Indira Gandhi National Open University
School of Sciences

BMTC-133
REAL ANALYSIS

Volume-I
INTRODUCTION TO REAL ANALYSIS

*****, 2020

ISBN-8]-

Further information on the Indira Gandhi National Open University courses, may be obtained from the University's office at Maidan Garhi, New Delhi-110 068 and IGNOU website www.ignou.ac.in.

Printed and published on behalf of the Indira Gandhi National Open University, New Delhi by Prof. Sujatha Varma, School of Sciences.

COURSE INTRODUCTION

Welcome to this third semester course through which we aim to introduce you to the branch of mathematics called "***analysis***". The word 'analysis', in general, means a detailed examination of the elements or structure of a substance or statement of a result. In mathematics, analysis involves the study of the structure of sets having different types of elements such as the set of real numbers, the set of rational numbers and also the set of certain functions.

Newton (1643-1727)

Real Analysis is the branch of mathematics that studies the structure of real numbers and the behavior of functions defined on the set of real numbers. You already have some idea of the structure of the set of real numbers, and of functions defined on it, from the 1st semester course, Calculus. In that course you were introduced to many concepts such as limit, continuity, differentiability and integrability for real-valued functions, though more from an algorithmic and computational aspect. However, when you apply the algorithms, it is important to know the logic behind them. The study of analysis provides the reasons behind these computation rules. The study will also help you to develop "analytical thinking" and the ability to apply mathematics more precisely and confidently.

Leibniz (1646-1716)

The analysis of real numbers and real functions played a fundamental role in the development of mathematics. Though many mathematicians like Newton and Leibniz contributed to the development of analysis, the origin of analysis took place in early 19th century with the work of Cauchy, who gave precise definitions of concepts such as limit and continuity. The convergence criterion introduced gave a path way to the techniques of approximation which led to the development of the theory of integration theory by the mathematicians, Riemann.

In this course we study the analysis of real numbers. The whole content is divided into six blocks.

In **Block 1**, comprising 4 units, we shall discuss the language of mathematics including mathematical symbols and the syntax used in expressing mathematical ideas. The most powerful tool used in analysis for giving justification is "methods of proof". In this block we also introduce you to some methods of proof. Then we discuss the algebraic and topological structure of real numbers, and some fundamental results about real numbers such as the order completeness properly and the Bolzano-Weierstrass theorem.

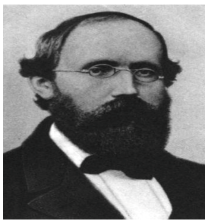
Cauchy (1789-1857)

In **Block 2**, comprising 2 units, we discuss the concepts of sequence and convergence in detail. You will also important theorems on limits of sequences here.

In **Block 3,** comprising 3 units, we familiarise you with the concept of an infinite series and its convergence. We shall give a few general tests for checking the convergence of a series with positive terms. After that, you will study some special tests like D'Alembert's Ratio test, Cauchy integral test, Raabe's Test and Gauss's test. Lastly, we discuss alternating series, i.e., a series whose terms are alternatively positive and negative.

In **Block 4,** comprising 4 units, we shall study two important concepts, namely, continuity and differentiability. Both these concepts involve the abstract notion of the limit of a function, of which you have some idea from the calculus course. There you were introduced to the epsilon-delta $(\varepsilon - \delta)$ definition, though not in a formal way. Here we shall give a precise meaning to the role of ε and δ in the definition, and explain how they can be used rigorously in the proofs. Then we define notion of continuity of a function at a point and over an interval. Next, you will study to the notion of a derivative of a function in a rigorous way. You have learnt many rules for differentiation from the calculus course. Here we explain the logic behind these rules. We prove some important theorems as applications of differentiation: the Inverse Function Theorem, Rolle's Theorem and the Mean Value Theorems.

In **Block 5,** comprising 5 units, we discuss the concept of Riemann integration, which you have studied in the Calculus course. In this course we will give a rigorous treatment to the theory of integration, using the concept of a limit. We give the proofs of the results wherever necessary.



Riemann
(1826-1866)

The last block, **Block 6**, comprises 6 units. It covers sequences and series, whose terms are functions defined on subsets of the set of real numbers. Such sequences and series are called sequence and series of functions. We introduce two types of convergence – Pointwise convergence and uniform convergence. Whenever the convergence is established for sequences or series, their limit is called the limit function. The fundamental question that interests many mathematicians is whether the properties of limit, continuity and differentiability are preserved by the limit function. In this block we discuss some partial answers to this question.

In all these blocks we have emphasized the importance of proofs. We give several proofs that involve different techniques that are explained in Unit 2. We advise you to work out the details by verifying each step in the proof. You need some practice in choosing δ for a given $\varepsilon > 0$. Similarly you should be able to choose an N corresponding to $\varepsilon > 0$ in the sequential definition of limits. Thus there is a great need to develop the ability to read and write the proofs by doing it yourself.

Throughout this course, we have tried to help you understand the results, methods and concepts with the help of several examples and exercises. Do solve the exercises as and when you encounter them, without referring to the "Solutions/Answers" given at the end of each unit. Only in case of some difficulty, you may look at the solution. You can also compare your solution, with the solutions given by us.

As we have said above, the course, comprises 6 blocks. Each block is divided into units and each unit is divided into sections. The sections of a unit are numbered sequentially as are the exercises, theorems, etc.

For your convenience, we have put

***	to show the end of an example,
■	to show the end of a proof,
E1, E2, E3…	for the exercises
Sec. x.y	to refer to section y of Unit x

We hope you enjoy studying this course. If you have any problems in understanding any portion of it, please ask your academic counsellor for help at your Learner Support Centre, or write to us at svarma@ignou.ac.in . Also, if you feel like studying any topic in greater detail, you may consult the following books:

1. R.G. Bartle and D. R Sherbert, *Introduction to Real Analysis*, John Wiley and Sons (Asia) P. Ltd., 2000.
2. K.A. Ross, Elementary Analysis – *The Theory of Calculus Series* – Undergraduate texts in Mathematics, Springer Verlag, 2003.
3. T.M. Apostal, *Calculus* (Vol. I), John Wiley and Sons (Asia) P. Ltd., 2002.
4. *Principles of Mathematical Analysis* by R. Walter Rudin, McGraw-Hill International Editions.
5. Trench, William F., *"Introduction to Real Analysis"*, Faculty Authored and Edited Books & CDs. 7, *https://digitalcommons.trinity.edu/mono/7* .
6. Elias Zakon, *Mathematical Analysis I*, published by The Trillia Group, 2004, http://www.trillia.com/zakon-analysisI.html.

Block

1

THE STRUCTURE OF $\mathbb{R}$

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Acknowledgement: To Prof. Sujatha Varma for her comments on the units, and to Sh. S.S. Chauhan and Sh. Santosh Kumar Pal for making the CRC of the block.

BLOCK INTRODUCTION

You must have read the Course Introduction, from which you know that this course unfolds in 6 blocks. This is the first block of the course, comprising of 4 units. In this block, the focus will be on understanding the language of mathematics, the thought processes involved in doing mathematics in general, and the analytic properties of $\mathbb{R}$.

Now, from your earlier studies, you would be aware of what a theorem is, and what it means to prove it. However, you need to study the formal grammar of mathematics to be able to communicate mathematics meaningfully. For this purpose, in Unit 1, we introduce you to various aspects of the language of mathematics. Here we shall look at how to express a 'mathematical statement' in different ways. You will also study ways of connecting statements, as well as statements with quantifying symbols.

In Unit 2, the focus is on different mathematical thought processes, the large domain of mathematical reasoning. Here is where you will study different ways of proving mathematical statements. You will also study ways of showing why a given statements is false. Here consider a remark about a phrase you may frequently come across in a 'proof' namely, "without loss of generality". This phrase means that it is enough to prove a particular case of the statement concerned, and **all the remaining cases** can be reduced to this case. For example, consider the statement "the square of every non-zero integer is a positive integer". To prove it, we can take, **without loss of generality**, an integer $n > 0$, and show that $n^2 > 0$. The remaining case is $n < 0$, which can be reduced to the case proved by taking $m = -n$.

Whatever you study in this unit, and in the previous one, will be needed by you not just in this course, but in all your further mathematics courses.

In Unit 3, we shall reacquaint you with the set of real numbers $\mathbb{R}$, with its algebraic and ordered properties. You will see how the order structure of $\mathbb{R}$ is different from that of $\mathbb{Q}$ due to the 'order completeness property' which $\mathbb{R}$ possesses, but $\mathbb{Q}$ lacks. You will also see a few consequences of this property. Finally, you will see the notions of 'finite', 'countably infinite' and 'uncountable' subsets of $\mathbb{R}$.

Unit 4 deals with the topological structure of $\mathbb{R}$. Specifically you will see the concepts of 'neighbourhood' of a point, and 'limit point' of a set. In this context, we shall present a theorem due to the mathematicians Bernhard Bolzano and Karl Weierstrass. Finally, we discuss the two important classes of subsets of $\mathbb{R}$, namely -- open sets and closed sets, and show you how they are related.

At the end of this block you will find **a set of miscellaneous examples and exercises** related to the concepts covered in this block. Please do study them, and try each exercise yourself. This will help you engage with the concepts concerned, and understand them better.

We hope you enjoy the course!

NOTATIONS AND SYMBOLS (used in Block 1)

(Also see the notations used in Calculus and Differential Equations)

$\mathbb{N}_{odd}\ (\mathbb{N}_{even})$	odd (even) natural numbers
$\mathbb{N}_{prime}$	set of prime numbers
$\mathbb{Z}^+\ (\mathbb{Z}^-)$	positive (negative) integers
$\mathbb{Q}^+\ (\mathbb{Q}^-)$	positive (negative) rationals
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+\ (\mathbb{R}^-)$	positive (negative) real numbers
$\Rightarrow (\Leftrightarrow)$	implies (implies and is implied by)
iff	if and only if
$< (\leq)$	is less than (is less than or equal to)
$> (\geq)$	is greater than (is greater than or equal to)
$\exists (\exists!)$	there exists (there exists a unique)
$\forall$	for all
$\sum_{i=1}^{n} a_i \left(\prod_{i=1}^{n} a_i\right)$	$a_1 + a_2 + \cdots + a_n\ (a_1 a_2 \ldots a_n)$
$\therefore$	therefore
w.r.t.	with respect to
$f : X \to Y$	f is a function from the set X to the set Y
s.t.	such that
$\{x \mid x \text{ satisfies } P\}$	the set of all x such that x satisfies the property P
$\wp(X)$	the power set of the set X
$\lvert x \rvert$	modulus of the real, or complex number, x
$\emptyset$	empty set
$N_\varepsilon(a)$	ε-neighbourhood of a
$\overline{S}$	closure of the set S
S°	interior of the set S
$\inf S$	the greatest lower bound (infimum) of S
$\sup S$	the least upper bound (supremum) of S

UNIT 1

COMMUNICATING MATHEMATICS

Structure **Page Nos.**

1.1 INTRODUCTION

You would, by now, have studied mathematics for at least 13 years. If you look back, you will see how the term 'mathematics' had different meanings for you when you were in Class 5, Class 10 and Class 12. What does it mean to you at this stage, after studying the first year courses? Does it seem to be merely a collection of facts? Or, do you see it as a language, with signs, symbols, and its own grammar?

It is such questions that we will consider in this unit. In Sec.1.2, you will see what a mathematically acceptable sentence is. Such a sentence is called a 'mathematical statement'.

In Sec.1.3, you will study the ways of connecting two simple mathematical statements to form compound statements. These statements form the basis of your further study in this unit and the next one.

In Sec.1.4, we focus on concepts involved in particular mathematical statements. These are 'quantifiers', which correspond to the English words 'all', 'some', 'only one'.

Throughout the unit, we shall be working towards helping you achieve the following objectives. To help you assess how much you have learnt, we have sprinkled exercises throughout the unit. Please try to solve them, as you come to them. Doing this will help you to understand the concepts concerned better.

Objectives

After studying this unit, you should be able to:

- distinguish between mathematical statements and non-statements;
- identify, and use, the logical connectives;
- identify, and use, the logical quantifiers.

1.2 MATHEMATICAL STATEMENTS

When you think about ten wholes and two hundredths multiplied with fifty two thousand six hundred and twenty eight, how would you arrive at the product? Don't you visualise the problem as numbers written in base 10, that is, the decimal system? If yes, you are actually looking at '10.02×52628'. Isn't this easier to comprehend and solve? This is just one example of how useful a system of symbols can be. In fact, this is one example of one aspect of mathematics as a language – symbols, and rules for working with them.

Similarly, when you work with sets or with variables, you do so with symbols, and rules (of grammar) for working with them. It is this kind of universally accepted symbols, and structured use of them, that helps any person working with mathematics, anywhere, read mathematics and communicate mathematical ideas to each other. This is true for any other language too, don't you agree?

Now, consider your own first language, or mother tongue. In the process of learning to use this language, you learnt its words and rules of grammar to construct sentences for communication with others. Similarly, for using the language of mathematics, there are rules of syntax and grammar that govern the use of words and symbols in it. A mathematical equation or inequality is a sentence in this language, and this sentence has nouns and verbs. For instance, the sentence 'one hundred divided by twenty is equal to 5' has the nouns one hundred, twenty and five; and the verbs are 'divided by' and 'is equal to'. Similarly, you have all the parts of grammar in the language of mathematics. What is interesting is that this language is universal, the same across the world. That is, any person, anywhere in the world would understand and follow the same rules of grammar and syntax when doing mathematics.

Let us now focus on the sentences in the language of mathematics. What is an "acceptable sentence"? If I say '$x \in \mathbb{N}$' is not a mathematically acceptable sentence, while 'All humans are mortal' is a mathematically acceptable sentence, you may wonder why. The following definition will give you an idea.

Definition: A **mathematical statement** is a sentence that is either true for all the cases covered by it or false for all the cases covered by it.

For instance, '$n^2 < 5$ for $n \in \{0, 1, 2\}$' is a mathematical statement which is true for all three cases covered by it. Note that it is not considering any n that is outside $\{0, 1, 2\}$.
Another example of a mathematical statement is 'The set of real numbers is a finite set', which is a false statement.
So, mathematical statements are the mathematically acceptable sentences. For example, all the sentences below are mathematical statements.

i) $|\cos x| \leq 1$ for all $x \in \mathbb{R}$.

ii) The set of stars in the sky is finite.

iii) $\frac{dy}{dx} = 1$ for $y : \mathbb{R} \to \mathbb{R} : y(x) = x^2$.

iv) All flowers have red petals.

v) Some flowers have red petals.

Note that the sentences in (i), (ii) and (v) are true for all the cases covered by them, and those in (iii) and (iv) are false. Hence they are examples of mathematical statements.

Remark 1: We shall usually just say 'statement' instead of 'mathematical statement' each time.

Next, let us see some examples of what are **not** statements.

i) Add four to five.

ii) If $x \in \mathbb{R}$ such that $x < 5$, is $x < 1$?

iii) $x \in \mathbb{N}$.

iv) $\alpha + \beta + \gamma$, where $\alpha, \beta, \gamma \in \mathbb{R}$.

Here, (i) is an imperative sentence, giving a command. It is not a statement as it is neither true nor false. Next, (ii) is a question, not a sentence; (iii) is ambiguous since it is not clear what x is, and hence we can't decide whether this is true or false; and (iv) is not a sentence, but an expression – an example of a phrase in the language of mathematics!

Why don't you try a related exercise now?

E1) Which of the following are mathematical statements? Give reasons for your answers.

i) $3a - b + c + 2$, for $a, b, c \in \mathbb{Z}$.

ii) $3a - b + c + 2 = 0.5$, for some $a, b, c \in \mathbb{Z}$.

iii) All human beings work in offices.

iv) $\frac{d^2y}{dx^2} = \frac{dy}{dx}$.

v) What is mathematics?

vi) The range of $f : \mathbb{R} \to \mathbb{R} : f(x) = x\sqrt{5}$ is $\mathbb{R}$.

Now, the underpinning of mathematics is the underlying thought process. This shows up through a series of statements that are formed by connecting each other. In the next section, you will study how the statements can be logically connected.

1.3 LOGICAL CONNECTIVES

While reading English, you would have come across compound sentences, that is, those made up of smaller sentences connected by 'or' or 'and'. For

example, consider the statement, 'There is a real number lying between 0 and 0.1'. This is actually made up of two statements connected with AND, namely,
'There is a real number greater than 0', and
'There is a real number less than 0.1'.

In the same way, most statements in mathematics are combinations of simpler statements joined by words and phrases like 'and', 'or', 'if …, then …', 'if and only if', etc. These words and phrases are called **logical connectives**. There are several such connectives, which we shall discuss one by one in this section.

1.3.1 Disjunction

Consider the sentence 'Ajay or Mustari went to the market'. This can be written as 'Ajay went to the market or Mustari went to the market'. So, this sentence is actually made up of two simple sentences connected by 'or'. When we connect two statements by 'or', we have a term for such a compound statement.

$p \vee q$ is the same as $q \vee p$.

Definition: The **disjunction** of two statements p and q is the compound statement ***p* or *q***, denoted by $\boldsymbol{p \vee q}$.

The symbol $\vee$ seems to have been first used by the famous polymath, Bertrand Russell, in an article he published in 1908.

$p : 0 < 1$ represents the fact that p denotes the statement '$0 < 1$'.

For example, 'The Discovery of India is a book or several women run their own businesses' is the disjunction of p and q, where

p: The Discovery of India is a book, and

q: Several women run their own businesses.

Similarly, if p denotes '$2 > 0$' and q denotes '$2 > 5$', then $p \vee q$ denotes the statement '2 is greater than 0 OR 2 is greater than 5'.

For any two statements p and q, $p \vee q$ is a statement. Hence, it must be either true or false. Let us now look at when $p \vee q$ is true and when it is false. For doing so, let us look at the examples given above. Since 'The Discovery of India is a book' is true, and q in that example is also true, $p \vee q$ is certainly true. Now, look at the next example above, about 2, 0 and 5. Since p is true, but q is not true, what about the truth value of $p \vee q$? Note that if even one of them is true, then the compound statement $p \vee q$ is true.

Thus, more generally, **if even one out of p and q is true, then '$p \vee q$' is true. Otherwise, $p \vee q$ is false**. This holds for any pair of statements p and q.

Let us consider an example.

Example 1: Is the disjunction of the statements '$\frac{d}{dt}(t^3 + 7t) = 3t^2 + 7$ for $t \in \mathbb{R}$' and '$3 + 5 = 2$' true or false? Give reasons for your answer.

Solution: Let p denote '$\frac{d}{dt}(t^3 + 7t) = 3t^2 + 7$ for $t \in \mathbb{R}$', and q denote '$3 + 5 = 2$'. Their disjunction is '$\frac{d}{dt}(t^3 + 7t) = 3t^2 + 7$ for $t \in \mathbb{R}$ or $3 + 5 = 2$'.
As you know that p is true and q is false, their disjunction $p \vee q$ is true.

***

Try some exercises now.

E2) Write down the disjunction of the following statements, and decide whether it is true or not.

i) $2+3=7$, ii) $\frac{1}{5} < \frac{1}{4}$.

E3) Give an example, with justification, of statements p and q, related to functions, such that

i) $p \vee q$ is false;

ii) $p \vee q$ is true.

Now let us look at the logical analogue of the conjunction, 'and'.

1.3.2 Conjunction

As in ordinary language, we use 'and' to combine simple statements to make compound ones. For instance, '$1+4 \neq 5$ and Prof. Rao teaches Chemistry' is formed by joining '$1+4 \neq 5$' and 'Prof. Rao teaches Chemistry' by 'and'. Let us define the formal terminology for such a compound statement.

Definition: Let p and q be two statements. The compound statement '***p* and *q***' is called the **conjunction** of the statements p and q. We denote this by $\boldsymbol{p \wedge q}$.

$p \wedge q$ is the same as $q \wedge p$.

For example, if

p: IGNOU is a mega-university, and

q: March 8th is International Women's Day,

then $p \wedge q$: IGNOU is a mega-university and March 8th is International Women's Day.

The symbol $\wedge$ seems to have first appeared in a book by the Russian logician, M.I. Schönfinkel, published in 1924.

Again, '$2+1=3 \wedge 3=5$' is the conjunction of '$2+1=3$' and '$3=5$'.

Now, when would $p \wedge q$ be true, and when would it be false? Do you agree that $p \wedge q$ will be true only when **both** p and q are true, and not otherwise? For instance, take the example above of IGNOU and March 8th. The conjunction will be true only if p is true and q is true, which is the case. So, $p \wedge q$ is true here.
Similarly, '$2+1=3 \wedge 3=5$' is false because '$3=5$' is false.

Consider another example.

Example 2: For which values of a and z will the conjunction of '$2 \div a = 1$, where $a \in \mathbb{R}$' and '$\text{Arg}\, z = \pi/3$, where $z \in \mathbb{C}$' be true?

The **truth value** of a statement p is 'true' or 'false', depending on whether p is true or false, respectively.

Solution: Let $p: 2 \div a = 1,$ where $a \in \mathbb{R}$, and
q: $\text{Arg}\, z = \pi/3$, where $z \in \mathbb{C}$.
$p \wedge q$ will be true only when p and q are both true.
p is true only for $a = 2$.

q is true for $z = \alpha + i\beta$, where $\frac{\beta}{\alpha} = \sqrt{3}, \beta > 0, \alpha > 0.$ Thus, q is true for infinitely many complex numbers lying on the ray, $y = \sqrt{3}\,x, x > 0.$

Thus, $p \wedge q$ is true for $a = 2$ and for every $z = x(1 + i\sqrt{3}),\ x > 0.$

***

Why don't you try some exercises now?

E4) Give the set of those real numbers x for which $p \wedge q$ will be true, where $p : x > -2,$ and $q : x + 3 \neq 7.$

E5) Give an example from calculus, of statements p and q, for which $p \vee q$ is true but $p \wedge q$ is false. Justify your choice of example.

The next connective actually relates 'conjunction' and 'disjunction', as you will see.

1.3.3 Negation

You must have come across young children who, when asked to do something, go ahead and do exactly the opposite! Or, when asked if they would like to eat something, will say 'No!', the 'negation' of yes! Now, if p denotes 'The child eats rice', how can we denote 'The child does not eat rice'? Let us define the connective that will help us do so.

The symbol ~, for negation, appeared in Bertrand Russell's article, "Mathematical Logic as based on the theory of types", in

Definition: The **negation** of a statement p is '**not** $\boldsymbol{p}$', denoted by **~** $\boldsymbol{p}$.

For example, if p is '$\mathbb{N}$ is a finite set', then $\sim p$ is '$\mathbb{N}$ is not a finite set'. Note that here p is false and $\sim p$ is true.
Let is consider another example.

Example 3: Write down the negation of the following statements and give the truth values of all the statements.

i) $\int x dx = 1.$

ii) No human can live without oxygen.

iii) $\frac{-7}{5} + \frac{9}{2} \geq \frac{2}{7}.$

Solution: i) The negation is $\int x dx \neq 1.$

Here, the given statement is false, as you know. Its negation is true.

ii) The negation is 'At least one human can live without oxygen'. Given the current level of scientific knowledge, the given statement is true, and its negation is false.

iii) The negation is $\frac{-7}{5} + \frac{9}{2} \ngeq \frac{2}{7}.$

As you know, this is equivalent to $\frac{-7}{5} + \frac{9}{2} < \frac{2}{7}.$

The given statement is true, while its negation is false.

***

From the discussion above regarding the truth value of $\sim p$, you would agree that **if p is true, $\sim p$ will be false, and vice versa**.

Let us now consider two laws, due to the logician Augustus De Morgan, relating conjuctions and disjunctions using negation.

De Morgan's Laws: Let p and q be statements. Then

(i) $\sim (p \vee q)$ is equivalent to $\sim p \wedge \sim q$;

(ii) $\sim (p \wedge q)$ is equivalent to $\sim p \vee \sim q$.

Compare these laws with the De Morgan laws given in Unit 1 of the course, Calculus.

We will not prove these laws here, but will give some examples of their use.

Example 4: Give a statement equivalent to '$f : \mathbb{R} \to \mathbb{R}$ is not continuous on $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R} : \lim_{x \to 1} f(x) \neq 1$'.

Solution: Let $p : f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$, $q : f : \mathbb{R} \to \mathbb{R} : \lim_{x \to 1} f(x) = 1$. Then the given statement is $\sim p \wedge \sim q$. This is equivalent to $\sim (p \vee q)$, that is, 'For $f : \mathbb{R} \to \mathbb{R}$, neither is f continuous on $\mathbb{R}$, nor is $\lim_{x \to 1} f(x) = 1$'.

***

Why don't you try the following exercise now?

E6) Write down the negation of each of the following statements. Also decide the truth value of each negation.

i) $0 - 5 \neq 5$.

ii) $n > 2$ for every $n \in \mathbb{N}$.

iii) All human beings can walk.

iv) $f : \mathbb{R} \to \mathbb{R} : f(x) = 2x$ is a periodic function.

v) $x \in \mathbb{R}$ such that $\sqrt{\dfrac{x-2}{3-x}}$ is **not** defined.

Let us now discuss the conditional connectives, representing 'If …, then …' and 'if and only if'.

1.3.4 Conditional Connectives

Consider the statement, 'If any student of Real Analysis gets 75% or more in the examination, then she will get an A grade for the course'. We can write this statement as 'If p is true, then q is true', where

p: Any student of Real Analysis gets 75% or more in the examination, and

q: Any student of Real Analysis will get an A grade for the course.

This compound statement is an example of the implication of q by p, which we now define.

Definitions: Given any two statements p and q, we denote the statement '**If p** is true**, then q** is true' by $\boldsymbol{p \Rightarrow q}$**.** We also read this as

'p is true **implies** q is true', or

'p being true **is sufficient** for q to be true', or

'p is true **only if** q is true', or

'q is true **if** p is true', or

'q being true **is necessary for** p to be true'.

Here, we call p the **hypothesis** and q the **conclusion**.
Further, a statement of the form $p \Rightarrow q$ is called a **conditional statement**, or an **implication**.

So, for example, in the conditional statement, 'If m is in $\mathbb{Z}$, then m belongs to $\mathbb{Q}$', the hypothesis is '$m \in \mathbb{Z}$' and the conclusion is '$m \in \mathbb{Q}$'.
Mathematically, we can write this statement as
$m \in \mathbb{Z} \Rightarrow m \in \mathbb{Q}$.

Let us analyse the statement $p \Rightarrow q$ for its truth value. That is, assuming that p is true, under what conditions is $p \Rightarrow q$ true or false? For instance, if $p : m \in \mathbb{Z}$, and $q : m \in \mathbb{Q}$, then whenever p is true, that is, $m \in \mathbb{Z}$, then q will automatically be true since $\mathbb{Z} \subseteq \mathbb{Q}$. So here, p being true implies that q is true, i.e., $p \Rightarrow q$ is true.
In other words, what you have seen in this example is that for every
p, q, p $\Rightarrow$ ***q*** **is true only when both *p* and *q* are true and the truth of *q* follows from the truth of *p*.** Otherwise, we cannot say that p implies q.

Remark 2: This is regarding the use of the terms 'sufficient' and 'only if'. We say 'p being true is **sufficient** for q to be true'. This means that if $p \Rightarrow q$ is true, then **it is enough** to know that p is true, because automatically then q will be true. Similarly, we say 'p is true **only if** q is true', or 'q being true is **necessary** for p to be true', when $p \Rightarrow q$ is true, because if q is not true, then p cannot be true in this situation.

Consider an example.

Example 5: Check whether the two statements $p \Rightarrow q$ and $q \Rightarrow p$ are true, where $p : x = \pi$ and $q : e^{ix} = -1$.

'q does not imply p' is symbolised as '$q \not\Rightarrow p$'

Solution: If we assume p is true, i.e., $x = \pi$, then $e^{i\pi} = -1$, that is, q is true. So $p \Rightarrow q$ is true.
Next, assume q is true, that is, $e^{ix} = -1$, i.e., $\cos x + i \sin x = -1$.
This happens when $x = \pi$, but also when $x = 3\pi, -\pi$, etc. So, p need not follow from q. Thus, $q \Rightarrow p$ is not true. Thus, $q \not\Rightarrow p$.

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Why don't you try a related exercise now?

E7) Which of the following statements are true, and why?

i) $p \Rightarrow q$, ii) $p \Rightarrow r$, iii) $(p \wedge r) \Rightarrow \sim q$,

iv) $r \Rightarrow p$, where

$p:\sqrt{17}\notin\mathbb{Q}$, $q:17\neq\frac{a^2}{b^2}$, for any $a,b\in\mathbb{Z}, b\neq 0$, $r:\sqrt{x}\in\mathbb{R}$, where x is a prime number.

In the example and exercise above, you noted that related to $p\Rightarrow q$ is the implication $q\Rightarrow p$. This is, in a sense, the 'reverse' implication. We have a name for this 'reverse' implication, which you may already know.

Definition: The **converse** of the implication $p\Rightarrow q$ is the implication $q\Rightarrow p$.

Note that this also means that the converse of $q\Rightarrow p$ is $p\Rightarrow q$, that is, **the converse of the converse of a given implication is the given implication itself**.

So, for example, the converse of '$m\in\mathbb{Z}\Rightarrow m\in\mathbb{Q}$' is '$m\in\mathbb{Q}\Rightarrow m\in\mathbb{Z}$'. Here, you can see that the first implication is true, while its converse is false. (Why?) However, consider the implication '$p\mid ab\Rightarrow p\mid a$ or $p\mid b$, where $p, a, b\in\mathbb{Z}$ and p is a prime'. What is its converse? Isn't it 'If $p, a, b\in\mathbb{Z}$, where p is a prime such that $p\mid a$ or $p\mid b$, then $p\mid ab$'? In this case, the implication, and its converse, are both true.

Now, what happens when we take the conjunction of an implication and its converse? We denote '$p\Rightarrow q$ and $q\Rightarrow p$' by a shorter notation, $\boldsymbol{p\Leftrightarrow q}$. This may or may not be a true statement. When it is true, we have the following definition.

Definition: Let p and q be two statements such that the conjunction $(p\Rightarrow q)\wedge(q\Rightarrow p)$ is true. Then p and q are called **logically equivalent**. We also read '$\boldsymbol{p\Leftrightarrow q}$ is true' as 'p is true **if and only if** q is true'. We also usually shorten 'if and only if' to **iff**.
In this case we also say that 'p being true **implies and is implied by** q being true', or 'p being true is **necessary and sufficient for** q to be true'.

Note that $p\Leftrightarrow q$ is the same as $q\Leftrightarrow p$.

Let us consider some examples.

Example 6: Check whether 'For $a, b, c\in\mathbb{R}$, $\int_a^b \mathrm{x}\ \mathrm{dx}=\mathrm{c}\Leftrightarrow \mathrm{b}^2-a^2=2c$' is true or false.

Solution: Here $p:\int_a^b xdx=c$ for $a,b,c\in\mathbb{R}$, and $q:b^2-a^2=2c$ for $a,b,c\in\mathbb{R}$.

Now to check if $p\Rightarrow q$, we assume p is true. That is, $\int_a^b x\,dx=c$.

Then $b^2-a^2=2c$, that is, q is true.

Conversely, assume q is true, that is, $b^2-a^2=2c$. Then $\int_a^b x\ dx=c$, that is, p is true.
Hence $p\Rightarrow q$ and $q\Rightarrow p$ are both true. Thus $p\Leftrightarrow q$ is true.

***

Example 7: Give an example, with justification, of two statements A and B, such that $B \Rightarrow A$ is true but A and B are not equivalent.

Solution: Let $A: f$ is a continuous function on $\mathbb{R}$, and
$B: f$ is a polynomial over $\mathbb{R}$.
Then $B \Rightarrow A$ is true, since every polynomial is a continuous function. However, A does not imply B, since if A is true then f can be any continuous function, like the exponential function, etc. So it need not be a polynomial. Hence, $A \Rightarrow B$ is false. Therefore, $A \Leftrightarrow B$ is false.

***

Here are some related exercises now.

E8) For each of the following compound statements, first identify the simple statements $p, q, r,$ etc., that are combined to make it. Then write it in symbols, using the connectives, and give its truth value.

i) If the triangle ABC is equilateral, then it is isosceles.

ii) The real numbers a and b are integers if and only if ab is a rational number.

iii) A child in India is in Class 1 or in Class 2 if she is 5 years old.

E9) Write down two statements p and q for which p being true is necessary for q to be true but p being true is not sufficient for q to be true.

Now we come to another important type of implication.
Consider the implication $m \in \mathbb{Z} \Rightarrow m \in \mathbb{Q}$. Now consider $\sim (m \in \mathbb{Q}) \Rightarrow \sim (m \in \mathbb{Z})$, that is, $m \notin \mathbb{Q} \Rightarrow m \notin \mathbb{Z}$.
So, given $p \Rightarrow q$, we have shown another related implication. Let us define this.

Definition: The **contrapositive** of the implication $p \Rightarrow q$ is the implication $\sim q \Rightarrow \sim p$.

Let us consider a simple example.

Example 8: Give the contrapositive of $A \Rightarrow B$, and the truth value of both the implications, where
$A: f: \mathbb{R} \to \mathbb{R}$ is monotonic on $[a,b]$,

$B: \int_a^b f(t)\,dt$ exists, where $f: \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$.

Solution: You know, from your course on Calculus, that $A \Rightarrow B$ is true.
Now consider $\sim B \Rightarrow \sim A$.

$\sim B:$ $\int_a^b f(t)\,dt$ does not exist, where $f: \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$.

$\sim A:$ $f: \mathbb{R} \to \mathbb{R}$ is not monotonic on $[a, b]$.

Then $\sim B \Rightarrow \sim A$ is also true.

***

This example leads us to the following remark.

Remark 3: Note that $p \Rightarrow q$ and $\sim q \Rightarrow \sim p$ are logically equivalent.
This fact is the basis of a method of proof you will study in Unit 2.

Try some exercises now.

E10) Write down the contrapositive of each statement given in E7. Also give the truth values of the contrapositives.

E11) Give the converse and the contrapositive of $p \Rightarrow q$, where p is '$f : \mathbb{R} \to \mathbb{R} : f(x) = \sin x$', and q is '$f : \mathbb{R} \to \mathbb{R}$ has a local extremum at $x = -\pi/2$'.

The contrapositive of an implication is NOT the same as its converse.

Let us now discuss another part of the language of mathematics.

1.4 LOGICAL QUANTIFIERS

Let me begin by asking you: Can all sentences be written in symbolic form by using only the logical connectives you have just studied? What about sentences like 'x is prime for some x in $\mathbb{N}$'? How would you symbolise the phrase 'for some x', which we can rephrase as 'there is an x', or 'there exists an x'? You must have come across these phrases often while studying mathematics.

We use the symbol '$\exists$' to denote the quantifier, 'there exists'. The way we use it is, for instance, to rewrite 'There is at least one child studying in Class 5 in India' as
'$(\exists x$ in $U) p(x)$',
where $p(x)$ is the sentence 'x is studying in Class 5 in India' and U is the set of all children.

$\exists$ is called the **existential quantifier.**

'There exists' means 'there is at least one'.

Another example of the use of the existential quantifier is the true statement '$\exists x \in \mathbb{R}$ such that $x+1>0$', which is read as 'There exists an x in $\mathbb{R}$ for which $x+1>0$'.

Yet another example is the false statement '$\exists x \in \mathbb{N}$ s.t. $x - \frac{1}{2} = 0$', which is read as 'There exists an x in $\mathbb{N}$ for which $x - \frac{1}{2} = 0$'.

Now suppose we take the negative of the statement about Class 5 children, given above. Wouldn't it be 'There is no child studying in Class 5 in India'? We could symbolise this as 'for all x in U, $q(x)$', where U is the set of all children and $q(x)$ denotes the sentence 'x is not studying in Class 5 in India', i.e., $q(x)$ is the same as $\sim p(x)$.
We have **a mathematical symbol for the quantifier 'for all'** or **'for every', which is '$\forall$'.**
So the statement above can be written as
'$(\forall x \in U)\, q(x)$', or '$q(x), \forall x \in U$'.

$\forall$ is called the **universal quantifier.**

Another example of the use of the universal quantifier is
'$\forall\ x \notin \mathbb{N},\ x^2 > x$', which is read as 'for every x which is not a natural number, $x^2 > x$'.

Of course, this is a false statement, because there is at least one $x \notin \mathbb{N}, x \in \mathbb{R}$, for which it is false, for example, $x = \frac{1}{2}$.

