\mathrm{z},\,\mathrm{p})}+\mathrm{q}\frac{\partial(\mathrm{f},\,\mathrm{g})}{\partial(\mathrm{z},\,\mathrm{q})}=0 \tag{50}$$

The expression on the left hand side of Eqn. (50) is denoted by $[f,\, g]$. Thus, we have

$$[f,\, g]=\frac{\partial(f,\, g)}{\partial(x,\, p)}+\frac{\partial(f,\, g)}{\partial(y,\, q)}+p\frac{\partial(f,g)}{\partial(z,\, p)}+q\frac{\partial(f,g)}{\partial(z,\, q)}=0 \tag{51}$$

Thus we see that the condition for integrability of Eqn. (40) is that

$[f,\, g]=0$

Eqn. (51) can also be written in the form

$$[f,g]=f_p\frac{\partial g}{\partial x}+f_q\frac{\partial g}{\partial y}+(pf_p+qf_q)\frac{\partial g}{\partial z}-(f_x+pf_z)\frac{\partial g}{\partial p}-(f_y+qf_z)\frac{\partial g}{\partial q}=0 \tag{52}$$

We can use Eqn. (52), which is a first order PDE, for finding the equation $g=0$, compatible with the given equation $f=0$. Once g is known we can find p and q and then integrate Eqn. (40) to obtain one parameter family of solutions in the form

$$h(x,\, y,\, z, b)=0 \tag{53}$$

where b is an arbitrary constant. The solution given by Eqn. (53) shall satisfy both the Eqns. (36) and (37). Thus, **compatible equations have a one-parameter family of common solutions**.

Note that, as we have mentioned above, compatible Eqns. (36) and (37) have a one-parameter family of common solutions. This does not mean that every solution of $f(x,\, y,\, z,\, p,\, q)=0$ is necessarily the solution of $g(x,\, y,\, z,\, p,\, q)=0$ or vice-versa. In Example 7, $z=xy$ satisfy Eqn. (41) and hence is a solution of Eqn. (41) but it does not satisfy Eqn. (42).

Let us now consider a few examples to illustrate the method discussed above.

Example 8: Show that the partial differential equation

$$p^2x+q^2y-z=0 \tag{54}$$

is compatible with

$$p^2x-q^2y=0 \tag{55}$$

and find their **one-parameter family of common solutions**.

Solution: Let $f=p^2x+q^2y-z=0$

and $g=p^2x-q^2y=0$

Here we have

$$f_x=p^2,\ f_y=q^2,\ f_p=2px,\ f_q=2qy,\ f_z=-1$$

$$g_x=p^2,\ g_y=-q^2,\ g_p=2px,\ g_q=-2qy,\ g_z=0$$

Using condition (52), we get

$$[f,g]=2xp^3-2yq^3-(p^2-p)\,2px+(q^2-q)\,2qy$$

$$=2(p^2x-q^2y)=0 \qquad \text{[using Eqn. (55)]}$$

Hence the given system of PDEs are compatible.

Solving Eqns. (54) and (55) for p and q, we get

$$p = \pm\left(\frac{z}{2x}\right)^{\frac{1}{2}},\ q = \pm\left(\frac{z}{2y}\right)^{\frac{1}{2}} \tag{56}$$

Considering only the positive values of p and q in Eqn. (56) and substituting in

$$dz = pdx + qdy\,,$$

we obtain

$$\frac{\sqrt{2}}{\sqrt{z}}dz = \frac{dx}{\sqrt{x}} + \frac{dy}{\sqrt{y}}$$

which on integration gives the one-parameter family of common solutions as

$$\sqrt{2z} = \sqrt{x} + \sqrt{y} + b$$

where b is an arbitrary parameter.

***

Let us consider another example.

Example 9: Show that the equation

$$z = px + qy \tag{57}$$

is compatible with any equation

$$f(x,\ y,\ z,\ p,\ q) = 0 \tag{58}$$

that is homogeneous in $x,\ y$ and z.

Solution: In this case, we have

$$[f,\ g] = xf_x + yf_y + (xp + yq)\ f_z\,, \tag{59}$$

where $g = px + qy - z = 0$

If f is homogeneous in $x,\ y,\ z$ and is of degree say n, then by Euler's theorem (ref Unit-4, Block-1), we have

$$xf_x + yf_y + zf_z = nf$$

Thus Eqn. (59) reduces to

$[f, g] = f_z(xp + yq - z) + nf = 0$ (by virtue of Eqns. (57) and (58)).

Thus the compatibility condition is satisfied and Eqns. (57) and (58) are compatible.

***

You may now try the following exercises.

E5) Show that the equations

$$f(x,\ y,\ p) = 0 \text{ and } g(x,\ y,\ q) = 0$$

are compatible if $f_p g_x - f_y g_q = 0$.

E6) Shot that the partial differential equations

$$f(x,\ y,\ p,\ q) = 0 \text{ and } g(x,\ y,\ p,\ q) = 0$$

are compatible if

$$\frac{\partial(f,\ g)}{\partial(x,\ p)} + \frac{\partial(f,\ g)}{\partial(y,\ q)} = 0.$$

E7) Show that the equations $(y-z)p+(z-x)q = x-y$ and $z-px-qy=0$ are compatible.

E8) Show that the equations $xp-yq = x$ and $x^2 p+q = xz$ are compatible and find their one-parameter family of common solutions.

We shall now discuss Charpit's method of finding the complete integral of first order non-linear PDEs.

17.3.2 Charpit's Method

As mentioned earlier, in the Charpit's method of finding the complete integral of non-linear PDE of the form (36), namely,

$$f(x, y, z, p,q) = 0,$$

we introduce another PDE of the first order of the type

$$F(x, y, z, p, q, a) = 0 \tag{60}$$

which contains an arbitrary parameter a and which is compatible with Eqn. (36). In other words, we try to find a function F such that

i) Eqns. (36) and (60) can be solved to obtain

$$p = p(x, y, z, a),\ q = (x, y, z, a) \tag{61}$$

ii) The p, q obtained in Eqn. (61) makes the equation

$$dz = p(x, y, z, a)\,dx + q(x, y, z, a)\,dy \tag{62}$$

integrable.

Once we are able to find such a function F, we can integrate Eqn. (62) and obtain the two parameter family of solution in the form

$$G(x, y, z, a, b) = 0, \tag{63}$$

which will be the complete integral of Eqn. (36).

Thus the main problem now is the determination of the second Eqn. (60). Infact, this problem has already been resolved in Sub-sec. 17.3.1 and we need only to obtain an equation $F = 0$ compatible with the given equation $f = 0$. You know that the conditions for the compatability of equations $F = 0$ and $f = 0$ as given by Eqns. (38) and (52) are:

$$J = \frac{\partial(f, F)}{\partial(p, q)} \neq 0$$

and

$$f_p \frac{\partial F}{\partial x} + f_q \frac{\partial F}{\partial y} + (pf_p + qf_q)\frac{\partial F}{\partial z} - (f_x + pf_z)\frac{\partial F}{\partial p} - (f_y + qf_z)\frac{\partial F}{\partial q} = 0 \tag{64}$$

Note that Eqn. (64) is a first order linear PDE for determining the function F which is considered to be a function of five variables x, y, z, p and q. Following the Lagrange's method of solving first order linear PDE, the auxiliary equations for Eqn. (64) are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)} \tag{65}$$

Eqns. (65) are known as the **Charpit's equations** and can be written at once using the given Eqn. (36).

Once we find a solution of the system of Eqns. (65) involving p or q or both in the form of Eqn. (60), viz.,

$$F(x, y, z, p, q, a) = 0,$$

then the problem reduces to solving Eqns. (36) and (60) for p and q and then integrating Eqn. (62) by using the methods of solving total differential equations which you have learnt in Unit 15.

We shall now illustrate the method through the examples.

Example 10: Find the complete integral of

$$z^2 = pqxy$$

by Charpit's method.

Solution: Let $f = pqxy - z^2 = 0$.
In this case, the auxiliary Eqns. (65) yield

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{2zp - pqy} = \frac{dq}{2zq - pqx}$$

$$\Rightarrow \quad \frac{dz}{2pqxy} = \frac{dz}{2z^2} = \frac{pdx + qdy + xdp + ydq}{2z(px + qy)} \text{ (using given equation)}$$

$$\Rightarrow \quad \frac{dz}{z} = \frac{d(xp + yq)}{xp + yq}$$

On integration, we obtain F which is compatible with f as

$$F = z - a(xp + yq) = 0, \tag{66}$$

where a is an arbitrary constant. Solving the given equation and Eqn. (66) for p and q, we obtain

$$p = \frac{z}{x}\left(\frac{2a}{1 \pm \sqrt{1 - 4a^2}}\right), \; q = \frac{z}{y}\left(\frac{1 \pm \sqrt{1 - 4a^2}}{2a}\right)$$

Let $c = \dfrac{1 \pm \sqrt{1 - 4a^2}}{2a}$

Then, we can write

$$p = \frac{z}{cx}, \; q = \frac{cz}{y}$$

where $a(c^{-1} + c) = 1$, c being a constant.

$$\therefore \quad dz = pdx + qdy = \left(\frac{1}{c}\frac{dx}{x} + c\frac{dy}{y}\right) z$$

Integrating, we obtain the complete integral of the given equation in the form $z = bx^{1/c} y^c$, b being the constant of integration.

***

Remark: For any given problem, it is not necessary to use all of the Charpit's Eqns. (65) for finding F. This is illustrated in our next example. What we have to be careful about is that p or q must occur in the solutions obtained.

Example 11: Using Charpit's method find the complete integral of the equation

$$(p^2+q^2)y = qz$$

Solution: The given equation can be expressed as

$$f = (p^2+q^2)\,y - qz = 0$$

The auxiliary equations in this case are

$$\frac{dx}{2py} = \frac{dy}{2qy-z} = \frac{dz}{2p^2y+2q^2y-qz} = \frac{dp}{pq} = \frac{dq}{-(p^2+q^2)+q^2}$$

The last two fractions in the above equations yield

$$pdp + qdq = 0$$

Integrating the above equation, we get

$$p^2+q^2 = a, \tag{67}$$

where a is a constant.
Solving Eqn. (67) and the given equation for p and q, we get

$$p = \pm\frac{\sqrt{az^2-a^2y^2}}{z} \text{ and } q = \frac{ay}{z}$$

Substituting these values of p and q in $dz = pdx + qdy$ and taking p with only positive sign, we get

$$dz = \frac{\sqrt{az^2-a^2y^2}}{z}\,dx + \frac{ay}{z}dy$$

$$\Rightarrow \quad \frac{zdz - ay\,dy}{\sqrt{az^2-a^2y^2}} = dx$$

$$\Rightarrow \quad \frac{1}{2a}\,\frac{2az\;dz - 2a^2ydy}{\sqrt{az^2-a^2y^2}} = dx$$

$$\Rightarrow \quad \frac{1}{2}\,\frac{d(az^2-a^2y^2)}{\sqrt{az^2-a^2y^2}} = a\;dx.$$

Integrating, we get

$\sqrt{az^2-a^2y^2} = ax+b$, b being the constant of integration.

$$\Rightarrow \quad az^2 - a^2y^2 = (ax+b)^2$$

which is the required complete integral.

