Indira Gandhi National Open University
School of Sciences

BMTC-133
REAL ANALYSIS

Block

2

SEQUENCES

Course Design Committee*

Prof. Rashmi Bhardwaj
G.G.S. Indraprastha University, Delhi

Dr. Sunita Gupta
University of Delhi

Prof. Amber Habib
Shiv Nadar University
Gautam Buddha Nagar

Prof. S. A. Katre
University of Pune

Prof. V. Krishna Kumar
NISER, Bhubaneswar

Dr. Amit Kulshreshtha
IISER, Mohali

Dr. Aparna Mehra
I.I.T. Delhi

Prof. Rahul Roy
Indian Statistical Institute, Delhi

Prof. Meena Sahai
University of Lucknow

Dr. Sachi Srivastava
University of Delhi

Faculty Members
SOS, IGNOU

Prof. M. S. Nathawat
(Director)

Dr. Deepika

Prof. Poornima Mital

Prof. Parvin Sinclair

Prof. Sujatha Varma

Dr. S. Venkataraman

* The course design is based on the recommendations of the Programme Expert Committee and the UGC-CBCS template

Block Preparation Team

Prof. H.L. Vasudeva (Editor)
Dept. of Mathematical Sciences
IISER, Mohali

Dr. Pawan Kumar
School of Sciences
IGNOU

Course Coordinator: Prof. Sujatha Varma

Acknowledgement: To Prof. Sujatha Varma and Prof. Parvin Sinclair for their comments on the units, and to S.S. Chauhan for making the CRC of the block.

BLOCK INTRODUCTION

In the last block, you were introduced to various aspects of the language of mathematics – emphasising on how to communicate mathematics. Then you studied the set of real numbers, and its algebraic and topological properties.
This block is devoted to the study of sequences of real numbers and the concept of "convergence" of such sequences. In Unit 5, you will study different kinds of sequences such as bounded sequences, monotone sequences and Cauchy sequences, and of course, the notion of subsequence. Then you will be introduced to the concept of convergence of sequences. Convergence, specifically, talks about what it means for a sequence to have a "limit". Once you understand this concept it will be easier for you to grasp the rest of the course material. This is because all the core concepts of real analysis such as continuity, differentiability and integrability employ, in some or other sense, the notion of limit. Therefore, we have put enough stress on the results that characterise convergent sequences. For instance, you will find the Cauchy's Criterion of Convergence quite helpful.

In Unit 6, we have collected some more results on limits of sequences. First you will study the Squeeze Theorem, which describes how the limits of sequences fit into the ordered structure of $\mathbb{R}$. The next result is the Monotone Convergence Theorem. It gives you a simple criterion of the convergence of monotone sequences. Finally, you will see two theorems due to Cauchy. These are Cauchy's First Theorem on Limits, and Cauchy's Second Theorem on Limits. These four theorems are applicable on a large class of sequences.

At the end of this block you will find **a set of miscellaneous examples and exercises** related to the concepts covered in this block. Please do study them, and try each exercise yourself. This will help you engage with the concepts concerned, and understand them better.

Notations and Symbols (used in Block 2 apart from Block 1)

$(a_n)_{n\in\mathbb{N}}$	Real sequence defined on $\mathbb{N}$
$\lim\limits_{n\to\infty} a_n$	the limit of the sequence $(a_n)_{n\in\mathbb{N}}$
$(a_{n_k})_{k\in\mathbb{N}}$	A subsequence of the sequence $(a_n)_{n\in\mathbb{N}}$
$a_n \to L$	the sequence $(a_n)_{n\in\mathbb{N}}$ converges to L

UNIT 5

SEQUENCE AND CONVERGENCE

Structure

5.1 INTRODUCTION

In the previous units you studied different subsets of real numbers. You were introduced to the algebraic and topological properties of the set of real numbers. You would also be able to distinguish between finite and infinite sets. In this unit we shall develop the notion of a sequence and its convergence, which as you will see will be used frequently in the rest of the course.

The concept of a sequence is fundamental to real analysis. Historically, sequences can be seen in the work of the Greek mathematician Archimedes, for example, in the approximation of π. Sequences also arise in numerical approximations of roots of equations in real variables.

We begin this unit by introducing to you the notion of a real sequence, in Section 5.2. In the course 'Calculus' you studied the bounded and monotone functions. In Sections 5.3 and 5.4, we shall discuss bounded and monotone sequences, respectively. Section 5.5 introduces the concept of subsequence of a sequence. Next, in Section 5.6 we shall talk about what it means for a sequence to converge, that is, we shall discuss the concept of limit of a sequence. In this section we shall also discuss the criterion of convergence given by an eminent mathematician, A.-L. Cauchy. Finally, in Section 5.7, we shall consider how the limits of sequences behave under addition, multiplication and reciprocals.

The above concepts are essential, as you will see in this course, to understand the behaviour of real valued functions. If you study this unit carefully and try each exercise as you come to it we expect that you will achieve the following objectives.

Objectives

After studying this unit, you should, be able to:

- define, and give examples of, a sequence and a subsequence;
- describe, and give examples of, a bounded sequence;
- describe, and give examples of, a monotone sequence;
- describe the notions of 'convergence' and 'limit' of a sequence;
- define a Cauchy sequence, and apply the Cauchy's Criterion for Sequence Convergence;
- use the algebra of limits of convergent sequences to compute the limit of sums and products of sequences.