We often use both quantifiers together, as in the statement,
'For every rational number x, there is a rational number lying strictly between x and $x+1$'.
In symbols, this is $(\forall x \in \mathbb{Q})(\exists y \in \mathbb{Q})(x < y < x+1)$.
What would the negation of this statement be? It would be 'There is a rational number x such that there is no rational number lying strictly between x and $x+1$'.
In symbols, this is
$(\exists x \in \mathbb{Q})(\forall y \in \mathbb{Q})\ (y \notin]x, x+1[)$.
This is according to the following rules for negation that relate $\forall$ and $\exists$.

The two rules are

i) $\sim (\forall x \in U)\, p(x)$ is equivalent to $(\exists x \in U)\ (\sim p(x))$, and

ii) $\sim (\exists x \in U)\, p(x)$ is equivalent to $(\forall x \in U)\ (\sim p(x))$,

where U is the set of values that x can take.

So far you have seen some examples in which the quantifiers occur singly, or together. Sometimes you may come across situations where you would use $\exists$ or $\forall$ twice or more in a statement. It is in situations like this, or worse, [say, $(\forall x_1 \in U_1)\ (\exists x_2 \in U_2)(\exists x_3 \in U_3)(\forall x_4 \in U_4)\ldots(\exists x_n \in U_n)\, p]$, where our rules for negation come in useful. In fact, applying them, in a second we can say that the negation of this seemingly complicated statement is
$(\exists x_1 \in U_1)(\forall x_2 \in U_2)(\forall x_3 \in U_3)(\exists x_4 \in U_4)\ldots(\forall x_n \in U_n)(\sim p)$.

Consider an example.

Example 9: Write the following statement, and its negation, using logical quantifiers. Also interpret its negation in words.
'Given $\varepsilon > 0$, there is a $\delta > 0$ s.t. $|x^2 - 1| < \varepsilon$ whenever $|x-1| < \delta$'.

Solution: The given statement says that for each positive real number ε, there is a positive real number δ for which, whenever $|x-1| < \delta$ is true then $|x^2 - 1| < \varepsilon$ is true. We could write this in symbols as
$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (|x-1| < \delta \Rightarrow |x^2 - 1| < \varepsilon)$.
What would its negation be? It would be
$\sim [(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (|x-1| < \delta \Rightarrow |x^2 - 1| < \varepsilon)]$,
which is equivalent to
$(\exists\, \varepsilon > 0)\ (\forall \delta > 0)\ [\sim (|x-1| < \delta \Rightarrow |x^2 - 1| < \varepsilon)]$.
That is, there is an $\varepsilon > 0$ s.t for every $\delta > 0$ and $x \in \mathbb{R}$ satisfying $|x-1| < \delta$, we cannot conclude that $|x^2 - 1| < \varepsilon$;
that is, there is an $\varepsilon > 0$ for which no $\delta > 0$ has the property that
$|x-1| < \delta \Rightarrow |x^2 - 1| < \varepsilon$.

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In the example above, you would have realised that the given statement says that $\lim\limits_{x \to 1} (x^2) = 1$, and its negation states that $\lim\limits_{x \to 1} x^2 \neq 1$.

Why don't you try an exercise now?

E12) How would you present the following statements, and their negations, using logical quantifiers? Also interpret the negations in words.

i) Some people can fool all other people all the time.

ii) Every real number is the square of some real number.

iii) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where $\theta \in \mathbb{R}$, $n \in \mathbb{Z}$.

And finally, let us look at a very useful quantifier, which is closely linked to $\exists$. You would need it for writing, for example, 'There is one and only one key that fits the given desk's lock', in symbols.
We use the symbol '$\exists$ **!** ' to denote '**there is one and only one**'. The way we use it is, for instance, to write the statement above as '$(\exists\, ! x \in A)\ p(x)$', where $p(x)$ is the sentence that x fits the desk's lock, and A is the set of keys.

The phrase 'there is one and only one x' means that there is at least one x satisfying the given condition, **and** there is only one such x.
We also read '$\exists\, !\ x \in A$' as '**there is a unique** x in A', or '**there is exactly one** x in A'.

For other examples, try and recall the statement of uniqueness in the mathematics that you've studied so far, for example,
'There is a unique circle that passes through three non-collinear points in a plane'.
How would you represent this in symbols? If x denotes a circle, and y denotes a set of 3 non-collinear points in a given plane, then the statement is $(\forall\ y \in P)\ (\exists\, !\, x \in C)$ (x passes through y).
Here C denotes the set of circles in a given plane, and P denotes the set of sets of 3 non-collinear points in the same plane.

And now, a short exercise for you!

E13) Which of the following statements are true (where x, y are in $\mathbb{R}$) ? Give reasons for your answers.

i) $\forall\, x \in \mathbb{R}, x \geq 0, \exists !\, y \in \mathbb{R}$ s.t. $y^2 = x$

ii) $\forall\ x \in \mathbb{R}, \exists\, !\, y \in \mathbb{R}$ s.t. $y^2 = x^3$

iii) $\exists\, x \in \mathbb{R}, \exists\, !\ y \in \mathbb{R}$ s.t. $xy = 0$

iv) $\sim (\exists\ x \in \mathbb{C})\ (\exists !\ y \in \mathbb{C})\ (x + y = 0)$.

What you have studied so far is the essence of communicating mathematical ideas, using the universal language of mathematics. In the next unit you will see how essential this is for the core of mathematical thinking. For now, let us summarise what you have studied in this unit.

1.5 SUMMARY

In this unit, we have considered the following points:

1. What a mathematical statement is.

2. The definition, and use of, logical connectives:
   Given statements p and q,

   i) their disjunction is 'p or q', denoted by $p \vee q$;
      $p \vee q$ is true if p is true, or q is true, or both p and q are true.

   ii) their conjunction is 'p and q', denoted by $p \wedge q$;
      $p \wedge q$ is true only when both p and q are true.

   iii) the negation of p is 'not p', denoted by $\sim p$;
      p is true if and only if $\sim p$ is false.

   iv) 'if p, then q' is denoted by $p \Rightarrow q$;
      $\sim(p \Rightarrow q)$ is equivalent to $(p \wedge \sim q)$.

   v) 'p if and only if q' is denoted by $p \Leftrightarrow q$;
      in this case, p and q are logically equivalent.

3. Logical quantifiers: 'For every', denoted by '$\forall$'; 'there exists', denoted by '$\exists$'; and 'there is one and only one', denoted by '$\exists\,!$'.

4. The rules of negation related to the quantifiers:
   $\sim(\forall\, x \in U)\, p(x)$ is equivalent to $(\exists\, x \in U)\,(\sim p(x))$,
   $\sim(\exists\, x \in U)\, p(x)$ is equivalent to $(\forall\, x \in U)\,(\sim p(x))$.

In the next section, we give solutions to the exercises of this unit. You should have tried to solve the exercises yourself before looking at these solutions.

1.6 SOLUTIONS / ANSWERS

E1) i) This is not a statement, since it is not even a sentence. It is only an expression.

ii) This is a statement, and is true since, for example, $3(0)-(0)+(-1.5)+2=0.5$.

iii) This is a mathematical statement since it is a sentence which is always false.

iv) This is not a statement, since we cannot decide whether it is true or false, as we do not know what y is.

v) This is a question, not a sentence, and hence not a statement.

vi) This is a true statement.

E2) $(2+3=7) \vee \left(\frac{1}{5} < \frac{1}{4}\right)$ is true since $\frac{1}{5} < \frac{1}{4}$ is true.

E3) i) Here both p and q need to be false. So, let's take
   p: Every real-valued function is continuous on $\mathbb{R}$,
   q: The greatest integer function is derivable at every point of $\mathbb{R}$.
   Since both p and q are false, $p \vee q$ is false.

ii) Take p as in (i) above, and
   $q: f: \mathbb{R} \to \mathbb{R}: f(x) = 0$ is well-defined.
   Since q is true, $p \vee q$ is true.

E4) $p \wedge q$ is true only when p is true and q is true. Here, $p \wedge q$ will be true for every $x \in \{r \in \mathbb{R} \mid r > -2, r \neq 4\}$.

E5) The example in E3 (ii) works here since $p \vee q$ is true, but $p \wedge q$ is false (because p is false – e.g., the greatest integer function is not continuous on $\mathbb{R}$).

E6) i) $\sim(0-5 \neq 5)$ is $(0-5=5)$, which is false.

ii) $\sim(n > 2$ for every $n \in \mathbb{N})$ is (there is some $n \in \mathbb{N}$ s.t. $n \leq 2)$, which is true.

iii) 'There is a human being who cannot walk', which is true.

iv) '$f : \mathbb{R} \to \mathbb{R} : f(x) = 2x$ is not a periodic function', which is true.

v) $x \notin \{(x \geq 2 \wedge x < 3) \vee (x \leq 2 \wedge x > 3)\}$
$\Leftrightarrow x \notin \{2 \leq x < 3\} \wedge x \notin \emptyset$
$\Leftrightarrow x \notin [2,3[$, by De Morgan's laws.

E7) i) $p \Rightarrow q$ is true, since if p is true then $\sqrt{17}$ cannot be written in the form $\frac{a}{b}$, by definition, and hence $17 \neq \frac{a^2}{b^2}$ for any $a, b \in \mathbb{Z}, b \neq 0$.

ii) This is false, since $\sqrt{17}$, being irrational, does not tell us about where $\sqrt{x}$ lies for $x \neq 17$.

iii) False, since $p \wedge r$ is true but $\sim q$ is false.

iv) False, since $\sqrt{17} \in \mathbb{R}$ does not imply that $\sqrt{17}$ is irrational.

E8) i) $p : \Delta ABC$ is equilateral, $q : \Delta ABC$ is isosceles.
$p \Rightarrow q$ is true.

ii) p: a and b are integers, q: $ab \in \mathbb{Q}$, where $a, b \in \mathbb{R}$.
$p \Leftrightarrow q$ is false since q does not imply p. For example,
$\left(\frac{1}{2}\right)\left(\frac{-7}{3}\right) \in \mathbb{Q}$ but $\frac{1}{2} \notin \mathbb{Z}, \frac{-7}{3} \notin \mathbb{Z}$.

iii) $p : x$ is a 5-year-old child in India.
$q : x$ is a child in Class 1 or Class 2.
$p \Rightarrow q$ is true, as per the Right to Education Act.

E9) We need to consider p and q, where $q \Rightarrow p$, $p \not\Rightarrow q$.
For example, take $q : x + 5 = \pi$, where $x \in \mathbb{R}$.
$p : x \in \mathbb{R} \setminus \mathbb{Q}$.
Then $q \Rightarrow p$, but $p \not\Rightarrow q$ since every irrational number is not $\pi - 5$.

E10) i) $\sim q \Rightarrow \sim p$, that is, 'if there are $a, b \in \mathbb{Z}, b \neq 0$ such that $17 = \frac{a^2}{b^2}$, then $\sqrt{17} \in \mathbb{Q}$'. This is true.

ii) $\sim r \Rightarrow \sim p$, that is, 'if there is a prime number x with $\sqrt{x} \notin \mathbb{R}$, then $\sqrt{17} \in \mathbb{Q}$', which is false. Here $\sim r$ is false, and $\sim p$ is false. Hence $\sim r \Rightarrow \sim p$ is false.

iii) $q \Rightarrow \sim (p \wedge r)$, that is, $q \Rightarrow (\sim p \vee \sim r)$, that is 'if $17 \neq \frac{a^2}{b^2}$ for any $a, b \in \mathbb{Z}, b \neq 0$, then $\sqrt{17} \in \mathbb{Q}$ or $\sqrt{x} \notin \mathbb{R}$ for some prime number x'.
This is false.

iv) $\sim p \Rightarrow \sim r$, that is, 'if $\sqrt{17} \in \mathbb{Q}$, then there is a prime number x for which $\sqrt{x} \notin \mathbb{R}$'.
This is false.

E11) The converse is $q \Rightarrow p$, that is, 'if f has a local extremum at $x = -\pi/2$, then $f(x) = \sin x$'.
The contrapositive is $\sim q \Rightarrow \sim p$, that is, 'if $f: \mathbb{R} \to \mathbb{R}$ does not have a local extremum at $x = -\pi/2$, then $f(x) \neq \sin x$ for some $x \in \mathbb{R}$'.

E12) i) $(\exists\, x \in P)\ (\forall\, y \in A)\ (x$ can fool $y)$, where P is the set of human beings, and A is the set $P \setminus \{x\}$.
Its negation is $(\forall\, x \in P)\ (\exists\, y \in A)\ (x$ can't fool $y)$, that is, 'For every person, there is some other human being who cannot be fooled by him/her'.

ii) $(\forall\, x \in \mathbb{R})\ (\exists\, y \in \mathbb{R})\ (x = y^2)$.
The negation is $(\exists x \in \mathbb{R})\ (\forall\, y \in \mathbb{R})\ (x \neq y^2)$, that is, 'There is a real number which cannot be written as the square of any real number'.

iii) $(\forall\, \theta \in \mathbb{R})\ (\forall\, n \in \mathbb{Z})\ [(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta]$.
The negation is
$(\exists\, \theta \in \mathbb{R})\ (\exists\, n \in \mathbb{Z})\ [(\cos\theta + i \sin\theta)^n \neq \cos n\theta + i \sin n\theta]$,
that is, for some real number θ and integer n,
$(\cos\theta + i \sin\theta)^n \neq \cos n\theta + i \sin n\theta$.

E13) i) True, since $y = \sqrt{x}$, and hence is unique.

ii) False, for example, for $x = -1$, there is no $y \in \mathbb{R}$ such that $y^2 = x^3$.

iii) True, for $x \neq 0$, only $y = 0$ satisfies $xy = 0$.

iv) The given statement is 'for every $x \in \mathbb{C}$, there is no unique $y \in \mathbb{C}$ for which $x + y = 0$'.
This is false, since for each $x \in \mathbb{C}$, $\exists\ y = -x$, which is unique, such that $x + y = 0$.

UNIT 2

MATHEMATICAL REASONING

Structure **Page Nos.**

2.1 INTRODUCTION

In the previous unit, you studied about the different components of the language of mathematics. In this unit, you will see how these play a significant role in proving a mathematical statement. Here you will also see that the concept of a mathematical proof is the core of mathematics.

To start with, in Sec.2.2, we will discuss what constitutes a proof. You will see how the statements that make up a proof are very carefully put together, in a well-defined reasoned way.

In the next two sections, Sec.2.3 and Sec.2.4, you will study different methods of proof. The variety in the methods comes from the kind of reasoning used, as you will see. You will also see that some statements can be proved by several diferent methods. Practice and experience helps one to decide which method of proof is best for the particular situation.

You should study this unit very carefully since it forms the basis of all the mathematics you will study. Throughout the unit you will be studying several examples. You will also get ample opportunity to create proofs, while doing the exercises. Please solve these exercises as you come to them. This will help you decide whether you have really understood what you have studied till that point.

The specific learning objectives of this unit now follow.

Objectives

After studying this unit, you should be able to:

- explain what a 'theorem', 'proof' and 'disproof' is;
- describe, and apply, the direct method of proof;
- describe, and apply, the methods of proof by contrapositive and proof by contradiction;
- explain what a counterexample is, and how it can be used to disprove a statement;
- state, and apply, both forms of the principle of mathematical induction.

2.2 WHAT IS A PROOF?

Suppose I tell somebody, "I am stronger than you." The person is quite likely to turn around, look menacingly at me, and say, "Prove it!" What she or he really wants is to be convinced of my statement by some evidence. (In this case it would probably be a big physical push!) Convincing evidence is also what the world asks for before accepting a scientist's prediction, or a historian's claim.

In the same way, if you want a mathematical statement to be accepted as true, you would need to provide **mathematically acceptable** evidence to support it. This means that you would need to show that the statement is **universally true**. And this would be done in the form of a 'logically valid proof'.

Let us see what a proof is.

Definitions: i) A **proof** of a statement p is a finite sequence of statements $p_1, \ldots, p_n$ such that p follows logically from $p_1 \wedge p_2 \wedge \ldots \wedge p_n$, i.e., $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Rightarrow p.$

Each p_i, for $i = 1, \ldots, n$, is called a **premise** (or an **assumption**, or a **hypothesis**).

ii) The statement p that is proved to be true is called a **theorem**.

In the next section you will read about some ways of disproving a statement.

iii) A **disproof** of a statement p is a proof of $(\sim p)$, or a proof that shows that p is false.

Let's consider an example of a proof:

Example 1: Prove the statement 'For any two sets A and B, $A \cap B \subseteq A$'.

Solution: One proof could be the following:
If $A \cap B = \emptyset$, then $A \cap B \subseteq A$ (since every set contains $\emptyset$).
If $A \cap B \neq \emptyset$, then let x be an arbitrary element of $A \cap B$.
Then $x \in A$ and $x \in B$ (by definition of '$\cap$').
Therefore, $x \in A$.

This is true for every x in $A \cap B$, since x was chosen arbitrarily.
Therefore, $A \cap B \subseteq A$, by definition of '$\subseteq$',

***

Why don't we analyse the argument in Example 1? The truth of each of the premises, and hence of its conclusion, follows from the truth of the earlier premises in it. We start by considering both possibilities.
In the first case, we get $p \Rightarrow q$, where p is '$A \cap B = \emptyset$', and q is '$A \cap B \subseteq A$'.
In the second case, we again assume that the first statement is true. Then, assuming the definition of 'intersection', the second statement is true. The third one is true, whenever the second one is true because of the properties of implication. The fourth statement is true whenever the first three are true, because of the definition and properties of the term 'for all'. And finally, the last statement, which is the statement to be proved, is true whenever all the earlier ones are true. In this way, we have shown that the given statement follows logically from the sequence of previous statements, and hence, is true. In other words, we have proved the given statement.

Sometimes it happens that we feel a certain statement is true, but we don't succeed in proving it. It may also happen that we can't disprove it. Such statements are called **conjectures**. If and when a conjecture is proved, it would be called a theorem. If it is disproved, then its negative will be a theorem!

See www.claymath.org/millennium-problems for a brief overview of some famous conjectures and their background.

In this context, there's a very famous conjecture which was made by a mathematician **Goldbach** in 1742. He stated that:
For every $n \in \mathbb{N}$, if n is even and $n > 2$, then n is the sum of two primes.
To this day, no one has been able to prove it or disprove it, though it appears simple to do. To disprove it several people have been hunting for an example for which the statement is not true. They have been looking for an even number $n > 2$ such that n cannot be written as the sum of two prime numbers.

There is no knowing how long it may take to turn a conjecture into a theorem, or to disprove it! A very important conjecture pertaining to topology, due to the famous French mathematician Henri Poincaré, was proved 100 years after it was made. The proof was by the Russian mathematician Grigori Perelman.

Now, as you have seen, a mathematical proof of a statement consists of one or more premises. These premises could be of four types:

i) a statement that has been proved earlier (e.g., In Unit 5, Block 1 of the course, Calculus, you have seen that to prove that the complex roots of a polynomial in $\mathbb{R}[x]$ occur in pairs, the division algorithm is used, which was proved earlier in the same unit.); or

ii) a statement that follows logically from the earlier statements given in the proof (as you have seen in Example 1); or

iii) a mathematical fact that has never been proved, but is universally accepted as true (e.g., two points determine a line uniquely) – such a fact is called an **axiom** (or **a postulate**); or

iv) the definition of a mathematical term (e.g., assuming the definition of '$\subseteq$' in the proof of $A \cap B \subseteq A$ in Example 1).

You will come across more examples of each type while doing the following exercises, and while studying and creating proofs in this course, and in other courses.

E1) Write down an example of a theorem, and its proof (of at least 4 steps), taken from the first or second semester courses you have studied. At each step, indicate which of the four types of premise it is.

E2) Is every statement a theorem? Why, or why not?

So far we have spoken about valid, or acceptable, proofs. Let us now look at a few erroneous proofs. So, what is NOT a proof? Here are a few examples.

1) One 'proof' submitted by a student for '$A \cap B \subseteq A,$ for any two sets A and B' is:
Let $A = \{1,2,3\}$, $B = \{3,4,5\}$, *then* $A \cap B = \{3\}$. *So* $A \cap B \subseteq A$.

   Let's see why this is not an acceptable proof. We want to prove a general statement, for any two sets A and B. But this student has only proved it for one particular case. So it has not been proved for **every** A and B. Therefore, this is not acceptable as a proof.

2) Another student began a proof for '$f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{3\} : f(x) = \dfrac{3x}{x-2}$ is $1-1$' as follows:
Since $f(x) = \dfrac{3x}{x-2}$ *is* $1-1, \ldots$.
This is not right because the student is assuming what is to be proved, instead of arriving at it by starting from the definition of f and the definition of a $1-1$ function.

There are several other errors that we must avoid to ensure that the proof we give is correct. You will realise some of them as you go through this unit and this course.

Why don't you check a proof for validity now?

E3) Check whether or not the following proof is acceptable.
Statement: If T is a set containing an infinite set $S,$ then $S \neq T.$
Proof: S is an infinite set.
T is a set containing $S.$
Therefore, $\forall\, t \in T, \exists\, s \in S$ s.t. $s \neq t.$
Hence, $S \neq T.$
[Also see E11(i).]

E4) Prove the statement '$f : \mathbb{R} \to \mathbb{R} : f(x) = 3x - 5$ is onto'.

You have seen that a proof of a statement is a logical argument that verifies the truth of that statement. There are several ways of proving a theorem, as you will see in the next two sections.

2.3 DIFFERENT METHODS OF PROOF

In this section we shall consider three different broad strategies for proving a statement. We will also discuss a method that is used only for disproving a statement.

2.3.1 Direct Method

This form of proof is based entirely on the argument that starts with a true premise and arrives at the required conclusion. Let us formally spell out the strategy.

Definition: A **direct proof** of $p \Rightarrow q$ is a logically valid argument that begins with the assumption that p is true. Then, in one or more steps of the form $p \Rightarrow q_1, q_1 \Rightarrow q_2, \ldots, q_n \Rightarrow q,$ we conclude that q is true.

Consider the following examples.

Example 2: Give a direct proof of the statement 'The product of two odd integers is odd'.

Solution: Let us clearly analyse what our hypotheses are, and what we have to prove.
We start by considering any two odd integers x and y. So our hypothesis is
$p : x$ and y are odd integers.
The conclusion we want to reach is
$q : xy$ is odd.
Here is the argument:
$x = 2m+1$ for some integer m (by definition of an odd number).
Similarly, $y = 2n+1$ for some integer n.
Therefore, $xy = (2m+1)\ (2n+1) = 2(2mn+m+n)+1$.
Therefore, xy is odd.
So we have shown that $p \Rightarrow q$, using three premises.

***

Example 3: Give a direct proof of the theorem '$|x+y| \le |x|+|y| \ \forall\, x, y \in \mathbb{R}$'.

Solution: First of all, we note that there are three possibilities for x and y :

i) either one of them is zero,

ii) both are positive or both are negative,

iii) one is positive and one is negative.

Let us prove the statement for each of these cases.

Case 1: Suppose $x = 0$. Then $|x| = 0$ also.
$\therefore\ |x+y| = |x|+|y|$
$\therefore$ The given statement is true.
In the same way, the statement is true if $y = 0$.

Case 2: Suppose $x > 0,\ y > 0$. Then $|x| = x, |y| = y$ and $|x+y| = x+y$.
Hence $|x+y| = |x|+|y|$.
Similarly, if $x < 0,\ y < 0$, then $|x| = -x$, $|y| = -y$ and $|x+y| = -(x+y)$.
$\therefore\ |x+y| = |x|+|y|$.
$\therefore$ The given statement is true.

Case 3: Suppose $x < 0,\ y > 0$. Then $|x| = -x, |y| = y$.
Now, if $x+y \ge 0$, then
$|x+y| = x+y$

$< -x + y$, since $x < 0$
$= |x| + |y|$.
So $|x + y| < |x| + |y|$.

Next, suppose $x + y < 0$. Then
$|x + y| = -(x + y) = (-x) + (-y) < -x + y = |x| + |y|$, since $y > 0$.
So $|x + y| < |x| + |y|$.
$\therefore |x + y| \le |x| + |y|$.
Similarly, if $x > 0,\ y < 0,\ \ |x + y| \le |x| + |y|$, i.e., in this case too, the given statement is proved to be true.

***

In the example above, note that we have proved the statement for every x and y since we have treated x and y as arbitrary real numbers, and not having a particular value.

Why don't you try an exercise now?

E5) Give a direct proof of the following statements:

i) For any two sets A and B, $(A \cap B)^c = A^c \cup B^c$.

ii) $f : \mathbb{R} \to \mathbb{R} : f(x) = -\dfrac{x}{2}$ is monotonic on $]0,3[$ (ref. Unit 6, Calculus).

Let us now consider two proof strategies that are different from the approach you have just studied.

2.3.2 Indirect Methods

In this sub-section we shall consider two roundabout methods for proving $p \Rightarrow q$.

PROOF BY CONTRAPOSITIVE:

Recall that the contrapositive is **not** the converse.

In this method, we use the fact that the statement $p \Rightarrow q$ is logically equivalent to its **contrapositive** $(\sim q \Rightarrow \sim p)$.
Because of this equivalence, to prove $p \Rightarrow q$, we can, instead, prove $\sim q \Rightarrow \sim p$. This means that we can assume that $\sim q$ is true, and then try to prove that $\sim p$ is true. In other words, **what we do to prove $p \Rightarrow q$ in this method is to assume that $\sim q$ is true, that is, q is false, and then show that p is false. Then $\sim p$ will be true.**

So, you see how roundabout a way this is to prove $p \Rightarrow q$. This is why it is called an indirect method of proof. Let us consider an example.

Example 4: Prove that 'If $x, y \in \mathbb{Z}$ such that x and y are odd, then xy is odd', by proving its contrapositive.

Solution: Let us name the statements involved as below.
p: both x and y are odd integers,
$q : xy$ is an odd integer.
So, $\sim q : xy$ is even, and
$\sim p : x$ is even, or y is even, or both are even, for $x, y \in \mathbb{Z}$.

We want to prove $p \Rightarrow q$, by proving that $\sim q \Rightarrow \sim p$.
So we start by assuming that $\sim q$ is true, i.e., we suppose that xy is even.
Then $xy = 2n$ for some $n \in \mathbb{N}$.
Therefore, $2|xy$.
Therefore, $2|x$ or $2|y$ (by definition of a prime number).
Therefore, x is even, or y is even, or both are even.
That is, $\sim p$ is true.
So, we have shown that $\sim q \Rightarrow \sim p$. Therefore, $p \Rightarrow q$.

***

This example leads to the point made in the following remark.

Remark 1: What you have proved in Example 4 by an indirect method, has been proved by the direct method in Example 2. Thus, sometimes a statement can be proved in several ways. It is for you to decide which is the best way to use in a given situation.

Consider another example.

Example 5: Prove, by the method of contrapositive, that if A, B, C are three non-empty sets such that $A \times C \subseteq B \times C$, then $A \subseteq B$.

Solution: The contrapositive of the given statement is 'if A and B are non-empty sets such that $A \not\subset B$, then for any non-empty set $C, A \times C \not\subset B \times C$'.
To prove this, we assume that $A \not\subset B$. So there is an $a \in A$ such that $a \notin B$. Now, take an arbitary non-empty set C. Then $(a,c) \in A \times C$, where $c \in C$, but $(a,c) \notin B \times C$. So, by definition, $A \times C \not\subset B \times C$. Thus, the contrapositive is proved. Hence the given statement is proved.

***

Why don't you try some related exercises now?

E6) Write down the contrapositive of the statement, 'If f is a $1-1$ function from a finite set X into itself, then f must be surjective.' Also, prove the contrapositive, and hence prove the given statement.

E7) Prove the statement 'If A, B are non-empty sets such that $A \subseteq B$, then for any non-empty set $C, A \times C \subseteq B \times C$', by proving its contrapositive.

And now let us consider another way of proving a statement indirectly.

PROOF BY CONTRADICTION:

In this method, to prove that a statement q is true, we start by assuming that q is false (i.e., $\sim q$ is true), as in the previous method. However, here we now use a logical argument to arrive at a situation where some statement is true as well as false.
For example, to prove 'x^2 is even whenever x is an even integer', we start by assuming that x^2 is **not** even for some **even** integer x. Since x^2 is not even, $2 \nmid x^2$. Hence $2 \nmid x$. But we started with assuming x is even. So, we reach a contradiction, namely, 'x is an even integer' and 'x is not an even integer'.

This can only happen if our original assumption is wrong. That is, 'x^2 is not even' is false. Hence, 'x^2 is even' is true.

So in this method, we reach a **contradiction** $r \wedge \sim r$ for some statement r. This means that the truth of $\sim q$ (that we started with) logically leads us to a contradiction, a situation that cannot be. This can only happen when our assumption is wrong, that is, $\sim q$ cannot be true, that is, $\sim q$ is false. Therefore, q must be true.

This method is called **proof by contradiction**. It is also called **reductio ad absurdum** (a Latin phrase) because it relies on reducing a given assumption to an absurdity. It is said to have been discovered by ancient Greek mathematicians. Let us consider some more examples of how this method is applied.

Example 6: Prove that $\sqrt{5}$ is irrational.

Solution: Here $q: \sqrt{5}$ is irrational.

To prove the given statement by contradiction, assume $\sim q$ is true, that is, $\sqrt{5}$ is rational. By definition of a rational number, there exist positive integers a and b such that $\sqrt{5} = \frac{a}{b}$, where a and b have no common factors.

$\therefore a = \sqrt{5}b$

$\therefore a^2 = 5b^2$

$\therefore 5 \mid a^2$

$\therefore 5 \mid a$, since 5 is a prime number.

$\therefore a = 5c$ for some $c \in \mathbb{Z}$.

$\therefore a^2 = 25c^2$

$\therefore 25c^2 = 5b^2$, since $a^2 = 5b^2$.

$\therefore 5c^2 = b^2$

$\therefore 5 \mid b^2$

$\therefore 5 \mid b$, since 5 is a prime number.

Hence, 5 divides both a and b, which **contradicts** our earlier assumption that a and b have no common factor.

Therefore, our assumption that $\sqrt{5}$ can be written as $\frac{a}{b}$, where a and b have no common factors is false, i.e., $\sqrt{5}$ is irrational.

***

Example 7: Prove that the greatest integer function, $f: \mathbb{R} \to \mathbb{Z}: f(x) = [x]$, is not continuous at any integer.

Solution: Let us assume that f is continuous at some integer k. Then, $\lim\limits_{x \to k} f(x) = k$, that is, $\lim\limits_{x \to k} [k] = k$. So, for $\varepsilon = 1$, $\exists\, \delta > 0$ s.t. for $|x - k| < \delta$, $|f(x) - f(k)| < 1$, that is, $|f(x) - k| < 1$.

Now, choose $\delta_1 = \min(\delta, 1)$ and $x_0 \in\,]k - \delta_1, k[$.

Then $f(x_0) = k - 1$. So, $|x_0 - k| < \delta_1$ (where $\delta_1 \le \delta$) and $|f(x_0) - k| = 1$, which contradicts the premise that $|f(x) - f(k)| < 1$ whenever $|x - k| < \delta$.

Hence, our assumption must be wrong.

Thus, f is not continuous at k.

Since k is an arbitrarily chosen integer, f is not continuous at any integer.

***

We can also use the method of contradiction to prove an implication $r \Rightarrow s$. Here we can use the fact that $\sim (r \Rightarrow s)$ is logically equivalent to $r \wedge \sim s$. So, to prove $r \Rightarrow s$, we can begin by assuming that $r \Rightarrow s$ is false, i.e., r is true and s is false. Then we can present a valid argument to arrive at a contradiction. Consider the following example from plane geometry.

Example 8: Prove that if two non-parallel lines L_1 and L_2 intersect, then their intersection consists of exactly one point.

Solution: To prove the given implication by contradiction, let us begin by assuming $\sim (r \Rightarrow s)$ is true, that is, $r \wedge \sim s$ is true, where

r: L_1 and L_2 are two non-parallel lines, and

s: L_1 and L_2 intersect in one point only.

So, we assume $\sim s$ is true, that is, the two non-parallel lines L_1 and L_2 intersect in more than one point. Let us call two of these distinct points A and B. Then, both L_1 and L_2 contain A and B. This contradicts the axiom from geometry that says 'Given two distinct points, there is exactly one line containing them.'

Therefore, our assumption is wrong, that is, if L_1 and L_2 intersect, then they must intersect in only one point.

***

The method of proof by contradiction is also used for solving many logical puzzles, by discarding all solutions that lead to contradictions. Consider the following example.

Example 9: There is a village that consists of two types of people – those who always tell the truth, and those who always lie. Suppose that you visit the village and two villagers A and B come up to you. Further, suppose

A says, "B always tells the truth." …(1)

And B says, "A and I are of opposite types." …(2)

What types of people are A and B?

Solution: Let us start by assuming A is a truth-teller, that is, what A says is true.

This implies that B is a truth-teller, using (1).

So, what B says is true.

This implies that A and B are of opposite types, using (2).

So we reach a contradiction, because our premises say that A and B are both truth-tellers.

$\therefore$ The assumption we started with is false.

$\therefore$ A always tells lies.

$\therefore$ What A has said is a lie, that is, 'B always tells the truth' is a false statement. That is, B lies sometimes. But, if a person in the village lies, then she always lies.

$\therefore$ B always tells lies.

$\therefore$ A and B are of the same type, i.e., both of them always lie.

***

Here are a few exercises for you now. While doing them you would realise that there are situations in which all the three methods of proof we have discussed so far can be used.

E8) Use the method of proof by contradiction to show that $\sqrt{17}$ is irrational.

E9) If you apply the 'proof by contradiction' method to prove the following statements, what is the assumption you would start with?

i) If f and g are two real valued functions over $\mathbb{R}$, such that $g(x) \geq f(x) \forall x \in \mathbb{R}$ and f is not bounded above, then g is not bounded above.

ii) For $n \in \mathbb{N}, n > 2$, there do not exist positive integers x, y, z such that $x^n + y^n = z^n$.

E10) Prove, by contradiction, that $\lim_{x \to 2} (x^2 - 1) \neq -1$.

Let us now consider a way of showing that a statement is false.

2.3.3 Counterexamples

Suppose I make the statement 'All human beings are 5 feet tall'. You are quite likely to show me an example of a human being standing nearby for whom the statement is not true. Similarly, to prove that $(\forall x)\ p(x)$ is false, we need to prove $\sim [(\forall x) p(x)]$ is true, that is, $(\exists x)\ (\sim p(x))$ is true (see Sec.1.4). Thus, we need only one x that satisfies $\sim p(x)$. This x is an example of what we now define.

Definition: An example that shows that a statement is false is a **counterexample** to the given statement. (The name itself suggests that it is an example to counter a given statement.)

A common situation in which we look for counterexamples is to disprove statements of the form $p \Rightarrow q$. For instance, to disprove the statement 'If n is an odd integer, then n is prime', we need to look for an odd integer which is not a prime number. For example, 15 is one such integer. So, $n = 15$ is a counterexample to the given statement.

Notice that **a counterexample to a statement p proves that** p is false, i.e., **$\sim p$ is true**.

Let us consider another example.

Example 10: Disprove the statement, 'For $a, b \in \mathbb{R}, a^2 = b^2$ implies $a = b$'.

Solution: In symbols, the given statement is
$(\forall\ a \in \mathbb{R})\ (\forall\ b \in \mathbb{R})\ [(a^2 = b^2) \Rightarrow (a = b)]$.
To disprove this statement, we need to prove its negation, namely,
$(\exists\ a \in \mathbb{R})\ (\exists\ b \in \mathbb{R})\ (a^2 = b^2$ with $a \neq b)$. So, we need to look for a counterexample, that is, a pair of real numbers a and b for which $a^2 = b^2$ but $a \neq b$. Can you think of such a pair? What about $a = 1$ and $b = -1$? They serve the purpose.
In fact, there are infinitely many counterexamples for the given statement. Think of at least five others.

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Now, an exercise!

E11) Disprove the following statements by providing a suitable counterexample each.

i) If T is a set containing an infinite set S, then $S \neq T$. (Also see E3.)

ii) $(x+y)^n = x^n + y^n \ \forall\ n \in \mathbb{N},\ x,\ y \in \mathbb{Z}.$

iii) $f : \mathbb{N} \to \mathbb{N}$ is $1-1$ iff f is onto.

iv) If $f : \mathbb{R} \to \mathbb{R} : \int f(x)\,dx$ exists, then f is continuous on $\mathbb{R}$.