***

Note that a first order PDE can have more than one complete integral as illustrated in the following example.

Example 12: Find the complete integral of the PDE

$$z^2(1+p^2+q^2) = 1$$

using Charpit's method.

Solution: Let $f = z^2(1+p^2+q^2) - 1 = 0$

The auxiliary Eqns. (65) in this case give

$$\frac{dx}{2pz^2} = \frac{dy}{2qz^2} = \frac{dz}{2p^2z^2+2q^2z^2} = \frac{dp}{-2zp(1+p^2+q^2)} = \frac{dq}{-2zq(1+p^2+q^2)} \tag{68}$$

From the last two fractions, we get

$$\frac{dp}{p} = \frac{dq}{q} \text{ or } p = aq \tag{69}$$

where a is an arbitrary constant.
Solving for p and q the given equation and Eqn. (69), we obtain

$$p = \frac{a}{\sqrt{1+a^2}}\sqrt{\frac{1-z^2}{z^2}} \text{ and } q = \frac{1}{\sqrt{1+a^2}}\sqrt{\frac{1-z^2}{z^2}}$$

$$\therefore \ dz = pdx + qdy = \frac{a}{\sqrt{1+a^2}}\sqrt{\frac{1-z^2}{z^2}}dx + \frac{1}{\sqrt{1+a^2}}\sqrt{\frac{1-z^2}{z^2}}dy$$

which on integration gives the complete integral of the given equation as

$$-\sqrt{1-z^2}\sqrt{1+a^2} = ax + y + c, \ c \text{ is a constant of integration}$$

or $\quad (ax+y+c)^2 = (1+a^2)(1-z^2) \quad (70)$

If in Eqns. (68), we consider 3rd and the 4th fractions, we obtain

$$\frac{dz}{z(1-z^2)} = \frac{dp}{-p} \text{ (using the given equation)}$$

$$\Rightarrow \left[\frac{1}{z} - \frac{1}{2(1-z)} - \frac{1}{2(1+z)}\right]dz + \frac{dp}{p} = 0$$

which on integration gives

$$z^2p^2 = b(1-z^2) \tag{71}$$

where b is an arbitrary constant.
Solving the given equation and Eqn. (71) for p and q, we obtain

$$p = b\sqrt{\frac{1-z^2}{z^2}} \text{ and } q = \sqrt{1-b^2}\sqrt{\frac{1-z^2}{z^2}}$$

$$\therefore \ dz = pdx + qdy = b\sqrt{\frac{1-z^2}{z^2}}dx + \sqrt{1-b^2}\sqrt{\frac{1-z^2}{z^2}}dy.$$

Integrating the above equation we get another complete integral of the given equation as

$$(bx + \sqrt{1-b^2}\,y + c_1)^2 + z^2 = 1 \tag{72}$$

***

We now take up an example where from the complete integral of the given equation we have also obtained its general and singular integrals.

Example 13: Use Charpit's method to find the complete integral of the equation

$$(p^2+q^2)y = qz \tag{73}$$

Also obtain its general and singular integrals.

Solution: Let $f = (p^2+q^2)y - qz = 0$

Charpit's auxiliary equations are

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{2p^2y + 2q^2y - qz} = \frac{dp}{qp} = \frac{dq}{-(p^2+q^2)+q^2}$$

Taking the last two fractions, we get

$$pdp + qdq = 0$$

Integrating, we get $p^2 + q^2 = a^2$, a is a constant. (74)

Solving Eqns.(73) and (74) for p and q, we obtain

$$p = \frac{a}{z}\sqrt{z^2 - a^2y^2} \text{ and } q = \frac{a^2y}{z}$$

$$\therefore\ dz = pdx + qdy = \frac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \frac{a^2y}{z}dy$$

or $$\frac{zdz - a^2ydy}{\sqrt{z^2 - a^2y^2}} = a\,dx\,.$$

Integrating the above equation we obtain the required complete integral as

$$(z^2 - a^2y^2)^{1/2} = ax + b$$

or $$z^2 - a^2y^2 = (ax+b)^2 \tag{75}$$

where b is an arbitrary constant.

Singular Integral

Differentiating Eqn. (75) partially w.r.t. a and b, we get

$$2ay^2 + 2(ax+b)x = 0 \tag{76}$$

$$2(ax+b) = 0 \tag{77}$$

Eliminating a and b between Eqns. (75), (76) and (77), we get $z = 0$, which satisfies Eqn. (73) and is its singular solution.

General Integral

Replacing b by $\phi(a)$ in Eqn. (75), we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2 \tag{78}$$

Differentiating Eqn. (78) partially w.r.t. a, we get

$$-2ay^2 = 2[ax+\phi(a)]\,[x+\phi'(a)] \tag{79}$$

General integral of Eqn. (73) is obtained by eliminating a from Eqns. (78) and (79).

***

You may now try the following exercise to check your understanding of the Charpit's method.

E9) Using Charpit's method, find the complete integrals of the following equations:

i) $p^2x + q^2y = z$

ii) $2(z + xp + yq) = yp^2$

iii) $2z + p^2 + qy + 2y^2 = 0$

iv) $2x(z^2q^2 + 1) = pz$

v) $pxy + pq + qy = yz$

After solving E9), you must have observed that the method given by Charpit for finding the complete integral of a non-linear first order PDE is usually quite lengthy and involved. However, there are some special types of first order non-linear PDEs whose complete integrals can be obtained easily by the Charpit's method. These special types of non-linear PDEs of first order are called **standard forms** of Eqn. (36). We now take up the methods of integrating these standard forms.

17.3.3 Standard Forms

Let us discuss various types of standard forms of Eqn. (36) one by one.

Type I: Equations involving only p and q:

Let us start by taking an example of the equation involving only p and q.

Example 14: Find the complete integral of the equation

$$pq = 1$$

Solution: For the given equation, the Charpit's auxiliary equations are

$$\frac{dx}{q} = \frac{dy}{p} = \frac{dz}{2pq} = \frac{dp}{0} = \frac{dq}{0} \tag{80}$$

From the fourth fraction, we get

$$dp = 0$$

$\Rightarrow \quad p =$ constant $= a$, say.

From the above equation and the given equation, we get

$$aq = 1 \Rightarrow q = \frac{1}{a}.$$

Substituting for p and q in $dz = pdx + qdy$, we obtain

$$dz = a\ dx + \frac{1}{a} dy$$

Integrating, we get the complete integral of the given equation as

$$z = ax + \frac{1}{a} y + b.$$

In the auxiliary system of Eqns. (80) if we consider the last fraction then we obtain

$$dq = 0$$

$\Rightarrow \quad q = a$ (constant)

and the complete integral is of the form

$$z = \frac{1}{a} x + ay + b.$$

***

In general, the non-linear PDEs of first order which do not contain the variables x, y, z explicitly and involve only p and q are of the form

$$f(p, q) = 0 \tag{81}$$

In this case the Charpit's equations (or auxiliary Eqns. (65)) are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

From the last two fractions in the above system of equations we get either $p = a$ or $q = a$, where a is a constant.

If we consider $p = a$, then Eqn. (81) takes the form $f(a, q) = 0$ which on solving gives $q = Q(a)$.

Therefore, the equation $\mathrm{dz} = \mathrm{pdx} + \mathrm{qdy}$ reduces to

$$\mathrm{dz} = \mathrm{adx} + \mathrm{Q(a)\,dy}$$

which on integration gives the complete integral of Eqn. (81) as

$$z = ax + Q(a)\, y + b, \tag{82}$$

where b is an arbitrary constant.

Similarly, if we consider $q = a$ then from Eqn. (81) we obtain $p = Q(a)$ and the complete integral of Eqn. (81) assumes the following form

$$z = Q(a)x + ay + b \tag{83}$$

We now illustrate the method through an example.

Example 15: Find the complete integral of the equation

$$p^2 - q^2 = 4 \tag{84}$$

Solution: The given equation is of the form $f(p, q) = 0$. Let us take $p = a$ (a constant).

Putting $p = a$ in Eqn. (84), we obtain

$$q = \pm\sqrt{a^2 - 4} = Q(a)$$

Substituting the values of a and $Q(a)$ in Eqn. (82), the complete integral of Eqn. (84) is

$$z = ax \pm \sqrt{a^2 - 4}\, y + c,\ c \text{ being a constant.}$$

***

We next take up an example to illustrate how a given equation can be easily solved if we first reduce it to the form (81).

Example 16: Find the complete integral of the equation

$$(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$$

Solution: Let $x + y = U^2$ and $x - y = V^2$

Then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial U}\frac{\partial U}{\partial x} + \frac{\partial z}{\partial V}\frac{\partial V}{\partial x}$$

$$= \frac{1}{2U}\frac{\partial z}{\partial U} + \frac{1}{2V}\frac{\partial z}{\partial V} \tag{85}$$

and

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial U}\frac{\partial U}{\partial y} + \frac{\partial z}{\partial V}\frac{\partial V}{\partial y}$$

$$= \frac{1}{2U}\frac{\partial z}{\partial U} - \frac{1}{2V}\frac{\partial z}{\partial V} \tag{86}$$

Substituting for p and q from Eqns. (85) and (86) in the given equation, it reduces to

$$\left(\frac{\partial z}{\partial U}\right)^2 + \left(\frac{\partial z}{\partial V}\right)^2 = 1$$

This equation is of Type I and we can write down its complete integral directly by using Eqn. (82) in the form

$$z = aU + \sqrt{1 - a^2}\,V + b$$

$$= a\sqrt{x+y} + \sqrt{1-a^2}\sqrt{x-y} + b$$ (substituting back the values of U and V in terms of x and y.)

where a and b are arbitrary constants.

***

You may now try the following exercise.

E10) Find the complete integrals of the following equations:

i) $p+q-pq=0$

ii) $p^2+q^2=1$

iii) $p=e^q$

iv) $(y-x)\,(qy-px)=(p-q)^2$

v) $(1-x^2)\;yp^2+x^2q=0$

We next take up those equations which do not involve the independent variables $x,\ y$ explicitly.