5.2 REAL SEQUENCES

Let us consider the case of a "bouncing ball" whose height (in meter) at each bounce reduces by some quantity (see Fig. 1). Assume that the heights at the first few bounces are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

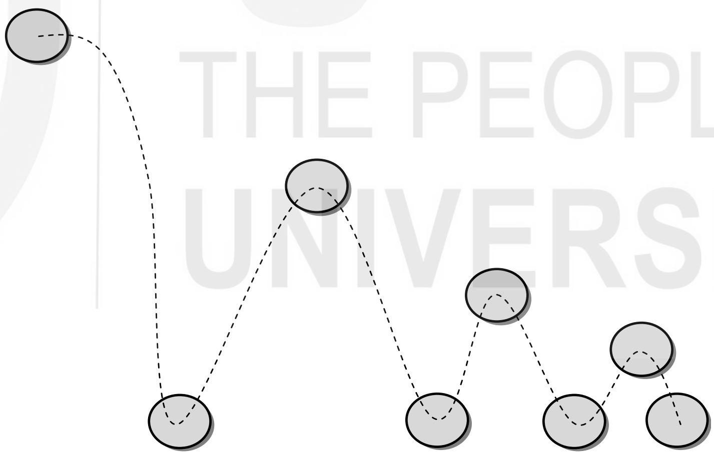


Fig. 1 A bouncing ball

Since the ball is a physical object, there are many forces like gravity and friction working on it, which finally take it into the resting position. If we ignore all the forces and assume that the height at each step becomes half the height at the previous step, then the height at successive steps can be represented as follows:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$$

We can read it as a function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) =$ height at the n^{th} bounce.

In fact, we can write this as an arrangement $(f(1), f(2), f(3), ..., f(n), ...)$ of countably infinite numbers. A sequence is just an arrangement of at most countably infinite numbers in a particular order. This order gives a unique place to each of the elements of a sequence. Let us record the formal definition.

Definition: A **real sequence** is a function from $\mathbb{N}$ to $\mathbb{R}$.

For example, the function $f:\mathbb{N}\to\mathbb{R}$, where $f(n)=\frac{1}{n^2}$ defines a sequence. The domain $\mathbb{N}$ makes it possible to list the elements of the range as $(f(1), f(2), f(3)\ldots, f(n),\ldots)$, i.e., $\left(1,\frac{1}{4},\frac{1}{9},\ldots,\frac{1}{n^2},\ldots\right)$. In general, for a sequence $a:\mathbb{N}\to\mathbb{R}$, we shall in place of $(a(1), a(2),\ldots, a(n),\ldots)$ use the notation $(a_1, a_2,\ldots, a_n,\ldots)$, which is often shortened as $(a_n)_{n\in\mathbb{N}}$. The numbers $a_1, a_2, a_3,...$ are called the **terms** of the sequence. Thus we call a_1 the first term, a_2 the second term, a_3 the third term and so on. In general, a_n is called the n^{th} term, for $n\in\mathbb{N}$. Occasionally, we shall use the notations x_n, y_n, u_n etc to denote the n^{th} term of a sequence. Also note that there is nothing special about the variable n in $(a_n)_{n\in\mathbb{N}}$. We can write $(a_k)_{k\in\mathbb{N}}$ as well to denote the same sequence.

Let us consider a few examples.

Example 1: Check whether the following are sequences or not.

i) $\left(\frac{1}{n^2-1}\right)_{n\in\mathbb{N}}$ ii) $(0)_{n\in\mathbb{N}}$

iii) $\left(\frac{1}{3^n}\right)_{n\in\mathbb{N}}$ iv) $\left(1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\cdots\right)$

Solution:

i) Did you notice that a_n is undefined for $n=1$? Hence $(a_n)_{n\in\mathbb{N}}$ is not a sequence. However, if we take $a_1=1$, and $a_n=\frac{1}{n^2-1}$ $\forall n>1$, then $(a_n)_{n\in\mathbb{N}}$ is a sequence.

ii) We have $a_n=0$ $\forall n\in\mathbb{N}$, which is well defined. Hence $(0)_{n\in\mathbb{N}}$ is a sequence.

iii) Here $a_n=\frac{1}{3^n}$, which is defined for each $n\in\mathbb{N}$. Thus $(a_n)_{n\in\mathbb{N}}$ is a sequence.

iv) In this case, we get $a_n=\frac{1}{2^{n-1}}$, which is defined for all $n\in\mathbb{N}$. Hence $\left(1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\cdots\right)$ is a sequence.