To disprove $p \Leftrightarrow q$ it is enough to prove that $p \Rightarrow q$ is false or $q \Rightarrow p$ is false.

There are some other strategies of proof, like a **constructive** proof, which you will come across later in this course and in other mathematics courses. We shall not discuss this method here. However, we will now discuss a very important technique of proof for sentences that are of the form $p(n), n \in \mathbb{N}$.

2.4 PRINCIPLE OF MATHEMATICAL INDUCTION

In a discussion with some students the other day, one of them told me that girls are better than boys at studies. I asked him how he had reached such a conclusion. As an argument he gave me instances of several girls who had topped their class in their exams. What he had done was to formulate his general opinion of girls on the basis of several particular instances. This is an example of **inductive logic**, a process of reasoning by which general rules are discovered by the observation of several individual cases. Inductive reasoning is used in all the social sciences and sciences, including mathematics. But in mathematics we use a more precise form.

Precision is required in mathematical induction because, as you know, a statement of the form $(\forall\ n \in S)\, p(n)$, where $S \subseteq \mathbb{N}$, is true only if it can be shown to be true for each n in S. (In the example above, even if the student is given an example of one girl who is not good at studies, he is not likely to change his general opinion.)

Here $p(n)$ is not a statement because we don't know if it is true or false unless we know the value of the variable n. We define such a sentence now.

Definition: A **predicate** is a sentence of the form $P(x)$, depending on a variable x, such that $P(x)$ may be true for some values of x, and false for some values of x. For a given value of x, $P(x)$ becomes a statement.

For example, if $P(n):\ n! < 2^n,\ n \in \mathbb{N}$, then $P(1),\ P(2), P(3)$ are true, but $P(4)$ is not true. So $P(n)$ is a predicate, but $P(1),\ P(2), \ldots,\ P(100), \ldots$ are all statements.

So, let us come back to seeing how we can make sure that the predicate $p(n)$ is true for each n that we are interested in. To answer this, let us consider an example.

Suppose we want to prove that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \text{ for each } n \in \mathbb{N}. \qquad \ldots(3)$$

Let $p(n)$ denote the predicate '$1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$'. Now, we can verify that it is true for a few values, say, $n = 1,\ n = 5,\ n = 10,\ n = 100,$ and so on. But we still can't be sure that it will be true for some value of n that we haven't checked for.

However, suppose we can show that whenever $p(n)$ is true for some $n,\ n = k$ say, then it will be true for $n = k + 1.$ Then we are in a very good position because we already know that $p(1)$ is true. And, since $p(1)$ is true, so is $p(1+1),$ i.e., $p(2)$, and so on. In this way we can show that $p(n)$ is true for every $n \in \mathbb{N}.$ So, our proof boils down to two broad stages, namely,

i) Checking that the statement $p(1)$ is true;

ii) Proving that whenever the statement $p(k)$ is true, then the statement $p(k+1)$ is true, where $k \in \mathbb{N}.$

This is the principle that we will now state formally, in a more general form.

Principle of Mathematical Induction (PMI): Let $p(n)$ be a predicate involving a natural number $n.$ Suppose the following two conditions hold:

i) $p(m)$ is true for some $m \in \mathbb{N}$;

ii) If $p(k)$ is true, then $p(k+1)$ is true, where $k(\geq m)$ is an arbitrary natural number.

Then $p(n)$ is true for every $n \geq m.$ ■

Looking at the two conditions in the principle, can you make out why it works? (As a hint, put $m = 1$ in our example above, of the sum of the first n natural numbers.)

Well, (i) tells us that the statement $p(m)$ is true. Then putting $k = m$ in (ii), we find that $p(m+1)$ is true. Again, since $p(m+1)$ is true, $p(m+2)$ is true, and so on.

Going back to the (3) above, let us complete the second step.

We assume that $p(k)$ is true, i.e., $1 + 2 + \cdots + k = \dfrac{k(k+1)}{2}.$

We want to check if $p(k+1)$ is true. So let us find

$$1 + 2 + \cdots + (k+1) = (1 + 2 + \cdots + k) + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1), \text{ since } p(k) \text{ is true}$$

$$= \frac{(k+1)\ (k+2)}{2}$$

So, $p(k+1)$ is true.

Hence, by applying the principle of mathematical induction, we find that $p(n)$ is true for every $n \in \mathbb{N}.$

What does the PMI really say? It says that if you can walk a few steps, say m steps, and if at each stage you can walk one more step, then you can walk any distance. It sounds very simple, but you may be surprised to know that the technique in this principle was first used only as late as the 16th century by the Italian mathematician and astronomer, F. Maurolycus (1494-1575). He used it to show that $1 + 3 + \cdots + (2n-1) = n^2\ \forall\, n \in \mathbb{N}.$ The mathematician, Pierre de Fermat (1601-1665), improved on the technique and proved that this principle is equivalent to the following often used principle of mathematics.

The Well-ordering Principle (WOP): Any non-empty subset of $\mathbb{N}$ contains a smallest (or least) element. ■

The modern form of the PMI was first used in the 19th century by the mathematicians De Morgan, Peano, Boole and Dedekind.

You may be able to see the relationship between the PMI and the WOP if we reword the PMI in the following form.

Principle of Mathematical Induction (Equivalent form)**:** Let $S \subseteq \mathbb{N}$ be such that

i) $m \in S$,

ii) for each $k \in \mathbb{N},\ k \geq m,$ whenever $k \in S,$ then $k+1 \in S.$

Then $S = \{m,\ m+1,\ m+2,\ \ldots\}.$ ■

Can you see the equivalence of the two forms of the PMI? If you take
$S = \{n \in \mathbb{N} \mid p(n)$ is true$\}$,
then you can see that the way we have written the principle above is a mere rewrite of the earlier form.

Now, let us consider an example of a proof using PMI.

Example 11: Use mathematical induction to prove that

'$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n+1)\ (2n+1), \forall\ n \in \mathbb{N}.$'

Solution: We will denote the predicate
$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n+1)\ (2n+1)$ by $p(n)$.

Since we want to prove $p(n)$ is true for every $n \in \mathbb{N},$ i.e., $\forall\ n \geq 1,$ we take $m = 1$ in PMI.

Step 1: $p(1)$ is $1^2 = \frac{1}{6}(1+1)\ (2+1),$ which is true.

Step 2: Suppose, for an arbitrary $k \in \mathbb{N},\ p(k)$ is true, i.e.,

$1^2 + 2^2 + \cdots + k^2 = \frac{k}{6}(k+1)\ (2k+1)$ is true.

Step 3: We need to check if the assumption in Step 2 implies that $p(k+1)$ is true.

$p(k+1)$ is $1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k+1}{6}(k+2)\ (2k+3)$

$\Leftrightarrow (1^2 + 2^2 + \cdots + k^2) + (k+1)^2 = \frac{k+1}{6}(k+2)\ (2k+3)$

Note the use of $\Leftrightarrow$ in the proof.

$\Leftrightarrow \frac{k}{6}(k+1)\ (2k+1) + (k+1)^2 = \frac{k+1}{6}(k+2)\ (2k+3),$ since $p(k)$ is true.

$\Leftrightarrow \frac{k+1}{6}[k(2k+1) + 6(k+1)] = \frac{k+1}{6}(k+2)\ (2k+3)$

$\Leftrightarrow 2k^2 + 7k + 6 = (k+2)\ (2k+3),$ dividing throughout by $\frac{k+1}{6}$.

This is a true statement.
So, $p(k)$ is true implies that $p(k+1)$ is true.
So, both the conditions of the principle of mathematical induction hold.

Step 4: Therefore, its conclusion must hold, i.e., $p(n)$ is true for every $n \in \mathbb{N}$.

***

Have you gone through Example 11 carefully? If so, you would have noticed that the proof consists of four steps:

Step 1 (**The basis of induction**)**:** Checking if $p(m)$ is true for some $m \in \mathbb{N}$.

Step 2 (**The induction hypothesis**)**:** Assuming that $p(k)$ is true for an arbitrary $k \in \mathbb{N},\ k \geq m$.

Note that deductive logic is used for proving Step 3.

Step 3 (**The induction step**)**:** Showing that $p(k+1)$ is true, by a direct, or an indirect, proof.

Step 4 (Conclusion): Hence concluding that $p(n)$ is true $\forall\ n \geq m$.

Now consider an example related to a formula you have applied several times in the course, Calculus.

You studied this in Block 3 of BMTC-131.

Example 12: Prove the **Leibniz formula**: Let u and v be functions from $\mathbb{R}$ to $\mathbb{R}$, having derivatives up to the *n*th order. Then, for every $n \in \mathbb{N}$,

$$(uv)^{(n)} = \sum_{i=0}^{n} {}^nC_i u^{(n-i)} v^{(i)}. \qquad \ldots(4)$$

Solution: Let $P(n)$ be the given predicate, (4).

Step 1: You know that $(uv)' = uv' + u'v$.
So $P(1)$ is true.

Step 2: Now assume that $P(k)$ is true, for some $k \in \mathbb{N}$.

Step 3: Now, we want to prove that $P(k+1)$ is true.
So, $(uv)^{(k+1)}$

$$= [(uv)^{(k)}]' = \left[\sum_{i=0}^{k} {}^kC_i u^{(k-i)} v^{(i)}\right]', \text{ where } u^{(0)} = u,\ v^{(0)} = v.$$

$$= \sum_{i=0}^{k} {}^kC_i u^{(k+1-i)} v^{(i)} + \sum_{i=0}^{k} {}^kC_i u^{(k-i)} v^{(i+1)}$$

$$= {}^kC_0 u^{(k+1)} v + {}^kC_0 u^{(k)} v^{(1)} + {}^kC_1 u^{(k)} v^{(1)} + \cdots$$
$$+ {}^kC_{m-1} u^{(k+1-m)} v^{(m)} + {}^kC_m u^{(k+1-m)} v^{(m)} + \cdots + {}^kC_k\, u\, v^{(k+1)}$$

$$= \sum_{i=1}^{k} \left({}^kC_{i-1} + {}^kC_i\right) u^{(k+1-i)} v^{(i)} + {}^kC_0\, u^{(k+1)}\, v + {}^kC_k\, u\, v^{(k+1)}$$

$$= \sum_{i=1}^{k} \left({}^kC_{i-1} + {}^kC_i\right) u^{(k+1-i)} v^{(i)} + {}^{k+1}C_0\, u^{(k+1)}\, v + {}^{k+1}C_{k+1}\, u\, v,$$

(since ${}^kC_0 = {}^{k+1}C_0$ and ${}^kC_k = {}^{k+1}C_{k+1}$)

$$= \sum_{i=0}^{k+1} {}^{k+1}C_i u^{(k+1-i)} v^{(i)}, \text{ since } {}^kC_{i-1} + {}^kC_i = {}^{k+1}C_i.$$

So $P(k+1)$ is true.

Step 4: Hence, we conclude that $P(n)$ is true $\forall\ n \in \mathbb{N}$.

***

Now consider an example in which $m \neq 1$.

Example 13: Show that $2^n > n^3$ for $n \geq 10$.

Solution: We write $p(n)$ for the predicate '$2^n > n^3$'.

Step 1: As we need to prove the result for $n \geq 10$, the basis of induction is $p(10)$.

For $n = 10,\ 2^{10} = 1024,$ which is greater than 10^3.

Therefore, $p(10)$ is true.

Step 2: We assume that $p(k)$ is true for an arbitrary $k \geq 10,$ i.e., $2^k > k^3$.

Step 3: Now, we want to prove $p(k+1)$ is true, that is, $2^{k+1} > (k+1)^3$. Note that $2^{k+1} = 2.2^k > 2.k^3,$ by our assumption in Step 2.

$$> \left(1 + \frac{1}{10}\right)^3 . k^3, \text{ since } 2 > \left(1 + \frac{1}{10}\right)^3.$$

$$\geq \left(1 + \frac{1}{k}\right)^3 . k^3, \text{ since } k \geq 10.$$

$$= (k+1)^3.$$

Thus, $p(k+1)$ is true.

Step 4: Therefore, we conclude that $p(n)$ is true $\forall\ n \geq 10$.

***

Why don't you try to apply the principle yourself now?

E12) Use mathematical induction to prove that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta\ \forall\ n \in \mathbb{Z}$ and $\forall\ \theta \in \mathbb{R}$.

[**Hint:** Use PMI to prove it for $n \in \mathbb{N}$ first.]

E13) Show that for any integer $n > 1,\ \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$

[**Hint:** Note that the basis of induction is $p(2)$.].

E14) Prove that $\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} . \frac{\pi}{2}\ \forall\ n \in \mathbb{N}$, a Wallis sine formula you have studied in Block 5 of the Calculus course.

Before going further, **a note of warning!** To prove that $p(n)$ is true $\forall\ n \geq m$, **both** – the basis of induction, **as well as** the induction step, must hold. If even one of these conditions does not hold, we cannot arrive at the conclusion that $p(n)$ is true $\forall\ n \geq m$.

For example, suppose $p(n)$ is $(x+y)^n \leq x^n + y^n\ \forall\, x,\ y \in \mathbb{R}$. Then $p(1)$ is true. But the inductive step, Step 3 does not hold. In fact, $p(n)$ is not true for every $n \in \mathbb{N}$. (Can you find a value of n for which $p(n)$ is false?)

Again, '$2^n > n^3\ \forall\ n \geq 2$' cannot be proved by PMI, since the basic of induction, $p(2)$, is not true, even though the induction step holds, as you have seen in

Example 13. In fact, the given statement is false. It is true for $n \geq 10$ but not for $n \geq 2$.

Now let us look at a situation in which we may expect the principle of induction to work, but it doesn't. Consider the sequence of numbers $1, 1, 2, 3, 5, 8, \ldots$. These are the **Fibonacci numbers**, named after the Italian mathematician of the medieval period, Fibonacci. Each term in the sequence, from the third term on, is obtained by adding the previous 2 terms. So, if a_n is the *n*th term, then $a_1 = 1,\ a_2 = 1$, and $a_n = a_{n-1} + a_{n-2} \forall\ n \geq 3.$

You will study about the Fibonacci sequence in Block 2.

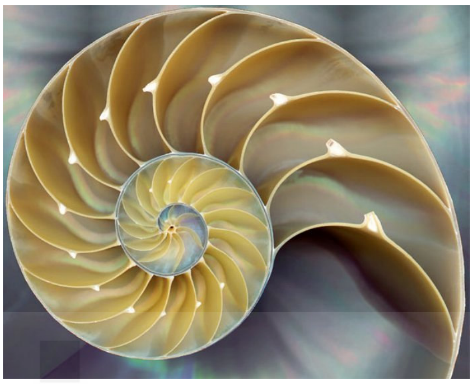

Fig. 1: The Fibonacci sequence shows up in nature in many ways including the way a nautilus is constructed.

Suppose we want to show that $a_n < 2^n\ \forall\ n \in \mathbb{N}$ using the PMI. Then, if $p(n)$ is the predicate, $a_n < 2^n$, we know that $p(1)$ is true.

Next, suppose we know that $p(k)$ is true for an arbitrary $k \in \mathbb{N}$, i.e., $a_k < 2^k$. We want to show that $a_{k+1} < 2^{k+1}$, i.e., $a_k + a_{k-1} < 2^{k+1}$. We know something about a_k, but we don't know anything about a_{k-1}. So, how can we apply the principle of induction in the form that we have stated it? In such a situation, a stronger, more powerful, version of the principle of induction comes in handy. Let's see what this is.

Principle of Mathematical Induction (Strong Form): Let $p(n)$ be a predicate that involves a natural number n. Suppose the following two conditions hold:

i) $p(m)$ is true for some $m \in \mathbb{N}$, and

ii) whenever $p(m),\ p(m+1), \ldots,\ p(k)$ are true, then $p(k+1)$ is true, where $k \geq m$ is an arbitrary natural number.

Then $p(n)$ is true for all natural numbers $n \geq m$. ■

Why do we call this principle stronger than the earlier one? This is because, in the induction step we are making more assumptions, i.e., that $\boldsymbol{p(n)}$ **is true for every $\boldsymbol{n}$ lying between $\boldsymbol{m}$ and $\boldsymbol{k}$**, not just that $p(k)$ is true.

Let us now go back to the Fibonacci sequence.

In using the strong form of PMI, we often need to check Step 1 for more than one value of *n*.

Step 1: To use the strong form of the PMI, we take $m = 1$. We have seen that $p(1)$ is true. We also need to see if $p(2)$ is true because we have to use the relation $a_n = a_{n-1} + a_{n-2}$, for $n \geq 3$. We find that both $p(1)$ and $p(2)$ are true.

Step 2: For an arbitrary $k \geq 2$, we assume that $p(n)$ is true for every n such that $1 \leq n \leq k$, i.e., $a_n < 2^n$ for $1 \leq n \leq k$.

Step 3: We must show that $p(k+1)$ is true, i.e., $a_{k+1} < 2^{k+1}$. Now

$a_{k+1} = a_k + a_{k-1}$
$< 2^k + 2^{k-1}$, by our assumption in Step 2.
$= 2^{k-1}(2+1)$
$< 2^{k-1}.2^2$
$= 2^{k+1}$
$\therefore p(k+1)$ is true.

Step 4: Hence $p(n)$ is true $\forall\ n \in \mathbb{N}$.

Though the "strong" form of the PMI appears to be different from the "weak" form, **the two are actually equivalent**. This is because each can be obtained from the other, which we shall not prove here. However, this means that we can use either form of mathematical induction. In a given problem we use the form that is more suitable. For instance, in the following example, as in the case of the Fibonacci sequence, you would agree that it is better to use the strong form of the PMI.

Example 14: Use the principle of mathematical induction to prove that any integer $n \geq 2$ is either a prime or a product of primes.

Solution: Here $p(n)$ is the predicate 'n is a prime or n is a product of primes'.

Step 1 (Basis of induction): Since 2 is prime, $p(2)$ is true.

Step 2 (Induction hypothesis): Assume that $p(n)$ is true for any integer n such that $2 \leq n \leq k$, i.e., $p(3)$, $p(4)$, …, $p(k)$ are true.

Step 3 (Induction step): Now consider $p(k+1)$.
If $k+1$ is a prime, then $p(k+1)$ is true.
If $k+1$ is not a prime, then $k+1 = rs$, where $2 \leq r < k$ and $2 \leq s < k$.
So, by the induction hypothesis, $p(r)$ is true and $p(s)$ is true.
Therefore, r and s are either primes or products of primes. Therefore, $k+1$ is a product of primes. So, $p(k+1)$ is true.

Step 4 (Conclusion): Therefore, $p(n)$ is true $\forall\ n \geq 2$.

***

Why don't you try some exercises now?

E15) If $a_1, a_2, \ldots$ are the terms in the Fibonacci sequence, use the principle of mathematical induction to show that $2 \mid a_{3n}$ for $n \geq 1$. Which form did you find more convenient, and why?

E16) A sequence of positive integers $\{a_1, a_2, \ldots, a_n, \ldots\}$ is defined by $a_1 = 1, a_2 = 4$ and $a_n = 2a_{n-1} - a_{n-2} + 2$ for $n \geq 3$.

Conjecture a formula for a_n, and prove it by the principle of mathematical induction. Which form of PMI would you use, and why?

With this we come to the end of our discussion on various techniques of proving or disproving mathematical statements. Let us take a brief look at what you have studied in this unit.

2.5 SUMMARY

In this unit you have studied the following points:

1. What constitutes a proof, and a disproof, of a mathematical statement.
2. The description, and examples, of a direct method of proof.
3. Two types of indirect methods of proof: proof by contrapositive and proof by contradiction.
4. The use of counterexamples for disproving a statement.
5. Statements, and the application of, the "weak" and "strong" forms of the principle of mathematical induction.

2.6 SOLUTIONS / ANSWERS

E1) There are several such examples. We give the following one.
Theorem: If A, B, C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
Proof: The proof comprises showing that
$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$, and
$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Step 1: $B \subseteq B \cup C$, $C \subseteq B \cup C$, by definition.

Step 2: $A \cap B \subseteq A \cap (B \cup C)$, and
$A \cap C \subseteq A \cap (B \cup C)$ (This follows from Step 1.)

Step 3: $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, by definition and Step 2.

Step 4: For an arbitrary element $x \in A \cap (B \cup C)$, $x \in A$ and $x \in B \cup C$, by definition.

Step 5: $x \in A$ and ($x \in B$ or $x \in C$) by Step 4.

Step 6: $x \in (A \cap B)$ or $x \in (A \cap C)$, from Step 5.

Step 7: $x \in (A \cap B) \cup (A \cap C)$, from Step 6.

Step 8: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$, from Step 4 and Step 7.

Step 9: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, by definition, Step 3 and Step 8.

E2) No. Only a statement that is proved to be true is a theorem.

E3) This is not a valid proof.
The third statement does not follow from the first two statements, or from any definition, or from any relevant axiom.

E4) Let y be an arbitrary real number.
f is surjective if $\exists\, r \in \mathbb{R}$ s.t. $f(r) = y$.

Now, $f(r) = y \Rightarrow 3r - 5 = y \Rightarrow r = \frac{y+5}{3}$.

Therefore, for any $y \in \mathbb{R}, \exists\, r = \frac{y+5}{3} \in \mathbb{R}$ s.t. $f(r) = y$.

Hence f is surjective.

E5) i) We will prove this in two stages, namely,
$(A \cap B)^c \subseteq A^c \cup B^c$ and $A^c \cup B^c \subseteq (A \cap B)^c$.
Using logical equivalence of statements at each step, we shall prove both stage simultaneously.

$x \in (A \cap B)^c$
$\Leftrightarrow x \notin A \cap B$
$\Leftrightarrow x \notin A$ or $x \notin B$
$\Leftrightarrow x \in A^c$ or $x \in B^c$
$\Leftrightarrow x \in A^c \cup B^c$.

Since x is an arbitrary element, $(A \cap B)^c = A^c \cup B^c$.

ii) For any $x, y \in\,]0,3[$,

$$x < y \Rightarrow \frac{-x}{2} > \frac{-y}{2} \Rightarrow f(x) > f(y).$$

Hence f is monotonic on $]0,3[$.

E6) The contrapositive is
'If X is a finite set and $f : X \to X$ is not surjective, then f cannot be injective'.
We shall prove this statement now.
Let $X = \{x_1, x_2, \ldots, x_n\}$.
Then $f(X) = \{f(x_1), f(x_2), \ldots, f(x_n)\}$.
Since f is not surjective, $|X| > |f(X)|$
Therefore, $f(x_i) = f(x_j)$ for some $i \neq j$.
Therefore, f is not injective.

E7) Its contrapositive is 'If A, B, C are sets such that $A \times C \not\subset B \times C$, then $A \not\subset B$'.
Proof: $A \times C \not\subset B \times C$
$\Rightarrow \exists\, (x, y) \in A \times C$ s.t. $(x, y) \notin B \times C$
$\Rightarrow \exists\, x \in A$ s.t. $x \notin B$
$\Rightarrow A \not\subset B$.

E8) Suppose $\sqrt{17}$ is rational.

Then $\sqrt{17} = \frac{a}{b}$, g.c.d $(a, b) = 1, a, b \in \mathbb{Z}$.

Then, as in Example 6, you can arrive at a contradiction to g.c.d $(a,b)=1,$ and hence conclude that $\sqrt{17}$ is irrational.

E9) i) Assume that g is bounded above.

ii) Given an $n\in \mathbb{N}, n>2,$ suppose there exist $x, y, z\in \mathbb{N}$ s.t $x^n + y^n = z^n.$

E10) Suppose that $\lim_{x\to 2}(x^2-1)=-1.$

Then given $\varepsilon>0, \exists\, \delta>0$ s.t.

$|x-2|<\delta \Rightarrow |x^2-1-(-1)|<\varepsilon,$ i.e., $|x^2|<\varepsilon,$ i.e. $x^2<\varepsilon$ …(5)

Thus, $2-\delta<x<2+\delta \Rightarrow x^2<\varepsilon$

Now, take $\varepsilon=0.1.$

For any $\delta>0, x=2$ lies in $]2-\delta, 2+\delta[,$ but $x^2=4\not<\varepsilon.$

So we reach a contradiction to (5). Hence, our assumption must be wrong.

Hence $\lim_{x\to 2}(x^2-1)\neq -1.$

E11) i) For example, $\mathbb{N}$ is an infinite set containing $\mathbb{N},$ and $\mathbb{N}=\mathbb{N}$.

ii) $(3-2)^2=1\neq 3^2+(-2)^2,$ for example.

iii) A counterexample to '$f:\mathbb{N}\to\mathbb{N}$ is $1-1 \Rightarrow f$ is onto' is $f:\mathbb{N}\to\mathbb{N}: f(n)=n+3.$

Firstly, $f(n_1)=f(n_2)\Rightarrow n_1+3=n_2+3\Rightarrow n_1=n_2$ for $n_1, n_2\in\mathbb{N}.$

Hence f is $1-1.$

However, there is no $n\in\mathbb{N}$ for which $f(n)=1.$

Hence f is not onto.

iv) You should check that the greatest integer function is a counterexample.

E12) Let $p(n)$: $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,$ for $n\in\mathbb{N}.$

$p(1)$ is true, as you can see.

Assume that $p(k)$ is true for some $k\in\mathbb{N}.$

Now, $p(k+1)=(\cos\theta+i\sin\theta)^{k+1}$

$=(\cos\theta+i\sin\theta)^k\,(\cos\theta+i\sin\theta)$

$=(\cos k\theta+i\sin k\theta)\,(\cos\theta+i\sin\theta),$ since $p(k)$ is true

$=\cos\overline{k+1}\,\theta+i\sin\overline{k+1}\,\theta,$ using the trigonometric formulae.

Thus, $p(k+1)$ is true.

Hence, $p(n)$ is true $\forall n\in\mathbb{N}.$

Also, $p(0)$ is trivially true.

Now, let n be a negative integer. Then $-n\in\mathbb{N}.$

Therefore, $p(-n)$ is true.

Now, $(\cos\theta+i\sin\theta)^n=[(\cos\theta+i\sin\theta)^{-n}]^{-1}$

$=[\cos(-n)\theta+i\sin(-n\theta)]^{-1},$ since $p(-n)$ is true.

$=(\cos n\theta-i\sin n\theta)^{-1}$

$= \cos n\theta + i \sin n\theta.$

Hence the given statement is true.

E13) $p(n): \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n},$ for $n \in \mathbb{N}.$

Since $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}, p(2)$ is true.

Assume that $p(k)$ is true for some $k \in \mathbb{N}.$

Then $\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$, as you can verify.

$\therefore p(k+1)$ is true.

Hence $p(n)$ is true $\forall n \geq 2.$

E14) Let $p(n): \int_0^{\pi/2} \sin^{2n} x\, dx = \frac{1.3.5...(2n-1)}{2.4.6...(2n)} . \frac{\pi}{2},$ for $n \in \mathbb{N}.$

Since $\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \frac{1-\cos 2x}{2} dx = \frac{1}{2}.\frac{\pi}{2}, p(1)$ is true.

Assume that $p(k)$ is true for some $k \in \mathbb{N}.$

Now you can use integration by parts to prove that

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx.$$

Therefore, $\int_0^{\pi/2} \sin^{2(k+1)} x dx = \frac{2k+1}{2(k+1)} \int_0^{\pi/2} \sin^{2k} x dx$

$= \frac{1.3.5...[2(k+1)-1]}{2.4.6...2(k+1)} . \frac{\pi}{2},$ since $p(k)$ is true.

Hence $p(n)$ is true $\forall n \geq 1.$

E15) Let us see if we can prove this using the 'weak' form of the PMI.

Let $p(n): 2 \mid a_{3n},$ for $n \in \mathbb{N}.$

$p(1)$ is true since $2 \mid a_3.$

Assume that $p(k)$ is true for some $k \in \mathbb{N}.$

Now $a_{3(k+1)} = a_{3(k+1)-1} + a_{3(k+1)-2}$

$= a_{3k+2} + a_{3k+1}$

$= (a_{3k+1} + a_{3k}) + (a_{3k} + a_{3k-1})$

$= (a_{3k} + a_{3k-1} + a_{3k}) + (a_{3k} + a_{3k-1})$

$= 3a_{3k} + 2a_{3k-1}$

Since a_{3k} is even and $2a_{3k-1}$ is even, $a_{3(k+1)}$ is even.

Thus, $p(k+1)$ is true.

Hence $p(n)$ is true $\forall n \geq 1.$

Note that even though the terms here are of the Fibonacci sequence, we did not required the strong form of the PMI for the proof.

E16) $a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16.$

Looking at these terms, we may conjecture that $a_n = n^2.$ Let us see if this is true. Since each term in the sequence requires the values of two previous terms, we need to apply the strong form of the PMI.

Let $p_n : a_n = n^2$, for $n \in \mathbb{N}$.

$p(1)$ is true.

Assume, for some $k \in \mathbb{N}$, $p(i)$ is true $\forall i \leq k$.

Now, $a_{k+1} = 2a_k - a_{k-1} + 2 = 2k^2 - (k-1)^2 + 2 = (k+1)^2$.

Thus, $p(k+1)$ is true.

Hence $p(n)$ is true $\forall n \geq 1$.

UNIT 3

ALGEBRAIC STRUCTURE OF $\mathbb{R}$

Structure | Page Nos.

3.1 INTRODUCTION

You are already familiar with the set $\mathbb{R}$ of real numbers, and the basic operations such as addition, subtraction, multiplication and division by non-zero elements on it. Have you ever wondered why division by zero is not allowed? What are the properties that are carried over to $\mathbb{R}$ from those of $\mathbb{Q}$, the set of rationals? What exactly is the property that $\mathbb{Q}$ does not have, but $\mathbb{R}$ has? How big is $\mathbb{R}$ in comparison to $\mathbb{Q}$? To answer such questions, we shall begin by presenting the essential properties that determine the "algebraic" and the "order" structure of $\mathbb{R}$. Also, we shall focus on the "order completeness" property of $\mathbb{R}$. This property is at the heart of real analysis. It is because in the absence of this property most of the results in real analysis would become invalid.

So, we begin Section 3.2 with an evolutionary aspect of real numbers and arrive at their algebraic and order properties which make $\mathbb{R}$ an 'ordered field'.

In Section 3.3 we shall discuss at length the order completeness property and show you why $\mathbb{R}$ is a 'complete' ordered field. You will see many applications of this property, some in this unit and others in the rest of the course. A few applications, for example, are the Archimedean and density properties of $\mathbb{R}$ which will be discussed in this section.

In Section 3.4, we shall address the notions of 'finite' and 'infinite' sets in order to estimate the size of different subsets of $\mathbb{R}$. Specifically, we shall see that there are many subsets of $\mathbb{R}$ which are infinite. Some, like $\mathbb{N}$ and $\mathbb{Q}$ are "countable" in the sense that they can be enumerated. On the other hand, we shall show you that $\mathbb{R}$ is "uncountable", i.e., there is no way to enumerate the elements of $\mathbb{R}$.

Specifically we expect to achieve the following objectives.

Objectives

After reading this unit, you should be able to

- describe and apply the algebraic properties of real numbers;
- describe the ordered structure on the set of real numbers, and its applications;
- show whether a subset of $\mathbb{R}$ is bounded below, bounded above, both or neither;
- explain the completeness property of the order on the real number system;
- describe and apply Archimedean property of $\mathbb{R}$;
- compute the infimum and supremum of subsets of $\mathbb{R}$;
- identify whether a subset of real numbers is countable or uncountable.

3.2 THE FIELD AND ORDER STRUCTURE OF $\mathbb{R}$

In this section we shall help you recall how real numbers evolved. We shall look at the real numbers as a set together with the operations of addition, subtraction, multiplication, and division by nonzero elements. We shall also see how real numbers can be represented on a line.

3.2.1 The Real Number Line

Recall that a **binary operation** on a set S is a mapping from the set $S \times S$ into S, that is, it maps each ordered pair of elements of S to an

We begin with the **natural numbers** $\mathbb{N} = \{1, 2, 3\}$. You know that if you add two natural numbers, the answer is a natural number. Thus, addition is a binary operation on $\mathbb{N}$, and so is multiplication. However, subtraction is not binary operation on it, because, for example, $1-1=0 \notin \mathbb{N}$. Not only this, people faced problems with solving the linear equations of the form $x+n=0$. Such problems led to the discovery of **integers** $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2 \ldots\}$. You can see that, on $\mathbb{Z}$, addition, multiplication and subtraction, all are binary operations. But, you know that there are problems with $\mathbb{Z}$ too. For example, if you have to distribute, 2 chocolates among three kids equally, each kid cannot get a whole part. The answer must be something different from an integer. For algebraists it were the equations $p - qx = 0$, with $q \neq 0$, that they could not solve in $\mathbb{Z}$. Specifically, the need arose to define a new operation called 'division by nonzero'. This operation gave us the set $\mathbb{Q}$ of **rational numbers** defined as

$$\mathbb{Q} = \left\{ \frac{p}{q} \middle| p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

Note that many rational numbers have the same representation in $\mathbb{Q}$. For example $\frac{2}{4}=\frac{8}{16}=\frac{1}{2}$. So when we write a rational number as p/q, we assume that there is no common factor between p and q, that is, p/q is in its **lowest form**. The operations '$+$' (addition), '$\cdot$' (multiplication) and $/$ (division by nonzero) on $\mathbb{Q}$ are defined as below.

For $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$,

i) $\frac{p}{q}+\frac{r}{s}=\frac{ps+rq}{qs}$

ii) $\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}$

iii) $\frac{\left(\frac{p}{q}\right)}{\left(\frac{r}{s}\right)}=\frac{ps}{qr},\ q\neq 0\ r\neq 0$

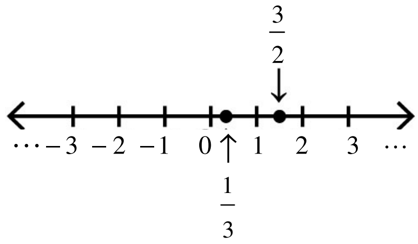

Fig. 1: Rational numbers on a line

You should note that the sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are defined in such a way that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}.$$

If we consider the distance between 0 and 1 as the unit of length, then we can represent all the rational numbers on a line. (See Fig. 1.)

For example $1/3$ is the point which is one-third of the way from 0 to 1. Similarly, $3/2$ is the point that is one-half of the way from 0 to 3. This is because the rational numbers possess a natural order inherited from $\mathbb{R}$, which you will see in Subsection 3.2.3. Another fact that you know about rational numbers is that they can be represented by decimals using only the digits $0,1,2,\ldots,9$. For example, we can write $-1/2=-0.5$ and $53/32=1.65625$ using the long division method. A **decimal** representation is an expression of the form

$$A_0.a_1\,a_2\,a_3...$$

where $A_0 \in \mathbb{Z}$ and $a_1, a_2, a_3...$ are digits from $\{0,1,2,\ldots,9\}$.

In the decimal representations above the number of nonzero digits after the decimal (.) is finite. Such a representation of a rational number is said to be **terminating**. Thus a terminating decimal representation, has the form $A_0.a_1\,a_2\,a_3\ldots a_n$, for some $n \in \mathbb{N}$. The corresponding rational form can be written as

$$A_0.a_1\,a_2\,a_3\ldots a_n = A_0+\frac{a_1}{10}+\frac{a_2}{100}+\frac{a_3}{1000}+\cdots+\frac{a_n}{10^n}.$$

Let us take an example.

Example 1: Find the rational number corresponding to 0.09375.

Solution: We write 0.09375 as

$$0.09375 = 0 + \frac{0}{10} + \frac{9}{100} + \frac{3}{1000} + \frac{7}{10000} + \frac{5}{100000} = \frac{3}{32}$$

So, the corresponding rational number is $3/32$.

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There are also rational numbers whose decimal representations do not terminate while carrying out long division. For example, when we divide 1 by 3 using long division method we get $0.3333\ldots$ Similarly dividing 499 by 330 results in $1.5121212\ldots$ These decimal representations, although non-terminating, have certain block of digits that keep on **recurring**. We can see that 3 recurs in $0.333\ldots$ and 12 recurs in $1.5121212\ldots$. We write such numbers briefly with a **bar** that covers the recurring digit or the block of digits.