Type II: Equations not Involving the Independent Variables

Equations of the type (36) which do not involve $x,\ y$ explicitly are of the form

$$f(z,\ p,\ q)=0 \tag{87}$$

Charpit's Eqns. (65) in this case assume the form

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-pf_z}=\frac{dq}{-qf_z}$$

the last two of which yield

$$\frac{dp}{p}=\frac{dq}{q} \tag{88}$$

Integrating the above equation, we obtain

$$p=aq \text{ or } q=ap \tag{89}$$

where a is a constant.

From Eqns. (87) and (89), we then have

$$f(z,\ aq,\ q)=0$$

$$\Rightarrow\quad q=Q(a,\ z) \tag{90}$$

On substituting for p and q, from Eqns. (89) and (90), in the equation

$dz=pdx+qdy$, we get

$$dz=aq\,dx+q\;dy=Q(a,\ z)\;(adx+dy)$$

Thus, the complete integral of Eqn. (87) is given by

$$\int\frac{dz}{Q(a,z)}=ax+y+b,$$

where b is an arbitrary constant.

On using an alternative solution of Eqn. (88) as $q=ap$ the complete integral of Eqn. (87) is given by

$$\int\frac{dz}{Q(a,z)}=x+ay+b$$

Let us consider the following examples.

Example 17: Find the complete integral of the equation

$$z\;pq-p-q=0$$

Solution: The given equation is

$$zpq-p-q=0 \tag{91}$$

Putting $p=aq$ in the given equation, we obtain

$$zaq^2 - aq - q = 0$$

$\Rightarrow$ $q = 0$ or $q = \dfrac{a+1}{az}$

Now, if $q = 0,\ p = 0$

and if $q = \dfrac{a+1}{az},\ p = \dfrac{a+1}{z}$

For the case $p = 0,\ q = 0$, the relation

$$dz = pdx + qdy,$$

yields

$$dz = 0$$

$\Rightarrow$ $z =$ constant.

which is obviously not the complete integral of the given equation.

Further, for $q = \dfrac{a+1}{az},\ p = \dfrac{a+1}{z}$, the relation

$$dz = pdx + qdy$$

yields

$$dz = \frac{a+1}{z}\left(dx + \frac{1}{a}\,dy\right)$$

which on integration gives the complete integral of the given equation in the form

$$z^2 = \frac{2(1+a)}{a}\,(ax + y) + b$$

b being an arbitrary constant.

***

We now take up an example where we have reduced the given equation to Type II equation and then obtain its complete integral.

Example 18: Find the complete integral of the equation

$$q^2 y^2 = z(z - px)$$

Solution: Let $X = \ln x$ and $Y = \ln y$ (91)

then $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X}\dfrac{\partial X}{\partial x} = \dfrac{1}{x}\dfrac{\partial z}{\partial X}$ (92)

and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial Y}\dfrac{\partial Y}{\partial y} = \dfrac{1}{y}\dfrac{\partial z}{\partial Y}$ (93)

If we denote $\dfrac{\partial z}{\partial X}$ by P and $\dfrac{\partial z}{\partial y}$ by Q then substituting $px = P$ and $qy = Q$ from Eqns. (92) and (93), respectively the given equation reduces to

$$Q^2 = z^2 - zP \tag{94}$$

which is of the Type II above. To solve Eqn. (94)
let us take $Q = aP$, a being a constant. (95)
Then from Eqns. (94) and (95), we obtain

$$a^2P^2 + zP - z^2 = 0$$

$$\Rightarrow \quad P = \frac{-z \pm \sqrt{z^2 + 4a^2z^2}}{2a^2} = kz \tag{96}$$

where $k = \dfrac{1}{2a^2}\left(-1 \pm \sqrt{1 + 4a^2}\right)$

$\therefore$ $dz = PdX + QdY$

$$= k\ zdX + a\ k\ zdY \text{ [using Eqns. (95) and (96)]}$$
$$= k\ z(dX + adY)$$

$$\Rightarrow \quad \frac{dz}{k\ z} = dX + a\,dY$$

Integrating, the above equation we obtain

$$X + aY + \ln b = \frac{1}{k}\ln z$$

or $\quad \ln x + a\ln y + \ln b = \frac{1}{k}\ln z$ [using Eqn. (91)]

or $\quad x\,b\,y^a = z^{1/k}$

which is the required complete integral.

***

You may now try the following exercise.

E11) Find the complete integrals of the following equations:

i) $4z = pq$

ii) $p^2 = zq$

iii) $p^3 + q^3 = 27z$

iv) $z = p^2 - q^2$

v) $p^2x^2 = z(z - 9y)$

We next take up the equations of the form (36) in which z does not occur explicitly and which are in variable separable form. In other words, we consider the equations of the form which can be written as

$$f(x, p) = g(y, q)$$

Type III: Variable Separable Equations

Let us start by considering a simple example.

Example 19: Find the complete integral of the equation

$$q - p + x - y = 0$$

Solution: The given equation can be written as

$$p - x = q - y \tag{97}$$

You may **note** that Eqn. (97) is of variable separable form where left hand side is a function of x and p only and right hand side is a function of y and q.

For Eqn. (97) the Charpit's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{p-q} = \frac{dp}{1} = \frac{dq}{-1}$$

From the first and the fourth fractions in the above equation, we obtain

$$\frac{dx}{1} = \frac{dp}{1}$$

Intewgrating, we get

$$p = x + a \text{ or } p - x = a$$

We then obtain from Eqn. (97)

$$q - y = a \text{ or } q = y + a$$

$\therefore \; dz = pdx + qdy$, yields

$$dz = (x+a)dx + (y+a)dy$$

or $2z = (x+a)^2 + (y+a)^2 + b$, where b is an arbitrary constant, is the required complete integral.

***

In general, we consider a first order PDE which can be written in the form

$$f(x,p) = g(y,q)$$

$$\Rightarrow \; f(x,p) - g(y,q) = 0$$

For such an equation, Charpit's equations are

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qg_q} = \frac{dp}{-f_x} = \frac{dq}{g_y} \tag{98}$$

The first and the fourth fractions of Eqns. (98) give

$$\frac{dp}{f_x} + \frac{dx}{f_p} = 0$$

$$\Rightarrow \; \frac{dp}{dx} + \frac{f_x}{f_p} = 0 \tag{99}$$

You may **note** that Eqn. (99) is an ODE in x and p. We can solve it by writing it in the form

$$f_x dx + f_p dp = 0$$

$$\Rightarrow \; d\, f(x,p) = 0$$

$$\Rightarrow \; f(x,p) = \text{constant} = a, \text{ say} \tag{100}$$

Similarly, from 2nd and 5th fractions of Eqns. (98) and by making use of the given equation, we get

$$g(y,q) = \text{ constant} = a \tag{101}$$

Note that in Eqn. (101) we have again taken the constant of integration as 'a'. This is because we are given $f(x,p) = g(y,q)$. Here f is a function of x and p whereas g is a function of y and q and if both are equal then each of f and g has to be separately equal to the same constant.

Eqns. (100) and (101) can now be solved to obtain

$$p = F(a,x) \text{ and } q = G(a,y)$$

$\therefore \; dz = pdx + qdy$ reduces to

$$dz = F(a,x)dx + G(a,y)dy$$

which on integration gives the complete integral as

$$z = \int F(a,x)dx + \int G(a,y)dy + b,$$

where b is another arbitrary constant.

Let us take up examples to illustrate the method discussed above.

Example 20: Find the complete integral of the equation

$$p^2 + q^2 = x + y$$

Solution: The given equation can be written as

$$p^2 - x = y - q^2$$

Hence each side must be equal to the same constant, say a.

$\therefore \; p^2 - x = a$ and $y - q^2 = a$

$\Rightarrow \quad p = \pm\sqrt{x+a}$ and $q = \pm\sqrt{y-a}$

Any combination of $\pm$ signs can be taken. We take here p and q with the positive signs only. Then

$dz = pdx + qdy$, yields

$$dz = \sqrt{a+x}\ dx + \sqrt{y-a}\ dy$$

Integrating, the complete integral is obtained as

$$z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2}.$$

***

We now take up an example where we have transformed the given equation to Type III equation to obtain its complete integral.

Example 21: Find the complete integral of the equation

$$y\, z\, p^2 = q$$

Solution: Let us take $Z = \dfrac{z^2}{2}$ and re-write the given equation as

$$y\, z^2 p^2 = z\, q \qquad (102)$$

Now $zp = z\dfrac{\partial z}{\partial x} = \dfrac{\partial Z}{\partial x} = P$, say and $zq = z\dfrac{\partial z}{\partial y} = \dfrac{\partial Z}{\partial y} = Q$, say.

With above substitutions Eqn. (102) reduces to

$$yP^2 = Q \text{ or } P^2 = \frac{Q}{y} \qquad (103)$$

You may **notice** that Eqn. (103) is now a variable separable equation so if we take $P = a$ then we get $Q = ya^2$

Putting these values of P and Q in $dZ = P\,dx + Qdy$, we get

$$dZ = adx + ya^2 dy$$

Integrating, $Z = ax + \dfrac{1}{2}a^2 y^2 + b$, b a constant.

Putting back the value of $Z = z^2/2$ in the above equation, we get

$$z^2 = 2ax + a^2 y^2 + 2b$$

as the required complete integral of the given equation.

***

How about doing an exercise now?

E12) Find the complete integrals of the following equations:

i) $q = xp + p^2$

ii) $\sqrt{p} + \sqrt{q} = 2x$

iii) $p^2 - y^3 q = x^2 - y^2$

iv) $p^2 y(1+x^2) = qx^2$

v) $p^2 q^2 + x^2 y^2 = x^2 q^2 (x^2 + y^2)$

vi) $p^2 q(x^2 + y^2) = p^2 + q$

You may recall that in Unit 6, Block 2, we defined the Clairaut's equation as the ODE of the type

$$y = xp + f(p).$$

In the case of the first order PDE, we can write the Clairaut's equation in the form

$$z = px + qy + f(p, q)$$

We now give the method of solving such equations by using the Charpit's equations.

Type IV: Clairaut's Equation

Let us consider a simple example of the PDE of the Clairaut's type.

Example 22: Find the complete integral of the equation

$$z = px + qy + pq.$$

Solution: The given equation is

$$px + qy + pq - z = 0$$

The auxiliary Eqns. (65) in this case take the form

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{px + qy} = \frac{dp}{0} = \frac{dq}{0}$$

From the last two fractions of the above equations, we get

$$p = a, \; q = b$$

where a and b are arbitrary constants.
If we substitute the above values of p and q in the given equation, we obtain

$$z = ax + by + ab$$

which is the required complete integral.