***

Example 2: Write the n^{th} term of the following sequences.

i) $(4,3,2,1,0,0,0,....)$ ii) $\left(1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},....\right)$

iii) $(1,-1,1,-1,1,-1,....)$ iv) $(1,5,6,10,11,15,\ldots)$

Solution: i) Here we have $a_1 = 4, a_2 = 3, a_3 = 2, a_4 = 1$ and $a_n = 0,$ for all $n \geq 5.$ We can, then, write the n^{th} term as

$$a_n = \begin{cases} 5-n, & \text{when } n < 5 \\ 0, & \text{when } n \geq 5 \end{cases}$$

ii) Here $a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, \ldots$ Then, the n^{th} term is $a_n = \frac{1}{n},$ for all $n \geq 1.$

iii) Observe that this sequence has odd terms 1 and even terms -1. That is,

$$a_n = \begin{cases} 1, & \text{when } n \text{ is odd} \\ -1, & \text{when } n \text{ is even} \end{cases}$$

Or equivalently, we can write $a_n = (-1)^{n-1}, \forall\, n \geq 1.$

iv) This sequence also has a pattern. To see this pattern, let us look at how each term, from second term onwards, is obtained

$$\begin{aligned} a_1 &= 1 \\ a_2 &= a_1 + 4 = 5 \\ a_3 &= a_2 + 1 = 6 \\ a_4 &= a_3 + 4 = 10 \\ a_5 &= a_4 + 1 = 11 \\ a_6 &= a_5 + 4 = 15 \\ &\vdots \end{aligned}$$

The explanation above reveals that each odd term is 1 more than the previous term, whereas each even term is 4 more than the previous term. This means, we can write the n^{th} term as

$$a_n = \begin{cases} a_{n-1} + 1, & \text{when } n \text{ is odd} \\ a_{n-1} + 4, & \text{when } n \text{ is even} \end{cases}$$

***

In Example 2 (iv), we saw that the n^{th} term of the sequence is not written explicitly in the terms of $n,$ but in the terms of a_{n-1}. A sequence whose n^{th} term depends on one or more previous terms is called a **recursive** sequence. A formal definition is given below.

Definition: A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be **recursive** if $a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k}) + g(n)$ for some $k \geq 1$ where $g(n)$ is a function of $n.$

For instance, the sequence in Example 2(iv) can be written as

$$a_n = a_{n-1} + g(n), \forall\, n \geq 2, a_1 = 1,$$

where $g(n) = \frac{1}{2}\left[5 + 3(-1)^n\right].$

A famous example of a recursive sequence is given below:

$$f_1 = f_2 = 1, \text{ and } f_n = f_{n-1} + f_{n-2}, n \geq 3$$

You can see that for $n \geq 3, f_n$ is defined as the sum of the previous two terms.

Thus, the first few terms of this sequence are

$$1, 1, 2, 3, 5, 8, 13, \cdots$$

The sequence $(f_n)_{n\in\mathbb{N}}$ is called the Fibonacci sequence after the Italian mathematician **Leonardo of Pisa,** better known as **Fibonacci** (1170–1250). The Fibonacci sequence is one of the most frequently occurring sequences in nature, in one or another form.

We shall see such sequences often in this course as they offer a convenient method for sequence representation.

Now we shall discuss about the algebra of sequences. Recall that two functions f and g are equal if they have the same domain, say S, and $f(x) = g(x)$ for all $x \in S$. In particular, two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are **equal** if $a_n = b_n, \forall n \in \mathbb{N}$ Consider, for example, the sequences $(a_n)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, \cdots\right)$ and $(b_n)_{n\in\mathbb{N}} = \left(1, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, \cdots\right)$

We have $a_1 = b_1$, but $a_2 \neq b_2$. So, $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are not equal.

We can construct new sequences from old ones just like we do in case of functions. For example, the sum of two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ is the sequence $(c_n)_{n\in\mathbb{N}}$, defined by $c_n = a_n + b_n \forall n \geq 1$, and the product is the sequence $(d_n)_{n\in\mathbb{N}}$ defined by $d_n = a_n\, b_n \; \forall\, n \geq 1$.

If each term of the sequence $(b_n)_{n\in\mathbb{N}}$ is nonzero, we can define the quotient of $(a_n)_{n\in\mathbb{N}}$ by $(b_n)_{n\in\mathbb{N}}$ as the sequence $(c_n)_{n\in\mathbb{N}}$ where $c_n = \frac{a_n}{b_n} \forall n \in \mathbb{N}$.

Let us look at an example.

Example 3: Find the sum, product and the quotient of the sequences $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}} = (n^2 - 1)_{n\in\mathbb{N}}$, wherever defined.

Solution: Then the sum of $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ is

$$(a_n + b_n)_{n\in\mathbb{N}} = \left(\frac{1}{n} + n^2 - 1\right)_{n\in\mathbb{N}} = \left(\frac{n^3 - n + 1}{n}\right)_{n\in\mathbb{N}}$$

The product of $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ is

$$(a_n b_n)_{n\in\mathbb{N}} = \left(\frac{n^2 - 1}{n}\right)_{n\in\mathbb{N}}$$

The quotient sequence $\left(\dfrac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ is not defined. This is because