For example

$$0.3333\ldots \quad = 0.\bar{3}$$

$$1.5121212\ldots = 1.5\overline{12}$$ (Note that the bar does not cover 5)

Now the question arises, how do we get back the number $1/3$ from $0.\bar{3}$ or $499/330$ from $1.5\overline{12}$? For the time being let us write:

$$0.\bar{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots \qquad \ldots (1)$$

The expression on the right hand side of Eq.(1) represents a sum of infinite terms whose precise meaning will only be clear in **Block 3**.

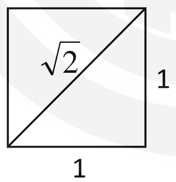

Fig. 2: The length of a diagonal of unit square is $\sqrt{2}$.

Now we show you how to get back $1/3$ from $0.\bar{3}$.

Let $x = 0.\bar{3}$. Then $10x = 3 \cdot 333\ldots$. So subtracting x from $10x$, we get $10x - x = 3\cdot 333\ldots - 0\cdot 333\ldots = 3$. So $9x = 3$, i.e. $x = \frac{1}{3}$.

To get the rational from of $1\cdot 5\overline{12}$, let $x = 1\cdot 5\overline{12}$. Then we can write

$$x = 1\cdot 5 + \frac{y}{10}, \text{ where } y = 0\cdot\overline{12} = 0\cdot 121212\ldots.$$

So, $100y = 12\cdot 1212\ldots$

$-y = -0\cdot 1212\ldots.$

Hence, $99y = 12$, i.e. $y = \frac{4}{33}$. Thus, we get $x = 1\cdot 5 + \frac{4}{330} = \frac{499}{330}$.

There are numbers whose decimal representations are non-terminating. Such numbers are called **irrational numbers**. For example, $\sqrt{2}$ which is the length of the diagonal of unit square is one such number. (See Fig. 2.) This is the content of the next Theorem.

Theorem 1: $\sqrt{2}$ is not a rational number.

Proof: We shall prove it by contradiction (see Section 2.3 of Unit 2). So, assume, if possible, $\sqrt{2} \in \mathbb{Q}$. Then we have $p, q \in \mathbb{Z}$ such that

$$\sqrt{2} = \frac{p}{q},$$

where $p(\neq 0)$ and $q(\neq 0)$ have no common factors. Squaring both sides of the equation above, gives us

$$2 = \frac{p^2}{q^2} \quad \text{i.e.} \quad p^2 = 2q^2.$$

This means p^2 is an even number, so is p. Let $p = 2k$ for some integer k.

Now we have

$$4k^2 = 2q^2 \text{ i.e. } p^2 = 2q^2.$$

This means q^2 is an even number; hence, so is q. Thus p and q have 2 as a common factor. This contradicts our assumption namely p and q have no common factor. Consequently, $\sqrt{2} \notin \mathbb{Q}$. ■

You just saw that $\sqrt{2}$ is not a rational number, that is, it is irrational. So its decimal representation must be **non-terminating** and **non-recurring**. Like $\sqrt{2}$, there are many other numbers such as $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}$ etc that are irrational.

Let us now see how to prove that $\sqrt{6}$ is irrational.

Example 2: Prove that $\sqrt{6}$ is irrational.

Solution: Assume, on the contrary, that $\sqrt{6} = \frac{p}{q}$ for some $p, q \in \mathbb{Z}, q \neq 0$ such that p and q have no common factor. Then $p^2 = 6q^2 = 2(3q^2)$. This implies p^2 is even and hence p is even. So, let $p = 2\ell$, Then

$$4\ell^2 = 6\ell^2 \Rightarrow 2\ell^2 = 3q^2.$$

This implies $3q^2$ is even, and hence q^2 is even. This implies q is even. Thus, we have found that 2 is a common factor of p and q. This contradicts our assumption that p and q have no common factor. Consequently, $\sqrt{6}$ is irrational.

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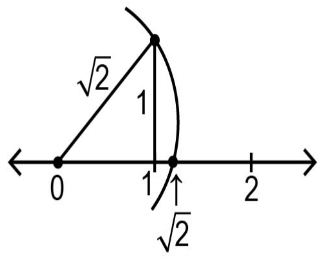

Fig. 3: Locating $\sqrt{2}$ on the number line.

The irrational numbers together with rational numbers are called **real** numbers. They are denoted by the symbol $\mathbb{R}$. Now we have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Now let us see how we can represent the irrational numbers of the form $\sqrt{p}$, on the numbers line. For $\sqrt{2}$, we describe the mechanism as below (see Fig. 3.)

Consider the unit square with corners at $(0,0),(1,0),(1,1)$ and $(0,1)$. The line segment joining $(0,0)$ and $(1,1)$ is one of its diagonals. Draw a circle with centre $(0,0)$ and diagonal as the radius. The point where it meets the x-axis represents the number $\sqrt{2}$.

Now try these exercises.

E1) Find the rational number corresponding to

i) 2.596306 ii) $4.76\overline{324}$

E2) Show that $\sqrt{p} \notin \mathbb{Q}$, where p is a prime number.

E3) Using compass and ruler determine the location of $\sqrt{3}$ on the number line. Can you determine the location of $\sqrt{p}$ using a compass and ruler, where p is a prime number?

If you have done E3) you would have understood how to represent an irrational number of the form $\sqrt{p}$ on the number line. However many irrational numbers such as π cannot be represented on the number line in this way. We shall now focus on the structure of $\mathbb{R}$ from the point of view of Algebra.

3.2.2 Algebraic Structure of $\mathbb{R}$

The relationship between real numbers is through 'addition', (and the inverse relation namely subtraction); multiplication (and the inverse operation namely division); and comparison. We begin by stating the properties addition, subtraction, multiplication and division. You should note that these are instances of binary operation mentioned in Subsection 3.2.1.

Definition: The binary operation which associates with $a,b \in \mathbb{R}$, the real number $a+b$ is called the **addition** of real numbers.

A) Properties of Addition:

i) (Commutativity) $a+b=b+a, \forall a,b \in \mathbb{R}$
ii) (Associativity) $(a+b)+c=a+(b+c), \forall a,b \in \mathbb{R}$
iii) (Existence of additive identity) $\exists 0 \in \mathbb{R}$ such that
$$a+0=a=0+a \forall a \mathbb{R}.$$
iv) (Existence of inverses) For each $a \in \mathbb{R}$, there exists $-a \in \mathbb{R}$ such that $a+(-a)=0=(-a)+a$. Read $-a$ as 'minus a'. We call $-a$ the additive inverse of a.

Definition: The binary operation which associates with $a,b \in \mathbb{R}$ the real number $a \cdot b$ is called the multiplication of real numbers.

M) Properties of Multiplication:

i) (Commutativity) $a.b = b.a,\ \forall a \in \mathbb{R}$
ii) (Associativity) $(a.b).c = a.(b.c),\ \forall a,b,c \in \mathbb{R}$
iii) (Existence of identity) $\exists 1 \in \mathbb{R}$ such that $1 \cdot a = a \cdot 1 = a,\ \forall a \in \mathbb{R}.$
iv) (Existence of inverses) For each $a \in \mathbb{R}, a \neq 0$ there exists $a^{-1} \in \mathbb{R}$ such that $a.a^{-1} = 1 = a^{-1}a.$ Read a^{-1} as 'a inverse of a'. We call a^{-1} the multiplicative inverse of $a.$

D) Distributivity:

$$a.(b+c) = a.b + a.c, \qquad \forall a,b,c \in \mathbb{R}.$$

You can see that addition and multiplication each satisfies four properties, namely commutativity, associativity, existence of identity, and the existence of inverse. In case of multiplication, inverses exist only for nonzero elements. The property distributivity connects addition and multiplication. Thus in all $\mathbb{R}$ satisfies 9 properties which are referred to as the **field properties** and any set that satisfies them is called a **field**. So, $\mathbb{R}$ is a field. Another example of a field is $\mathbb{Q}$. You should show that the rational numbers satisfy all the above properties.

Using the properties above you can derive many other algebraic properties of $\mathbb{R}$. which belong to the realm of algebra. To get the flavour of the properties we present two results below.

Theorem 2: Let $a,b \in \mathbb{R}.$ Then the following hold.

i) If $a+b=a,$ then $b=0,$ i.e. the additive identity is unique.
ii) If $b \neq 0$ and $a \cdot b = b,$ then $a=1,$ i.e. the multiplicative identity is unique.
iii) $a \cdot 0 = 0.$

Proof: i) We observe that

$b = 0 + b$ ($\because 0$ is an additive identity)
$= (-a+a)+b$ (Property A(iv))
$= -a + (a+b)$ (Property A(ii))
$= -a + a = 0$, using the Property A(iv). This completes the argument.

ii) $a = a\ .1$ (Property M (iii))
$= a(b.b^{-1})$ (Property M(iv))
$= (a.b).b^{-1}$ (Property M (ii))
$= b.b^{-1} = 1$, using the Property M(iv). This completes the argument.

iii) $a.0 + a.1 = a(0+1)$ (Distributivity)
$= a.1$

This gives $a.0 + a = a$. Hence by the uniqueness of the additive identity proved in (i) above, $a.0 = 0$. ■

Theorem 3: Let $a,b \in \mathbb{R}.$ Then the following holds.

i) If $a+b=0,$ then $b=-a,$ i.e., the additive inverse is unique.

ii) If $a \neq 0$ and $a.b = 1$, then $a.b = 1$, i.e. the multiplicative inverse is unique.

iii) If $a.b = 0$, then either $a = 0$ or $b = 0$.

Proof: i) Note that

$$
\begin{aligned}
b &= b + 0 && \text{(Property A(iii))} \\
&= b + (a + (-a)) && \text{(Property A(iv))} \\
&= (b + a) + (-a) && \text{(Property A (ii))} \\
&= (a + b) + (-a) && \text{(Property A(i))} \\
&= 0 + (-a) \\
&= -a
\end{aligned}
$$

ii) Note that

$$
\begin{aligned}
b &= 1.b && \text{(Property M(iii))} \\
&= (a.a^{-1})b && \text{(Property M (iv))} \\
&= (a^{-1}.a)b && \text{(Property M (i))} \\
&= a^{-1}.(ab) && \text{(Property M (ii))} \\
&= a^{-1}.1 = a^{-1}.
\end{aligned}
$$

ii) First assume that $a \neq 0$. We prove that $b = 0$.

Now $b = 1.b$

$$
\begin{aligned}
&= (a.a^{-1}).b && \text{(Which property?)} \\
&= (a^{-1}.a).\mathrm{b} && \text{(Which property?)} \\
&= a^{-1}.(a.b) && \text{(Which property?)} \\
&= a^{-1}.0 = 0 && \text{(Which property?)}
\end{aligned}
$$

Similarly you can show that if $b \neq 0$, then $a = 0$. ■

We can define the operation 'subtraction' as $a - b = a + (-b)$ and "division" as $a/b = a.b^{-1}$, for $b \neq 0$. Now we can perform all the algebraic manipulation on real numbers as we are used to, e.g. $2+5=5+2, 2+3-6=-1$, etc. Note the following definitions: For $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

When there is no confusion we shall write $a \cdot b$ as ab.

$$\frac{1}{a} = a^{-1}, \text{ if } a \neq 0$$

$$a^n = \underbrace{a.a.a....a}_{n \text{ times}}$$

$$na = \underbrace{a + a + a + ... + a}_{n \text{ terms}}$$

Let us consider a few examples.

Example 3: Prove that $(a+b)^2 = a^2 + 2ab + b^2$ for all $a, b \in \mathbb{R}$.

Solution: By definition

$$(a+b)^2 = (a+b)(a+b)$$

$= (a+b)a + (a+b)b$ (Distributivity)
$= a(a+b) + b(a+b)$ (Commutativity)
$= aa + ab + ba + bb$ (Distributivity)
$= aa + ab + ab + bb$ (Commutativity)
$= a^2 + 2ab + b^2$

***

Example 4: Show that $x^2 - y^2 = 0 \Leftrightarrow x = y$ or $x = -y$ for all $x, y \in \mathbb{R}$.

Solution: The result follows from the following:

$x^2 - y^2 = 0 \Leftrightarrow x^2 - xy + xy - y^2 = 0$

$\Leftrightarrow x(x-y) + y(x-y) = 0$ (Distributivity)

$\Leftrightarrow (x-y)(x+y) = 0$ (Distributivity)

$\Leftrightarrow x - y = 0$ or $x + y = 0$ (Theorem 3 iii))

$\Leftrightarrow x = y$ or $x = -y$

***

Why don't you try some exercises now?

E4) Let a and b be two elements of $\mathbb{R}$. Prove the following

i) $(-1)a = -a$ ii) $(-1)(-1) = 1$

E5) Show that $(a+b).c = a.c + b.c$ for all $x, y, c \in \mathbb{R}$.

E6) Solve the equation $x^2 + 5x - 6 = 0$, by clearly justifying which property you are using at each step.

E7) Using the principle of mathematical induction (Section 2.4 of Unit 2), prove the binomial theorem:

$$(a+b)^n = \sum_{i=0}^{n} {}^nC_i \, a^{n-i} b^i$$

where $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$.

Here we have discussed only some fundamental properties of $\mathbb{R}$ as a field, although a field has many other interesting properties. You can learn them in our course BMTC-104(Algebra). Next we shall see that $\mathbb{R}$, is a special kind of field.

3.2.3 $\mathbb{R}$ as an Ordered Field

Now we shall discuss the order properties of $\mathbb{R}$. First, we shall define the concept of order on a special subset of $\mathbb{R}$.

There is a subset of $\mathbb{R}$, denoted by $\mathbb{R}^+$, called the set of **positive real numbers** whose element satisfy the following properties:

i) If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$ and $ab \in \mathbb{R}^+$.

ii) For every $a \in \mathbb{R}$, exactly one of the following is true:
$a \in \mathbb{R}^+, -a \in \mathbb{R}^+, a = 0.$

Property i) makes the operations of addition and multiplication compatible to order. That is, the sum and product of two positive real numbers is positive. Property ii), on the other hand, classifies the elements of $\mathbb{R}$ into three distinct categories. That is, $\mathbb{R}$ satisfies the **law of trichotomy.**

The set of **negative real numbers**, denoted by $\mathbb{R}^-$, is defined as

$$\mathbb{R}^- = \{a \in \mathbb{R} \mid -a \in \mathbb{R}^+\}.$$

Then $\mathbb{R} = \mathbb{R}^- \cup \mathbb{R}^+ \cup \{0\}$. If $a \in \mathbb{R}^+ \cup \{0\}$, then a is called a **nonnegative** real number.

Using the properties of $\mathbb{R}^+$, we define the order $>$, called '**greater than**' on $\mathbb{R}$ as follows:

$$a > b \quad \text{iff} \quad a - b \in \mathbb{R}^+.$$

Likewise the order $<$ called '**less than**' is defined as

$$a < b \quad \text{iff} \quad b - a \in \mathbb{R}^+$$

In a similar way, the operators $\geq$ called '**greater than or equal to**' and $\leq$ called '**less than or equal to**' can be defined as:

$$a \geq b \quad \text{iff} \quad a > b \quad \text{or } a = b;$$
$$a \leq b \quad \text{iff} \quad a < b \quad \text{or } a = b.$$

The expressions such as $a < b, a > b, a \leq b$ and $a \geq b$ are called 'inequalities'. We shall discuss them in more detail in Section 3.3. The definitions above immediately tell us that $a \in \mathbb{R}^+$ iff $a > 0$ and $a \in \mathbb{R}^-$ iff $a < 0$.

Using these definitions we shall derive the order properties of $\mathbb{R}$.

Theorem 4 (Order Properties of $\mathbb{R}$) : Let $a, b, c \in \mathbb{R}$. Then the following hold.

i) If $a > b$ and $b > c$, then $a > c$.
ii) If $a > b$ then $a + c > b + c$.
iii) If $a > b$ and $c > 0$, then $ac > bc$.
iv) If $a > b$ and $c > 0$, then $ac < bc$.

Proof: i) Since $a > b$, we have $a - b > 0$. That is, $a - b \in \mathbb{R}^+$. Similarly $b - c \in \mathbb{R}^+$ Hence $a - c = (a - b) + (b - c) \in \mathbb{R}^+$. That is $a - c > 0$ which implies $a > c$.

ii) Note that

$$\begin{aligned}(a+c)-(b+c) &= a+c-b-c \\ &= a-b+c-c \\ &= a-b+0 \\ &= a-b > 0, \quad (\because a > b)\end{aligned}$$

Hence $a + c > b + c$.

iii) Since $a > b$, we have $a-b>0$. i.e $a-b \in \mathbb{R}^+$ Also $c>0$ means $c \in \mathbb{R}^+$. Hence $a-c=(a-b)+(b-c) \in \mathbb{R}^+$. But we have $(a-b)c = ac-bc$.

iv) Prove yourself on the lines of iii) above. ■

Since the field $\mathbb{R}$ possesses the order properties stated above, it is an **ordered field**. Since $\mathbb{Q}$ is a subset of $\mathbb{R}$, $\mathbb{Q}$ is also an ordered field. We shall denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Symbolically, $\mathbb{Q}^+ = \mathbb{Q} \cap \mathbb{R}^+$. Likewise, we write $\mathbb{Q}^- = \mathbb{Q} \cap \mathbb{R}^-$.

The order properties lead to many inequalities on real numbers. For example, using contradiction you can show that if x is positive then $\frac{1}{x}$ is also positive. Likewise, you can also show that if $x>1$, then $\frac{1}{x}<1$. A few more inequalities are given in the following examples.

Example 5: If $x>1$ and $y>1$, then show that $xy>1$.

Solution: Note that $y>1$ implies $y>0$. Now, since $x>1$ and $y>0$, Theorem 4(iii) implies, $xy>y$. But, $y>1$. Therefore, $xy>1$.

***

Example 6: Let $0<x<1$. Then show that $0<x^n<1$, for all $n \in \mathbb{N}$.

Solution: We prove it by the principle of mathematical induction which we have discussed in Unit 2, Section 2.4.

For $n=1$, the result holds. So, assume that it is true for some $n \in \mathbb{N}$. That is, $0<x^n<1$, for some $n \in \mathbb{N}$. Now, $x>0$ and $x^n>0$ implies $x.x^n>0$, i.e., $x^{n+1}>0$. Note also that $x>0$ and $x^n<1$ implies that $x.x^n<x$. But you also have $x<1$. Hence, $x^{n+1}<1$. Thus we have shown that $0<x^{n+1}<1$. The Principle of Mathematical Induction completes the proof.

***

Now try some exercises.

E8) Let $x, y \in \mathbb{R}$. Then show that

$$x>y>0 \Leftrightarrow \frac{1}{y}>\frac{1}{x}>0.$$

E9) Let a,b,c,d be positive real numbers. If $a<b$ and $c<d$, then show that $ac<bd$.

E10) Let $x>y>0$. Then, using the principle of induction, prove that $x^n>y^n$ for all $n \in \mathbb{N}$.

Can you conclude the following, from E10? $x>1 \Rightarrow x^n>1 \forall n \in \mathbb{N}$.

E11) Let $0<x<a<1$. Then show that $x^2<a$. Is it also true that $x^n<a$ for all natural numbers n?

E12) Let $a > 1$. Then show that $a^n > a^m \Leftrightarrow n > m$, for $m, n \in \mathbb{N}$. Does the result hold when $a < 1$?

Thus far you have seen that $\mathbb{R}$ is an ordered field. This characteristic of $\mathbb{R}$ was known to people of ancient civilizations such as Babylonia, Egypt and India. People used to compare quantities such as lengths and areas of different shapes, which often turn out to be irrational numbers. In the next section, you will learn more about the ordered structure of the real numbers.

3.3 $\mathbb{R}$ AS A COMPLETE ORDERED FIELD

In this section we shall discuss another crucial property of $\mathbb{R}$, the completeness of order, and see some of its applications.

3.3.1 Order Completeness Property

You know that every subset of $\mathbb{N}$ contains a least element. But in $\mathbb{Z}$ there are subsets that do not have least elements. For example, $\mathbb{Z}_{\text{even}} = \{\ldots -4, -2, 0, 2, 4, \ldots\}$, the set of all even integers has no least element. However, there are also certain subsets of $\mathbb{Z}$ that have least elements. For example, the set $S = \{-8, -6, -2, 4, 6, \ldots\}$ has -8 as its least element. Can you think of some more subsets of $\mathbb{Z}$ that, too, have a least element? Do they have any common property? Before answering this question we recall the notion of a lower bound.

Definition: Let $\emptyset \neq S \subseteq \mathbb{R}$. A number $\ell \in \mathbb{R}$ is called a **lower bound** of S, if $\ell \leq x, \quad \forall x \in S$. A set S is called **bounded below** if there is some lower bound of S.

You can see that if of ℓ is a lower bound of S, and if $\ell' < \ell$ then ℓ' is also a lower bound of S.

Now the answer of the question above is that every subset of $\mathbb{Z}$ that is bounded below has a least element. This is guaranteed by the following theorem, which we state without proof.

Theorem 5: Let $\emptyset \neq S \subseteq \mathbb{Z}$ and S be bounded below. Then $\min S$ exists.

Now let us turn to the set $\mathbb{Q}$. We ask you the question: **What is the least positive rational number?** The answer you should find is 'No such number exists.'. Why is it the case? Because if r is a positive rational number, then you can see that $\frac{r}{2}$ is also a positive rational number smaller than r. So we see that the notion of least element of a set becomes of little relevance when the set is a subset of $\mathbb{Q}$.

Therefore, we need to generalise the notion of the least element of a set. Let S be a nonempty subset of $\mathbb{Q}$ that is bounded below. This means S has a lower bound. In fact you can see that, if S has one lower bound then it has many lower bounds. The reason is very simple. Every number smaller than a lower bound of S is also a lower bound of S. So we can talk about '**the greatest lower bound**' of S which is a lower bound of S greater than every other lower bound of S. The greatest lower bound of S is also called the **infimum** of S, which in short is denoted as $\inf S$. It is straight forward to see that a set cannot have more than one infimum. For if ℓ_1 and ℓ_2 are two infimums of a nonempty set S, then by definition $\ell_1 \le \ell_2$ and $\ell_2 \le \ell_1$. This gives, $\ell_1 = \ell_2$.

You can see infimum as a generalisation of minimum.

Let us consider an example.

Example 7: Find $\inf S$, where $S = \left\{ \frac{p}{q} \middle| \, p \ge 10, q \le 8, p, q \in \mathbb{N} \right\}$.

Solution: Let $\frac{p}{q} \in S$. Note that $q \le 8$ implies $\frac{1}{q} \ge \frac{1}{8}$. Hence $\frac{p}{q} \ge \frac{10}{8} = \frac{5}{4}$.
Thus every element of S is greater than or equal to $5/4$. This means $5/4$ is a lower bound of S. Also you can see that $\frac{5}{4} \in S$. This means, $5/4$ is greater than every other lower bound of S. So, $\inf S = \frac{5}{4}$.

***

The set S in Example 7 above is nonempty, bounded below and contain its infimum too. However, not all subsets of $\mathbb{Q}$ are like this. The point we want to emphasize is that **there are subsets of $\mathbb{Q}$ which are nonempty and bounded below, but do not have the infimum in $\mathbb{Q}$.** A natural example comes from $\sqrt{2}$.

Consider the set $S = \{x \in \mathbb{Q}^+ | x^2 > 2\}$. It contains all the positive rational numbers whose square is greater than 2. Note that $S \ne \emptyset$. (Why?) It is also easy to see that S is bounded below, for example by 0. Also 1 is a lower bound of S as for every $x \in S$, we have $x^2 > 2 > 1$ i.e. $x > 1$. Similarly 1.4 is another lower bound of S as for every $x \in S$, we have $x^2 > 2 > (1.4)^2$ i.e. $x > 1.4$. In the same way, you can verify that the numbers

$$1.41, 1.414, 1.4142, 1.41421, \ldots$$

are also lower bounds of S. You can note that each of these lower bounds is greater than its predecessor. Does this sequence remind you to the approximations of $\sqrt{2}$? Indeed, it is the case. Then you should also realise that the sequence contains infinitely many elements.
What do we get from the discussion above?

The set $S = \{x \in \mathbb{Q}^+ | x^2 > 2\}$ is nonempty and bounded below but has no infimum **in** $\mathbb{Q}$. However, such a situation never arises in $\mathbb{R}$. That is, if S is a nonempty and bounded below subset of real numbers then, the infimum of S is always a real number. It is because of a fundamental property possessed by $\mathbb{R}$, but not by $\mathbb{Q}$. This property is stated below:

Greatest Lower Bound Property of $\mathbb{R}$
Let $\emptyset \ne S \subseteq \mathbb{R}$ and S be bounded below. Then $\inf S$ exists in $\mathbb{R}$.

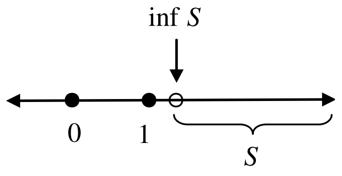

Fig. 4: A hole at $\inf S$ in $\mathbb{Q}$, where $S = \{x \in \mathbb{Q}^+ \mid x^2 > 2\}$.

The property above is also sometimes called the **Infimum Property** of $\mathbb{R}$. It is not possible to prove this property with what we have learned so far. Let us focus on how this property can be interpreted.

If we represent the rational numbers on a line we can see that the line contains many points which do not correspond to any rational number. At the moments let us call such points as "holes". One such hole is shown in Fig. 4. In set terminology, we can find a pair (A, B) of nonempty subsets of rational numbers such that

1) $A \cup B = \mathbb{Q}$, $A \cap B = \emptyset$.
2) Every element of A is smaller than every element of B.
3) A has no maximum element and B has no minimum element.

Such a pair (A, B) separates $\mathbb{Q}$ into two parts. An example is given below

$$A = \{x \in \mathbb{Q}^+ \mid x^2 < 2\} \cup \mathbb{Q}^- \text{ and } B = \{x \in \mathbb{Q}^+ \mid x^2 > 2\}$$

You can verify that the sets A and B satisfy all the three properties above. The main distinction between $\mathbb{Q}$ and $\mathbb{R}$ is that while there are infinitely many pairs (A, B) that separate $\mathbb{Q}$, there is not even a single pair that can separate $\mathbb{R}$. Thus the line representing $\mathbb{R}$, which we shall call the **real line** from now onwards, contains no holes.

Analogous to the lower bounds and greatest lower bounds we can define upper bounds and least upper bounds for the subsets of $\mathbb{R}$.

Definition: Let $\emptyset \neq S \subseteq \mathbb{R}$. A number $u \in \mathbb{R}$ is called an **upper bound** of S if $u \geq x$ for each $x \in S$. If there exists some upper bound of S, then S is called **bounded above**.

Definition: A set $S \subseteq \mathbb{R}$ is said to be **bounded** if it is both bounded below and bounded above.

Definition: Let $\emptyset \neq S \subseteq \mathbb{R}$. A number $u \in \mathbb{R}$ is called a **least upper bound** or **supremum** of S if

i) u is an upper bound of S.
ii) $u \leq v$ for all upper bounds v of S.

We denote supremum of S by $\sup S$. As in the case of greatest lower bounds, a set cannot have more than one least upper bound. (Apply the definition to prove it.)

Let us now consider an example.

Example 8: Find the supremum of the set $S = \left\{ \frac{n+1}{n} \middle| n \in \mathbb{N} \right\}$.

Solution: You can see that

$$\frac{n+1}{n} = 1 + \frac{1}{n} \leq 2, \text{ for all } n \in \mathbb{N}.$$

So, 2 is an upper bound of S. Also note that if u is any upper bound of S, then

$\frac{n+1}{n} \le u,$ for all $n \in \mathbb{N}.$

This gives for $n = 1, 2 \le u.$ Hence, $\sup S = 2.$

***

Now given a nonempty bounded above subset S of $\mathbb{R}$, you might ask whether sup S always exists or not.

To investigate, let us suppose $\emptyset \ne S \subseteq \mathbb{R}$ and S be bounded above. Define $T = \{-s \in \mathbb{R} \mid s \in S\}$. (See Fig. 5 below.)

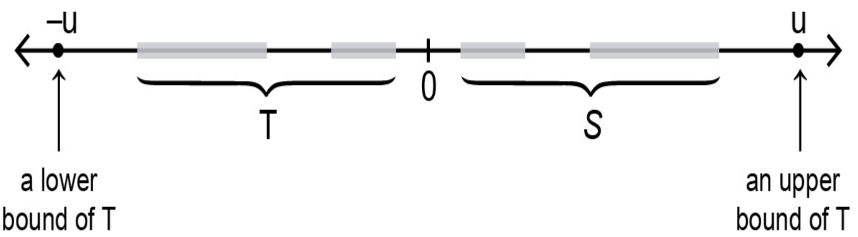

Fig. 5: Negatives of upper bounds of S are lower bounds of T

Let u be an upper bound of S. Then $-u$ is a lower bound of T (apply definition). Hence by the infimum property of $\mathbb{R}$, $\inf T$ exists. Let $t = \inf T$. We shall show that $-t = \sup S$. First note that $-t$ is an upper bound of S. Let u be another upper bound of S. Then $-u$ is a lower bound of T and hence $-u \le t$. This implies $u \ge -t$. Hence $-t = \sup S$. This proves the existence of $\sup S$.

Thus we have the following result.

Theorem 6: Let $\emptyset \ne S \subseteq \mathbb{R}$ and S be bound above. Then, $\sup S$ exists in $\mathbb{R}$.

Theorem 6 is called the **Least Upper Bound Property** (also the Supremum Property) of $\mathbb{R}$. It can be proved that the greatest lower bound property and the least upper bound property of $\mathbb{R}$ are equivalent. Now $\mathbb{R}$ being an ordered field, possesses the least upper bound property together with greatest lower bound property which make it a **complete ordered field.** It essentially means that every nonempty bounded subset of $\mathbb{R}$ has the infimum as well as the supremum **in** $\mathbb{R}$. This is called the **Completeness Property** of $\mathbb{R}$.

Let us look at an important result concerning the infimums and supremums of the subsets of $\mathbb{R}$.

Theorem 7: Let $\emptyset \ne S \subseteq \mathbb{R}$. Let ℓ be a lower bound and u an upper bound of S . Then

i) $\ell = \operatorname{int} S$ iff for every $\varepsilon > 0$, there exists some $x \in S$ such that $x < \ell + \varepsilon.$

ii) $u = \sup S$ iff for every $\varepsilon > 0$, there exists some $x \in S$ such that $x > u - \varepsilon.$

Proof: i) Let $\ell = \inf S.$ If possible assume that the statement "for every $\varepsilon > 0,$ there exists some $x \in S$ such that $x < \ell + \varepsilon$ " is false. This means, for some $\varepsilon > 0$ no element of S is smaller than $\ell + \varepsilon.$ That is $\ell + \varepsilon$ is a lower bound of $S.$ Since $\ell + \varepsilon$ is greater than ℓ, ℓ cannot be the greatest lower bound of $S.$ We have arrived at a contradiction.

Now we prove the converse. That is, we are given that for every $\varepsilon > 0$, there exists some $x \in S$ such that $x < \ell + \varepsilon$. To prove that $\ell = \inf S$, we need to show that if we pick any lower bound ℓ' of S, we must have $\ell' \le \ell$. But this is indeed the case. Because if $\ell' > \ell$ then take $\varepsilon = \ell' - \ell > 0$ and we have an element x of S such that $x < \ell + \varepsilon$, i.e., $x < \ell'$, which is not possible. Hence $\ell = \inf S$.

ii) Construct the proof yourself by suitably modifying the arguments in i) above. ■

Let us consider some examples.

Example 9: Find the supremum and infimum of $S = \left\{1 + \frac{(-1)^n}{n} \middle| n \in \mathbb{N}\right\}$.

Solution: Note that for all $n \in \mathbb{N}$.

$$1 + \frac{(-1)^n}{n} \le 1 + \frac{1}{2} = \frac{3}{2}$$

Also $\frac{3}{2} \in S$. Hence every upper bound of S must be greater than or equal to $\frac{3}{2}$. Hence $\frac{3}{2}$ is the supremum of S. For the infimum observe that

$$0 \le 1 + \frac{(-1)^n}{n}, \forall n \in \mathbb{N}.$$

Since $0 \in S$ every lower bound of S must be smaller than or equal to 0. Hence 0 is the infimum.

***

Example 10: For a given nonempty subset S of $\mathbb{R}$ and $a \in \mathbb{R}$, define the set $a + S = \{a + x \mid x \in S\}$. If S is bounded below, show that

$$\inf (a+S) = a + \inf S.$$

Solution: Let $\ell = \inf S$. This means,

$$\ell \le x, \ \forall x \in S \Rightarrow a + \ell \le a + x, \ \forall x \in S.$$

Hence, $a + \ell$ is a lower bound of $a + S$. Now let ℓ' be another lower bound of $a + S$. That is,

$$\ell' \le a + x, \ \forall x \in S \Rightarrow \ell' - a \le x, \ \forall x \in S.$$

But, this means $\ell' - a$ is a lower bound of S and hence $\ell' - a < \ell$. That is, $\ell' \le a + \ell$. This shows that $\inf (a+S) = a + \ell = a + \inf S$.

***

Using the arguments similar to Example 10, you can show that if S is bounded above then $\sup(a+S) = a + \sup S$.

You should try some exercises now.

E13) Let $\emptyset \ne S \subseteq \mathbb{R}$. Show that $\inf S \le \sup S$. When does the equality hold?

E14) Does the set given below has the infimum in $\mathbb{R}$? the supremum in $\mathbb{R}$?

$$\left\{\left(n+\frac{1}{n}\right)^n \middle| n\in\mathbb{N}\right\}$$

E15) For a given nonempty subset S of $\mathbb{R}$ and $a\in\mathbb{R}$, define the set $aS=\{as \mid s\in S\}$. Show that if S is bounded, then

$$\sup(aS)=\begin{cases} a\sup S, & \text{if } a>0 \\ a\inf S, & \text{if } a<0. \end{cases}$$

E16) Let S be a subset of non-negative real numbers that is bounded above. Prove that sup $\{x^2 \mid x\in S\}=(\sup S)^2$. Examine the case when S is a subset of negative real numbers.

There are several other properties of real numbers which can be derived from the Completeness Property. One such property of real numbers is due to Archimedes about which we shall talk next.

3.3.2 Archimedean Property

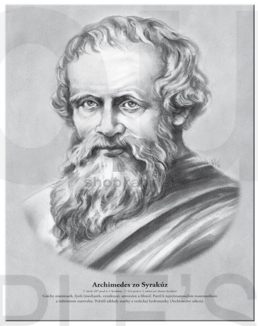

Fig. 6: Archimedes

Archimedes (287-212BC) was a Greek mathematician whose work is regarded as a foundation for real analysis. The concepts such as the limit of sequences (which you will see in Unit 5) were known to him centuries before they were rigorously introduced. The Archimedean property is one of his work that talks about the presence of arbitrarily large natural numbers and arbitrarily small real numbers.

Theorem 8(Archimedean Property): Let $x>0$. Then there exists some $n\in\mathbb{N}$ such that $x>\frac{1}{n}$.

Proof: We shall prove it by contradiction (see Unit 2, Section 2.2). Assume, if possible, that $x\le\frac{1}{n}$ for all $n\in\mathbb{N}$. But since $x>0$, this gives $n\le\frac{1}{x}$ for all $n\in\mathbb{N}$. This means that $1/x$ is an upper bound of $\mathbb{N}$. You also know that $\mathbb{N}\ne\emptyset$ and $\mathbb{N}\subseteq\mathbb{R}$. Hence, the supremum of $\mathbb{N}$ exists in $\mathbb{R}$. Let $u=\sup\mathbb{N}$. Now take $\varepsilon=1$. Then there exists some $n\in\mathbb{N}$ such that $u-1<n$. But this equivalent to $u < n+1$. Since $n\in\mathbb{N}$, $n+1\in\mathbb{N}$. Now $u=\sup\mathbb{N}$ and $u<n+1$ cannot hold together. Hence our assumption that $x\le\frac{1}{n}$ for all $n\in\mathbb{N}$ is false. This proves the theorem. ■

The following theorem is a consequence of the Archimedean property.

Theorem 9: Given two positive real numbers x and y such that $x<y$, there exists a rational number r such that $x<r<y$.