***

In general, consider the Clairaut's form of the first order PDE

$$px + qy + f(p, q) - z = 0 \tag{104}$$

For Eqn. (104), the Charpit's Eqns. (65) assume the form

$$\frac{dx}{x + f_p} = \frac{dy}{y + f_q} = \frac{dz}{px + qy + pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

Obvious solution of this system of equations are

$p = a$ and $q = b$

If we substitute these values of p and q in Eqn. (104), we get

$$z = ax + by + f(a, b),$$

which is the complete integral of Eqn. (104).

Let us now consider an example where we have solved the given equation by first reducing it to the Clairaut's form.

Example 23: Find the complete integral of the equation

$$p\,q\,z = p^2(xq + p^2) + q^2(yp + q^2)$$

Solution: The given equation is

$$p\,q\,z = p^2(xq + p^2) + q^2(yp + q^2)$$

Dividing both sides by pq, we get the equation

$$z = px + \frac{p^3}{q} + yq + \frac{q^3}{p}$$

$$\Rightarrow \quad z = xp + yq + \left(\frac{p^3}{q} + \frac{q^3}{p}\right)$$

which is in the Clairaut's form.
Hence substituting in the above equation

$p = a$ and $q = b$,

where a and b are arbitrary constants, we obtain the complete integral of the given equation as

$$z = ax + by + \left(\frac{a^3}{b} + \frac{b^3}{a}\right).$$

***

You may now try the following exercise.

E13) Find the complete integrals of the following equations:

i) $(p+q)\,(z - xp - yq) = 1$

ii) $z = xp + yq + \sqrt{\alpha p^2 + \beta q^2 + \gamma}$

iii) $z = px + qy + 3(pq)^{1/3}$

We now end this unit by giving a summary of what we have covered in it.

17.4 SUMMARY

In this unit we have covered the following:

1. **The Lagrange's method** for solving quasi-linear PDE of first order yields the general integral $F(u, v) = 0$ of an equation

   $P(x, y, z)\,p + Q(x, y, z)\,q = R(x, y, z)$

   where $u(x, y, z) =$ constant and $v(x, y, z) =$ constant are two independent integrals of the auxiliary equations

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)}$$

2. First order PDEs

   $f(x, y, z, p, q) = 0$, (see Eqn. (37))

   and

   $g(x, y, z, p, q) = 0$, (see Eqn. (38))

   are compatible, if and only if,

   i) $J = \dfrac{\partial(f, g)}{\partial(p, q)} \neq 0$.

   ii) $p = \phi(x, y, z),\ q = \psi(x, y, z)$ obtained by solving $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ makes $dz = \phi(x, y, z)dx + \psi(x, y, z)dz$ integrable.

3. Condition ii) in Step 2 above, alternatively yields, the condition of compatibility of Eqns. (36) and (37) as

$$[f, g] = \frac{\partial(f, g)}{\partial(x, p)} + \frac{\partial(f, g)}{\partial(y, q)} + p\frac{\partial(f, g)}{\partial(z, p)} + q\frac{\partial(f, g)}{\partial(z, q)} = 0 .$$

4. The **Charpit's method** of solving PDE of the form (36) consists of the following steps.

   i) determine one solution of the auxiliary equations

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

   involving p or q or both which contains an arbitrary constant and is of the form $F(x, y, z, p, q, a) = 0$, a being constant.

   ii) solve $f = 0$ and $F = 0$ for p and q in terms of x, y, z.

   iii) substitute the above values of p and q in the relation

   $dz = pdx + qdy$

   and integrate it to obtain the complete integral of Eqn. (36).

5. There are four special forms of Eqn. (36) called **standard forms** to which Charpit's method can be applied easily. These forms are

   I Equations involving p and q only, for which the auxiliary equations reduces to

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

   II Equations not involving the independent variables explicitly, for which the auxiliary equations reduces to

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}$$

   III Variable separable equations of the form $f(x, p) = g(y, p)$ for which auxiliary equations yield the solution of the form

   $f(x, p) = a = g(y, q)$

   where a is a constant.

   IV For the first order PDE of the Clairaut's form, namely, $z = xp + yq + f(p, q)$ the auxiliary equations yield the solution of the form

   $p = a$ and $q = b$, giving

   $z = ax + by + f(a, b)$

   where a, b are arbitrary constants.

17.5 SOLUTIONS/ANSWERS

E1) i) The given equation can be written as

$zxp - zyp = y^2 - x^2$

The auxiliary equations are

$$\frac{dx}{xz} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2} = \frac{xdx + ydy + zdz}{0}$$

The integral of the above system of equations are

$u = xy = c_1$ and $v = x^2 + y^2 + z^2 = c_2$

Hence the general integral of given PDE is

$F(xy, x^2 + y^2 + z^2) = 0$,

where F is an arbitrary function.

ii) The given equation is

$y^2 p - xyq = x(z - 2y)$

The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Integrating the above system of equation, we get

$u = x^2 + y^2 = c_1$ and $v = yz - y^2 = c_2$

Hence the general integral of the given PDE is

$x^2 + y^2 = \phi(yz - y^2)$

iii) The auxiliary equations corresponding to the given equation are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)} = \frac{xdx + ydy + zdz}{0}$$

Integrating above system of equations, we have the two families of surfaces

$u = y^2 - 2yz - z^2 = c_1,\ v = x^2 + y^2 + z^2 = c_2$

Hence the general integral of given PDE is

$x^2 + y^2 + z^2 = f(y^2 - 2yz - z^2)$,

where f is an arbitrary function.

iv) The auxiliary equations corresponding to the given equation are

$$\frac{zx\ dx}{y-z} = \frac{zx\, dy}{z-x} = \frac{xy\, dz}{x-y} = \frac{dx + dy + dz}{0} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}$$

Integrating the last two fractions, we get the two families of surfaces as

$x + y + z = c_1$ and $xyz = c_2$

Thus the general integral of the given PDE is

$F(x + y + z, xyz) = 0$

v) Auxiliary equations are

$$\frac{dx}{x^3 + 3xy^2} = \frac{dy}{y^3 + 3x^2 y} = \frac{dz}{2(x^2 + y^2)z} \qquad \text{(i)}$$

Choosing $\frac{1}{x}, \frac{1}{y}, 0$ as multipliers, each fraction of (i)

$$= \frac{\frac{dx}{x} + \frac{dy}{y}}{4(x^2 + y^2)} \qquad \text{(ii)}$$

Taking the last fraction of (i) and fraction (ii), we get $\frac{xy}{z^2} = c_1$ (iii)

Choosing $1, 1, 0$ as multipliers, each fraction of (i) is

$$= \frac{dx + dy}{x^3 + 3xy^2 + y^3 + 3x^2 y} = \frac{dx + dy}{(x+y)^3} \qquad \text{(iv)}$$

Choosing $1, -1, 0$ as multipliers, each fraction of (i)

$$= \frac{dx - dy}{x^3 + 3xy^2 - y^3 - 3x^3 y} = \frac{dx - dy}{(x - y)^3} \qquad \text{(v)}$$

From (iv) and (v) $(x + y)^{-3}(dx + dy) = (x - y)^{-3}(dx - dy)$

Integrating, $-(x + y)^2 + (x - y)^{-2} = c_2$

Thus, the required general solution is

$$F\left[(x - y)^{-2} - (x + y)^{-2}, \frac{xy}{z^2}\right] = 0.$$

E2) The auxiliary equations are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{(x^2 - y^2)z}$$

$$= \frac{yzdx + zxdy + xydz}{0} = \frac{xdx + ydy - dz}{0}$$

Integrating, we have

$$xyz = c_1 \text{ and } x^2 + y^2 - 2z = c_2 \qquad (105)$$

The given curve is $x + y = 0,\ z = 1$, whose parametric equation is

$$x = t,\ y = -t,\ z = 1$$

Substituting these values in Eqn. (105), we get

$$-t^2 = c_1 \text{ and } 2t^2 - 2 = c_2$$

Eliminating t from the above equations, we get

$$2c_1 + 2 = -c_2$$

In the above relation substituting for c_1 and c_2 from Eqn. (105), we get

$$x^2 + y^2 + 2xyz - 2z + 2 = 0,$$

which is the desired integral surface.

E3) The auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z} = \frac{dx + dy + dz}{x + y + z}$$

Integrating $\frac{dx}{y} = \frac{dy}{x}$, we get

$$x^2 - y^2 = c_1 \qquad (106)$$

And integrating, $\frac{dz}{z} = \frac{d(x + y + z)}{x + y + z}$, we obtain

$$\frac{x + y + z}{z} = c_2 \qquad (107)$$

The given curve $z = x^3,\ y = 0$ has the parametric equations as

$$x = t,\ y = 0,\ z = t^3$$

Substituting these values of x, y, z in Eqns. (106) and (107), we get

$$t^2 = c_1 \text{ and } \frac{t + t^3}{t^3} = c_2$$

Eliminating t from these equations, we get

$$\frac{1 + c_1}{c_1} = c_2$$

Substituting in the above equation the values for c_1 and c_2 from Eqns. (106) and (107), we get

$z(1 + x^2 - y^2) = (x^2 - y^2)\,(x + y + z)$ as the desired integral surface.

E4) The auxiliary equations are

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}=\frac{dx+dy+dz}{0}=\frac{xdx+ydy+zdz}{0}$$

Integrating, we get

$$x+y+z=c_1 \text{ and } x^2+y^2+z^2=c_2 \qquad (108)$$

The given curve $z=0,\ xy=1$ has parametric equations as

$$x=t,\ y=\frac{1}{t},\ z=0$$

Substituting these values of $x,\ y,\ z$ in Eqn. (108), we obtain

$$t+\frac{1}{t}=c_1 \text{ and } t^2+\frac{1}{t^2}=c_2$$

Eliminating t from these equation, we get

$$c_1^2-2=c_2$$

In this equation substituting the values of c_1 and c_2 from Eqn. (108), we have

$$xy+yz+zx-1=0$$

$$\Rightarrow \quad z=\frac{1-xy}{x+y}$$

which is the desired integral surface.

E5) From Eqn. (52), the compatibility condition for two PDEs of first order is

$$[f,\ g]=f_p\frac{\partial g}{\partial x}+f_p\frac{\partial g}{\partial y}+(pf_p+qf_q)\frac{\partial g}{\partial z}-(f_x+pf_z)\frac{\partial g}{\partial p}-(f_y+qf_z)\frac{\partial g}{\partial q}=0$$

Since our PDEs are

$$f(x,\ y,\ p)=0 \text{ and } g(x,\ y,\ q)=0$$

$$\therefore \quad f_q=0,\ f_z=0,\ g_p=0,\ g_z=0$$

Then Eqn. (52) reduces to

$$f_p\ g_x-f_y\ g_q=0$$

which is the required condition.