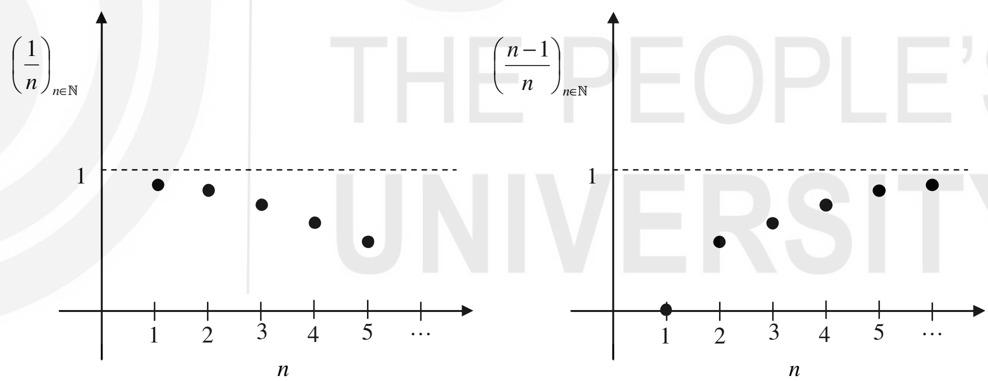
$\dfrac{a_n}{b_n} = \dfrac{1}{n(n^2-1)}$ is undefined at $n=1$.

But the quotient sequence $\left(\dfrac{b_n}{a_n}\right)_{n\in\mathbb{N}}$ is defined, where $\dfrac{b_n}{a_n} = n(n^2-1)$.

***

The definition of product of two sequences leads us to the following definition. The k^{th} **power** of a sequence $(a_n)_{n\in\mathbb{N}}$ is the sequence $((a_n)_{n\in\mathbb{N}})^k$ whose n^{th} term is a_n^k. For example, the square of the sequence $(1,2,3,4,\ldots)$ is the sequence $(1,4,9,16,\ldots)$. If $a_n \geq 0, \forall n\in\mathbb{N}$, we can define the k^{th} **root** of $(a_n)_{n\in\mathbb{N}}$ as the sequence $((a_n)_{n\in N})^{\frac{1}{k}}$ whose n^{th} term is $a_n^{\frac{1}{k}}$ for all $n\in\mathbb{N}$. For example the square root of the sequence $(1,2,3,4,\ldots)$ is $(1,\sqrt{2},\sqrt{3},\sqrt{4},\ldots)$.

Now we shall discuss the geometrical representation of a sequence. Since a sequence $(a_n)_{n\in\mathbb{N}}$ is a function, each term a_n corresponds to a point $(n,\ a_n)$ in the Cartesian plane. Thus plotting the points $(n,\ a_n),\ n\in\mathbb{N}$ gives a geometrical representation of the sequence. For example, the sequences $\left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$ and $\left(\dfrac{n-1}{n}\right)_{n\in\mathbb{N}}$, can be plotted as in Fig. 2.



(a) (b)

Fig. 2: Plots of the sequences $\left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$ **and** $\left(\dfrac{n-1}{n}\right)_{n\in\mathbb{N}}$

An advantage of geometrical representation is that it instantly gives a hint about the behaviour of the terms a_n as n grows.

Example 4: Plot the sequence $(a_n)_{n\in\mathbb{N}}$, where

$$a_n = \sqrt{1+\frac{1}{n}}.$$

Solution: To compute the first few terms we put $n = 1,2,3,4,5,$ and we get

$$a_1 = \sqrt{1+1} = \sqrt{2} \approx 1.41$$

$$a_2 = \sqrt{1+\frac{1}{2}} = \sqrt{\frac{3}{2}} \approx 1.22$$

$$a_3 = \sqrt{1+\frac{1}{3}} = \frac{2}{\sqrt{3}} \approx 1.15$$

$$a_4 = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2} \approx 1.12$$

$$a_5 = \sqrt{1+\frac{1}{5}} = \sqrt{\frac{6}{5}} \approx 1.09$$

The plot is given below

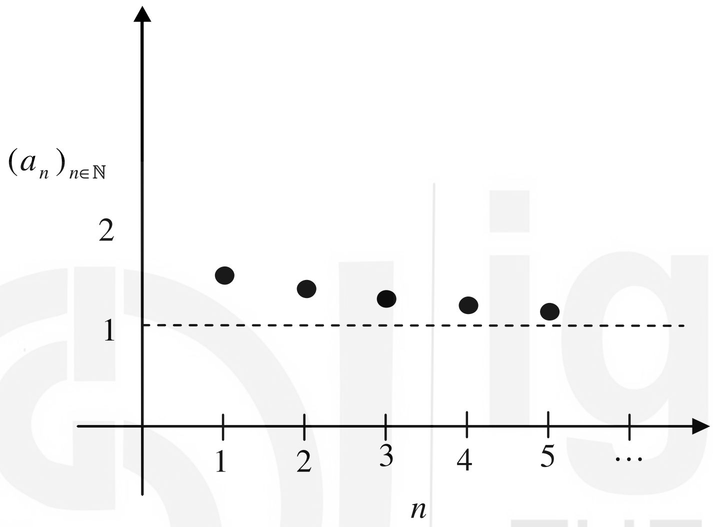


Fig. 3: Plot of $\left(\sqrt{1+\frac{1}{n}}\right)_{n\in\mathbb{N}}$.

Now try to solve some exercises.