Proof: Let us first assume that $x>0$. Since $y-x>0$. by the Archimedean property, there exists some $q\in\mathbb{N}$ such that $y-x>\frac{1}{q}$. This implies

$qx < qy - 1$. Now, consider the set $S = \{n \in \mathbb{N} \mid qy \le n\}$. S contains all the natural numbers greater than or equal to qy. By the Archimedean property, there exists at least one such natural number. Hence, $S \neq \emptyset$. But, S is a subset of $\mathbb{N}$. Hence, by the Well -ordering Principle (see Unit 2, Section 2.4), S contains a least element, say p. This means

$$
\begin{array}{lll}
qy \le p & \text{and} & p - 1 < qy \\
\Rightarrow qy - 1 \le p - 1 & \text{and} & \dfrac{p-1}{q} < y \\
\Rightarrow qx < p - 1 & \text{and} & \dfrac{p-1}{q} < y \qquad (\because qx < qy - 1) \\
\Rightarrow x < \dfrac{p-1}{q} & \text{and} & \dfrac{p-1}{q} < y \\
\Rightarrow x < \dfrac{p-1}{q} < y & &
\end{array}
$$

Since $p-1$ and q are natural numbers, $\dfrac{p-1}{q}$ is a rational number, which lies between x and y.

If $x < 0$, the Archimedean property implies that there is some $n \in \mathbb{N}$ such that $n > -x$. Then $0 < n + x < n + y$. Now we can infer from the first case that there is a rational number r such that $n + x < r < n + y$. This implies that the rational number $r - n$ lies between x and y. ■

Let us consider an example.

Example 11: Let $a \in \mathbb{Q}$. What is the infimum of the set of all rational numbers greater than a?

Solution: The given set can be written as $S = \{x \in \mathbb{Q} \mid x \ge a\}$. Note that a is a lower bound of S. Let $\ell \in \mathbb{R}$ be another lower bound of S. First, assume that $\ell \in \mathbb{Q}$. Now, if $\ell > a$, then $\dfrac{\ell + a}{2}$ is a rational number lying between a and ℓ.

This implies, $\dfrac{\ell + a}{2} \in S$, which is a contradiction to the definition of ℓ.

Now assume that $\ell \notin \mathbb{Q}$. If $\ell > a$, then by Theorem 9 there exists some $r \in \mathbb{Q}$ such that $a < r < \ell$. This means, $r \in S$. This again contradicts the definition of a.

Thus in both the cases, $\ell \le a$. Therefore, $\inf S = a$.

***

Another consequence of the Archimedean property is the existence of square roots of positive real numbers.

Theorem 10: For every real number $a > 0$, there exists a positive real number whose square is a.

Proof: Let $S = \{x \in \mathbb{R}^+ \mid x^2 \le a\}$. Note that $S \neq \emptyset$. (Why?) Also, S is bounded above. (Why? Find an upper bound.) Now the Completeness Property of $\mathbb{R}$ implies that the supremum of S exists. So, let $u = \sup S$. Note that $u > 0$. (Why?)

To prove the result, we need to show that $u^2 = a$. We shall do so by contradiction. Assume if possible, that $u^2 \neq a$. Then by the Law of Trichotomy, either $u^2 > a$ or $u^2 < a$.

First assume $u^2 < a$. Then $u^2 - a > 0$ and $u > 0$ imply $\frac{u^2 - a}{2u} > 0$. Hence $\frac{2u}{u^2 - a} \in \mathbb{R}$, so by the Archimedean property there exists some $n \in \mathbb{N}$ such that $n > \frac{2u}{u^2 - a}$ and $n > \frac{1}{u}$. This implies
$u^2 - \frac{2u}{n} > a$ and $u - \frac{1}{n} > 0.$

Therefore, $\left(u - \frac{1}{n}\right)^2 = u^2 - \frac{2u}{n} + \frac{1}{n^2} > a + \frac{1}{n^2} > a$.

This means, $\left(u - \frac{1}{n}\right)^2 > x^2, \forall\, x \in S$. That is $u - \frac{1}{n} > x, \forall\, x \in S$, which means $u - \frac{1}{n}$ is an upper bound of S. But this is not possible as $u = \sup S$ and $u - \frac{1}{n} < u$. Hence, our assumption $u^2 > a$ is false.

A similar approach can be used to show that $u^2 < a$ is also not true (see E20). Hence, by contradiction, $u^2 = a$. ■

Now, given $a \in \mathbb{R}^+$, we can define the **positive square root** of a as $\sqrt{a} = a^{\frac{1}{2}} = \sup\{x \in \mathbb{R}^+ \mid x^2 \leq a\}$. Similarly, the **positive n^th root** of a is defined as $a^{\frac{1}{n}} = \sup\{x \in \mathbb{R}^+ \mid x^n \leq a\}$.

Next, we state the following theorem, without proof.

Theorem 11: Let $a \in \mathbb{R}^+$. Then $a^{\frac{1}{n}} \in \mathbb{R}^+$ for all $n \in \mathbb{N}$.

Theorem 12: Let $a \leq \frac{1}{n}$, for all $n \in \mathbb{N}$. Then, $a \leq 0$.

Proof: Assume, if possible, that $a > 0$. Then the given inequality is

$$a \leq \frac{1}{n}, \forall\, n \in \mathbb{N}.$$

But this contradicts the Archimedean property. Hence, our assumption is wrong. Thus, $a \leq 0$. ■

Let us see some applications of Theorem 12.

Example 12: Find $\inf S$, where $S = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$.

Solution: Note that $0 < \frac{1}{n}$ for all $n \in \mathbb{N}$. Thus 0 is a lower bound of S. Let ℓ be another lower bound of S. Then

$\ell \le \frac{1}{n}$ for all $n \in \mathbb{N}$. So, by Theorem 12, $\ell \le 0$. Hence, $\inf S = 0$.

***

Example 13: What is $\inf S$, if $S = \left\{ \frac{n+1}{n} \middle| n \in \mathbb{N} \right\}$?

Solution: You can see that

$$1 < 1 + \frac{1}{n} = \frac{n+1}{n}, \forall n \in \mathbb{N}.$$

This means 1 is a lower bound of S. Now, let ℓ be another lower bound of S. Then,

$$\begin{aligned} \ell \le \frac{n+1}{n}, \ \forall n \in \mathbb{N} &\Rightarrow \ell \le 1 + \frac{1}{n}, \ \forall n \in \mathbb{N} \\ &\Rightarrow \ell - 1 \le \frac{1}{n}, \ \forall n \in \mathbb{N} \\ &\Rightarrow \ell - 1 \le 0, \qquad \text{by Theorem 15} \\ &\Rightarrow \ell \le 1 \end{aligned}$$

Hence $\inf S = 0$.

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Now we turn our attention to learn about the absolute values and inequalities associated with the real numbers.

3.3.2 Absolute Values and Inequalities

First we define what we mean by the absolute value of a real number.

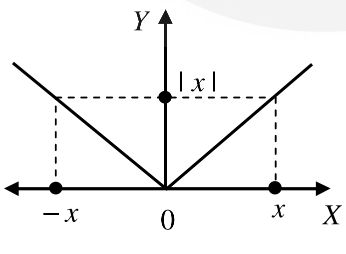


Fig. 7: Graph of $|x|$

Definition: For $x \in \mathbb{R}$, the **absolute value** of x, denoted by $|x|$ (read as $'\text{mod } x'$) is defined as

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0. \end{cases}$$

Absolute value is also sometimes called the **magnitude.** This is because $|x|$ is always a positive number for $x \ne 0$ and $|0| = 0$. Actually, $|\ |$ is a function from $\mathbb{R}$ to $\mathbb{R}^+ \cup \{0\}$ whose graph is shown in Fig. 7.

Observe that $|5| = 5$, $|-3| = 3$, and $|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$.
In the following examples we present a a few results concerning the absolute values.

Example 14: Show that $|xy| = |x| \cdot |y|$, for all real numbers x and y.

Solution: An elegant proof is the following. For any $x, y \in \mathbb{R}$

$$|xy| = \sqrt{(xy)^2} = \sqrt{x^2\, y^2} = \sqrt{x^2}\,\sqrt{y^2} = |x||y|.$$

***

You can regard $|a|$ as the **distance** of a from 0. Using this we can define the distance between any two elements a, b of $\mathbb{R}$ as $|a-b|$. (See Fig. 8). We shall formally define what we mean by the distance in Unit 4.

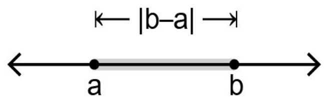


Fig. 8: Distance between a and b.

Now let us consider the following result.

Example 15: Show that $|x+y| \le |x| + |y|, \qquad \forall x, y \in \mathbb{R}.$

Solution: If possible, let us assume that $|x+y| > |x| + |y|$. Then,

$$\begin{aligned} |x+y|^2 &= (x+y)^2 = x^2 + y^2 + 2xy \\ &\le x^2 + y^2 + 2|x||y| \\ &= |x|^2 + |y|^2 + 2|x||y| \\ &= (|x| + |y|)^2 \end{aligned}$$

Note that $|a-b| = |b-a|$ for all $a, b \in \mathbb{R}$.

(See Example 14 and the observation above Example 14.) Since both the sides are positive, we have, $|x+y| \le |x| + |y|$.

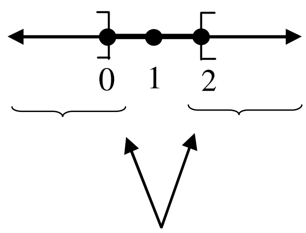
***

Now we discuss another consequence of the completeness property – the inequalities. You have seen some inequalities just above. Now, we discuss them in detail, algebraically and geometrically.

Consider, for instance, the inequality $2x \le x^2$, where $x \in \mathbb{R}$. Certainly it is not true for all $x \in \mathbb{R}$. For example $x = 1$ doesn't satisfy it. However, our aim is to find all those $x \in \mathbb{R}$ for which it is true. We can do this by using the definition of $\le$ and Theorem 4.

Using definition of $\le$, we have

$$\begin{aligned} 2x < x^2 &\Leftrightarrow 2x - x^2 \le 0 \Leftrightarrow x(2-x) \le 0 \\ &\Leftrightarrow (x \le 0 \text{ and } 2 - x \ge 0) \text{ or } (x \ge 0 \text{ and } 2 - x \le 0) \\ &\Leftrightarrow (x \le 0 \text{ and } x \le 2) \quad \text{or } (x \ge 0 \text{ and } x \ge 2) \\ &\Leftrightarrow x \le 0 \text{ or } x \ge 2 \end{aligned}$$



$\{x \in \mathbb{R} \mid x \le 0 \text{ or } x \ge 2\}$

Fig. 9

Thus we find that the inequality $2x \le x^2$ is satisfied by only those real numbers that are either less than or equal to 0 or greater than or equal to 2. In set notation, this means

$$\{x \in \mathbb{R} \mid 2x \le x^2\} = \{x \in \mathbb{R} \mid x \le 0 \text{ or } x \ge 2\}.$$

You can see that the set in the right hand side is much easier to visualize that the one in the left hand side. (See the geometrical representation of this set in Fig. 9.)

Let us consider some more examples.

Example 16: Let $a > 0$. How are the inequalities $|x| < a$ and $|x| \geq a$ related? Describe geometrically.

Solution: First let us consider the inequality $|x| < a$. When $x > 0$, it gives $x < a$. On the other hand, when $x \leq 0$, it gives $-x < a$, i.e., $-a < x$. Thus we get $-a < x < a$. Thus $|x| < a$ means x lies strictly between $-a$ and a.
Now we consider the inequality $|x| \geq a$. When $x > 0$, it gives $x \geq a$, otherwise $-x \geq a$, i.e., $x \leq -a$. Thus $|x| \geq a$ is equivalent to $x \leq -a$ or $x \geq a$.
In set terminology, we find that the sets

$$\{x \in \mathbb{R} \mid |x| < a\} \text{ and } \{x \in \mathbb{R} \mid |x| \geq a\}$$

are the complements of each other. Geometrically, they are described in Fig. 10.

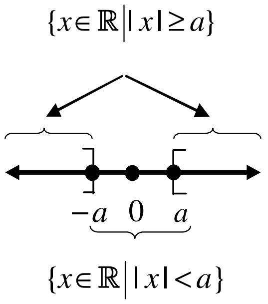


Fig. 10

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Example 17: Describe the set $S = \{x \in \mathbb{R} \mid |x| - 4 \leq |x+5|\}$ geometrically.

Solution: We write the inequality as

$$|x+5| \geq |x| - 4$$

$\Leftrightarrow$ either $x+5 \geq |x| - 4$ or $x+5 \leq -|x| + 4$

$\Leftrightarrow$ either $|x| \leq x+9$ or $|x| \leq -x-1$

$\Leftrightarrow$ either $-x-9 \leq x \leq x+9$ or $x+1 \leq x \leq -x-1$

$\Leftrightarrow$ either $-x-9 \leq x$ and $x \leq x+9$ or $x+1 \leq x$ and $x \leq -x-1$

$\Leftrightarrow$ either $x \geq -\dfrac{9}{2}$ and $0 \leq 9$ or $1 \leq 0$ and $x \leq -\dfrac{1}{2}$

$\Leftrightarrow x \geq -\dfrac{9}{2}$

The last inequality follows from the fact that "$1 \leq 0$ and $x \leq -\frac{1}{2}$" is false.

Hence $S = \left\{x \in \mathbb{R} \,\middle|\, x \geq -\frac{9}{2}\right\}$. See Fig. 11 for a geometrical description of S.

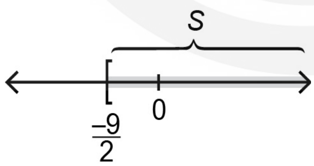

Fig. 11

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Example 18: Let p be a prime number. Show that $\{x \in \mathbb{Q}^+ \mid x^2 > p\} = \sqrt{p}$.

Solution: Let $\ell = \inf S$, where $S = \{x \in \mathbb{Q}^+ \mid x^2 > p\}$. First assume, if possible that $\ell < \sqrt{p}$. Then, to arrive at a contradiction, we have to find a natural number n such that $\ell + \frac{1}{n}$ is a lower bound of S. That is, we have to find a natural number n such that for all $x \in S$

$$\left(\ell + \frac{1}{n}\right)^2 < x^2.$$

But $x^2 > p$ for all $x \in S$. So, it is sufficient to find n such that

$$\left(\ell + \frac{1}{n}\right)^2 < p \Leftrightarrow \frac{1}{n^2} + \frac{2\ell}{n} < p - \ell^2.$$

Now you know that $\frac{1}{n^2} < \frac{1}{n}$ which implies

$$\frac{1}{n^2} + \frac{2\ell}{n} < \frac{1+2\ell}{n}.$$

Hence, it is sufficient to find n such that

$$\frac{1+2\ell}{n} < p - \ell^2, \text{ that is, } n > \frac{1+2\ell}{p-\ell^2}.$$

Since such an n always exists by the Archimedean Property, so our assumption that $\ell < \sqrt{p}$ is false.

Now let us assume that $\ell > \sqrt{p}$. This time we need to find a natural number n such that $\ell - \frac{1}{n} \in S$. That is, we need to find a natural number n such that

$$\left(\ell - \frac{1}{n}\right)^2 > p.$$

But, note that

$$\left(\ell - \frac{1}{n}\right)^2 = \ell^2 - \frac{2\ell}{n} + \frac{1}{n^2} > \ell^2 - \frac{2\ell}{n}.$$

So, it is sufficient to find n such that $\ell^2 - \frac{2\ell}{n} > p$, i.e., $n > \frac{2\ell}{\ell^2 - p}$. But such an n always exists by the Archimedean Property. Hence we have arrived at a contradiction to the definition of ℓ. Therefore, our assumption that $\ell > \sqrt{p}$ is false. Thus we conclude that $\ell = \sqrt{p}$.

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You might be now willing to do some exercises.

E17) Find the infimum and supremum of the set $\left\{\frac{1}{n^2} \middle| n \in \mathbb{N}\right\}$. Use them to find the infimum and supremum of the set $\left(\frac{an^2+1}{n^2} \middle| n \in \mathbb{N}\right)$.

E18) Justify whether the following statements are true or false:

i) $\{x \in \mathbb{R} \mid x < 1\} = \{x \in \mathbb{R} \mid x^2 < 1\}$

ii) $\{x \in \mathbb{R} \mid x|x| > 4\} = \{x \in \mathbb{R} \mid x \geq 2\}$

iii) $|x| + |x-1| \geq |2x-1|, \forall x \in \mathbb{R}$

iv) $\left(1 + \frac{1}{n}\right)^n \geq n, \forall n \in \mathbb{N}$

E19) Describe the set $\left\{ x \in \mathbb{R} \,\middle|\, \frac{2x+1}{3x-4} < 1 \right\}$ geometrically.

E20) Let $a \in \mathbb{R}^+$. Consider the set $S = \{x \in \mathbb{R}^+ \mid x^2 \le a\}$. If $u = \sup S$, then show that $u^2 \ge a$.

E21) Show that for every $x \in \mathbb{R}^+$, there exists some $n \in \mathbb{N}$ such that $n - 1 \le x < n$.

E22) Let x and y be two real numbers such that $x < y$. Show that there exists an irrational number ξ such that $x < \xi < y$.

Thus far we have seen that $\mathbb{R}$ is an ordered field that is complete with respect to the order. We have seen many consequences of the completeness property of $\mathbb{R}$. We expect from you to devote an **ample amount of time on the inequalities** as they are the fundamental to every concept in real analysis. You can also look at our course BMTC-131 (Calculus) to learn many other important inequalities.

3.4 COUNTABLE AND UNCOUNTABLE SUBSETS OF $\mathbb{R}$

How many stars are there in our galaxy? How many sand particles are there in a desert land? How many drops are there in an ocean? These numbers are enormously large. Yet, you will be surprised to know that they are all finite. In this section we shall talk about the size of different subsets of $\mathbb{R}$ and of $\mathbb{R}$ itself. In particular we shall give precise meaning to the terms such as 'finite', 'countable' and 'uncountable'.

People in the old civilizations used to count the sizes of different sets by putting them into one-to-one correspondence (bijection). For example, you know that the sets $\{1, 2, 3, ..., 26\}$ and $\{a, b, c, ..., z\}$ have the same number of elements. We can assign $1 \to a, 2 \to b, 3 \to c$ and so on to get an explicit bijection. Consider a set S. If we can establish a bijection from the set $\{1, 2, 3, ..., n\}$ to S, for some natural number n, then it means that we can count and get the total number of elements of S. Let us extend this idea to the sets $\mathbb{N}$ and $\mathbb{Z}$. Look at the following illustration.

$$\begin{array}{l} \mathbb{N} = \{1, \;\; 2, \;\; 3, \;\; 4, \ldots\} \\ \quad\;\;\; \downarrow \;\; \downarrow \;\; \downarrow \;\; \downarrow \;\; \ldots \\ \mathbb{Z} = \{0, -1, \;\; 1, -2, 2, \ldots\} \end{array}$$

Although in this we cannot get the total number of elements of $\mathbb{Z}$, counting them still makes sense. We can list them. For example, the illustration above provides a list of the elements of $\mathbb{Z}$ with 0 as its 1st element, -1 as the 2nd and so on. You might be thinking some other way of listing them. For example you can regard 0 as the 1st, 1 as the 2nd, -1 as the 3rd, 2 as the 4th and so on. This is possible. The only thing that you have to take care while listing the elements of a set is that each element is listed exactly once and no element is skipped.

Now consider the set $[0,1]$. From your calculus course you know that it has infinitely many points. Can you list its elements? You can say that 0 is the first one. But then what is the second? Is it $0.1, 0.01$ or 0.001? Soon you will realize that you have no way to list its elements.

The discussion above tells us that we can put together the sets like $\{a,b,c,..,z\}$ and $\mathbb{Z}$ and distinguish them from $[0,1]$ by saying that the first two are 'countable' i.e. their elements can be listed and the third one is 'uncountable' because its elements cannot be listed. We give a formal meaning to the terms 'coutable' and 'uncountable' in the following definition.

Definitions:

1) A set S is said to be **finite** if there is a bijection from $\{1,2,3,...,n\}$ to S, for some $n \in \mathbb{N}$.
2) A set S is said to be **countably infinite** if there is a bijection from $\mathbb{N}$ to S.
3) A set is **countable** if it is either finite or countably infinite.
4) A set is **infinite** if it is not finite.
5) A set is **uncountable** if it is not countable.

The empty set is countable as it is finite. The power set of $\{0,1\}$ is also countable as it is finite having the elements $\emptyset, \{0\}, \{1\}$ and $\{0,1\}$. The set $\mathbb{N}$ is also countable as the identity function serves as a bijection from $\mathbb{N}$ to itself.

Now consider the following example.

Example 19: Show that the set $\mathbb{N}_{\text{odd}} = \{1,3,5,7,...\}$ of odd natural numbers is countable.

Solution: Look at the following illustration.

$$\begin{array}{ll} \mathbb{N} & = \{1,\ \ 2,\ \ 3,\ \ 4,\ldots\} \\ & \quad\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ldots \\ \mathbb{N}_{\text{odd}} & = \{1,\ \ 3,\ \ 5,\ \ 7,\ldots\} \end{array}$$

This gives us the function $f : \mathbb{N} \to \mathbb{N}_{\text{odd}}$ defined by $f(n) = 2n-1$. It is easy to check that f is a bijection. Hence, $\mathbb{N}_{\text{odd}}$ is a countable set.

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You can give arguments similar to Example 16 to show that $\mathbb{N}_{\text{even}}$, the set of even natural numbers, is countable.

Is the set $\mathbb{N}_{\text{prime}} = \{2,3,5,7,11,...\}$ of prime numbers countable? You may know that $\mathbb{N}_{\text{prime}}$ is infinite. (Suppose, if possible that there are only finitely many prime numbers, say $p_1, p_2, ..., p_k$. Then $P = p_1 p_2 ... p_k + 1$ is again a prime because P is not divisible by any of $p_1, p_2, ..., p_k$. Therefore, $\mathbb{N}_{\text{prime}}$ is infinite.) So, you need to check whether or not $\mathbb{N}_{\text{prime}}$ is countably infinite. But finding a bijection from $\mathbb{N}$ to $\mathbb{N}_{\text{prime}}$ would be waste of time as it is an 'unsolved problem'. That is, for any arbitrary number n, till date we do not know what the n^{th} prime number is. So, let us look for other ways to show that $\mathbb{N}_{\text{prime}}$ is countably infinite. You know that $\mathbb{N}_{\text{prime}} \subseteq \mathbb{N}$ and also that $\mathbb{N}$ is countable. Does it imply that $\mathbb{N}_{\text{prime}}$ is countably infinite? The answer lies in the following result.

Theorem 13: Every subset of $\mathbb{N}$ is countable.

Proof: You know that every finite subset is countable by definition. So, let S be an infinite subset of $\mathbb{N}$. Then, $S \neq \emptyset$. So, by the well ordering principle, S must have a least element. Let $x_1 = \min(S)$. Again, $S \setminus \{x_1\} \neq \emptyset$, as S is infinite. So, $S \setminus \{x_1\}$ must also have a least element. Let $x_2 = \min(\sin \setminus \{x_1\})$.

Now define

$$x_k = \min(S \setminus \{x_1, x_2, ..., x_{k-1}\}).$$

Again, if $S \setminus \{x_1, x_2, ..., x_k\} = \emptyset$, then $S = \{x_1, x_2, ..., x_k\}$, which is not possible. So, $S \setminus \{x_1, x_2, ..., x_k\} \neq \emptyset$. Hence by well ordering principle $x_{k+1} = \min(S \setminus \{x_1, x_2, .., x_k\})$ is well defined.

This gives us a function $f : \mathbb{N} \to S$ defined by $f(k) = x_k, \forall k \in \mathbb{N}$.

Note that the way we have constructed x_k, we see that

$$x_1 < x_2 < x_3 < ...$$

This proves that f is one-one.

To show that f is onto, take $x \in S$. Then $x > x_1$. So, let k be the largest natural number such that $x > x_k$. This means $x \leq x_{k+1}$. Now $x < x_{k+1}$ means $\min(S \setminus \{x_1, x_2, ..., x_k\}) = x$, which contradicts the definition of x_{k+1}. Hence $x = x_{k+1}$.

This shows that every element of S has some pre-image in $\mathbb{N}$. That is f is onto. Hence, S is countable. ■

Theorem 13 tells us that there is a bijection from $\mathbb{N}$ to ever nonempty infinite subset of $\mathbb{N}$. It has an important consequence. Consider any nonempty set S. To prove that S is countable we need not find a bijection from $\mathbb{N}$ to S. An injection from S to $\mathbb{N}$ (or a surjection from $\mathbb{N}$ to S) is sufficient. This is what the following theorem states.

Theorem 14: Let S be an infinite subset of $\mathbb{R}$.

i) If $f : S \to \mathbb{N}$ is an injection, then S is countable.
ii) If $f : \mathbb{N} \to S$ is a surjection, then S is countable.

Proof: i) Let $M = f(S)$. Since $f : S \to \mathbb{N}$ is an injection, $f : S \to M$ is a bijection. But $\mathrm{M} \subseteq \mathbb{N}$, so by Theorem 13, M is countable. That is, there is a bijection $g : M \to \mathbb{N}$. Now from your knowledge of calculus, you know that $g \circ f : S \to \mathbb{N}$ is a bijection. Hence S is countable. (Why?)

ii) Since $f:\mathbb{N}\to S$ is a surjection, for each element $x\in S$, there is some element $n\in\mathbb{N}$ such that $f(n)=x$. Then $f^{-1}(x)$ is a nonempty subset of $\mathbb{N}$. Hence by well ordering principle $f^{-1}(x)$ has a least element.

Recall that $f^{-1}(x)$ is a set.

So define $g:S\to\mathbb{N}$ by $g(x)=\min(f^{-1}(x))$,

Now we show that g is $1-1$. Let $x_1,x_2\in S, x_1\neq x_2$. Since f is a function, $f^{-1}(x_1)$ and $f^{-1}(x_2)$ must have no element in common. This implies $\min(f^{-1}(x_1))$ and $\min(f^{-1}(x_2))$ cannot be equal. That is $g(x_1)\neq g(x_2)$. Now, by part (i), S is countable. ■

As an application of Theorem 14, consider the following example.

Example 20: Show that $\mathbb{N}\times\mathbb{N}$ is countable.

Solution: We define a function $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ by $f(m,n)=(m+n)^2+n$. If we prove that f is an injection our task is over. So, for $(m,n),(p,q)\in\mathbb{N}\times\mathbb{N}$. we have

$$\begin{aligned} f(m,n)=f(p,q) &\Leftrightarrow (m+n)^2+n=(p+q)^2+q \\ &\Leftrightarrow (m+n)^2-(p+q)^2=q-n \end{aligned}$$

That is, $(m+n+p+q)(m+n-p-q)=q-n$, which implies $(m+n+p+q)|m+n-p-q|=|q-n|$. Now, if $q\neq n$, then $m+n+p+q$ divides $|q-n|$. But this means $m+n+p+q\leq|q-n|$, which is impossible.

Hence, $q=n$. Now,

$$\begin{aligned} (m+n+p+q)|m+n-p-q|=0 &\Rightarrow |m+n-p-q|=0 \\ &\Rightarrow m+n-p-q=0 \\ &\Rightarrow m+n=p+q \\ &\Rightarrow m=p \qquad (\because q=n) \end{aligned}$$

Thus we have proved that f is injective. Hence by Theorem 14(i), $\mathbb{N}\times\mathbb{N}$ is countable.

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Theorem 15: A countable union of countable sets is countable. In other words, if $\{S_1,S_2,S_3,....\}$ is a collection of countable sets then $\bigcup_{n=1}^{\infty}S_n$ is countable.

Proof: Since S_n's are countable, we can list their elements as follows:

$$\begin{aligned} &S_1: x_{11},x_{12},x_{13},x_{14},.... \\ &S_2: x_{21},x_{22},x_{13},x_{24},.... \\ &S_3: x_{31},x_{32},x_{33},x_{34},.... \\ &\vdots \end{aligned}$$

We define $f:\mathbb{N}\times\mathbb{N}\to\bigcup_{n=1}^{\infty}S_n$ by $f(m,n)=x_{mn}$. Note that f is surjection. Also, since $\mathbb{N}\times\mathbb{N}$ is countable, there is some bijection $g:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$. Now $f\circ g:\mathbb{N}\to\bigcup_{n=1}^{\infty}S_n$ is surjection. (Why? Use your calculus knowledge.) Hence, by Theorem 15(ii), $\bigcup_{n=1}^{\infty}S_n$ is countable. ■

Example 21: Show that $\mathbb{Q}$ is countable.

Solution: Define $f:\mathbb{Q}\to\mathbb{N}$ by $f\left(\frac{m}{n}\right)=(m+n)^2+n,$

where m and n have no common factors.

Then similar to Example 17, you can prove that f is an injection. Therefore, by Theorem 14(i), $\mathbb{Q}$ is countable.

***

Thus far we have seen examples of countable subsets of $\mathbb{R}$ only. However many subsets of $\mathbb{R}$ are uncountable. First, we shall see that $\mathbb{R}$, itself, is uncountable.

Theorem 16: $\mathbb{R}$ is uncountable.

Proof: We prove it by contradiction. Assume that $\mathbb{R}$ is countable. Let $L=(x_1,x_2,x_3,...)$ be the list of all the elements of $\mathbb{R}$. Recall that the decimal expansion of a real number x is

$$x=A.a_1a_2a_3a_4a_5\ldots$$

with the exception that only a finite number of a_i's can be 0. Then we can write

$$\begin{aligned}
x_1 &= A_1.a_{11}\,a_{12}\,a_{13}\,a_{14}\,a_{15}\ldots\\
x_2 &= A_2.a_{21}\,a_{22}\,a_{23}\,a_{24}\,a_{25}\ldots.\\
x_3 &= A_3.a_{31}\,a_{32}\,a_{33}\,a_{34}\,a_{35}\ldots.\\
x_4 &= A_4.a_{41}\,a_{42}\,a_{43}\,a_{44}\,a_{45}\ldots.\\
x_5 &= A_5.a_{51}\,a_{52}\,a_{53}\,a_{54}\,a_{55}\ldots.\\
&\vdots
\end{aligned}$$

Now we construct a real number $y=0.b_1b_2b_3\ldots$ such that $b_k\neq a_{kk},\ \forall\ k=1,2,3,\ldots$ This means that y cannot be equal to x_1 as $b_1\neq a_{11}$. Similarly y cannot be equal to x_2 as $b_2\neq a_{22}$. Indeed $y\neq x_i,\ \forall\ i=1,2,\ldots$

This means that y is not listed in L. This contradicts the assumption that L contains all the elements of $\mathbb{R}$. Hence $\mathbb{R}$ is uncountable. ■

Let us consider an example.

Example 22: Show that the set of all irrational numbers is uncountable.

Solution: We prove it by contradiction. Assume that the set $\mathbb{R}\setminus\mathbb{Q}$ is countable. Then, $\mathbb{R}$, being the union of two countable sets $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$, is also countable. But this is impossible. Hence $\mathbb{R}\setminus\mathbb{Q}$ is uncountable.

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You can do now following exercises.

E23) Let S be a countable set and there is a bijection from S to T. Show that T is countable.

E24) Let $a\in\mathbb{R}$. Check whether the set $\{x\in\mathbb{R}\,|\,x>a\}$ is countable or uncountable.

E25) Show that every infinite set contains a countable subset.

E26) Let S be a finite set and T be a countably infinite set such that $S\cap T=\emptyset$. Show that $S\cup T$ is a countably infinite set.

We have seen that many familiar subsets of $\mathbb{R}$ such as $\mathbb{N},\mathbb{Z}$ and $\mathbb{Q}$ are countable, while $\mathbb{R}$ and $\mathbb{R}\setminus\mathbb{Q}$ are uncountable. We shall consider many other uncountable subsets of $\mathbb{R}$ in the next unit. We end this unit here.

3.5 SUMMARY

In this unit we have considered the following points.

1. Both $\mathbb{Q}$ and $\mathbb{R}$ as ordered fields.
2. Inequalities as a consequence of order properties of $\mathbb{R}$.
3. Greatest lower bound and least upper bound properties of $\mathbb{R}$.
4. Archimedean property of $\mathbb{R}$ and its applications such as in
   i) showing the existence of a rational number between every two real numbers.
   ii) showing the existence of an irrational number between every two real numbers.
   iii) computing infimums and supremums of subsets of $\mathbb{R}$
5. The concept of finite, countable and uncountable sets
   i) Examples of these concepts from subsets of $\mathbb{R}$.
   ii) Some results related to these concepts.

3.6 SOLUTIONS/ANSWERS

E1) i) $2.596306=2+\dfrac{5}{10}+\dfrac{9}{100}+\dfrac{6}{1000}+\dfrac{3}{10000}+\dfrac{0}{100000}+\dfrac{6}{1000000}=\dfrac{1298153}{500000}$

ii) Let $x=4\cdot 76\overline{324}=4\cdot 76+\dfrac{y}{100}$, where $y=0\cdot\overline{324}=0\cdot 324324\ldots$

Then $1000y = 324\cdot\overline{324}$. So, subtracting y from $1000y$, we get $1000y - y = 324\cdot\overline{324} - 0\cdot\overline{324} = 324$.

Then $y = \frac{324}{999} = \frac{36}{14}$. Thus $x = 4\cdot 76 + \frac{36}{111000} = \frac{52872}{11100}$.

Let $m^2 = kp$, for some $k \in \mathbb{Z}$. Now m divides m^2. Hence m divides kp. But p is a prime, so m divides k. Then $k = m\ell$ for some $\ell \in \mathbb{Z}$. Therefore, $m^2 = m\ell p$ which implies p divides m.

E2) Let $\sqrt{p} = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, where m and n have no common factor. Then

$$p = \frac{m^2}{n^2} \Rightarrow \qquad m^2 = pn^2.$$

This means m^2 is a multiple of p, and hence m is a multiple of p. (why?, see margin). That is, $m = pk$ for some $k \in \mathbb{Z}$. Now we have

$$(pk)^2 = pn^2 \Rightarrow \qquad n^2 = pk^2.$$

Again using the same argument, we find that n is a multiple of p. Thus p is a common factor of both m and n, which is a contradiction. Hence, $\sqrt{p} \notin \mathbb{Q}$.

E3) Draw a straight line and mark on it 0 at some point. Then mark O at 0 and 1,2,3 and so on at equal intervals. Draw a perpendicular of length 1 unit at 1 (See Fig 12 below).

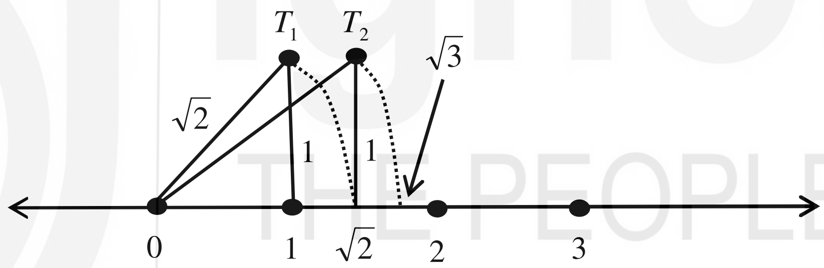

Fig.12: Representation of $\sqrt{3}$ on the real line

Let T_1 be the top of this perpendicular. The length of OT_1 is $\sqrt{2}$. Take a compass and put its foot at O and pencil at T_1 and draw an arch. It cuts the line at the point $\sqrt{2}$. Now draw a perpendicular of length 1 unit at $\sqrt{2}$. Let T_2 be the top of this perpendicular. Fix the foot of the compass at O and pencil at T_2 and draw an arch. It cuts the line at $\sqrt{3}$ because $OT_2 = \sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$.

You can extend this procedure to locate $\sqrt{p}$, for any prime number p.

E4) i) We have

$$\begin{aligned}
(-1)a + a &= (-1)a + 1.a && \text{(Property M(iii))}\\
&= a.(-1) + a.1 && \text{(Property M(i))}\\
&= a.(-1+1) && \text{(Distributivity)}\\
&= a.0 && \text{(Property A(iii))}\\
&= 0 && \text{(Theorem 2 (iii))}
\end{aligned}$$

Since additive inverse is unique, $(-1)a = -a$.

ii) Put $a=-1$ in i) and use the fact that $-(-1)=1$.