E6) Here $f(x,\ y,\ p,\ q)=0$ and $g(x, y,\ p,\ q)=0$

$$\therefore \quad f_z=0 \text{ and } g_z=0$$

Then Eqn. (52) reduces to

$$f_pg_x+f_qg_y-f_xg_p-f_yg_q=0$$

$$\Rightarrow \quad (f_pg_x-f_xg_p)+(f_qg_y-f_yg_q)=0$$

$$\Rightarrow \quad \frac{\partial(f,\ g)}{\partial(x,\ p)}+\frac{\partial(f,\ g)}{\partial(y,\ q)}=0$$

which is the required condition.

E7) Here let, $f=(y-z)p+(z-x)q-(x-y)=0$

and $g=z-px-qy=0$

Using Eqn. (52), we get

$$\begin{aligned}[f,\ g]&=(y-z)\ (-p)+(z-x)\ (-q)+[(y-z)p+(z-x)q].1\\ &\quad -[-1-q+p(q-p)]\ (-x)-[1+p+q(q-p)]\ (-y)\\ &=(q-p)\ [xp+yq]-(x-y)+yp-qx\end{aligned}$$

$$= (q-p)z - (x-y) + yp - qx \qquad \text{[since } xp + yq = z \text{ from } g = 0]$$
$$= 0 \qquad \text{[since } z(q-p) = (x-y) - yp + qx \text{ from } f = 0]$$

Hence the given PDEs are compatible.

E8) Let $f = xp - yq = 0$

and $g = x^2 p + q - xz = 0$

Using Eqn. (52), we get

$$[f, g] = x(2xp - z) - y.(0) + [(xp - yq](-x) - [p + p.0]x^2 - [-q + q.0].1$$
$$= 2x^2.p - xz - x^2 p + xyq - px^2 + q$$
$$= -xz + q + xyq$$
$$= -x^2 p + xyq + x^2 \qquad \text{[since } -xz + q = -x^2 p \text{ from } g = 0]$$
$$= x(yq - xp)$$
$$= 0 \qquad \text{[using } f = 0]$$

Hence the given PDEs are compatible.

On solving $f = 0$ and $g = 0$ for p and q, we get

$$p = \frac{1+yz}{1+xy},\ q = \frac{x(z-x)}{1+xy}$$

Substituting the above values of p and q in $dz = pdx + qdy$, we get

$$dz = \frac{1+yz}{1+xy}dx + \frac{x(z-x)}{1+xy}dy$$

$\Rightarrow \quad (1+xy)dz = (1+yz)dx + x(z-x)dy$

$\Rightarrow \quad (1+xy)dz = (1+xy+yz)dx + xzdy - x^2dy - xydx$ (adding and subtracting $xy\,dx$)

$\Rightarrow \quad (1+xy)dz = (1+xy)dx + z(ydx + xdy) - x(xdy + ydx)$

$\Rightarrow \quad (1+xy)\, d(z-x) - (z-x)\, (xdy + ydx) = 0$

$\Rightarrow \quad (1+xy)\,d(z-x) - (z-x)\,d(1+xy) = 0$

$\Rightarrow \quad \dfrac{(1+xy)\,d(z-x) - (z-x)\; d(1+xy)}{(1+xy)^2} = 0$, provided $1 + xy \neq 0$

$\Rightarrow \quad d\left(\dfrac{z-x}{1+xy}\right) = 0$

Integrating, we get

$$\frac{z-x}{1+xy} = b$$

$\Rightarrow \quad z = x + b(1+xy)$, for b being an arbitrary constant, is a one-parameter family of commong solutions of the given PDEs.

E9) i) The given PDE is

$$p^2 x + q^2 y = z \tag{109}$$

The auxiliarly equations are

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2(p^2 x + q^2 y)} = \frac{dp}{p - p^2} = \frac{dq}{q - q^2}$$

$$\Rightarrow \quad \frac{p^2 dx + 2pxdp}{p^2 x} = \frac{q^2 dy + 2qydy}{q^2 y}$$

$$\Rightarrow \quad p^2 x = aq^2 y. \tag{110}$$

where a is a constant.

On solving Eqns. (109) and (110) for p and q, we get

$$p=\left\{\frac{az}{(1+a)x}\right\}^{1/2} \text{ and } q=\left\{\frac{z}{(1+a)y}\right\}^{1/2}$$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$\left(\frac{1+a}{z}\right)^{1/2} dz = \left(\frac{a}{x}\right)^{1/2} dx + \left(\frac{1}{y}\right)^{1/2} dy$$

Integrating, we get

$$\{(1+a)z\}^{1/2} = (ax)^{1/2} + (y)^{1/2} + b,$$

which is the required complete integral of the given PDE.

ii) The given PDE is

$$f = 2(z + xp + yq) - yp^2 = 0 \tag{111}$$

The auxiliary equations corresponding to Eqns. (65) are

$$\frac{dx}{2x-2yp} = \frac{dy}{2y} = \frac{dz}{2xp-2yp^2+2yq} = \frac{dp}{-(2p+p.2)} = \frac{dq}{-(2q-p^2+q.2)}$$

From 2[nd] and 4[th] fractions, we get

$$2\frac{dy}{y} + \frac{dp}{p} = 0$$

Integrating, we get

$$y^2 p = a \tag{112}$$

Solving Eqns. (111) and (112) for p and q, we get

$$p = \frac{a}{y^2},\ q = \frac{a^2}{2y^4} - \frac{ax}{y^3} - \frac{z}{y}$$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \frac{a}{y^2}dx + \frac{a^2}{2y^4}dy - \frac{ax}{y^3}dy - \frac{z}{y}dy$$

$$\Rightarrow \quad ydz + zdy = \frac{a}{y}dx - \frac{ax}{y^2}dy + \frac{a^2}{2y^3}dy$$

Integrating, we get

$$yz = \frac{ax}{y} - \frac{a^2}{4y^2} + b$$

$$\Rightarrow \quad z = \frac{ax}{y^2} + \frac{b}{y} - \frac{a^2}{4y^3},$$

which is the complete integral of the given PDE.

iii) The given PDE is

$$f = 2z + p^2 + qy + 2y^2 = 0 \tag{113}$$

The auxiliary equations corresponding to Eqn. (65) are

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{dz}{2p^2+qy} = \frac{dp}{-(0+p.2)} = \frac{dq}{-(q+4y+q.2)}$$

Here 1[st] and 4[th] fractions yield

$$dx + dp = 0$$

Integrating, $x + p = a$

$$\Rightarrow \quad p = a - x \tag{114}$$

From Eqns. (113) and (114), we get

$$q = \frac{-(2z + 2y^2 + (a-x)^2)}{y}$$

Substituting for p and q in $dz = pdx + qdy$, we get

$$dz = (a-x)\ dx - \frac{(a-x)^2 + 2y^2 + 2z}{y} dy$$

$$\Rightarrow \quad y^2 dz = y^2 (a-x)dx - (a-x)^2\ ydy - 2y^3 dy - 2z\ ydy$$

$$\Rightarrow \quad y^2 dz + 2zydy + (a-x)^2\ ydy - y^2 (a-x)dx + 2y^3 dy = 0$$

or

$$d\left(y^2 z + \frac{1}{2}(a-x)^2 y^2 + 2\frac{y^4}{4} \right) = 0$$

Integrating, we get

$$y^2\ [2z + (a-x)^2 + y^2] = b,$$

which is the complete integral of the given PDE.

iv) $z^2 = 2(a^2+1)x^2 + 2ay + b$

v) $z = ax + be^y (y+a)^{-a}$

E10) i) The given PDE is

$$p + q - pq = 0$$

which is an equation in p and q only

Let $p = a$

We then get from the given equation

$$q = \frac{a}{a-1}$$

Substituting for p and q, from above, in $dz = pdx + qdy$, we get

$$dz = a\ dx + \frac{a}{a-1} dy$$

Integrating, we get the complete integral of the given PDE as

$$z = ax + \frac{a}{a-1} y + b$$

ii) $z = ax \pm \sqrt{1-a^2}\ \ y + b$

iii) $z = e^a x + ay + b$

iv) The given PDE is

$$(y-x)\ (qy - px) = (p-q)^2$$

Let $y + x = U$, and $xy = V$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial U}\ \frac{\partial U}{\partial x} + \frac{\partial z}{\partial V}\ \frac{\partial V}{\partial x}$$

$$= \frac{\partial z}{\partial U} + y\frac{\partial z}{\partial V}$$

Similarly, $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial U} + x\dfrac{\partial z}{\partial V}$

The given equation then reduces to

$$\frac{\partial z}{\partial U} = \left(\frac{\partial z}{\partial V} \right)^2$$

This equation is now of the Type 1 and we can write its complete integral directly, using Eqn. (82), in the form

$$z = aU + \sqrt{a}\ V + b$$

$$\Rightarrow \quad z = a(y+x) + \sqrt{a}\ xy + b$$

v) The given PDE can be written as

$$\frac{1-x^2}{x^2} p^2 + \frac{q}{y} = 0$$

Let $\sqrt{1-x^2} = X$ and $\frac{y^2}{2} = Y$

Then the given PDE reduces to

$$\left(\frac{\partial z}{\partial X}\right)^2 + \frac{\partial z}{\partial Y} = 0$$

and its complete integral, using Eqn. (82), is

$$z = aX + (-a^2)Y + b$$

$$= a\sqrt{1+x^2} - \frac{a^2 y^2}{2} + b$$

E11) i) The given PDE is

$$f = 4z - pq = 0 \qquad (115)$$

which is Type II equation

Putting $p = aq$ in the given equation, we get

$$4z = aq^2 \Rightarrow q = \pm\sqrt{\frac{4z}{a}}$$

Considering only the positive signs and substituting the values of p and q, from above, in the equation $dz = pdx + qdy$, we get

$$dz = a\sqrt{\frac{4z}{a}}\, dx + \sqrt{\frac{4z}{a}}\ dy$$

$$\Rightarrow \quad \sqrt{\frac{a}{4z}}\ dz = adx + dy$$

Integrating, we get the complete integral of the given PDE as

$$\frac{\sqrt{az}}{\sqrt{4}.\frac{1}{2}} = (ax+y) + b$$

$$\Rightarrow \quad \pm\sqrt{az} = ax + y + b$$

$$\Rightarrow \quad az = (ax + y + b)^2$$

ii) $a^2 \ln | z | = (ax + b + y)^2$

iii) $8(x + ay + b)^3 = (1 + a^3)z^2$

iv) $4(a^2 - 1)\ z = (ax + y + b)^2$

v) Proceed as in Example 18 and obtain

$$x\ y^a\ b = z^{1/k}$$

where $k = \frac{1}{2}[-a + \sqrt{a^2+4}\,]$

E12) i) The given PDE is

$$q = xp + p^2$$

Hence each side must be equal to the same constant, say a.