E1) Identify the n^{th} terms of the following sequences:

i) $\left(\frac{1}{3}, \frac{1}{15}, \frac{1}{35}, \frac{1}{63}, \ldots\right)$

ii) $(-1, 1, -1, 2, -1, 3, \ldots)$

iii) $\left(1, 2^{\frac{1}{2}}, 3^{\frac{1}{3}}, 4^{\frac{1}{4}}, \ldots\right)$

E2) Find the first five terms of the following sequences

i) $\left(\left(1+\frac{1}{n}\right)^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ ii) $\left(\frac{10^n}{n!}\right)_{n\in\mathbb{N}}$ iii) $\left(\frac{n-1}{n+1}\right)_{n\in\mathbb{N}}$

iv) $(a_n)_{n\in\mathbb{N}}$, where $a_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is odd} \\ 1-\frac{1}{n}, & \text{if } n \text{ is even} \end{cases}$

E3) Are the sequences $(1,0,1,0,1,\ldots)$ and $(0,1,0,1,0,\ldots)$ equal? Why?

E4) Write the cube, and the cube root of the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$.

E5) Represent the sequence $\left(\left(1+\frac{1}{n}\right)^{1-\frac{1}{n}}\right)_{n\in\mathbb{N}}$ geometrically.

If you have gone through the exercises above, you would have found some sequences whose terms do not cross a certain number. We shall discuss such sequences in the next section.

5.3 BOUNDED SEQUENCES

Let us consider the sequence $\left((-1)^n\right)_{n\in\mathbb{N}}$. Its each term is either 1 or -1. Thus no term goes beyond the interval $[-1,1]$. Can we say something similar about the sequence $\left(n+(-1)^n\right)_{n\in\mathbb{N}}$? That means, do all its terms lie between two fixed numbers ? (Think!) Now we formally define the concept namely, "Bounded Sequences".

Definitions: i) A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be **bounded below** if there exists a number ℓ such that $\ell \le a_n, \ \forall\ n \in \mathbb{N}$. Such a number ℓ is called a **lower bound** of $(a_n)_{n\in\mathbb{N}}$.

ii) A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be **bounded above** if there exists a number u such that $a_n \le u, \ \forall\ n\in \mathbb{N}.$ Such a number u is called an upper bound of $(a_n)_{n\in\mathbb{N}}$.

iii) When a sequence $(a_n)_{n\in\mathbb{N}}$ is both bounded below and bounded above, it is called a **bounded sequence.**

iv) A sequence is said to be **unbounded** if it is not bounded.

Thus for a bounded sequence $(a_n)_{n\in\mathbb{N}}$ we have two numbers ℓ and u such that

$$\ell \le a_n \le u, \qquad \forall n\in \mathbb{N}.$$

You can relate these definitions with the definition of bounded below, bounded above and bounded sets. Note that an unbounded sequence may be bounded above or bounded below. For example the sequence of natural numbers $(1,2,3,4,\ldots)$ is unbounded, because it is not bounded above. But it is bounded below by 1. Similarly, the sequence of negative integers $(-1,-2,-3,-4,\ldots)$ is unbounded, but is bounded above by -1.

Consider the following examples to gain more clarity about the above definitions.

Example 5: Check whether the sequence $\left(n+(-1)^n\right)_{n\in\mathbb{N}}$ is bounded below or bounded above or both.

Solution: The n^{th} term of the given sequence is $a_n = n+(-1)^n$.
You can see that a_n is either $n-1$ or $n+1$. In either case $0 \le a_n, \forall n \in \mathbb{N}$.
Hence 0 is a lower bound of $(a_n)_{n\in N}$, and so $(a_n)_{n\in N}$ is bounded below.
Now if u is an upper bound of $(a_n)_{n\in N}$ then

$$\begin{aligned} a_n \le u \ ,\forall n\in\mathbb{N} &\Rightarrow n+1 \le u \ ,\forall n\in\mathbb{N} \\ &\Rightarrow \quad n \le u-1, \forall n\in\mathbb{N} \end{aligned}$$

What does this last inequality mean? It says that all natural numbers are smaller than or equal to $u-1$. But this is not true, and hence $(a_n)_{n\in\mathbb{N}}$ is not bounded above.

***

Example 6: Check whether the sequence $\left(\left(1+\frac{1}{n}\right)^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ is bounded or not.

Solution: First let us check the boundedness of the sequence from below.
Note that $\forall n\in\mathbb{N}$

$$1+\frac{1}{n}>1 \qquad \Rightarrow \qquad \left(1+\frac{1}{n}\right)^{\frac{1}{n}}>1.$$

Hence the sequence is bounded below. Now we check the boundedness of the sequence from above. You can see that for all $n\in\mathbb{N}$

$$\frac{1}{n}\le 1 \Rightarrow 1+\frac{1}{n}\le 2 \le 2^n \Rightarrow \left(1+\frac{1}{n}\right)^{\frac{1}{n}} \le 2.$$

Thus the sequence is bounded above also. Hence the sequence is bounded.

***

There are sequences which are neither bounded below nor are bounded above. See the following example for one such sequence.

Example 7: Show that the sequence $\left((-1)^n n\right)_{n\in\mathbb{N}}$ is neither bounded below nor is bounded above.