E5) We have

$$\begin{aligned}(a+b).c &= c.(a+b) && \text{(Property M(i))}\\ &= c.a+c.b && \text{(Distributivity)}\\ &= a.c+b.c && \text{(Property M(i))}\end{aligned}$$

E6) We can see that

$$\begin{aligned}x^2+5x+6=0 &\Rightarrow (2+3)x+6=0\\ &\Rightarrow x^2+2x+3x+6=0 && \text{(Distributivity)}\\ &\Rightarrow x(x+2)+3(x+2) && \text{(Distributivity)}\\ &\Rightarrow (x+3)(x+2)=0 && \text{(Distributivity)}\\ &\Rightarrow x+3=0 \text{ or } x+2=0 && \text{(Theorem 3)}\\ &\Rightarrow x=-3 \text{ or } x=-2 && \text{(Property A(iv))}\end{aligned}$$

E7) For $n=1$, the statement is

$$(a+b)^1 = {}^1C_0 a^1 b^0 + {}^1C_1 a^0 b^1 ,$$

Recall that ${}^nC_r = \frac{n!}{r!(n-r)!}$

which is true. Now, assume that the statement is true for some $n\in\mathbb{N}$. Then

$$\begin{aligned}(a+b)^{n+1} &= (a+b)^n(a+b)\\ &= \left(\sum_{i=0}^{n} {}^nC_i\, a^{n-i} b^i\right)(a+b)\\ &= \sum_{i=0}^{n} {}^nC_i\, a^{n+1-i} b^i + \sum_{i=0}^{n} {}^nC_i a^{n-i} b^{i+1}\\ &= a^{n+1} + {}^nC_1\, a^n b + {}^nC_2 a^{n-1} b^2 + \cdots + {}^nC_n ab^n\\ &\qquad + {}^nC_0\, a^n b + {}^nC_1 a^{n-1} b^2 + \cdots + {}^nC_{n-1} a b^n + b^{n+1}\\ &= a^{n+1} + \left({}^nC_1 + {}^nC_0\right)a^n b + \left({}^nC_2 + {}^nC_1\right)a^{n-1}b^2 + \cdots + \left({}^nC_n + {}^nC_{n-1}\right)ab^n + b^{n+1}\\ &= a^{n+1} + \sum_{i=1}^{n}\left({}^nC_i + {}^nC_{i-1}\right)a^{n-1}b^i + b^{n+1}\\ &= a^{n+1} + \sum_{i=1}^{n} {}^{n+1}C_i\, a^{n+1-i} b^i + b^{n+1}\\ &= \sum_{i=0}^{n+1} {}^nC_i\, a^{n+1-i} b^i\end{aligned}$$

Hence, the statement is true for $n+1$ also. Thus by the PMI, the statement of the Binomial Theorem is true for all $n\in\mathbb{N}$.

E8) Since $x>0$ and $y>0$, we have $\frac{1}{x}>0$ and $\frac{1}{y}>0$. Now

$$x>y \Rightarrow \frac{1}{y}\cdot x > \frac{1}{y}\cdot y \qquad \text{(by Theorem 4 (iii))}$$

$$\Rightarrow \frac{1}{y}\cdot x > 1 \qquad \left(\frac{1}{y}.y=1.\right)$$

Consequently, $\frac{1}{y}.x.\frac{1}{x} > 1.\frac{1}{x}$. That is, $\frac{1}{y} > \frac{1}{x}$.

E9) Assume, if possible, $ac \geq bd$. Then, since $c > 0$, we have $\frac{1}{c} > 0$. Then

$$ac \cdot \frac{1}{c} > bd \cdot \frac{1}{c} \Rightarrow a > b \cdot \frac{d}{c}$$

Now $d > c$ and $c > 0$ imply $\frac{d}{c} > 1$. (Why?) Then $b > 0$ implies $b \cdot \frac{d}{c} > b$. So $a > b$. But, this is a contradiction. Therefore, $ac < bd$.

E10) We are give that $x > y > 0$. Let $P(n): x^n > y^n, n \in \mathbb{N}$. Then, $P(1): x > y$ is true, as it is given. Assume, now, that $P(n)$ is true for some n. Then

$$x^{n+1} = x^n.x > y^n.x > y^n.y = y^{n+1}.$$

Hence $P(n+1)$ is true. Thus, by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$.

E11) Let us first note that $x < a$ implies that $x^2 < ax$ because $x > 0$. Also, from the hypothesis we know that $x < 1$. This implies $ax < a$ because $a > 0$. Next, we prove that $x^n < a$, for all $n \in \mathbb{N}$ using the PMI. So, let $P(n): x^n < a, \quad n \in \mathbb{N}$. Then $P(1)$ is true as given. Now, let $P(n)$ be true for some $n \in \mathbb{N}$. Then $x^{n+1} = x^n.x < ax < a$ using the induction hypothesis, and the fact that $0 < x < 1$. This means $P(n+1)$ is true. Hence, by PMI $P(n)$ is true for all $n \in \mathbb{N}$.

E12) We have

$$a > 1 \Rightarrow a > 0 \Rightarrow a^m > 0 \Rightarrow \frac{1}{a^m} > 0.$$

Now

$$\begin{aligned} a^n > a^m &\Leftrightarrow a^n.\frac{1}{a^m} > a^m.\frac{1}{a^m} \\ &\Leftrightarrow a^{n-m} > 1 \\ &\Leftrightarrow n - m > 0 \\ &\Leftrightarrow n > m \end{aligned}$$

The result does not hold when $a < 1$. For instance, take $a = \frac{1}{2}, n = 2, m = 1$. Then we get $\frac{1}{2^2} > \frac{1}{2}$ which is false.

E13) We know that for all $x \in S$

$$\inf S \leq x \leq \sup S.$$

Therefore, $\inf S \leq \sup S$.

When S is a singleton set, i.e. $S = \{x\}$, then $\inf S = x = \sup S$.

E14) Let $S = \left\{ \left(n + \frac{1}{n} \right)^n \middle| n \in \mathbb{N} \right\}$.

For $n = 1$ we have

$$\left(n + \frac{1}{n} \right)^n = \left(1 + \frac{1}{1} \right)^1 = 2 \in S.$$

Hence $S \neq \emptyset$. Also for all $n \geq 1$

$$n+\frac{1}{n} \geq 1 \Rightarrow \left(n+\frac{1}{n}\right)^n \geq 1.$$

Hence S is bounded below. Therefore, by the Greatest Lower Bound Property of $\mathbb{R}$, $\inf S$ exists in $\mathbb{R}$,

For supremum, note that for all $n \in \mathbb{N}$,

$$\left(n+1+\frac{1}{n+1}\right)^{n+1} > \left(n+1+\frac{1}{n+1}\right)^n > \left(n+\frac{1}{n}\right)^n \qquad \left(\because 1+\frac{1}{n+1} > \frac{1}{n}\right)$$

(Using E12 and E10 in this order.) This means $\left(n+\frac{1}{n}\right)^n$ increases with n. Therefore, S is not bounded above. Consequently, $\sup S$ does not exist in $\mathbb{R}$.

E15) Let $\ell = \inf S$ and $u = \sup S$.

Case i) $a \geq 0$: Since, u is an upper bound of S, for each $x \in S$,

$$x \leq u \Rightarrow ax \leq au \qquad (\because a \geq 0)$$

Hence, au is an upper bound of aS. Now, if v is any upper bound of aS, then

$$ax \leq v \,\forall x \in S \Rightarrow x \leq \frac{v}{a} \,\forall x \in S$$

$$\Rightarrow \frac{v}{a} \text{ is an upper bound of } S.$$

$$\Rightarrow u \leq \frac{v}{a} \Rightarrow au \leq v$$

Therefore, $\sup(aS) = au = a\sup S$.

Case ii) $a < 0$: Since ℓ is a lower bound of S, for each $x \in S$, $\ell \leq x \Rightarrow a\ell \geq ax$. This means, $a\ell$ is an upper bound of aS.

Now, if v is any upper bound of aS, then

$$ax \leq v \,\forall x \in S \Rightarrow x \geq \frac{v}{a} \,\forall x \in S$$

$$\Rightarrow \frac{v}{a} \text{ is a lower bound of } S$$

$$\Rightarrow \frac{v}{a} \leq \ell \quad (\because \ell = \inf S)$$

$$\Rightarrow v \geq a\ell \quad (\because a < 0)$$

This implies, $\sup(aS) = a\ell = a\inf S$.

E16) Since, S is nonempty and bounded above, $\sup S$ exists in $\mathbb{R}$. So, let $u = \sup S$. Then, since u is an upper bounded of S, for each $x \in S$,

$$x \leq u \Rightarrow x^2 \leq u^2 \qquad (\because x \geq 0)$$

(Using E10.) Hence u^2 is an upper bound of S^2.

Now, let v be any upper bound of S^2. Then, for all $x \in S$,

$$
\begin{aligned}
x^2 \le v \Rightarrow v - x^2 &\ge 0\\
&\Rightarrow \left(\sqrt{v}-x\right)\left(\sqrt{v}+x\right) \ge 0\\
&\Rightarrow \sqrt{v} - x \ge 0 \qquad\qquad (\because \sqrt{v}+x \ge 0)\\
&\Rightarrow \sqrt{v} \ge x.
\end{aligned}
$$

This implies, $\sqrt{v}$ is an upper bound of S. But $u = \sup S$. So, $u \le \sqrt{v}$, which implies $u^2 \le v$ again using E10. This proves that $\sup S^2 = u^2 = (\sup S)^2$.

For $\emptyset \ne S \subseteq \mathbb{R}^-$, do yourself.

E17) Let $S = \left\{\frac{1}{n^2}\middle| n \in \mathbb{N}\right\}$.

On the lines of Example 12, you can show that $\inf S = 0$ and $\sup S = 1$.

Let $T = \left\{\frac{an^2+1}{n^2}\middle| n \in \mathbb{N}\right\} = \left\{a + \frac{1}{n^2}\middle| n \in \mathbb{N}\right\} = a + S$.

Hence, using Example 10, you can see that

$$\inf T = \inf(a+S) = a + \inf S = a + 0 = a.$$

Similarly, $\sup T = \sup(a+S) = a + \sup S = a + 1$.

E18) i) False. Because $-2 \in \{x \in \mathbb{R} | x < 1\}$ but $-2 \notin \{x \in \mathbb{R} | x^2 < 1\}$.

ii) True. This is because for $x > 0$,

$$x|x| \ge 4 \Leftrightarrow x^2 \ge 4 \Leftrightarrow x \ge 2.$$

And, for $x \le 0$, $x|x| \ge 4 \Leftrightarrow -x^2 \ge 4 \Leftrightarrow x^2 \le -4$, which is impossible.

Thus, $\{x \in \mathbb{R} | x|x|x \ge 4\} = \{x \in \mathbb{R} | x \ge 2\}$.

iii) True. By the triangle inequality, you can see that

$$|2x-1| = |x + x - 1| \le |x| + |x-1|, \qquad \forall x \in \mathbb{R}.$$

iv) False. Because for $n = 3$ we get $\left(\frac{4}{3}\right)^3 \ge 3$, which is false.

E19) Consider the inequality

$$\frac{2x+1}{3x-4} < 1 \qquad \dots (2)$$

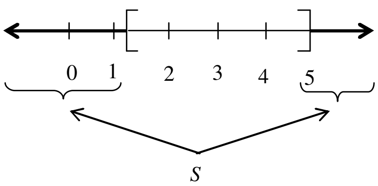

Fig.13: $S = \left\{x \in \mathbb{R} \middle| x > 5 \text{ or } x < \frac{4}{3}\right\}$

There are two cases i) $3x - 4 \ge 0$, ii) $3x - 4 < 0$

Case i): $3x - 4 \ge 0$

This means $x \ge \frac{4}{3}$, Now, Eq. (2) can be written as

$2x + 1 < 3x - 4 \Leftrightarrow 5 < x$.

That is, we have $x \geq \frac{4}{3}$ and $x > 5$, which is $x > 5$.

Case ii): $3x - 4 < 0$

This means $x < \frac{4}{3}$ in this case the inequality (2) reduces to

$$2x + 1 > 3x - 4 \Leftrightarrow 5 > x$$

That is, we have $x < \frac{4}{3}$ and $x > 5$, which is $x < \frac{4}{3}$.

Combining both the cases we get

$$\left\{x \in \mathbb{R} \left| \frac{2x+1}{3x-4} < 4 \right.\right\} \Leftrightarrow \left\{x \in \mathbb{R} \left| x > 5 \text{ or } x < \frac{4}{3} \right.\right\}$$

In Fig. 13, the shaded portion of the real line represents this set.

E20) Let us assume that $u^2 < a.$ To reach a contradiction, we need to find a natural number n such that $u + \frac{1}{n} \in S.$ That is, we need to find an $n \in \mathbb{N}$ such that

$$\left(u + \frac{1}{n}\right)^2 \leq a \Leftrightarrow u^2 + \frac{1}{n^2} + \frac{2u}{n} \leq a.$$

But you know that

$$\frac{1}{n^2} < \frac{1}{n} \Rightarrow u^2 + \frac{1}{n^2} + \frac{2u}{n} < u^2 + \frac{1+2u}{n}.$$

Thus if we can find an $n \in \mathbb{N}$ such that

$$u^2 + \frac{1+2u}{n} \leq a,$$

our task is over. This is equivalent to finding an $n \in \mathbb{N}$ such that $n \geq \frac{1+2u}{a-u^2}.$ Such an n always exists, by the Archimedean property. Thus we have arrived at a contradiction. Therefore, $u^2 \geq a.$

E21) Assume that the statement is false. That is, for all $n \in \mathbb{N},$ either $x < n-1$ or $x \geq n.$ Now, if $x < n-1, \forall n \in \mathbb{N},$ then for $n = 1$ we get $x < 0,$ which is a contradiction as $x \in \mathbb{R}^+.$

The case $x \geq n, \forall n \in \mathbb{N}$ is also not possible due to the Archimedean Property. Therefore, we must have some $n \in \mathbb{N}$ such that $n - 1 \leq x < n.$

E22) You are given $x < y.$ This implies $\sqrt{2}x < r < \sqrt{2}y.$ Hence, by Theorem 9, there exists a rational r such that $\sqrt{2}x < r < \sqrt{2}y.$ This implies $x < \frac{r}{\sqrt{2}} < y.$ Now show that $r/\sqrt{2}$ is an irrational number.

E23) Since S is countable, there exists a bijection $f : \mathbb{N} \to S.$ Let $g : S \to T$

be the bijection given. Then $g \circ f : \mathbb{N} \to T$ is also a bijection. This means, T is countable.

E24) Let $S = \{x \in \mathbb{R} \mid x > a\}$. The set S is uncountable. If possible, assume that S is countable. Let $T = \{x \in \mathbb{R} \mid x < a\}$. Define $g : S \to T$ as $g(x) = 2a - x$. We can see that g is a bijection. Then by E23, T is countable. Now $\mathbb{R} = S \cup T \cup \{a\}$ is a union of countable sets. Hence, $\mathbb{R}$ is countable, which is a contradiction.

Therefore, S is uncountable.

E25) Let S be infinite. Then $S \neq \emptyset$. So, pick $x_1 \in S$. Define $A_1 = \{x_1\}$. Again, $S \setminus A \neq \emptyset$ as S is infinite. Pick $x_2 \in S \setminus A$. Define $A_2 = A_1 \cup \{x_2\}$. Continuing this way, let us assume that A_n is defined.

Then, $S \setminus A_n \neq \emptyset$ as S is infinite. So, pick $x_{n+1} \in S \setminus A_n$. Then define $A_{n+1} = A_n \cup \{x_{n+1}\}$. This shows that A_n is well defined for all $n \in \mathbb{N}$. Now, we know that for each $n \in \mathbb{N}$, A_n is finite and hence countable.

Therefore, $A = \bigcup_{n=1}^{\infty} A_n$ is a countable union of countable subsets of S.

This implies A is a countable subset of S.

E26) Since S is finite let $S = \{a_1, a_2, a_3, \ldots, a_n\}$ and $T = \{b_1, b_2, b_3, \cdots\}$. Define $f : \mathbb{N} \to S \cup T$ by

$$f(k) = \begin{cases} a_k, & \text{if } 1 \le k \le n \\ b_{k-n}, & \text{if } \quad k > n \end{cases}$$

Now, let $k, \ell \in \mathbb{N}$ such that $k \neq \ell$. Then $a_k \neq a_\ell$ and $b_k \neq b_\ell$. Also since $S \cap T = \emptyset$, we have $a_k \neq b_\ell$ for any $k, \ell \in \mathbb{N}$.

Thus, $k \neq \ell \Rightarrow f(k) \neq f(\ell)$. Hence f is $1-1$. To show that f is onto, pick $a_k \in S$, then $k \in \mathbb{N}$ such that $f(k) = a_k$. Similarly. If you pick $b_k \in T$, then $n + k \in \mathbb{N}$ such that

$$f(n+k) = b_{n+k-n} = b_k.$$

Thus, f is onto. This proves that f is a bijection. Hence $S \cup T$ is countably infinite.

UNIT 4

TOPOLOGICAL STRUCTURE OF $\mathbb{R}$

Structure **Page Nos.**

4.1 INTRODUCTION

You are quite familiar with an elastic string or a rubber tube or a spring. Suppose you have an elastic string. If you first stretch it and then release the pressure, then the string will come back to its original length. This is a physical phenomenon but in mathematics, we interpret it differently. According to geometry, the unstretched string and the stretched string are different since there is a change in the length. But you will be surprised to know that according to another branch of mathematics, the two positions of the string are identical and there is no change. This branch is known as Topology, one of the most exciting areas of mathematics.

The word "topology" is a combination of the two Greek words "topos" and "logos". The term "topos" means the top or the surface of an object and "logos" means the study. Thus "topology" means the study of surfaces. Since the surfaces in one dimensional space such as $\mathbb{R}$ are just points and intervals and their unions, we shall study the topological characteristics of such subsets of $\mathbb{R}$.

We begin with intervals in Section 4.2 and study their types and discuss whether the union, intersections or complements of intervals are intervals or not. Next, in Section 4.3 we shall introduce the notion of the neighborhood of a point and the notion of limit points of a set. In Section 4.4, we shall see what kinds of sets have limit points. Specifically, we shall discuss the Bolzano Weierstrass Theorem. In Section 4.5 and 4.6 we shall show you how these notions give rise to two different types of subsets of $\mathbb{R}$ namely -- closed sets and opens sets. We shall finally look at how the closed sets and open sets are related to each other.

Objectives

After reading this unit, you should, be able to

- describe intervals as special subsets of $\mathbb{R}$ and show when the union, intersection or complement of an interval is an interval;
- describe the notion of a neighborhood of a point on the line, and the notion of limit points of a set;
- describe and apply Bolzano Weierstrass Theorem;
- find the limit points of a set;
- describe when a set is closed and understand the properties of closed sets;
- describe when a set is open and establish the relationship between open and closed sets.

4.2 INTERVALS

You can recall from your calculus course that an interval is a set that contains every point lying between any two points of it. Formally it can be defined as follows:

$[a,b]=\{x\in\mathbb{R}\,|\,a\le x\le b\}$

$[a,\infty]=\{x\in\mathbb{R}\,|\,a\le x\}$

$]-\infty,b[=\{x\in\mathbb{R}\,|\,a\le x\}$

$]a,b[=\{x\in\mathbb{R}\,|\,a<x<b\}$

$]a,\infty[=\{x\in\mathbb{R}\,|\,a<x\}$

$]-\infty,a[=\{x\in\mathbb{R}\,|\,x<a\}$

Definition: A set $S\subseteq\mathbb{R}$ is said to be an **interval** if for any $x,y\in S, x<y,$ and r is a real number such that $x<r<y$, then $r\in S$.

Let us recall different kinds of intervals you have studied such as open and closed.

Some examples of closed intervals are $[a,b]$, $[a,\infty[$ and $]-\infty,a]$, where $a,b\in\mathbb{R}$. Of course, in the interval $[a,b]$ it is assumed that $a\le b$. So, you can see that $[a,a]$ is just the singleton $\{a.\}$

The examples of open intervals, include the sets $]a,b[$, $]a,\infty[$ and $]-\infty,a[$. Again, in $]a,b[$ it is assumed that $a\le b$. So, what does $]a,a[$ contain ? It contains all those real numbers greater than a and less than a. But there is no such number, hence $]a,a[=\emptyset$.

Is $\mathbb{R}$ an interval? Of course, it is. $\mathbb{R}$ contains every number lying between any two real numbers. We shall often write $\mathbb{R}=]-\infty,\infty[$. Here we must tell you that not all intervals can be classified as open or closed. For example, $[a,b[$ and $]a,b]$ which are called semi-open (or semi-closed) intervals.

Are you puzzled why the intervals such as $[a,\infty[$ and $]-\infty,a]$ are not semi-closed? We shall discuss more about the terminology 'open' and 'closed' in later sections.

Let us look at some properties of intervals.

Is $S\cap T$ an interval, when it contains just a single point?

Theorem 1: The intersection of two intervals is an interval.

Proof: Let us assume that S and T are intervals with at least two points in

$S \cap T$. Choose $x, y \in S \cap T$, with $x < y$. Then since $x, y \in S$ and S is an interval, $]x, y[\subseteq S$. Similarly, $]x, y[\subseteq T$. Thus $]x, y[\subseteq S \cap T$, and hence $S \cap T$ is an interval. ■

For example, you can see that $]2,5[\cap [4,9[$ is an interval because $]2,5[$ and $[4,9[$ are intervals and $]2,5[\cap]4,9[=]4,5[$

However, the union of two intervals is not necessarily an interval. Take, for instance, $[0,1]$ and $[2,3]$. You can see that $1,2 \in [0,1] \cup [2,3]$ but $1.5 \notin [0,1] \cup [2,3]$. However, if we consider two intervals with nonempty intersection, will their union be an interval? The following theorem answers this question.

Theorem 2: If the intersection of two intervals is nonempty, then their union is an interval.

We shall not prove it here. (See E3.)

Let us consider the following example.

Example 1: Identify the following sets as open, closed or semi-open intervals.

i) $S_1 = \left\{ x \in \mathbb{R} \,\middle|\, 1 \le \dfrac{x^2 - 3x + 2}{x - 2} \le 4 \right\}$ ii) $S_2 = \left\{ x \in \mathbb{R} \,\middle|\, x + \dfrac{1}{x} > 0 \right\}$

iii) $S_3 = \{x \in \mathbb{R} \mid x^2 + x - 6 < 0\}$ iv) $S_4 = \{x \in \mathbb{R} \mid |x - 1| \le 1\}$

Solution: i) The elements of S_1 satisfy the inequality

$$
\begin{aligned}
1 \le \frac{x^2 - 3x + 2}{x - 2} \le 4 &\Leftrightarrow 1 \le \frac{(x-1)(x-2)}{x-2} \le 4 \\
&\Leftrightarrow 1 \le x - 1 \le 4 \text{ and } x - 2 \ne 0 \\
&\Leftrightarrow 2 \le x \le 5 \quad \text{ and } x \ne 2 \\
&\Leftrightarrow 2 < x \le 5
\end{aligned}
$$

Thus, S_1 is the semi-open interval $]2,5]$.

ii) The elements of S_2 satisfy the inequality $x + \dfrac{1}{x} > 0$. Note that

$$x + \frac{1}{x} > 0 \Leftrightarrow \frac{x^2 + 1}{x} > 0 \Leftrightarrow x > 0$$

So, we have $S_2 = \{x \in \mathbb{R} : x > 0\} =]0, \infty[$. Thus, S_2 is an open interval.

iii) We know that

$$
\begin{aligned}
x^2 + x - 6 < 0 &\Leftrightarrow (x-2)(x+3) < 0 \\
&\Leftrightarrow (2-x)(x+3) > 0 \\
&\Leftrightarrow (2 - x > 0, x + 3 > 0) \text{ or } (2 - x < 0, x + 3 < 0) \\
&\Leftrightarrow -3 < x < 2 \qquad \text{or } 2 < x < -3 \\
&\Leftrightarrow -3 < x < 2 \quad (\because 2 < x < -3 \text{ is false.})
\end{aligned}
$$

Thus $S_3 =]-3, 2[$. Hence S_3 is an open interval.

iv) The inequality $|x - 1| \le 1$ is equivalent to $-1 \le x - 1 \le 1$, which is equivalent to $0 \le x \le 2$. Hence $S_4 = [0, 2]$ is a closed interval.

***

Now we talk about the complement of an interval. Look at the interval $]a,b[$. It contains all those real numbers that lie strictly between a and b. So its complement must contains all those real numbers that are either smaller than or equal to a, or greater than or equal to b. Observe that a and b lie in the complement as they do not lie in $]a,b[$.

Thus the complement of $]a,b[$ is $]-\infty,a]\cup[b,\infty[$. This implies that the complement of the interval $]a,b[$ is not an interval. Can you think of an interval whose complement is also an interval?

The following exercises would help you comprehend the concept of intervals.

E1) Let $A=\left\{x\in\mathbb{R}\left|\frac{x}{x+1}<1\right.\right\}$ and $B=\{x\in\mathbb{R}|\ x+1<|x|\}$. Express A and B as intervals. Check whether $\mathrm{A}\cap\mathrm{B}$ is an open interval or not.

E2) Express the sets $A=\{x\in\mathbb{R}\mid x^2\le x\}$ and $B=\{x\in\mathbb{R}\mid 4(x^2-2x)\le -3\}$ as intervals. Find $A\cup B$, $A\cup B^c$, and $A^c\cup B$ and $A^c\cup B^c$.

E3) Prove that if S and T are intervals with $S\cap T\ne\emptyset$ then $S\cup T$ is also an interval.

Remark: Even when the intersection is empty, the union may still be an interval. For example, take $S=]1,2]$ and $T=]2,3]$. Then $S\cup T=[1,3]$, which is an interval, but $S\cap T=\emptyset$.

E4) Is $\mathbb{Q}$ an interval? Is any subset of $\mathbb{Q}$ an interval? Justify.

You have seen that intervals are special kind of subsets of $\mathbb{R}$ which do not exist in $\mathbb{Q}$. (See Theorem 9 and E22 of Unit 3.) Let us now discuss two closely related concepts.

4.3 NEIGHBOURHOODS AND LIMIT POINTS

In this section we shall introduce to you the notion of 'closeness'. Then we shall talk about the points that are 'arbitrarily' close to a given set.
We begin with the definition of distance between real numbers. Let $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+\cup\{0\}$ be a mapping. Suppose d has the following properties:

i) $d(x,y)\ge 0,\quad \forall x,y\in\mathbb{R}$
ii) $d(x,y)=0$ iff $x=y,\quad \forall x,y\in\mathbb{R}$
iii) $d(x,y)=d(y,x),\quad \forall x,y\in\mathbb{R}$
iv) $d(x,y)\le d(x,z)+d(z,y),\quad \forall x,y\in\mathbb{R}$

Then $d(x,y)$ is called the **distance** between the points x and y. An important example of distance which we shall deal with is the usual distance between points on the line, i.e., $d(x,y)=|x-y|,\ \forall x,y\in\mathbb{R}$.

From Unit 3, you know that every real number can be represented by a point on the number line and vice-versa .So, let us consider the real number 1 and some points near it on the number line. (see Fig.1 below).

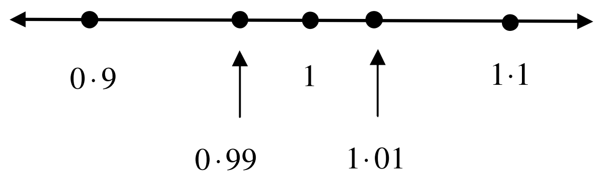

Fig. 1: Points close to 1

Look at the points $0\cdot 9$ and $1\cdot 1$, both of which lie at a distance of 0.1 from 1. That is, $|1-(0\cdot 9)|=|1-(1\cdot 1)|=0.1$. So we can say that $0\cdot 9$ and $1\cdot 1$ are close to 1. However, you can see that the points $0\cdot 99$ and $1\cdot 01$ are even more close to 1 as $|1-(0\cdot 99)|=|1-(1\cdot 01)|=0\cdot 01$. The terms close, closer, more close, much close etc do not give clarity to what closeness means. So, we fix a distance, $\varepsilon\,(>0)$, and say that x is close to 1 if the distance between 1 and x is less than ε. In this sense we can call x lies in the ε – neighbourhood of 1. Below we define a neighbourhood formally.

Definition: Let $a\in\mathbb{R}$ and $\varepsilon>0$. Then an ε - **neighborhood** of a is the set $N_\varepsilon(a)$ defined as:

$$N_\varepsilon(a)=\{x\in\mathbb{R}\mid |x-a|<\varepsilon\}=\{x\in\mathbb{R}\mid a-\varepsilon<x<a+\varepsilon\}.$$

Thus $N_\varepsilon(a)$ is an interval around a. That is, $N_\varepsilon(a)=]a-\varepsilon,a+\varepsilon[$. You can see what this means geometrically in. Fig. 2.

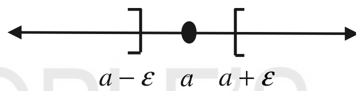

Fig. 2: $N_\varepsilon(a)$

Now let us consider a few examples.

Example 2: Represent the set $N_{\frac{1}{100}}(0)$ on the number line.

Solution: The set $N_{\frac{1}{100}}(0)$ contains all those points that are within a distance of $\dfrac{1}{100}$ from 0, i.e.,

$$N_{\frac{1}{100}}(0)=\left\{x\in\mathbb{R}\,\middle|\,|x|<\frac{1}{100}\right\}=\left]-\frac{1}{100},\frac{1}{100}\right[.$$

Geometrically it is represented in Fig. 3.

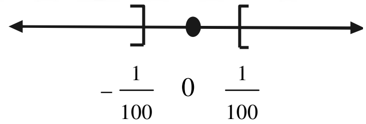

Fig. 3: $N_{\frac{1}{100}}(0)$

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Example 3: Show that if $0<\varepsilon'<\varepsilon$, then $N_{\varepsilon'}(a)\subset N_\varepsilon(a)$.

Solution: Let $x\in N_{\varepsilon'}(a)$. Then $|x-a|<\varepsilon'$. Since $\varepsilon'<\varepsilon$, it follows that $|x-a|<\varepsilon$. Hence $x\in N_\varepsilon(a)$.

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Now, look at the set $S=\left\{\dfrac{1}{n}\,\middle|\,n\in\mathbb{N}\right\}$. (See Fig. 4).

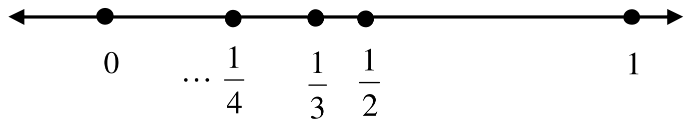


Fig. 4: $S = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$

Its elements are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{10^{10}}, \ldots, \frac{1}{10^{100}}, \ldots$.

Does $0 \in S$? No. Still you will find that 0 cannot be separated from S. Let us see what it means. Consider the neighbourhood $N_{\frac{1}{10}}(0)$. Then $\frac{1}{11} \in N_{\frac{1}{10}}(0)$ and $\frac{1}{11} \in S$. That is, $\frac{1}{11} \in N_{\frac{1}{10}}(0) \cap S$. Similarly if you consider the neighbourhood $N_{\frac{1}{100}}(0)$ of 0, you will see that $\frac{1}{101} \in N_{\frac{1}{100}}(0) \cap S$. In fact, you consider any ε-neighbourhood $N_\varepsilon(0)$ of 0, by Archimedean property you will find some $n \in \mathbb{N}$, such that $0 < \frac{1}{n} < \varepsilon$, i.e, $\frac{1}{n} \in N_\varepsilon(0) \cap S$. Thus, we see that although 0 lies outside S, every neighbourhood of 0 contains points of S. In other words, we can say that 0 is arbitrarily close to S.

The point 0 is an example of a point which we shall define now.

Definition: Let $\emptyset \neq S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a **limit point** of S if for every $\varepsilon > 0, N_\varepsilon(x) \cap S$ contains an element of S other than x.
To clarify the definition above, note that x need not belong to S. So it does not matter whether x belongs to S or not, but for x to be a limit point of $S, N_\varepsilon(x) \cap S$ must contain at least one element different from x.

Now you can see why 0 is a limit point of the set $S = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$.

Let us consider a few more examples.

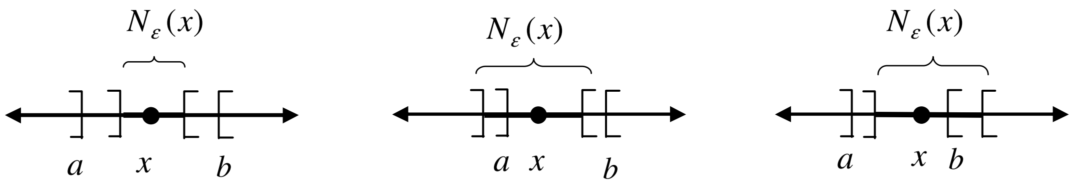
Example 4: Find all the limit points of $\mathbb{Q}$.

Solution: Let $x \in \mathbb{R}$ be arbitrary. Consider the neighbourhood $N_\varepsilon(x)$ for some $\varepsilon > 0$. Since you know that there are many rational numbers between $x - \varepsilon$ and $x + \varepsilon$, the set $N_\varepsilon(x) \cap \mathbb{Q}$ contains an element other than x. Hence x is a limit point of $\mathbb{Q}$. Since x is arbitrary in $\mathbb{R}$, all the real numbers are the limit points of $\mathbb{Q}$.

***

Example 5: Show that every element of $]a,b[$ is a limit point of $]a,b[$.

Solution: Let $x \in\,]a,b[$ be arbitrary. Consider the neighbourhood $N_\varepsilon(x)$ of x for some $\varepsilon > 0$. We have to find some $p \in N_\varepsilon(x) \cap\,]a,b[$ such that $p \neq x$. Then there are three cases. (See Fig. 5.)



(a) **(b)** **(c)**

Fig. 5

Case 1: $\varepsilon \le \min\{x-a, b-x\}$. Now observe that

$$a < x < x + \frac{\varepsilon}{2} < x + \varepsilon \le b \text{ (Why?)}.$$

Take $p = x + \frac{\varepsilon}{2}$. Then $p \in]a,b[$ and $p \in N_\varepsilon(x)$. Thus $p \in N_\varepsilon(x) \cap]a,b[$.

Case 2: $\varepsilon > x - a$. Then take $p = \frac{a+x}{2}$. Note that

$$x - \varepsilon < a < \frac{a+x}{2} < x < b,$$

which implies $p \in N_\varepsilon(x) \cap]a,b[$.

Case 3: $\varepsilon > b - x$. Then take $p = (x+b)/2$. Then using the arguments similar to the previous case, you can see that $p \in N_\varepsilon(x) \cap]a,b[$.

Thus, for all $\varepsilon > 0$, we have shown that $N_\varepsilon(x) \cap]a,b[$ contains a point different from x. Hence x is a limit point of $]a,b[$. Since x is arbitrary, every point of $]a,b[$ is a limit point of $]a,b[$.

***

You may have observed that if a is a limit point of S, then $N_\varepsilon(a)$ for any $\varepsilon > 0$, contains many points of S, other than a. In fact, $N_\varepsilon(a)$ contains infinitely many points of S distinct from a. This is, precisely the content of the next theorem.

Theorem 3: Let $\emptyset \ne S \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then a is a limit point of S iff for every $\varepsilon > 0, N_\varepsilon(a)$ contains infinitely many points of S other than a.

The idea of the proof is to begin with an arbitrary neighbourhood of a, and get smaller and smaller neighbourhoods successively picking a point from each neighbourhood distinct from the previous point.

Proof: First assume that a is a limit point of S. Take $\varepsilon > 0$. Then $N_\varepsilon(a) \cap S$ contains some element p and $p \ne a$. Let $x_1 = p$. Choose $0 < \varepsilon_1 < \varepsilon$ such that $x_1 \notin N_{\varepsilon_1}(a)$. (See Fig. 6.)

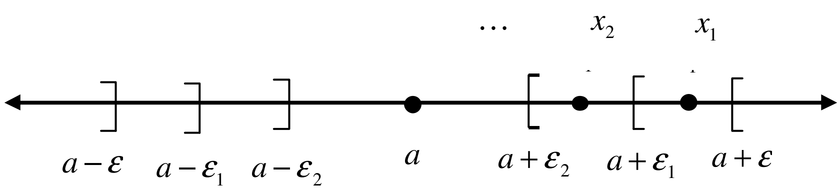


Fig. 6: Points $x_1, x_2, x_3....$ of S lying in $N_\varepsilon(a)$.