$\therefore \quad xp + p^2 = a$ and $q = a$

$\Rightarrow \quad p = \dfrac{-x \pm \sqrt{x^2 + 4a}}{2}$

Substituting the values of p and q, from above, in the relation $dz = pdx + qdy$, we get

$$dz = \frac{-x \pm \sqrt{x^2 + 4a}}{2}\, dx + a\, dy$$

On integration, we get the complete integral of the given PDE as

$$z = -\frac{x^2}{4} \pm \left(\frac{x\sqrt{x^2 + 4a}}{4} \pm a \ln \left| x + \sqrt{x^2 + 4a} \right| \right) + ay + b$$

ii) $\quad z = \dfrac{1}{6}(2x - a)^3 + a^2 y + b$

iii) $\quad z = \dfrac{\sqrt{x^2 + a}}{2} + \dfrac{a}{2} \ln \left| x + \sqrt{x^2 + a} \right| - \dfrac{a}{2} \dfrac{1}{y^2} + \ln | y | + b$

iv) $\quad z = a\sqrt{1 + x^2} + \dfrac{1}{2} a^2 y^2 + b$

where a and b are arbitrary constants.

Hint: The given PDE can be written as

$$\frac{p^2(1 + x^2)}{x^2} = \frac{q}{y}$$

v) $\quad z = \dfrac{1}{3}(x^2 + a^2)^{3/2} + (y^2 - a^2)^{1/2} + b$

Hint: Dividing the given PDE by $q^2 x^2$, we get

$$\frac{p^2 - x^4}{x^2} = \frac{y^2 q^2 - y^2}{q^2}$$

which is in variable separable form.

vi) $\quad z + b = \ln \left| x + \sqrt{a + x^2} \right| + \dfrac{1}{2\sqrt{a}} \ln \left| \dfrac{y - \sqrt{a}}{y + \sqrt{a}} \right|$

Hint: Dividing the given PDE by $p^2 q$, we get

$$x^2 + y^2 = \frac{1}{q} + \frac{1}{p^2}$$

$$\Rightarrow \quad x^2 - \frac{1}{p^2} = \frac{1}{q} - y^2$$

E13) i) The given PDE is

$$(p + q)\,(z - xp - yq) = 1$$

It can be written in the form

$$z = xp + yq + \frac{1}{p + q}$$

which is a Clairaut's equation

Hence the complete integral of the given equation is

$$z = ax + by + \frac{1}{a+b}$$

ii) $z = ax + by + \sqrt{\alpha a^2 + \beta b^2 + \gamma}$

iii) $z = ax + by + 3(ab)^{1/3}$

– x –

MISCELLANEOUS EXERCISES

1. State whether the following statements are true or false. Justify your answers with the help of a short proof or a counter example.

   i) The equation $yz\,dx + (x^2y - xz)dy + (x^2z - xy)dz = 0$ is integrable.

   ii) The solution of the PDE $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z^2$, is $z = -[y + f(x - y)]$.

   iii) The complete solution of the second order PDE $\frac{\partial^2 u}{\partial x \partial y} = x - y$,

   involves two arbitrary constants.

   iv) The PDEs $p^2x + q^2y = z$ and $p^2x - q^2y = 0$ are compatible.

   v) The complete integral of the PDE $(\sqrt{p} + \sqrt{q}) = 2x$ is given by

   $z = \frac{(2x-a)^2}{6} + a^2y + b$, a and b are arbitrary constants.

   vi) One of the solution of the simultaneous differential equation

   $\frac{dx}{x(y^3 - 2x^3)} = \frac{dy}{y(2y^3 - x^3)} = \frac{dz}{z(x^3 - y^3)}$ is $xyz^{3/2} = c$.

   vii) The equation $(6x + yz)dx + (xz - 2y)dy + (xy + 2z)dz = 0$ is exact.

2. Find the integral curves of the following system of equations

   i) $\frac{dx}{3} = \frac{dy}{5} = \frac{dz}{3z + \tan(3y - 5x)}$

   ii) $\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}$

   iii) $\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$

   iv) $\frac{dx}{y^2(x - y)} = \frac{dy}{x^2(x - y)} = \frac{dz}{z(x^2 + y^2)}$

   v) $\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$.

3. Find the differential equation of the space curves in which the following two families of surfaces $u = c_1$ and $v = c_2$ intersect.

   i) $u = x^2 + y^2 + z^2,\ v = xyz$

   ii) $u = x^2 + y^2,\ v = 3x + 2y + z$

   iii) $u = xyz,\ v = x^2 + y^2 - 2z$.

4. Find the equations of the system of curves on the cylinder $2y = x^2$ orthogonal to its intersection with the hyperboloids of one-parameter system $xy = z + c$.

5. Show that there is no set of surfaces orthogonal to the curves given by $\frac{dx}{z} = \frac{dy}{z + x} = \frac{dz}{x}$.

6. An electric circuit consists of an inductance of 0.1 henry, a resistance of 10 ohms, a condenser of capacitance 25 microfarads and electromagnetic force of $400 \cos 200t$ volts. Write the system of simultaneous equtions governing the charge and the current at time t.

7. Classify the following equations into linear, semi-linear, quasi-linear and non-linear equations.
   i) $(x^2+z^2)p-xyq=z^3x+y^2$
   ii) $xp-yxq=xz^2$
   iii) $yp-xq=xyz+x$
   iv) $z^2(1+p^2+q^2)=1$
   v) $(2xy-1)p+(z-x^2)q=2(x-yz)$
   vi) $uu_x=e^y+\sin x,\ u=u(x,\ y)$
   vii) $p+q=x+y+z$
   viii) $z(p-q)=z^2+(x+y)^2$
   ix) $p+q=pq$
   x) $x^2p+(x+y)^2q=(x+y)\ (z^2+1)$

8. Determine the total differential equations for the following family of surfaces, c being the defining parameter in each case.
   i) $y(x+z)=c(y+z)$
   ii) $(yz+1)\ (cy+1)=(xy+1)$
   iii) $xyz=c$
   iv) $2zy-x(y^2+z^2)=2cx$
   v) $3x^2-y^2+z^2+xyz=c$.

9. Verify that the total differential equatons are exact/integrable and find the corresponding integrals.
   i) $2yz\,dx+zx\,dy-xy(1+z)dz=0$
   ii) $zy\ dx=zx\ dy+y^2dz$
   iii) $(x^2-y^2-z^2+2xy+2xz)dx+(y^2-z^2-x^2+2yz+2yx)dy$ $+(z^2-x^2-y^2+2zx+2zy)dz=0$
   iv) $(y+z)dx+(z+x)dy+(x+y)dz=0$
   v) $(yz+z^2)dx-xz\,dy+xy\,dz=0$
   vi) $z^2dx+(z^2-2yz)dy+(2y^2-yz-xz)dz=0$.

10. Find $f(y)$ such that the total differential equation

$$\{(yz+z)/x\}dx-z\,dy+f(y)dz=0$$

is integrable. Hence solve it.

11. Eliminate the arbitrary constants a and b from the following equations and obtain the corrresponing PDEs.
   i) $z=(x+a)\ (y+b)$
   ii) $z=(x^2+a)\ (x^2+b)$
   iii) $z=xy+y\sqrt{x^2-a^2}+b$
   iv) $z=ax+by+ab$
   v) $z=ax+a^2y^2+b$
   vi) $z=ax+(1-a)\,y+b$

12. Find the PDE arising from each of the following surfaces:
   i) $z=x+y+f(xy)$
   ii) $z=f(x-y)$

iii) $z = y^2 + 2f\left(\frac{1}{x} + \ln y\right)$

iv) $f(x^2 + y^2,\ z - xy) = 0$

v) $x + y + z = f(x^2 + y^2 + z^2)$

vi) $y = f(x - at) + g(x + at)$

vii) $z = f(x + iy) + g(x - iy)$

13. Find the general integral of the following PDEs:

i) $xzp + yzq = xy$

ii) $p\tan x + q\tan y = \tan z$

iii) $z(p - q) = z^2 + (x + y)^2$

iv) $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$

v) $(y + zx)p - (x + yz)q = x^2 - y^2$.

14. Find the integral surface of the following PDEs passing through the given curves.

i) $(y - z)p + (z - x)q = x - y;\ z = 0,\ y = 2x$

ii) $(x - y)p + (y - x - z)q = z;\ z = 1,\ x^2 + y^2 = 1$

15. Verify that the equations

i) $z = \sqrt{2x + a} + \sqrt{2y + b}$, and

ii) $z^2 + \mu = 2(1 + \lambda^{-1})(x + \lambda y)$

are both complete integrals of the PDE $z = \frac{1}{p} + \frac{1}{q}$. Show further, that the complete integral ii) is the envelope of the one parameter subsystem obtained by taking $b = -\frac{a}{\lambda} - \frac{\mu}{1 + \lambda}$ in integral i).

16. Show that $z = ax + (y/a) + b$ is a complete integral of $pq = 1$. Find the particular solution corresponding to the sub-family $b = a$. Also find the singular integral, if it exists.

17. Find the complete integrals of the following PDEs:

i) $(x^2 + y^2)(p^2 + q^2) = 1$

ii) $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$

iii) $z = px + qy + \sqrt{\alpha p^2 + \beta q^2 + \gamma}$

iv) $q^2 = z^2 p^2 (1 - p^2)$

v) $z^2 = 1 + p^2 + q^2$

vi) $z(p^2 - q^2) = x - y$

vii) $2x(z^2 q^2 + 1) = pz$

viii) $p^2 + q^2 - 2px - 2qy + 1 = 0$

ix) $2(z + px + qy) = yp^2$

x) $zpq = p + q$.