Solution: Let $a_n = (-1)^n n.$ In a more explicit form, we can write

$$a_n = \begin{cases} -n, & \text{if } n \text{ is odd} \\ n, & \text{if } n \text{ is even} \end{cases}$$

If we assume, for some $u\in\mathbb{R}$ that

$$a_n \le u \ \forall n\in\mathbb{N},$$

then $n \le u \quad \forall n \in \mathbb{N}$ which is impossible.

Hence $(a_n)_{n\in\mathbb{N}}$ is not bounded above. Think of a similar argument for showing that $(a_n)_{n\in\mathbb{N}}$ is not bounded below.

***

Now you may try the following exercises.

E6) Which of the following sequences are bounded below? Which are bounded above? Which are both?

i) $\left(\frac{n^2+n}{n^2+1}\right)_{n\in\mathbb{N}}$ ii) $\left(\frac{(-1)^n n}{n+1}\right)_{n\in\mathbb{N}}$

iii) $\left(\sqrt{n}-\sqrt{n+1}\right)_{n\in\mathbb{N}}$ iv) $\left(1+(-1)^n\right)_{n\in\mathbb{N}}$

E7) Show that $(a_n)_{n\in\mathbb{N}}$ is bounded iff $(|a_n|)_{n\in\mathbb{N}}$ is bounded.

E8) Explain why the following sequences are unbounded.

i) $\left((-1)^n n^2\right)_{n\in\mathbb{N}}$ ii) $\left(2^n+(-1)^n\right)_{n\in\mathbb{N}}$

iii) $(a_n)_{n\in\mathbb{N}}$, where $a_n = \begin{cases} 1, & \text{if } n = 2^m, \text{for some } m \in \mathbb{N} \\ -n, & \text{otherwise} \end{cases}$.

E9) If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded sequences, what can you say about $(a_n+b_n)_{n\in\mathbb{N}}$ and $(a_n b_n)_{n\in\mathbb{N}}$? Justify.

E10) If $(a_n)_n$ is a bounded sequence, and $a_n \ne 0 \ \forall n \in \mathbb{N}$, then $\left(\frac{1}{a_n}\right)_{n\in\mathbb{N}}$ is also bounded. True or false? Justify.

In the next section we shall discuss another kind of sequences namely, monotone sequences and their properties.

5.4 MONOTONE SEQUENCES

In the course Calculus you have studied monotone functions. Monotone sequences are defined in the same manner.

Consider the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$. Its terms are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ which keep on decreasing as n increases . This is because $m < n$ implies $\frac{1}{m} > \frac{1}{n}$. Now consider the sequence $\left(1-\frac{1}{n}\right)_{n\in\mathbb{N}}$. Its terms are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ Do they behave in the same way as the terms of $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$? The answer is no. In fact, the behaviour is reversed. The terms of $\left(1-\dfrac{1}{n}\right)_{n\in\mathbb{N}}$ increase with n. That is, when $m<n$, we find $1-\dfrac{1}{m}<1-\dfrac{1}{n}$. These sequences lead us to the following definitions.

Definitions: A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be

i) **increasing** if $m<n$ implies $a_m \le a_n$..

ii) **strictly increasing** if $m<n$ implies $a_m < a_n$.

iii) **decreasing** if $m<n$ implies $a_m \ge a_n$.

iv) **strictly decreasing** if $m<n$ implies $a_m > a_n$.

v) **monotone** if it is either increasing or decreasing.

vi) **strictly monotone** if it is either strictly increasing or strictly decreasing.

vii) **constant** if all its terms are equal.

Let us look at some examples.

Example 8: Show that the sequence $\left(\dfrac{1}{n^2}\right)_{n\in\mathbb{N}}$ is monotone.

Solution: You can see that when $m<n$, we have $\dfrac{1}{m^2}>\dfrac{1}{n^2}$. So the sequence is decreasing, and hence monotone.

***

Example 9: Show that the sequence $\left(\dfrac{n^2-1}{n^2+1}\right)_{n\in\mathbb{N}}$ is monotone.

Solution: Let $a_n = \dfrac{n^2-1}{n^2+1} = \dfrac{n^2+1-2}{n^2+1} = 1-\dfrac{2}{n^2+1}$

Now let $m<n$. Then

$$
\begin{aligned}
m^2<n^2 \Rightarrow\ & m^2+1<n^2+1 \\
&\Rightarrow \frac{1}{m^2+1} > \frac{1}{n^2+1} \\
&\Rightarrow \frac{2}{m^2+1} > \frac{2}{n^2+1} \\
&\Rightarrow -\frac{2}{m^2+1} < -\frac{2}{n^2+1} \\
&\Rightarrow 1-\frac{2}{m^2+1} < 1-\frac{2}{n^2+1} \\
&\Rightarrow a_m < a_n
\end{aligned}
$$

Thus $(a_n)_{n\in\mathbb{N}}$ is increasing, and hence monotone.