Now $N_{\varepsilon_1}(a)$ is a neighbourhood of a, hence there is some $x_2 \in N_{\varepsilon_1}(a) \cap S$

and $x_2 \neq a$. Also, $x_2 \neq x_1$ as $x_1 \notin N_{\varepsilon_1}(a)$. Again we can choose $0 < \varepsilon_2 < \varepsilon_1$ such that $x_2 \notin N_{\varepsilon_2}(a)$. Since $N_{\varepsilon_2}(a)$ is a neighborhood of a, there is some $x_3 \in N_{\varepsilon_2}(a) \cap S$ and $x_3 \neq a$. Also $x_3 \notin \{x_1, x_2\}$. This process can be continued as given ε_{n-1}, and $\{x_1, x_2 ..., x_n\}$ we can always select ε_n such that $0 < \varepsilon_n < \varepsilon_{n-1}$ and $x_n \notin N_{\varepsilon_n}(a)$. Then there exists an $x_{n+1} \in N_{\varepsilon_n}(a) \cap S$ such that $x_{n+1} \neq a$ and $x_{n+1} \notin \{x_1, x_2, ..., x_n\}$. Thus we get infinitely many points $x_1, x_2, x_3, ..., x_n, x_{n+1}, ...$ of S that lie in $N_\varepsilon(a)$.

The converse holds by definition. ■

Now try solving some exercises.

E5) Let x be a limit point of a set S and $S \subseteq T$. Is x a limit point of T? Justify.

E6) Find the limit points of $\mathbb{R}$.

E7) Does a finite set have any limit points? Justify.

E8) Does the set $\mathbb{N}$ have any limit points? What about $\mathbb{Z}$? Justify.

E9) Show that 0 is a limit point of the set $\left\{ \frac{1+(-1)^n}{2} - \frac{1}{n} \middle| n \in \mathbb{N} \right\}$.

If you have gone through the exercises above, you might have understood why the limit points of finite sets do not exist. You must have also seen that the limit points of some infinite sets like $\mathbb{N}$ and $\mathbb{Z}$ do not exist. This raises a general question, what are the conditions that ensure the existence of a limit point of a set. This is the topic of discussion in the next section.

4.4 BOLZANO WEIERSTRASS THEOREM

Fig. 7: Bernhard Bolzano

In this section, we shall discuss the Bolzano-Weierstrass Theorem. Bernhard (1781-1848) Bolzano discovered this theorem. He was, however, debarred from publishing his work. Karl Weierstrass (1815-1897) discovered the result independently.

The theorem says that there is a limit point for a bounded infinite set S. You may note that boundedness or infiniteness of S alone is insufficient for the existence of a limit point. For instance, the unbounded and infinite sets $\mathbb{N}$ and $\mathbb{Z}$ have no limit points. Likewise, the finite sets are bounded, but have no limit points.

Now think of a set S which is bounded and infinite both. For example, take $S = \{1, 2, 3\} \cup]5, 6[$. The limit points of S is the closed interval $[5, 6]$.

Now we state the theorem formally.

Theorem 4 (Bolzano Weierstrass Theorem): Let $\emptyset \neq S \subseteq \mathbb{R}$ and S be

bounded and infinite. Then there exists a limit point of S (in $\mathbb{R}$).

Proof: Consider the set A_S as defined below.

$$A_S = \{x \in \mathbb{R} \mid x \geq s \text{ for finitely many } s \in S\}$$

Fig. 8: Karl Weierstrass

We shall complete the proof in two steps.

Step 1: $\sup A_S$ exists.

Since S is bounded, let ℓ be a lower bound and u an upper bound of S. Then $\ell \leq s, \forall s \in S$. Hence $\ell > s$, for no $s \in S$. This implies $\ell \in A_S$. Hence $A_S \neq \emptyset$. Now for any $x \in A_S$, we have $x > s$, for finitely many $s \in S$. This means the rest of the infinitely many elements of S are larger than x. But since u is larger than all the elements of S, hence $x \leq u$. This shows A_S is bounded above. Now the least upper bound property of $\mathbb{R}$ implies $\sup A_S$ exists.

Step 2: $\sup A_S$ is a limit point of S.

Let $u_0 = \sup A_S$. Then we have to show that u_0 is a limit point of S. So, for $\varepsilon > 0$, consider the set $N_\varepsilon(u_0) =]u_0 - \varepsilon, u_0 + \varepsilon[$. Since u_0 is the supremum of A_S, $u_0 + \varepsilon \notin A_S$. This means $u_0 + \varepsilon > s$, for infinitely many $s \in S$. Again, since u_0 **is the supremum of** A_S**, there is some** $a \in A_S$ **such that** $u_0 - \varepsilon < a$**. But** $a \in A_S$ means $a > s$, for finitely many $s \in S$. That is the rest of the infinitely many elements of S are greater than a, and hence greater than $u_0 - \varepsilon$. Thus we have infinitely many elements of S lying between $u_0 - \varepsilon$ and $u_0 + \varepsilon$. That is u_0 is a limit point of S. ■

Let us consider a few examples.

Example 6: Show that there exists a limit point of the set

$$S = \left\{ 2 + \frac{\cos n\pi}{n} \,\middle|\, n \in \mathbb{N} \right\}.$$

Solution: Notice that, for all $n \in \mathbb{N}$,

$$-1 \leq \cos n\pi \leq 1 \Rightarrow -\frac{1}{n} \leq \frac{\cos n\pi}{n} \leq \frac{1}{n} \Rightarrow 2 - \frac{1}{n} \leq 2 + \frac{\cos n\pi}{n} \leq 2 + \frac{1}{n}.$$

Thus all the elements of S lie between 1 and 3, hence, S is bounded. Next to show that S is infinite, we consider the function $f : \mathbb{N} \to S$ defined by

$$f(n) = 2 + \frac{\cos n\pi}{n}.$$

Now let $m, n \in \mathbb{N}$. Then

$$f(n) = f(m) \Rightarrow 2 + \frac{\cos n\pi}{n} = 2 + \frac{\cos m\pi}{m} \Rightarrow \frac{(-1)^n}{n} = \frac{(-1)^m}{m} \Rightarrow \frac{m}{n} = (-1)^{m-n}$$

Now $\frac{m}{n}$ is positive, hence $(-1)^{m-n}$ must be positive. So $m-n$ must be even, say $m-n=2k,$ for some integer $k.$ Then

$$\frac{m}{n}=(-1)^{m-n} \Rightarrow \frac{n+2k}{n}=1 \Rightarrow k=0 \Rightarrow m=n.$$

Thus f is one-one. This means, S is countably infinite, and hence infinite. Then, by the Bolzano-Weierstrass Theorem, there exists a limit point of S.

***

Example 7: Show that there exists a limit point of the set

$$S=\left\{\frac{1}{m}-\frac{1}{n}\,\middle|\, m,n\in \mathbb{N}\right\}.$$

Solution: You can notice that $\forall m,n\in \mathbb{N}$

$$\left|\frac{1}{m}-\frac{1}{n}\right|\le \frac{1}{m}+\frac{1}{n}\le 2 \quad\Rightarrow\quad -2\le \frac{1}{m}-\frac{1}{n}\le 2.$$

Hence S is bounded. Now notice that $\left\{1-\frac{1}{n}\,\middle|\, n\in \mathbb{N}\right\}$ is a subset of S. Since $\left\{1-\frac{1}{n}\,\middle|\, n\in \mathbb{N}\right\}$ is infinite, S too is infinite. (Why?) Hence by the Bolzano-Weierstrass Theorem, a limit point of S exists.

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The Bolzano Weierstrass Theorem is a fundamental result in real analysis. Note that the theorem guarantees only the existence of a limit point of course, in many cases we need only that much information. Now try the following exercise.

E10) Using Bolzano Weierstrass Theorem, show that each of the following sets has at least one limit point in $\mathbb{R}$.

(i) $\left\{\frac{11}{2^{3n}}\,\middle|\, n\in \mathbb{N}\right\}$ (ii) $\left\{\frac{(-1)^n}{6^n}\,\middle|\, n\in \mathbb{N}\right\}$ iii) $\left\{1+\frac{1}{n}\,\middle|\, n\in \mathbb{N}\right\}$

By now you must have understood that some sets do not have limit points at all even if they have infinitely many points, e.g. the set $\mathbb{N}$ of natural numbers. Another thing is that the limit points of a set may exist, but do not all lie in the set. For instance, 0 is a limit point of $\left\{\frac{1}{n}\,\middle|\, n\in \mathbb{N}\right\}$. But $0\notin \left\{\frac{1}{n}\,\middle|\, n\in \mathbb{N}\right\}$. In the next section we study sets that contain all their limit points or have no limit point at all.

4.5 CLOSED SETS

In this section we shall discuss closed subsets of $\mathbb{R}$. Let us consider the closed interval $[a,b]$. We have seen that every point of this interval other than a and b is a limit point of it. Take the point a. Let $\varepsilon > 0$ be arbitrary. Does $N_\varepsilon(a) \cap [a,b]$ have any point other than a? There are, in fact, two cases arises depending on whether $a+\varepsilon \le b$ or $a+\varepsilon > b$. (See Fig. 9.)

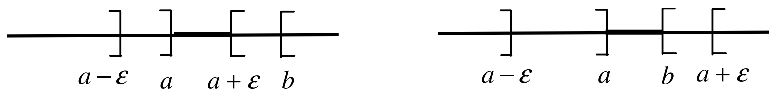

Fig. 9: $N_\varepsilon(a) \cap [a,b]$

So, we can write

$$N_\varepsilon(a) \cap [a,b] = \begin{cases} [a, a+\varepsilon[& \text{, if } a+\varepsilon \le b \\ [a,b] & \text{, if } a+\varepsilon > b \end{cases}$$

In either case, there is certainly a point other than a in $N_\varepsilon(a) \cap [a,b]$. (Of course, we have assumed that $a < b$.)This means a is a limit point of $[a,b]$. On similar lines you can show that b is also a limit point of $[a,b]$. Thus all the elements of $[a,b]$ are limit points of $[a,b]$.

Does $[a,b]$ have any other limit points? To answer it, let us take $x < a$. Then taking $\varepsilon = a - x > 0,$ we find that the neighborhood $N_\varepsilon(x) =]2x-a, a[$ contains no point of $[a,b]$. Hence x is not a limit point of $[a,b]$. Similarly, you can show that if $x > b$, then also x is not a limit point of $[a,b]$. Thus we see that the interval $[a,b]$ contains all its limit points.

The interval $[a,b]$ is an example of a set which will be termed a 'closed set'.
Definition: Let $S \subseteq \mathbb{R}$. Then S is called a **closed set** if S contains all its limit points.

Thus you can see that $[a,b]$is a closed set , when $a < b.$ Again assuming $a < b,$ consider the interval $]a,b]$. For any $\varepsilon > 0$, the interval $]a-\varepsilon, a+\varepsilon[$ contains infinitely many points of $]a,b]$. So, a is a limit point of $]a,b]$. Since $a \notin]a,b]$, the interval $]a,b]$ is not closed. Similar reasons can be given to conclude that $]a,b[$ and $[a,b[$ are not closed.

Are the sets Ø and $\mathbb{R}$ closed? You can see that Ø is a closed set as it has no limit points. $\mathbb{R}$ is a closed set as its limit points are in $\mathbb{R}$.

Now consider the following example.

Example 8: Show that every singleton set in $\mathbb{R}$ is a closed set.

Solution: Let $S = \{a\},$ for some $a \in \mathbb{R}$. If $x \in \mathbb{R}$ is a limit point of S then for every $\varepsilon > 0,$ $N_\varepsilon(x) \cap S \subseteq S,$ so $N_\varepsilon(x) \cap S$ contains at most one point. This is a contradiction. Hence, no real number is a limit point of $S.$ That is, the set of limit points of S is empty. Consequently, S contains all its limit points. Hence S is closed.

***

You have to be little careful while using the definition of a closed set. It does not imply that every point of a closed set has to be a limit point, for instance, see the singleton set in Example 8 above. To show that a set is closed you need to find all the limit points of the set and then show that they all belong to the set. On the contrary to prove that a set is not closed, you need to find just one limit point of the set that lies outside it.
Let us consider a few examples.

Example 9: Check whether $S = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$ is closed or not.

Solution: Earlier you have seen that 0 is a limit point of S. Since $0 \notin S$, S is not a closed set.

***

Example 10: Show that $[a, \infty[$ is closed.

Solution: Let x be a limit point of $[a, \infty[$. Then for each $\varepsilon > 0$, there exists some $y \neq x$ such that

$$y \in N_\varepsilon(x) \cap [a, \infty[\Rightarrow \varepsilon - x < y < x + \varepsilon \text{ and } y \geq a$$
$$\Rightarrow a < x + \varepsilon$$
$$\Rightarrow a - x < \varepsilon$$

Since the last inequality holds for each $\varepsilon > 0$, we get $a \leq x$, i.e., $x \in [a, \infty[$. This shows that every limit point of $[a, \infty[$ is in $[a, \infty[$. Hence $[a, \infty[$ is closed.

***

Example 11: Show that the following set is not closed.

$$S = \left\{ \frac{1 + (-1)^n}{2} + \frac{1}{n} \middle| n \in \mathbb{N} \right\}$$

Solution: Consider the neighbourhood $N_\varepsilon(0) =]-\varepsilon, \varepsilon[$ of 0. Then, by Archimedean property, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. So, we have

$$-\varepsilon < \frac{1}{2n+1} < \frac{1}{n} < \varepsilon.$$

That is, $\frac{1}{2n+1} \in N_\varepsilon(0)$. Also, observe

$$\frac{1}{2n+1} = \frac{1 + (-1)^{2n+1}}{2} + \frac{1}{2n+1} \in S.$$

So, $\frac{1}{2n+1} \in N_\varepsilon(0) \cap S$. Hence 0 is a limit point of S. Since $0 \notin S$, S is not a closed set.

***

Let us now look at a property of closed sets.

Theorem 5: If S and T are closed subsets of $\mathbb{R}$ then $S \cup T$ is also a closed subset of $\mathbb{R}$.

Proof: Let x be a limit point of $S \cup T$. Then for every $\varepsilon > 0$, there exists some $y \neq x$ such that

$$\begin{aligned} y \in N_\varepsilon(x) \cap (S \cup T) &\Rightarrow y \in (N_\varepsilon(x) \cap S) \cup (N_\varepsilon(x) \cap T) \\ &\Rightarrow y \in N_\varepsilon(x) \cap S \text{ or } y \in N_\varepsilon(x) \cap T \end{aligned}$$

This implies, x is a limit point of S or x is a limit point of T. Hence $x \in S$ or $x \in T$, as S and T both are closed. This implies $x \in S \cup T$. Thus, $S \cup T$ is closed. ■

To see an application of Theorem 5 consider the sets $[3,4]$ and $[5,6]$, both of which are closed. Then $[3,4] \cup [5,6]$ is also closed. In fact, if you are given any finite number of closed sets, then their union is also closed as you can see from the following theorem.

Theorem 6: If $S_1, S_2, ..., S_n$ are closed sets, then $\bigcup_{i=1}^{n} S_i$ is also closed.

Proof: We prove it by the Principle of Mathematical Induction. Note that the statement of the theorem can be rewritten as

$P(n)$: If $S_1, S_2, ..., S_n$ are closed, then $\bigcup_{i=1}^{n} S_i$ is closed, $n \geq 2$.

Clearly $P(2)$ is true, by Theorem 5. Let $P(k)$ be true for some $k < n$. Let $S = \bigcup_{i=1}^{k} S_i$. Then S is closed, which implies $S \cup S_{k+1}$ is closed (using Theorem 5). This means $P(k+1)$ is true. Hence, the Principle of Mathematical Induction implies that $P(n)$ is true for all $n \geq 2$. ■

Now let us consider an example.

Example 12: Check whether the set $S = \{x_1, x_2, ..., x_n\}$, where $x_i \in \mathbb{R}, \forall 1 \leq i \leq n$, and $n \in \mathbb{N}$, is closed or not.

Solution: Note that we can write S as the union of its individual elements i.e. $S = \bigcup_{i=1}^{n} \{x_i\}$. Since each $\{x_i\}$ is closed, by Theorem 6 S is closed.

***

You must have noted that in Theorem 6 the sets S_is were finite in number. The result cannot hold if we take infinitely many S_is. The next example shows this.

Example 13: Let $S_n = \left[\frac{1}{n}, 1\right]$ for $n \in \mathbb{N}$. Check whether $\bigcup_{n=1}^{\infty} S_n$ is closed or not.

Solution: We note that

$$\bigcup_{n=1}^{\infty} S_n = [1,1] \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \cup ... \subseteq \;]0,1].$$

Now take $x \in]0,1]$. Then there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < x \leq 1$. So $x \in \left]\frac{1}{n}, 1\right]$. Hence $x \in \bigcup_{n=1}^{\infty} S_n$. This shows that $\bigcup_{n=1}^{\infty} S_n =]0,1]$. You can show that 0 is a limit point of $]0,1]$. Since $0 \notin]0,1]$, it follows that $]0,1]$ is not a closed set. Hence $\bigcup_{n=1}^{\infty} S_n$ is not closed.

***

Example 13 shows that the union of infinite number of closed sets need not be closed. However if we take the intersection of an infinite number of closed sets, we get a closed set. In fact, it is true that, the intersection of an arbitrary collection of closed sets is closed. Note that an arbitrary collection of sets is a set of the form $\{S_\alpha \mid \alpha \in \Lambda\}$, where Λ can be any set finite, countably infinite or even uncountable. Now we state the result below.

Theorem 7: Let $\{S_\alpha \mid \alpha \in \Lambda\}$ be an arbitrary collection of closed sets, i.e., S_α is closed for each $\alpha \in \Lambda$. Then $\bigcap_{\alpha \in \Lambda} S_\alpha$ is also closed.

Proof: Let x be a limit point of $\bigcap_{\alpha \in \Lambda} S_\alpha$. Then for each $\varepsilon > 0$, there exists $y \neq x$ such that

$$y \in N_\varepsilon(x) \cap \left(\bigcap_{\alpha \in \Lambda} S_\alpha\right) \Rightarrow y \in \bigcap_{\alpha \in \Lambda} \left(N_\varepsilon(x) \cap S_\alpha\right)$$
$$\Rightarrow y \in N_\varepsilon(x) \cap S_\alpha \qquad \text{for all } \alpha \in \Lambda.$$

This implies, x is a limit point of S_α for all $\alpha \in \Lambda$. But S_α is closed, hence $x \in S_\alpha$ for all $\alpha \in \Lambda$.

That is, $x \in \bigcap_{\alpha \in \Lambda} S_\alpha$. Therefore, $\bigcap_{\alpha \in \Lambda} S_\alpha$ is closed. ■

It is often convenient to write the union of a set and all its limit points as a new set. So, we have the following definition.

Definition: Let $S \subseteq \mathbb{R}$. The **closure** of S is the union of S and all the limit points of S. We write $\overline{S}$ to denote the closure of S.

By definition, you can see that $S \subseteq \overline{S}$. However $\overline{S} \subseteq S$ is true only when S is closed. Thus we can say $S = \overline{S}$ iff S is closed. Let us find the closure of some subsets of $\mathbb{R}$.

In Example 4 you have seen that every real number is a limit point of $\mathbb{Q}$.

Hence, $\overline{\mathbb{Q}} = \mathbb{R}$. Similarly, you can see that $\overline{]a,b[} = [a,b]$.

Example 14: If $S \subseteq T$ and T is closed, then show that $\overline{S} \subseteq T$.

Solution: Assume, if possible, that $\overline{S} \not\subseteq T$. So, let $x \in \overline{S}$ such that $x \notin T$. Then, $x \notin S$. Since T is closed, x cannot be a limit point of T. This means,

$N_\varepsilon(x) \cap T \setminus \{x\} = \varnothing$, for some $\varepsilon > 0 \Leftrightarrow N_\varepsilon(x) \cap S \setminus \{x\} = \varnothing$, for some $\varepsilon > 0$

$\Leftrightarrow x$ is not a limit point of S

$\Leftrightarrow x \notin \overline{S}$

This is a contradiction. Hence $\overline{S} \subseteq T$.

***

Now try doing the following exercises to reinforce your learning about closed sets.

E11) Check whether $\bigcup_{a>0} [a, \infty[$ is closed or not. If it is not closed, what is its closure?

E12) Examine whether the set $\bigcup_{n=1}^{\infty}]-n, n[$ is a closed set or not.

E13) Prove that the interval $]-\infty, a[$ is not a closed set, for any $a \in \mathbb{R}$. Is $\bigcup_{a \in \mathbb{R}}]-\infty, a[$ closed?

E14) Check whether the set $S = \{x \in \mathbb{R} \,|\, x^5 + 3x^4 - x^3 + x + 1 = 0\}$ closed set or not.

E15) Examine whether the set $\bigcap_{n=1}^{\infty} \left[1 + \frac{1}{n}, 5\right]$ is closed or not.

E16) Find $\overline{\mathbb{N}}$ and $\overline{\mathbb{Z}}$.

E17) Find the closure of the set of irrational numbers.

We have seen that closed sets contain all those points that are arbitrarily close to them. On the other hand, there are many subsets of $\mathbb{R}$ whose all elements are interior to them. We shall discuss such sets in the next section.

4.6 OPEN SETS

Consider the open interval $]1,2[$. Take the point $1 \cdot 5$ in this interval. Then we see that $N_{0 \cdot 1}(1 \cdot 5) =]1 \cdot 4, 1 \cdot 6[\subseteq]1,2[$. Now take $1.001 \in]1,2[$. Then we know that $1.001 - 1 = 0.001$. So the neighbourhood $N_{0.001}(1.001) =]1, 1.002[\subseteq]1,2[$. Now take any point $x \in]1,2[$. You will find some neighbourhood of x lying in $]1,2[$. (See Fig. 10). We can say x lies in the 'interior' of S

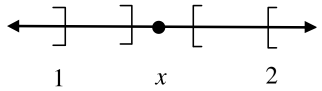

Fig. 10: x is an interior point of $]1,2[$.

Definition: Let $S \subseteq \mathbb{R}$ and $x \in S$. Then x is said to be an **interior point** of S if there exists some $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq S$. The set of all interior points of S is called the **interior** of S, which we shall denote S°.

As you can see, from the definition, an interior point of S always lies in S. This means $S^\circ \subseteq S$. You may wonder if $S^\circ = S$ occurs at all. As an example, note that $\varnothing^\circ = \varnothing$. Another example is $\mathbb{R}$ itself. To see why, let $a \in \mathbb{R}$. Then we can always find some $\varepsilon > 0$ s.t. $N_\varepsilon(a) =]a - \varepsilon, a + \varepsilon[\subseteq \mathbb{R}$. Thus $\mathbb{R}^\circ = \mathbb{R}$.

Let us think of some other examples.

Example15: Find the interior of a finite set.

Solution: Since you know that every neighbourhood contains infinitely many points, no neighbourhood of a point can be a subset of a finite set. Therefore a finite set has no interior points.

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Definition: Let $S \subseteq \mathbb{R}$. Then S is called an **open set** if $S^\circ = S$.
In other words, the definition above says that a set is open if and only if its every point is an interior point. Now consider the following example.

Example 16: Show that $]a,b[$ is an open set.

Solution: You know that for each $x \in]a,b[$, there exists $\varepsilon = \min\{x-a, b-x\} > 0$ such that $N_\varepsilon(x) \subseteq]a,b[$. Thus x is an interior point of $]a,b[$. Hence $]a,b[$ is an open set.

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In case of the closed interval $[a,b]$ finding an ε as above is not possible for every $x \in [a,b]$. Particularly the condition is violated at the points a and b. For example, we see that every neighborhood $N_\varepsilon(a)$ of a contains some points outside $[a,b]$. (See Fig.11.) Hence, $N_\varepsilon(a) \subseteq [a,b]$ is not possible for any $\varepsilon > 0$. Thus a is not an interior point of $[a,b]$. Therefore, $[a,b]$ is not an open set.

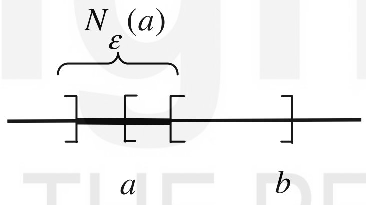

Fig. 11: $N_\varepsilon(a)$ **contains points outside** $[a,b]$.

Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then $N_\varepsilon(a) =]a-\varepsilon, a+\varepsilon[$. Hence $N_\varepsilon(a)$ is an open set, by Example 16. Thus every neighbourhood of a real number is an open set.

Recall that the complement of a set S is the set $S^c = \{x \in \mathbb{R} \mid x \notin S\}$.

Let us now state an important relationship between the open and closed sets.

Theorem 8: Let $S \subseteq \mathbb{R}$. Then S is open iff S^c is closed.

Proof: First we prove that if S is open then S^c is closed.

Let x be a limit point of S^c. Then for each $\varepsilon > 0$, the set $N_\varepsilon(x) \cap S^c$ contains a point other than x. This implies $N_\varepsilon(x) \not\subseteq S$. Hence $x \notin S^\circ$. But S is open, i.e, $S^\circ = S$, so, $x \notin S$. This means $x \in S^c$. Since x is arbitrary every limit point of S^c lies in S^c. That is, S^c is closed.

Now let us prove that if S^c is closed then S is open.

So, let $x \in S$. Then $x \notin S^c$. This means x cannot be a limit point of S^c. For if x is a limit point of S^c, then S^c being closed must contain x, which is a contradiction. Hence for some $\varepsilon > 0$, $N_\varepsilon(x) \cap S^c = \{x\}$ or $N_\varepsilon(x) \cap S^c = \emptyset$. But $x \notin S^c$ implies that $N_\varepsilon(x) \cap S^c = \{x\}$ is not possible. Hence $N_\varepsilon(x) \cap S^c = \emptyset$, which implies $N_\varepsilon(x) \subseteq S$. So x is an interior point of S. Since x is arbitrary, every element of S is an interior point of S. Hence S is an open set. ■

The topological relationship between a set and its complement stated in Theorem 8 is remarkable and has many applications. For example, given a finite number of open sets, we can show that their intersection is an open set. See the following result.

Theorem 9: If $S_1, S_2, ..., S_n$ are open sets, then $\bigcap_{i=1}^{n} S_i$ is also an open set

Proof: Let $S_1, S_2, ..., S_n$ be open sets. Then $S_1^c, S_2^c, ..., S_n^c$ are closed sets (by Theorem 8). Hence $\bigcup_{i=1}^{n} S_i^c$ is a closed set (by Theorem 6). But you know that, using De Morgan's law, we can write

$$\bigcup_{i=1}^{n} S_i^c = \left(\bigcap_{i=1}^{n} S_i\right)^c.$$

Hence $\left(\bigcap_{i=1}^{n} S_i\right)^c$ is closed, which implies $\bigcap_{i=1}^{n} S_i$ is open. ■

Another application of Theorem 8 is that given an arbitrary collection of open sets, we can show that their union is an open set. Formally, it is stated below.

Theorem 10: If $\{S_\alpha \mid \alpha \in \Lambda\}$ is an arbitrary collection of open sets, i.e., S_α is open for each $\alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} S_\alpha$ is an open set.

Proof: Since S_α is open for each $\alpha \in \Lambda$, S_α^c is closed for each $\alpha \in \Lambda$. Hence $\bigcap_{\alpha \in \Lambda} S_\alpha^c$ is closed. But, by De Morgan's laws you know that

$$\bigcap_{\alpha \in \Lambda} S_\alpha^c = \left(\bigcup_{\alpha \in \Lambda} S_\alpha\right)^c.$$

This means $\left(\bigcup_{\alpha \in \Lambda} S_\alpha\right)^c$ is closed, which implies $\bigcup_{\alpha \in \Lambda} S_\alpha$ is open. ■

Theorem 10 can also be proved directly, by applying the definition of open sets (see E23).

Let us now prove some results about open sets in the following examples.

Example 17: Prove that arbitrary intersection of open sets need not be open.

Solution: Consider the family of open sets $\left\{ \left]-\frac{1}{n}, \frac{1}{n}\right[\,\middle|\, n \in \mathbb{N} \right\}$.

Then $\bigcap_{n=1}^{\infty} \left]-\frac{1}{n}, \frac{1}{n}\right[= \{0\}$, which is not an open set. This proves the result.

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Example 18: Show that the set $]3, \pi[\cup]4.5, 10[\cup]\sqrt{2}, \sqrt{3}[$ is an open set.

Solution: You know that union of open sets is open, and the set $]3, \pi[\cup]4 \cdot 5, 10[\cup]\sqrt{2}, \sqrt{3}[$ is union of open intervals, which are open sets, therefore, this set is an open set.

Now you try doing some exercises.

E18) Show that $]a, \infty[$ is an open set, where $a \in \mathbb{R}$.

E19) Examine whether the set $\bigcup_{n=1}^{\infty}]-n, n[$ is an open set or not.

E20) Let $S \subseteq \mathbb{R}$. Show that S° is open and $\overline{S}$ is closed.

E21) Check whether the set $]-\infty, 25] \cap]15, \infty[$ is an open set or not.

E22) Examine whether the set $\bigcup_{n=1}^{\infty} \left]1+\frac{1}{n}, 7\right[$ is an open set or not

E23) Prove Theorem 10 using the definition of open sets only.

We end this unit here. Let us summarise what we have covered in this unit.

4.7 SUMMARY

In this unit we have covered the following points.

1. Described the intervals in $\mathbb{R}$ and their properties in terms of the union, intersection and complement;
2. Introduced the notion of neighbourhood of a point and the limit point of a set;
3. Described a necessary and sufficient condition for a real number to be a limit point of a set;
4. Stated and proved Bolzano Weierstrass theorem;
5. Introduced the notion of a closed set and the closure of a set;
6. Introduced the notion of the interior of set and an open set;
7. Explained the relationship between the closed sets and open sets.

4.8 SOLUTIONS/ANSWERS

E1) Assume that $x < -1$. Then $x+1 < 0$. So, multiplying both the sides of the inequality $\frac{x}{x+1} < 1$ with $x+1$, we get $x > x+1$. This means $0 > 1$, which is false. Now assume that $x > -1$. Then $x+1 > 0$. In this case, if we multiply both the sides of the inequality $\frac{x}{x+1} < 1$ with $x+1$, we get $x < x+1$. This is always true. Hence, $A =]-1, \infty[$.

The inequality describing B is $x+1 < |x|$. From Example 16 of Unit 3, we know that this is equivalent to $x > x+1$ or $x < -(x+1)$. But $x > x+1$ is not possible. Therefore, the given inequality becomes $x < -(x+1)$, i.e., $x < -\frac{1}{2}$. Hence, $B = \left]-\infty, -\frac{1}{2}\right[$.

Now, $A \cap B =]-1, \infty[\cap \left]-\infty, -\frac{1}{2}\right[= \left]-1, -\frac{1}{2}\right[$. Thus $A \cap B$ is an open interval.

E2) The inequality describing A is

$$\begin{aligned} x^2 \le x &\Leftrightarrow x - x^2 \ge 0 \\ &\Leftrightarrow x(1-x) \ge 0 \\ &\Leftrightarrow (x \ge 0 \text{ and } 1\text{-}x \ge 0) \text{ or } (x \le 0 \text{ and } 1 - x \le 0) \\ &\Leftrightarrow (0 \le x \le 1) \text{ or } (1 \le x \le 0) \\ &\Leftrightarrow 0 \le x \le 1 \qquad (\because 1 \le x \le 0 \text{ is false.}) \end{aligned}$$

Therefore, $A = [0,1]$. The inequality describing B is

$$\begin{aligned} 4(x^2 - 2x) \le -3 &\Leftrightarrow 4x^2 - 8x + 3 \le 0 \\ &\Leftrightarrow (2x-3)(2x-1) \le 0 \\ &\Leftrightarrow \frac{1}{2} \le x \le \frac{3}{2} \text{ (Why ?)} \end{aligned}$$

Therefore, $B = \left[\frac{1}{2}, \frac{3}{2}\right]$.

Now, $A \cup B = [0,1] \bigcup \left[\frac{1}{2}, \frac{3}{2}\right] = \left[0, \frac{3}{2}\right]$.

Note that $A^c =]-\infty, 0[\cup]1, \infty[$ and $B^c = \left]-\infty, \frac{1}{2}\right[\cup \left]\frac{3}{2}, \infty\right[$.

Hence,

$$A \cup B^c = [0,1] \bigcup \left\{ \left]-\infty, \frac{1}{2}\right[\bigcup \left]\frac{3}{2}, \infty\right[\right\} =]-\infty, 1] \bigcup \left]\frac{3}{2}, \infty\right[$$

$$A^c \cup B = \{]-\infty, 0[\bigcup]1, \infty[\} \bigcup \left[\frac{1}{2}, \frac{3}{2}\right] =]-\infty, 0[\bigcup \left[\frac{1}{2}, \infty\right[$$

$$A^c \cup B^c =]-\infty, 0[\bigcup]1, \infty[\bigcup \left]-\infty, \frac{1}{2}\right[\bigcup \left]\frac{3}{2}, \infty\right[= \left]-\infty \frac{1}{2}\right[\bigcup]1, \infty[.$$

E3) Let $x, y \in S \cup T$ be such that $x \in S, y \in T$ and $x < y$. Since $S \cap T \ne \emptyset$, let $z \in S \cap T$. Take $r \in \mathbb{R}$ such that $x < r < y$. Then, by the Law of Trichotomy, exactly one of the following holds

$$r = z, \quad r < z, \quad r > z.$$

If $r = z$, we are done. If $r < z$, then $x < r < z$. Then, since S is an interval $r \in S$. If $r > z$, then $z < r < y$. Then, since T is an interval $r \in T$. Thus in either case $r \in S \cup T$. Hence, $S \cup T$ is an interval.

E4) We know that there is an irrational number between any two rational numbers. Hence, $\mathbb{Q}$ is not an interval. The same reasoning implies that no subset of $\mathbb{Q}$ is an interval.

E5) Since x is a limit point of S, for each $\varepsilon > 0$, $N_\varepsilon(x) \cap S \setminus \{x\} \neq \emptyset$. Then

$$S \subseteq T \Rightarrow S \setminus \{x\} \subseteq T \setminus \{x\}$$
$$\Rightarrow N_\varepsilon(x) \cap S \setminus \{x\} \subseteq N_\varepsilon(x) \cap T \setminus \{x\} \qquad \text{(Why?)}$$
$$\Rightarrow N_\varepsilon(x) \cap T \setminus \{x\} \neq \emptyset$$

Hence, x is a limit point of T.

E6) Let $x \in \mathbb{R}$. Then $N_\varepsilon(x) \cap \mathbb{R} \setminus \{x\} \neq \emptyset$ for all $\varepsilon > 0$.
Hence x is a limit point of $\mathbb{R}$. Since x is arbitrary, all the real numbers are the limit points of $\mathbb{R}$.

E7) From Theorem 3, we know that if a is a limit point of a finite set S, then $N_\varepsilon(a) \cap S$ must contain infinitely many points different from a.

This is impossible. Hence, finite sets have no limit points.

E8) Let $n \in \mathbb{N}$. Then

$$N_{\frac{1}{2}}(n) \bigcap \mathbb{N} \setminus \{n\} = \left]n - \frac{1}{2}, n + \frac{1}{2}\right[\bigcap \mathbb{N} \setminus \{n\} = \emptyset$$

Hence n is not a limit point of $\mathbb{N}$. Since $n \in \mathbb{N}$ is arbitrary, no point of $\mathbb{N}$ is a limit point of $\mathbb{N}$.

Now, let $x \in \mathbb{R} \setminus \mathbb{N}$. Then there exists some $n \in \mathbb{N}$ such that $n - 1 < x < n$. (Why?) Let $\varepsilon = \min\{n - x, x - n + 1\}$. Then we can see that $N_\varepsilon(x) \cap \mathbb{N} \setminus \{x\} = \emptyset$. Hence, x is not a limit point of $\mathbb{N}$.

Thus we have shown that the limit points of $\mathbb{N}$ do not exist.

A similar reasoning can be used to show that $\mathbb{Z}$, too, has no limit points.

E9) Let $S = \left\{ \frac{1 + (-1)^n}{2} - \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$.

We can see that when n is odd, $-\frac{1}{n} \in S$. Now, for each $\varepsilon > 0$ there exists some $n \in \mathbb{N}$ such that

$$\varepsilon > \frac{1}{n} \Rightarrow -\varepsilon < -\frac{1}{n} \Rightarrow N_\varepsilon(0) \cap S \setminus \{0\} \neq \emptyset$$

This implies, 0 is a limit point of S.