– x –

SOLUTIONS/ANSWERS TO MISCELLANEOUS EXERCISES

1. i) **True**, verify the condition of integrability.

   ii) **False**, $\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{z^2} \Rightarrow x-y=c_1; \frac{-1}{z}=y+c_2=y+f(x-y)$.

   iii) **False**,
   $$\frac{\partial^2 u}{\partial x \partial y}=x-y \Rightarrow \frac{\partial u}{\partial y}=\frac{x^2}{2}-yx+\phi(y) \Rightarrow u=\frac{x^2 y}{2}-\frac{y^2 x}{2}+f(y)+g(x)$$
   solution involves two arbitrary functions.

   iv) **True**, Here $f=p^2x+q^2y-z=0$ and $g=p^2x-q^2y=0$
   Verify that the compatibility condition $[f, g]=0$ is satisfied.

   v) **False**, $\sqrt{p}-2x=-\sqrt{q}=-a$ (say)
   $\Rightarrow \quad p=(2x-a)^2, q=a^2$
   $\therefore \quad dz=p\,dx+q\,dy \Rightarrow z=\frac{(2x-a)^3}{6}+a^2y+b$.

   vi) **True**, $\dfrac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^3-x^3}=\dfrac{dz}{2z(x^3-y^3)}$ yields $xyz^{3/2}=c$.

   vii) **True**, verify the condition of exactness.

2. i) The first two terms of auxiliary equations yield
   $$3y-5x=c_1$$
   Combining 1st and 3rd fraction $dx=\frac{3dz}{3z+\tan c_1}$
   $\Rightarrow \quad \ln c_2+x=\ln(3z+\tan c_1)$
   $\Rightarrow \quad 3z+\tan c_1=c_2e^x$
   The integral curves are the intersection of two families of surfaces $3y-5x=c_1$ and $e^{-x}\{3z+\tan(3y-5x)\}=c_2$.

   ii) $$\frac{dx}{x(y^2-z^2)}=\frac{dy}{-y(z^2+x^2)}=\frac{dz}{z(x^2+y^2)}$$
   $$=\frac{\frac{dx}{x}-\frac{dy}{y}-\frac{dz}{z}}{0}=\frac{x\,dx+y\,dy+z\,dz}{0}$$
   Integrating $\frac{x}{yz}=c_1$ and $x^2+y^2+z^2=c_2$
   $\therefore$ Required solution is $f\left(\frac{x}{yz}, x^2+y^2+z^2\right)=0$.

   iii) $$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$

$$\therefore \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)}$$

From first two fractions $\ln(x-y) = \ln(y-z) + \ln c_1$

$$\Rightarrow \quad \frac{x-y}{y-z} = c_1$$

From the second and third fraction $\ln(y-z) = \ln(z-x) + \ln c_2$

$$\Rightarrow \quad \frac{y-z}{z-x} = c_2$$

$\therefore$ Required solution is $f\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$.

iv) $$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)}$$

From the first two fractions

$x^2dx = -y^2dy$ or $3x^2dx + 3y^2dy = 0$

Integrating $x^3 + y^3 = c_1$

Choosing $1, -1, 0$ as multipliers, each fraction

$$= \frac{dx-dy}{y^2(x-y)+x^2(x-y)} = \frac{dx-dy}{(x-y)(x^2+y^2)}$$

From the third fraction and the above fraction.

$$\frac{dz}{z(x^2+y^2)} = \frac{dx-dy}{(x-y)\,(x^2+y^2)} \text{ or } \frac{dz}{z} = \frac{dx-dy}{x-y}$$

Integrating $(x-y)/z = c_2$

$\therefore$ Required solution is $f\left(x^3+y^3, \frac{(x-y)}{z}\right) = 0$.

v) Given system can be written as

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

From the last two fractions $y/z = c_1$

Choosing x, y, z as multipliers

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

From above and third fraction $\frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)} = \frac{dz}{2xz}$.

Integrating $(x^2+y^2+z^2)/z = c_2$

$\therefore$ Required solution is $f\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0$.

3. i) $$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}.$$

ii) $$\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{2x-3y}.$$

iii) $\dfrac{dx}{x(y^2+z)} = \dfrac{dy}{-y(x^2+z)} = \dfrac{dz}{z(x^2-y^2)}$.

4. $(x+y)(z+1) = c_1$ and $(x-y)(z-1) = c_2$.

5. The surface orthogonal to the curves given by

$$\frac{dx}{z} = \frac{dy}{z+x} = \frac{dz}{x}$$

has the differential equation

$$z\,dx + (z+x)dy + x\,dz = 0 .$$

Check that the condition of integrability is not satisfied for this differential equation, hence it cannot be solved. Hence no set of surfaces orthogonal to the given curve exists.

6. Here $R=10,\ L=0.1,\ C=25\times10^{-6},\ E(t)=400\cos 200t$

Governing equations are

$$\frac{di}{dt} - 100i + 400000q = 400\cos 200t$$

and $\dfrac{dq}{dt} = i$

$$\Rightarrow \frac{di}{400\cos 200t + (100i - 400000q)} = \frac{dq}{i} = dt .$$

7. i) quasi-linear
ii) semi-linear
iii) linear
iv) non-linear
v) quasi-linear
vi) non-linear
vii) linear
viii) quasi-linear
ix) non-linear
x) semi-linear

8. i) $(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0$

ii) $(1+yz)dx + x(z-x)dy - (1+xy)dz = 0$

iii) $yz\,dx + xz\,dy + xy\ dz = 0$

iv) $yz\ dx + (x^2y - zx)dy + (x^2z - xy)dz = 0$

v) $(6x+yz)dx + (xz-2y)dy + (xy+2z)dz = 0$.

9. i) $2yz\ dx + zx\ dy - xy(1+z)dz = 0$

Dividing by xyz throughout, we get

$$\frac{2dx}{x} + \frac{dy}{y} - \left(1+\frac{1}{z}\right)dz = 0$$

Integrating, $2\ln x + \ln y - (z + \ln z) = \ln c$

or $\ln x\ + \ln y - \ln c - \ln z = z$ or $x^2 y = cze^z$.

ii) Given equation can be written as

$$\frac{y\,dx - x\,dy}{y^2} = \frac{dz}{z} \text{ or } d\left(\frac{x}{y}\right) = \frac{dz}{z}$$

Integrating, $\frac{x}{y} = \ln z - \ln c$, or $z = ce^{x/y}$.

iii) Adding and subtracting $x^2dx,\ y^2dy,\ z^2dz$ in the first, second and the third term, respectively of the given equation and simplifying, we get

$$[-(x^2+y^2+z^2)+2x(x+y+z)]dx$$
$$+[-(x^2+y^2+z^2)+2y(x+y+z)]dy$$
$$+[-(x^2+y^2+z^2)+2z(x+y+z)]dz = 0$$

or,

$$-(x^2+y^2+z^2)(dx+dy+dz)+2(x+y+z)(x\,dx+y\,dy+z\,dz)=0$$

or, $\frac{dx+dy+dz}{x+y+z} = \frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2}$

Integrating, $\ln(x+y+z) = \ln(x^2+y^2+z^2)+\ln c$

or, $(x+y+z) = c(x^2+y^2+z^2),\ c$ being arbitrary.

iv) $xy + yz + zx = c$.

v) $(yz+z^2)dx - xz\,dy + xy\,dz = 0$

Let $x = uz,\ \therefore dx = u\,dz + z\,du$ and $y = vz,\ \therefore dy = v\,dz + z\,dv$

Putting the above values in the given equation, we get

$$(vz^2+z^2)(u\,dz+z\,du) - uz^2(v\,dz+z\,dv) + vu\,z^2dz = 0$$

or, $(v+1)z^3du - u\,z^3dv + (v+1)u\,z^2dz = 0$

Dividing by $(v+1)uz^3$, we get

$$\frac{du}{u} - \frac{dv}{v+1} + \frac{dz}{z} = 0$$

Integrating, we get

$$uz = c(v+1)$$

or, $xz = c(y+z)$.

vi) Substitution $x = uz$ and $y = vz$ reduces the given equation to

$$z^3du + z^3(1-2v)\,dv = 0$$

or, $du + (1-2v)\,dv = 0$

Integrating, $u + v - v^2 = c$

or, $(x+y)z - y^2 = cz^2$.

10. Mutliplying the given equation throughout by x, we get

$$(yz+z)dx - xz\,dy + xf(y)dz = 0 \qquad \text{(i)}$$

If Eqn. (i) is integrable, then intgrability condition yields,

$$(yz+z)(-x-xf') - xz[f-(y+1)] + xf[z-(-z)] = 0$$

or, $xz(1+y)f' = xzf \qquad \left(f' = \frac{df}{dy}\right)$

or, $\frac{df}{f} = \frac{dy}{1+y}$

Integrating, we get $f = c(y+1),\ c$ being a constant.

Putting this value of f in (i), we get

$$z(y+1)dx - xz\,dy + xc(y+1)dz = 0$$

or $\frac{dx}{x} - \frac{dy}{y+1} + \frac{cdz}{z} = 0$

Integrating, we get the required solution as

$xz^c = c_1(y+1)$, c and c_1 are arbitrary constants.

11. i) $pq = z$
 ii) $pq = 4xy$
 iii) $px + qy = pq$
 iv) $z = px + qy + pq$
 v) $q = 2yp^2$
 vi) $p + q = 1$.

12. i) Differentiating the given equation w.r.t. x and y, we get

$p = 1 + f'(xy).y$

$q = 1 + f'(xy).x$

Eliminating $f'(xy)$ from the above two equations, the required PDE is $xp - yq = x - y$.

 ii) $p + q = 0$
 iii) $px^3 + qx = 2y^2$
 iv) $py - qx = y^2 - x^2$

 v) Differentiating the given equation partially w.r.t. x and y, we get

$$1 + p = f'(x^2 + y^2 + z^2)(2x + 2zp)$$

$$1 + q = f'(x^2 + y^2 + z^2)\ (2y + 2zq)$$

Eliminating f' from the above two equations

$$\frac{(1+p)}{2x+2zp} = \frac{1+q}{2y+2zq}$$

$$\Rightarrow\ (y-z)p + (z-x)q = x - y.$$

 vi) $$y = f(x-at) + g(x+at)$$

$$\frac{\partial y}{\partial x} = f'(x-at) + g'(x+at)$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x-at) + g''(x+at)$$

$$\frac{\partial y}{\partial t} = f'(x-at)\ (-a) + g'(x+at)\ (a)$$

$$\frac{\partial^2 y}{\partial t^2} = f''(x-at)\ (-a)^2 + g''(x+at)\, a^2$$

$$= a^2[f''(x-at) + g''(x+at)]$$

$$= a^2 \frac{\partial^2 y}{\partial x^2}$$

$\therefore$ Required PDE is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.

 vii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

13. i) $xy - z^2 = \phi\left(\frac{x}{y}\right)$.

ii) $\frac{\sin z}{\sin y} = \phi\left(\frac{\sin x}{\sin y}\right)$.

iii) Given $zp - zq = z^2 + (x+y)^2$

Lagrange's auxiliary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

From the first two fractions, we get $x + y = c_1$.