***

Example 10: Check whether the sequence $(a_n)_{n\in\mathbb{N}}$ is monotone, where

$$a_n = \begin{cases} n, & \text{if } n \text{ is odd} \\ n\text{-}1, & \text{if } n \text{ is even} \end{cases}.$$

Solution: Here there are 4 possibilities, as given in the following table

m	n	a_m	a_n	
odd	odd	m	n	$m < n \Rightarrow a_m < a_n$
odd	even	m	$n-1$	$m < n \Rightarrow a_m \le a_n$
even	odd	$m-1$	n	$m < n \Rightarrow a_m < a_n$
even	even	$m-1$	$n-1$	$m < n \Rightarrow a_m < a_n$

Thus in all the four cases, we have seen that $m < n$ implies $a_m \le a_n$ or $a_m < a_n$. That is, $m < n$ implies $a_m \le a_n$. Thus $(a_n)_{n\in\mathbb{N}}$ is increasing, and hence monotone.

***

Now let us look at the following example which gives us a nonmonotone sequence.

Example 11: Show that the sequence $\left(\frac{3^n}{n!}\right)_{n\in\mathbb{N}}$ is not monotone.

Solution: Let us look at the first few terms of this sequence. These are

$$3, \frac{9}{2}, \frac{9}{2}, \frac{27}{8}, \frac{27}{8}, \frac{81}{80}, \frac{243}{560}, \ldots$$

You can see that the first 5 terms of this sequence are in increasing order However, the 6^{th} term is less than the 5^{th} term. Hence, the sequence is neither increasing nor decreasing. That is the sequence is not monotone.

***

Now you should do some exercises.

E11) Every constant sequence is monotone? True or false? Justify.

E12) Is every strictly monotone sequence monotone? What about the converse?

E13) Show that every sequence that is increasing as well as decreasing is constant.

E14) Which of the following sequences are monotone and which are not?

i) $\left(1 - \frac{1}{10^n}\right)_{n\in\mathbb{N}}$ ii) $\left(\frac{n^3 + n^2 - n + 2}{n^3 + 1}\right)_{n\in\mathbb{N}}$

iii) $\left(\dfrac{(-1)^n}{2^n}\right)_{n\in\mathbb{N}}$ iv) $\left(c^n\right)_{n\in\mathbb{N}}, 0 < c < 1$

v) $(a_n)_{n\in\mathbb{N}}$, where $a_1 = 1, a_{n+1} = 1 + \dfrac{1}{a_n} \quad \forall n \geq 1$

vi) $(f_n)_{n\in\mathbb{N}}$, where $f_n = f_{n-1} + f_{n-2}, \quad n \geq 3, \quad f_1 = f_2 = 1.$

E15) Prove that if $(a_n)_{n\in\mathbb{N}}$ is increasing then $(-a_n)_{n\in\mathbb{N}}$ is decreasing.

E16) Give an example for each of the following:

i) an increasing sequence that is bounded above

ii) a decreasing sequence that is bounded below.

E17) Give an example of a sequence that is not monotone, and is neither bounded below nor bounded above .

E18) If $(a_n)_{n\in\mathbb{N}}$ is increasing and $a_n > 0 \ \forall n \in \mathbb{N}$, then $\left(a_n + \dfrac{1}{a_n}\right)_{n\in\mathbb{N}}$ is increasing. True or false? Justify.

What do you get when you extract the odd terms of a sequence without changing the order of the terms? You get a "subsequence", a concept which we shall discuss next.

5.5 SUBSEQUENCES

We shall here discuss the notion of a subsequence of a sequence. We shall use it frequently to prove or disprove a statement about a sequence.

Consider the sequence

$$(a_n)_{n\in\mathbb{N}} = \left(1,\ 2,\ \frac{1}{2},\ 3,\ \frac{1}{3},\ 4,\ \frac{1}{4},\ 5,\ \frac{1}{5},\ 6,\ \frac{1}{6},\cdots\right)$$

If we pick the odd terms of this sequence, we get a new sequence

$$(b_k)_{k\in\mathbb{N}} = \left(1,\ \frac{1}{2}, \frac{1}{3}, \frac{1}{4},\cdots\right)$$

You can see that

$$b_1 = a_1, b_2 = a_3, b_3 = a_5, \cdots$$

This means for each $k \in \mathbb{N}$, we have $b_k = a_{2k-1}$. Since each term of $(b_k)_{k\in\mathbb{N}}$ has been picked from $(a_n)_{n\in\mathbb{N}}$ without changing the order, we can call $(b_k)_{k\in\mathbb{N}}$ a "sub"sequence of $(a_n)_{n\in\mathbb{N}}$.

Now let us look at the sequence

$$(c_k)_{k\in\mathbb{N}} = \left(2, \frac{1}{2}, 4, \frac{1}{4}, 6, \frac{1}{6}, \ldots\right)$$

You can see that all the terms of $(c_k)_{k\in\mathbb{N}}$ are also picked from $(a_n)_{n\in\mathbb{N}}$, keeping the order unchanged, but now with a different combination. Explicitly we have,

$$c_1 = a_2, c_2 = a_3, c_3 = a_6, c_4 = a_7, \ldots$$

Thus for each $k \in \mathbb{N}$, we can write $c_k = a_{n_k}$, for some $n_k \in \mathbb{N}$. Here n_k depends on k. For example,

$$n_1 = 2, n_2 = 3, n_3 = 6, n_4 = 7, \ldots$$

If you put in some effort, you can find a general form of n_k in terms of k.

Now let us define what a subsequence is, formally.