E10) i) Let $S = \left\{ \frac{11}{2^{3n}} \,\middle|\, n \in \mathbb{N} \right\}$. We can see that for all $n \in \mathbb{N}$,

$$0 < \frac{1}{2^{3n}} < 1 \Rightarrow 0 < \frac{11}{2^{3n}} < 11.$$

Hence S is bounded. Now we define $f : \mathbb{N} \to S$ by $f(n) = \frac{11}{2^{3n}}$.

We shall prove that f is one-one. So, let $m, n \in \mathbb{N}$ such that $m \neq n$.

Then

$$3m \neq 3n \Rightarrow 2^{3m} \neq 2^{3n} \Rightarrow \frac{1}{2^{3m}} \neq \frac{1}{2^{3n}} \Rightarrow \frac{11}{2^{3m}} \neq \frac{11}{2^{3n}}$$

Thus f is one-one. This implies S is countably infinite (why?), and hence infinite. Therefore, by the Bolzano Weierstrass Theorem, S has a limit point.

ii) Let $S = \left\{ \frac{(-1)^n}{6^n} \middle| \; n \in \mathbb{N} \right\}$. Then

$$-1 \le \frac{(-1)^n}{6^n} < 1 \quad \forall n \in \mathbb{N}.$$

So, S is bounded. Now define $f : \mathbb{N} \to S$ by $f(n) = \frac{(-1)^n}{6^n}$.

We prove that f is one-one. Let $m, n \in \mathbb{N}$ such that $f(m) = f(n)$. Then

$$\frac{(-1)^m}{6^m} = \frac{(-1)^n}{6^n} \Rightarrow (-1)^{m-n} = 6^{m-n}$$
$$\Rightarrow m - n = 0 \text{ (Why?)}$$
$$\Rightarrow m = n$$

Thus, f is one-one. Hence, S is (countably) infinite. Therefore, by the Bolzano Weierstrass Theorem S has at least one limit point.

iii) Here $S = \left\{ 1 + \frac{1}{n} \middle| \; n \in \mathbb{N} \right\}$.

Show that S is bounded and infinite. Then apply the Bolzano Weierstrass Theorem.

E11) First we show that $]0, \infty[= \bigcup_{a>0} [a, \infty[$. So, let $x \in]0, \infty[$. Then

$$x > 0 \Rightarrow x \in [x, \infty[\Rightarrow x \in \bigcup_{a>0} [a, \infty[.$$

Now, let $x \in \bigcup_{a>0} [a, \infty[$. Then $x \in [a, \infty[$ for some $a > 0$, which implies $x > a > 0$. Consequently, $x \in]0, \infty[$.

Now, we show that 0 is a limit point of $]0, \infty[$. Let $\varepsilon > 0$ be arbitrary.

Then there exists some $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < \varepsilon \Rightarrow \frac{1}{n} \in]-\varepsilon, \varepsilon[= N_\varepsilon(0).$$

But $\frac{1}{n} \in]0, \infty[$ also. Thus $N_\varepsilon(0) \cap]0, \infty[\neq \emptyset$. Therefore 0 is a limit point of $]0, \infty[$. Since $0 \notin]0, \infty[$, we conclude that $]0, \infty[$ is not closed.

E12) We know that for each $x \in \mathbb{R}$, there exists some $n \in \mathbb{N}$ such that

$-n < x < n.$ Hence $\mathbb{R} \subseteq \bigcup_{n=1}^{\infty}]-n, n[$. But $]-n, n[\subseteq \mathbb{R}$ for each $n \in \mathbb{N}$, so $\bigcup_{n=1}^{\infty}]-n, n[\subseteq \mathbb{R}$.

Therefore, $\mathbb{R} = \bigcup_{n=1}^{\infty}]-n, n[$. Since $\mathbb{R}$ is closed, the given set is also closed.

E13) Let $\varepsilon > 0$ be arbitrary. Then

$$N_\varepsilon(a) \cap]-\infty, a[\; = \;]a-\varepsilon, a[.$$

Since $\varepsilon > 0,$ there exists some $x \in \mathbb{R}$ such that $a - \varepsilon < x < a.$ That is,

$$N_\varepsilon(a) \cap]-\infty, a[\setminus \{a\} \neq \varnothing$$

Hence, a is a limit point of $]-\infty, a[$. But $a \notin \;]-\infty, a[,$ hence $]-\infty, a[$ is not closed.

Now, let us see, what the set $\bigcup_{a\in\mathbb{R}}]-\infty, a[$ is equal to. Take $x \in \mathbb{R}.$ Then $x < a$ for some $a \in \mathbb{R}$. That is $x \in \;]-\infty, a[$ for some $a \in \mathbb{R}.$ Therefore, $x \in \bigcup_{a\in\mathbb{R}}]-\infty, a[,$ which implies $\mathbb{R} \subseteq \bigcup_{a\in\mathbb{R}}]-\infty, a[$. But we already know that $]-\infty, a[\subseteq \mathbb{R}$ for all $a \in \mathbb{R}$. Therefore, $\bigcup_{a\in\mathbb{R}}]-\infty, a[\subseteq \mathbb{R}.$

This proves that $\mathbb{R} = \bigcup_{a\in\mathbb{R}}]-\infty, a[$. Since $\mathbb{R}$ is closed, $\bigcup_{a\in\mathbb{R}}]-\infty, a[$ is also closed.

E14) We know that $x^5 + 3x^4 - x^3 + x + 1$ is a polynomial of degree 5, and hence has at most 5 real roots. Thus

$$S = \{x \in \mathbb{R} \mid x^5 + 3x^4 - x^3 + x + 1 = 0\}$$

has at most 5 points. Therefore, S, being a finite set, is closed.

E15) We know that $\left[1 + \frac{1}{n}, 5\right]$ is closed for each $n \in \mathbb{N}$. Therefore, by Theorem 7, $\bigcap_{n=1}^{\infty} \left[1 + \frac{1}{n}, 5\right]$ is closed.

E16) From E8 we know that the limit points of $\mathbb{N}$ do not exist. Therefore, $\overline{\mathbb{N}} = \mathbb{N}$. Similarly, $\overline{\mathbb{Z}} = \mathbb{Z}.$

E17) We have to find $\overline{\mathbb{Q}^c}$. So, let $x \in \mathbb{Q}$. Then we know (from E22) of Unit 3) that for each $\varepsilon > 0,$ there exists an irrational number between $x - \varepsilon$ and $x + \varepsilon.$ That is, for each $\varepsilon > 0$, the set $N_\varepsilon(x) \cap \mathbb{Q}^c$ contains an element different from $x.$ Therefore, x is a limit point of $\mathbb{Q}^c.$ Since x is arbitrary, every rational number is a limit point of $\mathbb{Q}^c$. Hence $\overline{\mathbb{Q}^c} = \mathbb{Q} \cup \mathbb{Q}^c = \mathbb{R}$.

E18) Let $x \in]a, \infty[$. Choose $\varepsilon = x - a$. Then

$$N_\varepsilon(x) =]x - \varepsilon, x + \varepsilon[=]a, a + 2\varepsilon[\subseteq]a, \infty[.$$

This implies x is an interior point of $]a, \infty[$. Since x is arbitrary, every element of $]a, \infty[$ is an interior point of $]a, \infty[$. That is $]a, \infty[\subseteq]a, \infty[$ holds always. Hence $]a, \infty[=]a, \infty[$, and $]a, \infty[$ is open.

E19) Since $]-n, n[$ is open for each $n \in \mathbb{N}$, by Theorem 10, $\bigcup_{n=1}^{\infty}]-n, n[$ is open.

E20) i) Let $x \in S^\circ$. Then there exists an $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq S$. Let $y \in N_\varepsilon(x)$. Then there exists a $\delta > 0$ such that $N_\delta(y) \subseteq N_\varepsilon(x) \subseteq S$ So, $y \in S^\circ$. Therefore $N_\varepsilon(x) \subseteq S^\circ$. Thus x is an interior point of S°. Since x is arbitrary every element of S° is an interior point of S°. Hence S° is open.

ii) Let x be a limit point of $\overline{S}$. Assume, if possible, that $x \notin \overline{S}$. Then $x \notin S$ and x is not a limit point of S. So, there exists an $\varepsilon > 0$ such that

$$\begin{aligned} N_\varepsilon(x) \cap S \setminus \{x\} = \emptyset &\Leftrightarrow N_\varepsilon(x) \cap S = \emptyset \\ &\Leftrightarrow S \subseteq N_\varepsilon(x)^c \\ &\Leftrightarrow \overline{S} \subseteq N_\varepsilon(x)^c \qquad (\because N_\varepsilon(x)^c \text{ is closed}) \\ &\Leftrightarrow N_\varepsilon(x) \cap \overline{S} = \emptyset \\ &\Leftrightarrow N_\varepsilon(x) \cap \overline{S} \setminus \{x\} = \emptyset \end{aligned}$$

This proves that x is not a limit point of $\overline{S}$, a contradiction.

Therefore, $x \in \overline{S}$, and hence $\overline{S}$ is closed.

E21) We know that $]-\infty, 25] \cap]15, \infty[=]15, 25]$,

E22) You know that $\left]1 + \frac{1}{n}, 7\right[$ is open for each $n \in \mathbb{N}$ hence by Theorem 10 $\bigcup_{n=1}^{\infty} \left]1 + \frac{1}{n}, 7\right[$ is open.

E23) To prove that $\bigcup_{\alpha \in \Lambda} S_\alpha$ is open, we need to show that $\left(\bigcup_{\alpha \in \Lambda} S_\alpha\right)^\circ = \bigcup_{\alpha \in \Lambda} S_\alpha$.

We already know that $\left(\bigcup_{\alpha \in \Lambda} S_\alpha\right)^\circ \subseteq \bigcup_{\alpha \in \Lambda} S_\alpha$. This means, we have to show that $\bigcup_{\alpha \in \Lambda} S_\alpha \subseteq \left(\bigcup_{\alpha \in \Lambda} S_\alpha\right)^\circ$. So, let $x \in \bigcup_{\alpha \in \Lambda} S_\alpha$. Then $x \in S_\alpha$ for some

$\alpha \in \Lambda$. But S_α is open i.e. $S_\alpha = S_\alpha^\circ$. Therefore, $x \in S_\alpha^\circ$. Now there exists some $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq S_\alpha$. This implies $N_\varepsilon(x) \subseteq \bigcup_{\alpha\in\Lambda} S_\alpha$.

Hence $x \in \left(\bigcup_{\alpha\in\Lambda} S_\alpha\right)^\circ$. This completes the proof.

MISCELLANEOUS EXAMPLES AND EXERCISES

The examples and exercises given below cover the concepts and processes you have studied in this block. Doing them will give you a better understanding of the concepts concerned, as well as practice in solving such problems.

Miscellaneous Examples

Example 1: Given two sets S and T of $\mathbb{R}$, define $S+T=\{x+y \mid x\in S, y\in T\}$. Suppose S and T are bounded. Check whether $S+T$ is bounded or not.

Solution : Let ℓ_1 be a lower bound of S, and u_1 an upper bound of S. Similarly, let ℓ_2 be a lower bound of T, and u_2 an upper bound of T. Then we have

$$\forall x\in S, \qquad \ell_1 \le x \le u_1,$$

and

$$\forall y\in T, \qquad \ell_2 \le y \le u_2.$$

Therefore, $\forall x\in S, \forall y\in T, \ell_1+\ell_2 \le x+y \le u_1+u_2$.

This implies $\ell_1+\ell_2$ is a lower bound of $S+T$ and u_1+u_2 is an upper bound of $S+T$. Hence, $S+T$ is bounded.

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Example 2: Show that if $a>1$, then $1<a^{\frac{1}{n}}<a$ for all natural numbers n.

Solution: We shall prove this, in two parts, by the method of contradiction. So, suppose that $a^{\frac{1}{n}} \le 1$ for some $n\in\mathbb{N}$. You also know, from Theorem 14 of Unit 3, that $a^{\frac{1}{n}}>0$ for all $n\in\mathbb{N}$.

Then from E9 of Unit 3, you know that $\left(a^{\frac{1}{n}}\right)^n \le 1^n$ i.e. $a\le 1$. This is a contradiction to the hypothesis that $a>1$. Hence $a^{\frac{1}{n}}>1$ for all $n\in\mathbb{N}$.

Again, suppose that $a^{\frac{1}{n}} \ge a$ for some $n\in\mathbb{N}$. Then

$$\left(a^{\frac{1}{n}}\right)^n \ge a^n \Rightarrow a \ge a^n \Rightarrow 1 \ge a^{n-1}$$

But you know, from E10 of Unit 3, that if $a>1$, then $a^n>1$ for all $n\in\mathbb{N}$. So, again, we have arrived at a contradiction. Hence, $a^{\frac{1}{n}}<a$, for all $n\in\mathbb{N}$.

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Example 3: Show that if $a,b\in\mathbb{R}$ are such that $|a|^2\ge b^2$, then $|a|\ge b$.

Solution: When b is negative, then $|a| \geq b \, \forall a \in \mathbb{R}$. So, let us assume that $b \geq 0$. Now

$$\begin{aligned} |a|^2 \geq b^2 &\Rightarrow |a|^2 - b^2 \geq 0 \\ &\Rightarrow |a|^2 - |a|b + |a|b - b^2 \geq 0 \\ &\Rightarrow |a|(|a| - b) + b(|a| - b) \geq 0 \\ &\Rightarrow (|a| + b)\,(|a| - b) \geq 0 \\ &\Rightarrow |a| - b \geq 0 \qquad (\because |a| + b \geq 0) \\ &\Rightarrow |a| \geq b \end{aligned}$$

***

Example 4: Which of the following statements are true, and which are false? Justify your answers with a short proof or a counter-example.

i) Every subset of a bounded set is bounded.
ii) Every point of a bounded set is its interior point.
iii) The set of all limit points of a countable set is countable.
iv) ''No subset of $\mathbb{N}$ is uncountable.'' is the negation of "Every subset of $\mathbb{N}$ is countable."

Solution: i) Let S be a bounded set, and $T \subseteq S$. Then, there exist real numbers ℓ and u such that $\ell \leq x \leq u$ for all $x \in S$. This implies $\ell \leq x \leq u$ for all $x \in T$. Therefore, T is bounded. This shows that every subset of a bounded set is bounded. Hence the statement is true.

ii) Consider the set $\{1\}$ which is bounded. We have $1 \in \{1\}$, but there is no $\varepsilon > 0$ such that $]1-\varepsilon, 1+\varepsilon[\subseteq \{1\}$. Therefore, 1 is not an interior point of $\{1\}$. Hence, the statement is false.

iii) Consider the set $\mathbb{Q}$ which is countable. You also know that $\mathbb{R}$ is the set of all limit points of $\mathbb{Q}$, and that $\mathbb{R}$ is uncountable. Therefore, the statement is false.

iv) Note that the given statement can symbolically be written as $\sim p \equiv q$, where

p: Every subset of $\mathbb{N}$ is countable.
q: No subset of $\mathbb{N}$ is uncountable.

But you know that

$\sim p$: Some subset of $\mathbb{N}$ is uncountable.

So, $\sim p \not\equiv q$. Hence the statement is false.

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Example 5: Prove that $\big||x| - |y|\big| \leq |x - y| \, \forall x, y \in \mathbb{R}$.

Solution: For any $x, y \in \mathbb{R}$ we have $|x| = |x - y + y| \leq |x - y| + |y|$. This implies

$$|x| - |y| \leq |x - y| \qquad \ldots (1)$$

Now interchanging the role of x and y, in Eq. (1) we get $|y|-|x| \le |y-x|$, i.e.,

$$-(|x|-|y|) \le |x-y| \qquad \dots (2)$$

From Eqs. (1) and (2), we get $\left||x|-|y|\right| \le |x-y|$.

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Example 6: Show that every subset of a countable set is countable.

Solution: Let S be a countable set, and $T \subseteq S$. Since $T \subseteq S$, there exists an injection $f : T \to S$. Since S is countable, there exists a bijection $g : S \to \mathbb{N}$. Then the composition $g \circ f : T \to \mathbb{N}$ is an injection. Hence by Theorem 17 (i) of Unit 3, T is countable.

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Example 7: Let S and T be two finite sets. Show that $S \times T$ is also finite.

Solution: Since S is finite, let $\{x_1, x_2, \dots, x_m\}$ be the list of elements of S. Similarly, let $\{y_1, y_2, \dots, y_n\}$ be the list of elements of T. Then $\{(x_i, y_j) \mid 1 \le i \le m, 1 \le j \le n\}$ is the list of elements of $S \times T$. Thus $S \times T$ has mn elements. Hence $S \times T$ is finite.

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Miscellaneous Exercises

E1) Prove or disprove: If $x, y \in \mathbb{R} \setminus \mathbb{Q}$, then $x + y \in \mathbb{R} \setminus \mathbb{Q}$.

E2) Using the Principle of Mathematical Induction show that if S contains n elements, then $\wp(S)$, the power set of S, contains 2^n elements.

E3) Use the Well-ordering Principle to show that there is no $n \in \mathbb{N}$ such that $0 < n < 1$.

E4) Which of the following sets are finite, and which are infinite? Justify your answers.

i) $\left\{x \in \mathbb{R} \,\middle|\, 0 \le x < \frac{1}{n} \; \forall n \in \mathbb{N}\right\}$ ii) $\{\sqrt{10}, \sqrt{11}, \sqrt{12}, \dots, \sqrt{100^{100}}\}$

iii) $\left\{\frac{1}{n} \,\middle|\, n \in \mathbb{Z}, n \ne 0\right\}$ iv) $\bigcup_{n \in \mathbb{Z}} \{n^2\}$

E5) Check whether the following sets are bounded below, bounded above or both. Accordingly, show whether they have the infimum, the supremum or both in $\mathbb{R}$.

i) $S = \left\{2^{1-\frac{1}{n}} + 3^{\frac{1}{n}} \,\middle|\, n \in \mathbb{N}\right\}$ ii) $S = \left\{\frac{x}{x+4} \,\middle|\, x > 0\right\}$

iii) $S = \{n + (-1)^n \mid n \in \mathbb{N}\}$ iv) $S = \{\sin x \mid x \in \mathbb{R}\}$

E6) Union of two bounded sets is bounded. True or False? Justify your answer.

E7) Find the infimum and supremum of the following sets.

i) $S = \{x \in \mathbb{R} \mid x^2 < 3x - 2\}$ ii) $S = \{x \in \mathbb{R} \mid |x-1| \le 1\}$

E8) Which of the following sets are countable and which are uncountable? Justify your answers.

i) $\{0.1, 0.01, 0.001, 0.0001, \cdots\}$ ii) $\left\{\frac{x^2-1}{x^2+2} \middle| x \in \mathbb{Q}\right\}$

iii) $\{x \in \mathbb{R} \mid x < 3\} \cup \{x \in \mathbb{N} \mid x \ge 3\}$ iv) $\bigcup_{n=1}^{\infty} \{x \in \mathbb{R} \mid n-1 < x < n\}$

v) $\mathbb{Q} \times \mathbb{N}$ vi) $[0,1]$

E9) Identify which of the following sets are closed, which are open and which are neither.

i) $\{x \in \mathbb{R} \mid x^2 \le x+1\}$ ii) $\mathbb{Q} \cap [0,1]$

iii) $\bigcup_{n=1}^{\infty} \left[\frac{n}{1+n}, \frac{n+1}{n}\right]$ iv) $\left\{m + \frac{1}{n} \middle| m, n, \in \mathbb{N}\right\}$

E10) Show that for any subsets S and T of $\mathbb{R}$, $\overline{S \cup T} = \overline{S} \cup \overline{T}$.

E11) Show that if S is open and T is closed, then

i) $S \setminus T$ is open, ii) $T \setminus S$ is closed.

SOLUTIONS/ANSWERS

E1) The statement is false. For a counter-example, let $x = \pi$ and $y = 1 - \pi$. Then $x + y = 1$ which is rational.

E2) Write $P(n)$: $|S| = n \Rightarrow |\wp(S)| = 2^n$.
So, $P(1)$: $|S| = 1 \Rightarrow |\wp(S)| = 2$.

We know that when S is a singleton, $\wp(S)$ contains just two elements, namely $\emptyset$ and S. Therefore, $P(1)$ is true.

Now let us assume that for some n $P(k)$ is true for all $k, 1 \le k \le n$. We have to show that $P(n+1)$ is true. So, let $S = \{x_1, x_2, \ldots, x_n, x_{n+1}\}$. Look at the element x_{n+1}. Every subset of S either contains x_{n+1} or does not. This means, for any $A \subseteq S$, we can write

$$A = B \text{ or } A = B \cup \{x_{n+1}\},$$

where B is a subset of $\{x_1, x_2, \ldots, x_n\}$. Since $\{x_1, x_2, \ldots, x_n\}$ contains n elements, by the induction hypothesis, the number of Bs is 2^n. Hence the number of As is $2^n + 2^n = 2^{n+1}$. This means $P(n+1)$ is true. Therefore, by PMI $P(n)$ holds for all $n \in \mathbb{N}$.

E3) We shall prove it by contradiction. Let $S = \{n \in \mathbb{N} \mid 0 < n < 1\}$. Assume, if possible, that $S \neq \emptyset$. Since S is a subset of $\mathbb{N}$, the Well Ordering principle implies that S contains a least element, say p . So, $0 < p < 1$. But then $0 < p^2 < p < 1$, which implies $p^2 \in S$. This is a contradiction to the definition of p. Therefore, $S = \emptyset$.

E4) i) By the Archimedean Property we know that if $0 \le x < \frac{1}{n}$ for all $n \in \mathbb{N}$, then $x = 0$. Then the given set is $\{0\}$, which is finite.

ii) The given set is $S = \{\sqrt{n} \mid n \in \mathbb{N}, 10 \le n \le 100^{100}\}$. Let $N = \{1, 2, 3, \ldots, 100^{100} - 9\}$. Let us define the function $f : N \to S$ by $f(n) = \sqrt{n+9}$.

Now let us show that f is a bijection. Take $m, n \in N$ such that $f(m) = f(n)$. Then

$$\sqrt{m+9} = \sqrt{n+9} \Rightarrow m+9 = n+9 \Rightarrow m = n.$$

So f is $1-1$. Now let $x \in S$. Then $x = \sqrt{n}$ for some natural number n lying in $\{10, 11, \ldots, 100^{100}\}$. This implies $x = \sqrt{n+9}$ for some natural number n lying in $\{1, 2, 3, \ldots, 100^{100} - 9\} = N$. This means $x = f(n)$ for some $n \in N$. Hence f is onto. Therefore, f is a bijection. Thus S is a finite set.

iii) Let $S = \left\{ \frac{1}{n} \middle| n \in \mathbb{Z}, n \neq 0 \right\}$. Let us define $f : \mathbb{Z} \to S$ by

$$f(n) = \begin{cases} \dfrac{1}{n}, & \text{if } n > 0 \text{ else.} \\ \dfrac{1}{n-1}, & \end{cases}$$

Now you can show that f is a bijection. Since $\mathbb{Z}$ is countable, and $f : \mathbb{Z} \to S$ is a bijection, it follows that S is countable.

iv) Note that $\bigcup_{n \in \mathbb{Z}} \{n^2\} = \{0, 1, 4, 9, \ldots\}$, which is the set of the squares of whole numbers. The set of whole numbers is infinite, and so is the set of their squares.

E5) i) We know that for all $n \in \mathbb{N}, 0 < 3^{\frac{1}{n}} < 3$. For all $n \in \mathbb{N}$

$$2^{1-\frac{1}{n}} < 2 \Rightarrow 1 < \frac{2}{2^{1-\frac{1}{n}}} \Rightarrow 1 < 2^{\frac{1}{n}},$$

which is true. Thus for all $n \in \mathbb{N}$, $2^{1-\frac{1}{n}} < 2$. This implies for all $n \in \mathbb{N}$

$0 < 2^{1-\frac{1}{n}} + 3^{\frac{1}{n}} < 5$.

Hence S is bounded below as well as bounded above, i.e. S is bounded. Therefore, by the completeness properly of $\mathbb{R}$, the infimum and the supremum of S exist in $\mathbb{R}$.

ii) We have for all $x \in S$,

$$0 < x < x+4 \Rightarrow 0 < \frac{x}{x+4} < 1$$

This means S is bounded. Hence by the Completeness Property of $\mathbb{R}$, the infimum and the supremum of S exist in $\mathbb{R}$.

iii) We can write

$$n + (-1)^n = \begin{cases} n-1, & \text{when } n \text{ is odd} \\ n+1, & \text{when } n \text{ is even} \end{cases}$$

So,

$$\begin{aligned} S &= \{n-1 \mid n \in \mathbb{N}, n \text{ is odd}\} \cup \{n+1 \mid n \in \mathbb{N}, n \text{ is even}\} \\ &= \{0,2,4,\ldots\} \cup \{1,3,5,\ldots\} = \mathbb{N} \cup \{0\} \end{aligned}$$

So, S is bounded below by 0, but has no upper bound (Why?). Since S is nonempty, by the Greatest Lower Bound Property of $\mathbb{R}$, S has infimum in $\mathbb{R}$.

iv) We know that $-1 \le \sin x \le 1$ for all $x \in \mathbb{R}$. Thus S is bounded, and nonempty subset of $\mathbb{R}$. Hence, by the Completeness Property of $\mathbb{R}$, S has both infimum and supremum in $\mathbb{R}$.

E6) Let S and T be two bounded sets. Then there exist real numbers ℓ_1, ℓ_2, u_1 and u_2 such that $\ell_1 \le x \le u_1$ for all $x \in S$, and $\ell_2 \le x \le u_2$ for all $x \in T$. Let $\ell = \min\{\ell_1, \ell_2\}$ and $u = \max\{u_1, u_2\}$. Then we have $\ell \le x \le u$ for all $x \in S \cup T$. Hence $S \cup T$ is bounded.

E7) i) We have

$$x^2 < 3x - 2 \Rightarrow x^2 - 3x + 2 < 0 \Rightarrow (x-1)(x-2) < 0 \Rightarrow 1 < x < 2$$

Thus $S = \{x \in \mathbb{R} \mid 1 < x < 2\}$. So, 1 is a lower bound of S, and 2 an upper bound of S. Let u be some upper bound of S. If $u < 2$, then $u < \frac{u+2}{2} < 2$. Since we know that an upper bound is never smaller than a lower bound, $1 \le u$. Then $1 < \frac{u+2}{2} < 2$, which implies $\frac{u+2}{2} \in S$. This is a contradiction to the definition of u. Hence $u \ge 2$. Thus $\sup S = 2$.

Now let ℓ be some lower bound of S. If $\ell > 1$ then

$1 < \frac{\ell+1}{2} < \ell < 2$, which implies that $\frac{\ell+1}{2} \in S$. This is a contradiction to the definition of ℓ. Hence $\ell \le 1$. This implies $\inf S = 1$.

ii) We have $|x-1| \le 1 \Leftrightarrow -1 \le x-1 \le 1 \Leftrightarrow 0 \le x \le 2$.

Therefore, $S = \{x \in \mathbb{R} | 0 \le x \le 2\}$. So, 0 and 2 are, respectively, lower and upper bounds of S. Also, $0,2 \in S$. Hence $\inf S = 0$ and $\sup S = 2$.

E8) i) Let $S = \left\{ \frac{1}{10^n} \middle| n \in \mathbb{N} \right\}$. Now look at the function $f : \mathbb{N} \to S$ defined by $f(n) = \frac{1}{10^n}$.

We can see that for each $y \in S$, $y = \frac{1}{10^n}$ for some $n \in \mathbb{N}$. That is, f is onto. Hence S is countable. (Why?)

ii) Let S be the given set. We can see that $S \subseteq \mathbb{Q}$. Since $\mathbb{Q}$ is countable, S too is countable (Why?).

iii) If possible, assume that S is countable. Then $A = \{x \in \mathbb{R} | x < 3\}$ must be countable, because $A \subseteq S$. Now, let $B = \{x \in \mathbb{R} | x > 3\}$. Define $f : A \to B$ by $f(x) = 6 - x$. We can show that f is a bijection. Hence B must be countable. Now $\mathbb{R} = A \cup B \cup \{3\}$ is also countable, which is a contradiction. Hence S is uncountable.

iv) Let $S = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} | \, n-1 < x < n\}$. Then we can write $S = \mathbb{R}^+ \setminus \mathbb{N}$, or equivalently, $\mathbb{R}^+ = S \cup \mathbb{N}$. If S is countable, then $\mathbb{R}^+$ must be countable, which is not possible. Hence, S is uncountable.

v) Define the function $f : \mathbb{Q} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by

$f\left(\frac{m}{n}, k\right) = \left((m+n)^2 + n, k\right)$. Show that f is a bijection from $\mathbb{Q} \times \mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. Since $\mathbb{N} \times \mathbb{N}$ is countable, by E23 of Unit 3, $\mathbb{Q} \times \mathbb{N}$ is countable.

vi) Assume, if possible, $[0,1]$ is countable. Then $f(x) = x + n$ is a bijection from $[0,1]$ to $[n-1, n]$ for any $n \in \mathbb{Z}$. This means, $[n-1, n]$ is countable for any $n \in \mathbb{Z}$. Now we have $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n-1, n]$ (Why?)

On the right hand side is a countable union of countable sets, and hence a countable set. But on the left hand is $\mathbb{R}$ which we know is uncountable. This is a contradiction. Therefore, $[0,1]$ is uncountable.

E9) i) We have $\{x \in \mathbb{R} | x^2 \le x + 1\} = \{x \in \mathbb{R} | x^2 - x - 1 \le 0\}$.

We can factorise $x^2 - x - 1$ as

$$x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right).$$

Now

$$x^2 - x - 1 \le 0 \Leftrightarrow \left(\frac{1-\sqrt{5}}{2} \le x \le \frac{1+\sqrt{5}}{2}\right)$$

Thus the given set becomes equal to $\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$, which is closed.

ii) Let $x \in \mathbb{Q} \cap [0,1]$. Then for every $\varepsilon > 0$ there exists an irrational number between x and $x+\varepsilon$. (See …. of Unit 3.) So, for every $\varepsilon > 0, N_\varepsilon(x) \not\subseteq \mathbb{Q} \cap [0,1]$. Hence $\mathbb{Q} \cap [0,1]$ is not open.

We know that $\frac{1}{\sqrt{2}} \notin \mathbb{Q} \cap [0,1]$. In order to prove that $\mathbb{Q} \cap [0,1]$ is not closed, it is sufficient to show that $\frac{1}{\sqrt{2}}$ is a limit point of $\mathbb{Q} \cap [0,1]$. So, let $\varepsilon > 0$, and consider the neighbourhood $N_\varepsilon\left(\frac{1}{\sqrt{2}}\right)$ of $\frac{1}{\sqrt{2}}$. There are three cases:

Case i) $\varepsilon > \frac{1}{\sqrt{2}}$

When $\varepsilon > \frac{1}{\sqrt{2}}$, we have $\frac{1}{\sqrt{2}} - \varepsilon < 0$. So $0 \in N_\varepsilon\left(\frac{1}{\sqrt{2}}\right)$. Also $0 \in \mathbb{Q} \cap [0,1]$. Hence $N_\varepsilon\left(\frac{1}{\sqrt{2}}\right) \cap \mathbb{Q} \cap [0,1]$ contains a point other than $\frac{1}{\sqrt{2}}$.

Case ii) $\varepsilon > 1 - \frac{1}{\sqrt{2}}$

In this case $1 < \frac{1}{\sqrt{2}} + \varepsilon$. Also $\frac{1}{\sqrt{2}} < 1$. This means $1 \in N_\varepsilon\left(\frac{1}{\sqrt{2}}\right)$. Hence $N_\varepsilon\left(\frac{1}{\sqrt{2}}\right) \cap \mathbb{Q} \cap [0,1]$ contains a point other than $\frac{1}{\sqrt{2}}$.

Case iii) $\varepsilon \le \min\left\{\frac{1}{\sqrt{2}},\ 1 - \frac{1}{\sqrt{2}}\right\}$

In this case $0 \le \frac{1}{\sqrt{2}} - \varepsilon < \frac{1}{\sqrt{2}} < \frac{1}{\sqrt{2}} + \varepsilon \le 1$

Now by Theorem 12 of Unit 3, we know that there exists a rational number between $\frac{1}{\sqrt{2}} - \varepsilon$ and $\frac{1}{\sqrt{2}}$. Hence $N_\varepsilon\left(\frac{1}{\sqrt{2}}\right) \cap \mathbb{Q} \cap [0,1]$ contains a point other than $\frac{1}{\sqrt{2}}$.

Thus, for every $\varepsilon > 0, N_\varepsilon\left(\frac{1}{\sqrt{2}}\right) \cap \mathbb{Q} \cap [0,1]$ contains a point other than $\frac{1}{\sqrt{2}}$.

Hence, $\frac{1}{\sqrt{2}}$ is a limit point of $\mathbb{Q} \cap [0,1]$. Since $\frac{1}{\sqrt{2}} \notin \mathbb{Q} \cap [0,1]$, it is proved that $\mathbb{Q} \cap [0,1]$ is not closed.

iii) We know that for all $n \in \mathbb{N}$

$$\frac{n}{n+1} \geq \frac{1}{2} \text{ and } \frac{n+1}{n} \leq 2$$

That is, for all $n \in \mathbb{N}$

$$\frac{1}{2} \leq \frac{n}{n+1} < \frac{n+1}{n} \leq 2 \Rightarrow \left[\frac{n}{n+1}, \frac{n+1}{n}\right] \subseteq \left[\frac{1}{2}, 2\right]$$

This implies $\bigcup_{n=1}^{\infty}\left[\frac{n}{n+1}, \frac{n+1}{n}\right] \subseteq \left[\frac{1}{2}, 2\right]$

Also for $n = 1$,

$$\left[\frac{n}{n+1}, \frac{n+1}{n}\right] = \left[\frac{1}{2}, 2\right]$$

Hence $\bigcup_{n=1}^{\infty}\left[\frac{n}{n+1}, \frac{n+1}{n}\right] = \left[\frac{1}{2}, 2\right]$ which is closed, but not open.

iv) Let $S = \left\{m + \frac{1}{n} \middle| m, n \in \mathbb{N}\right\}$. First we check whether S is closed or not. Taking $m = 1$, and $n \in \mathbb{N}$, we have $1 + \frac{1}{n} \in S$. Now we shall show that 1 is a limit point of S. So, take $\varepsilon > 0$. Then, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. This implies

$$1 < 1 + \frac{1}{n} < 1 + \varepsilon \Rightarrow 1 + \frac{1}{n} \in N_\varepsilon(1).$$

So for each $\varepsilon > 0, N_\varepsilon(1) \cap S$ contains an element other than 1.

That is, 1 is a limit point of S. But $1 \notin S$. Therefore, S is not closed.

Now we check whether S is open or not. Pick $1+\frac{1}{2}=\frac{3}{2}\in S$. Then for each $\varepsilon>0$ there exists an irrational number between $\frac{3}{2}-\varepsilon$ and ε. That is, for each $\varepsilon>0$, $N_\varepsilon\left(\frac{3}{2}\right)\not\subseteq S$. Hence $\frac{3}{2}$ is not an interior point of S. Therefore, S is not open.

E10) Note that $S\subseteq\overline{S}$ and $T\subseteq\overline{T}$. Therefore, $S\cup T\subseteq\overline{S}\cup\overline{T}$, which implies $\overline{S\cup T}\subseteq\overline{S}\cup\overline{T}$. (See Example 14 of Unit 4.) On the other hand, $S\subseteq S\cup T$ and $T\subseteq S\cup T$. Therefore, $\overline{S}\subseteq\overline{S\cup T}$ and $\overline{T}\subseteq\overline{S\cup T}$. This implies $\overline{S}\cup\overline{T}\subseteq\overline{S\cup T}$. Consequently, $\overline{S}\cup\overline{T}=\overline{S\cup T}$.

Interchanging the roles of S and T, we have $x\in\overline{T}\Rightarrow x\in\overline{T\cup S}$. Thus $\overline{S}\cup\overline{T}\subseteq\overline{S\cup T}$. The proof is over.

E11) i) Write $S\setminus T=S\cap T^c$. Since S and T^c both are open, $S\setminus T$ is open.

ii) Write $T\setminus S=T\cap S^c$. Since T and S^c both are closed, $T\setminus S$ is closed.