From the second and third fractions, we get

$$dy = \frac{-z\,dz}{z^2 + c_1^2} \text{ or } \frac{2z\,dz}{z^2 + c_1^2} = -2dy$$

Integrating, $\ln(z^2 + c_1^2) - \ln c^2 = -2y$ or $\frac{z^2 + c_1^2}{c^2} = e^{-2y}$

or, $e^{2y}[z^2 + (x+y)^2] = c_2$

Thus, the required general integral is $e^{2y}[z^2 + (x+y)^2] = \phi(x+y)$.

iv) $\phi\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$

v) $\phi(x^2 + y^2 - z^2,\ xy + z) = 0$

14. i) $5(x+y+z)^2 = 9(x^2 + y^2 + z^2)$

ii) Given equation is $(x-y)p + (y-x-z)q = z$ (i)

Lagrange's auxiliary equations are

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z} = \frac{dx+dy+dz}{0} \quad \text{(ii)}$$

$\therefore$ From the last fraction $x + y + z = c_1$ (iii)

Taking $\frac{dy}{y-x-z} = \frac{dz}{z}$ or $\frac{2dy}{2y - c_1} - \frac{2dz}{z} = 0$

Integrating $\ln(2y - c_1) - 2\ln z = \ln c_2$ or $(2y - c_1) = c_2 z^2$

or, $(y - x - z)/z^2 = c_2$ (iv)

Given curve is $z = 1,\ x^2 + y^2 = 1$ (v)

Putting $z = 1$ in (iii) and (iv), we get

$x + y = c_1 - 1$ and $y - x = c_2 + 1$ (vi)

But $2(x^2 + y^2) = (x+y)^2 + (y-x)^2$ (vii)

Using (v) and (vi), (vii) becomes

$$2 = (c_1 - 1)^2 + (c_2 + 1)^2 \quad \text{(viii)}$$

Putting the values of c_1 and c_2 from (iii) and (iv) in (viii), we get the required integral surface as

$$z^4(x+y+z)^2 + (y-x-z)^2 - 2z^4(x+y+z) - 2z^2(y-x-z) = 0$$

15. The given equation is

$$z = \sqrt{2x+a} + \sqrt{2y+b} \quad \text{(i)}$$

Differentiating w.r.t. x and y, we get

$$p = \frac{1}{\sqrt{2x+a}} \text{ and } q = \frac{1}{\sqrt{2y+b}}$$

$$\therefore \quad \frac{1}{p} + \frac{1}{q} = \sqrt{2x+a} + \sqrt{2y+b} = z$$

which shows that (i) is a solution of

$$\frac{1}{p} + \frac{1}{q} = z$$

Since it contains two arbitrary constants a and b, it is a complete integral.

Similarly verify that ii) is a complete integral.

When $b = -\frac{a}{\lambda} - \frac{\mu}{1+\lambda}$, then

$$z = \sqrt{2x+a} + \sqrt{2y - \frac{a}{\lambda} - \frac{\mu}{1+\lambda}} \qquad \text{(ii)}$$

Differentiating the above equation w.r.t. a, we get

$$\sqrt{2y - \frac{a}{\lambda} - \frac{\mu}{1+\lambda}} = \sqrt{\frac{2x+a}{\lambda}} \qquad \text{(iii)}$$

Adding Eqns. (ii) and (iii), we get

$$2x + a = \frac{\lambda^2 z^2}{(1+\lambda)^2} \qquad \text{(iv)}$$

Also $\sqrt{2x+a} - \lambda\sqrt{2y - \frac{a}{\lambda} - \frac{\mu}{1+\lambda}} = 0$ (v)

Subtracting Eqn. (v) from Eqn. (ii), we get

$$z = (1+\lambda)\sqrt{2y - \frac{a}{\lambda} - \frac{\mu}{1+\lambda}}$$

or $\quad 2y - \frac{a}{\lambda} - \frac{\mu}{1+\lambda} = \frac{z^2}{(1+\lambda)^2}$ (vi)

Adding Eqns. (ii) and (v), we get

$$2(x + \lambda y) = \frac{\mu}{1+\frac{1}{\lambda}} + \frac{z^2}{1+\frac{1}{\lambda}}$$

$$\Rightarrow \quad 2(1+\lambda^{-1})\,(x+\lambda y) = z^2 + \mu$$

which is our solution ii).

16. $z = ax + \frac{y}{a} + b$ (i)

Differentiating w.r.t. x and y, we get

$$p = a \text{ and } q = \frac{1}{a}$$

$$\therefore \quad pq = 1$$

Thus (i) is a solution of $pq = 1$. It involves two arbitary constants so it is a complete integral. For $b = a$, we obtain the particular integral

$$z = ax + \frac{y}{a} + a$$

The problem has no singular solution.

17. i) Given $(x^2+y^2)\,(p^2+q^2) = 1$ (i)

Let $x = r\cos\theta,\ y = r\sin\theta$ (ii)

then $x^2 + y^2 = r^2,\ \theta = \tan^{-1}(y/x)$

$$\therefore\quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial z}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\left(\because \frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2}\right)$$

Similarly, $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial r}\sin\theta + \dfrac{\cos\theta}{r}\dfrac{\partial z}{\partial \theta}$

Putting these values of p and q in Eqn. (i) and simplifying, we get

$$r^2\left[\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2\right] = 1 \text{ i.e., } \left(r\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = 1 \qquad \text{(iii)}$$

Let $\ln r = R$ then Eqn. (iii) reduces to

$$\left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = 1$$

which is Type-I (standard form) Charpti's equation whose solution is $z = aR + b\theta + c$; a, b, c being arbitrary constants.

where $a^2 + b^2 = 1$ i.e., $b = \sqrt{1-a^2}$

$\therefore\quad z = a\ln r + \sqrt{1-a^2}\ \theta + c$

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$.

ii) Take $x + y = X^2$ and $x - y = Y^2$ and proceed as in i) above.
The required complete integral is

$$z = a\sqrt{x+y} + \sqrt{1-a^2}\ \sqrt{x-y} + c.$$

iii) $z = ax + by + \sqrt{\alpha a^2 + \beta b^2 + \gamma}$.

iv) Given equation is $q^2 = z^2 p^2(1-p^2)$

Let $q = ap$ then we have $a^2p^2 - z^2p^2(1-p^2) = 0$

$\Rightarrow\quad p^2(a^2 - z^2 + z^2p^2) = 0$

$\Rightarrow$ either $p = 0$ or $p = \pm\sqrt{(z^2 - a^2)}/z$

Now $dz = p\,dx + q\,dy = p(dx + ady)$

When $p = 0$ then $dz = 0 \Rightarrow z = c$

when $p = \pm\sqrt{(z^2-a^2)}/z$ then

$$dz = \pm[\sqrt{(z^2-a^2)}/z]\,(dx + ady)$$

$$\therefore\quad dx + a\,dy = \pm\frac{1}{2}(z^2-a^2)^{-1/2}2z\,dz$$

Integrating $x + ay + b = (z^2-a^2)^{1/2}$

or $(x + ay + b)^2 = z^2 - a^2$

Hence the required complete integral is

$$z^2 - a^2 = (x + ay + b)^2 \text{ or } z = c.$$

v) $z = \pm\cosh[(x + ay + b)/\sqrt{(1+a^2)}]$.

vi) The given equation can be written in the form

$$\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2 = x - y \qquad \text{(i)}$$

Let $\sqrt{z}\,dz = dZ$ so that $\frac{2}{3}z^{3/2} = Z$

With above substitution Eqn. (i) becomes

$$P^2 - Q^2 = x - y \text{ where } P = \frac{\partial Z}{\partial x},\ Q = \frac{\partial Z}{\partial y}$$

Separating variables we can write

$$P^2 - x = Q^2 - y = a \quad \text{(say)}$$

then $P = (x+a)^{1/2}$, $Q = (a+y)^{1/2}$

$\therefore\ dZ = (x+a)^{1/2}dx + (a+y)^{1/2}dy$

Integrating, $z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + b$

$\Rightarrow\quad z^{3/2} = (a+x)^{3/2} + (a+y)^{3/2} + c$ where $c = \frac{3b}{2}$.

vii) Write the given equation as

$$2x\left[\left(z\frac{\partial z}{\partial y}\right)^2 + 1\right] = z\frac{\partial z}{\partial x}$$

Take $z\,dz = dZ$ so that $z^2/2 = Z$ and given equation becomes

$$Q^2 = \frac{P}{2x} - 1 \text{ where } P = \frac{\partial Z}{\partial x},\ Q = \frac{\partial Z}{\partial y}$$

Now variables are separated. Proceed as in vi) above and integrate to obtain

$$z^2 = 2x^2(1+a) + 2\sqrt{a}\,y + 2b$$

where a, b are arbitrary constants.

viii) Given equation is $p^2 + q^2 - 2px - 2qy + 1 = 0$

Here $f = p^2 + q^2 - 2px - 2qy + 1 = 0$ (i)

Charpit's auxiliary equations are

$$\frac{dp}{-2p} = \frac{dq}{-2q} = \frac{dz}{-p(2p-2x) - q(2q-2y)} = \frac{dx}{-(2p-2y)} = \frac{dy}{-(2q-2y)}$$

From the first two fractions, we get $p = aq$

Putting $p = aq$ in (i), we get

$$(a^2+1)q^2 - 2(ax+y)q + 1 = 0$$

$$\Rightarrow\quad q = \frac{2(ax+y) \pm \sqrt{4(ax+y)^2 - 4(a^2+1)}}{2(a^2+1)}$$

Putting the valus of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{(ax+y) \pm \sqrt{(ax+y)^2 - (a^2+1)}}{(a^2+1)}(a\,dx + dy) \qquad \text{(ii)}$$

Put $ax + y = v$ so that $a\,dx + dy = dv$ and Eqn. (ii) gives

$$(a^2+1)dz = [v \pm \sqrt{v^2 - (a^2+1)}\,]dv$$

Integrating,

$$(a^2+1)z = \frac{1}{2}v^2 + \frac{1}{2}v\sqrt{v^2-(a^2+1)} \pm \frac{1}{2}(a^2+1)\ln\left|v+\sqrt{v^2-(a^2+1)}\right| + b$$

is the complete integral where $v = ax + y$ and a, b are constants.

ix) $z = \dfrac{ax}{y^2} + \dfrac{b}{y} - \dfrac{a^2}{4y^3}$.

x) $z^2 = 2(a+1)(x + y/a) + b$.

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