Definition: A sequence $(b_k)_{k\in\mathbb{N}}$ is called a **subsequence** of a sequence $(a_n)_{n\in\mathbb{N}}$ if

i) for each $k \in \mathbb{N}$, there is some $n_k \in \mathbb{N}$ such that $b_k = a_{n_k}$.

ii) $n_k < n_{k+1} \; \forall k \in \mathbb{N}$.

Remember (i) says that every term in $(b_k)_{k\in\mathbb{N}}$ is a term of $(a_n)_{n\in\mathbb{N}}$, and (ii) says that the terms of $(b_k)_{k\in\mathbb{N}}$ preserve the order in $(a_n)_{n\in\mathbb{N}}$.

Let us consider an example.

Example 12: Identify which of the following are subsequences of $\left(\frac{n}{n^2+1}\right)_{n\in\mathbb{N}}$.

i) $\left(\frac{1}{k^2+1}\right)_{k\in\mathbb{N}}$ ii) $\left(\frac{2k+1}{2(2k^2+2k+1)}\right)_{k\in\mathbb{N}}$

iii) $\left(\frac{k}{3k^2-1}\right)_{n\in\mathbb{N}}$ iv) $\left(\frac{k^2}{k+1}\right)_{k\in\mathbb{N}}$

Solution: Let $a_n = \frac{n}{n^2+1}, \quad n \in \mathbb{N}$. The first few terms of $(a_n)_{n\in\mathbb{N}}$ can be computed as

$$\left(\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}, \frac{6}{37}, \frac{7}{50}, \frac{8}{65}, \frac{9}{82}, \ldots\right)$$

i) Let $b_k = \frac{1}{k^2+1}, k \in \mathbb{N}$. Then $b_2 = \frac{1}{5}$ which is not a term of $(a_n)_{n\in\mathbb{N}}$. That is, $b_2 \neq a_n$, for any $n \in \mathbb{N}$. (You must prove it.) Hence $(b_n)_{n\in\mathbb{N}}$ is not a subsequence of $(a_n)_{n\in\mathbb{N}}$.

ii) Let

$$\begin{aligned} b_k &= \frac{2k+1}{2(2k^2+2k+1)} \\ &= \frac{2k+1}{4k^2+4k+1+1} = \frac{2k+1}{(2k+1)^2+1} \\ &= \frac{n_k}{n_k{}^2+1} = a_{n_k}, \ \text{where } n_k = 2k+1 \end{aligned}$$

Thus for each $k \in \mathbb{N}$, we have some n_k (here $n_k = 2k+1$) such that $b_k = a_{n_k}$. Also $n_k < n_{k+1} \ \forall k \in \mathbb{N}$. Hence $(b_k)_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

iii) Let $b_k = \dfrac{k}{3k^2-1}$. Here, $b_1 = \dfrac{1}{2} = a_1$, but $b_2 = \dfrac{2}{11}$. So $b_2 \neq a_n$ for any $n \in \mathbb{N}$. Hence $(b_k)_{k\in\mathbb{N}}$ is not a subsequence of $(a_n)_{n\in\mathbb{N}}$.

iv) Let $b_k = \dfrac{k^2}{k^4+1}$. If we take $n_k = k^2$, then we can easily see that for each $k \in \mathbb{N}, \ b_k = a_{n_k}$. Also $n_k < n_{k+1}, \forall k \in \mathbb{N}$. Hence $(b_k)_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

***

Let us now look at the following result.

Theorem 1: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. If $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, then $(a_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

Proof: Let $b_k = a_{n_k} \ \forall k \in \mathbb{N}$. Since $(n_k)_{k\in\mathbb{N}}$ is strictly increasing, we have $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Therefore, $(b_k)_{k\in\mathbb{N}}$, i.e., $(a_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$. ■

Due to Theorem 1, we shall use the notation $(a_{n_k})_{k\in\mathbb{N}}$ to denote a subsequence of $(a_n)_{n\in\mathbb{N}}$. Now consider an increasing sequence $(a_n)_{n\in\mathbb{N}}$. Let $(a_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(a_n)_{n\in\mathbb{N}}$. What can you say about $(a_{n_k})_{k\in\mathbb{N}}$? That is, will $(a_{n_k})_{k\in\mathbb{N}}$ be increasing? To see, let $k, \ell \in \mathbb{N}$ be such that $k < \ell$. Then

$$k < \ell \Rightarrow n_k < n_\ell \Rightarrow a_{n_k} \leq a_{n_\ell}.$$

This proves that $(a_{n_k})_{k\in\mathbb{N}}$ is increasing. Thus every subsequence of an increasing sequence is increasing. Can you write a similar statement about decreasing sequences?

Let us now look at one more example.

Example 13: Consider the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = n^{(-1)^n}$ for all natural numbers n. Answer the following.

i) Find a subsequence of $(a_n)_{n\in\mathbb{N}}$ that is decreasing and bounded below.

ii) Find a subsequence of $(a_n)_{n\in\mathbb{N}}$ that is increasing, but not bounded above.

iii) Write the first 5 terms of the subsequence $(a_{n^2})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$. Is $(a_{n^2})_{n\in\mathbb{N}}$ monotone? Strictly monotone?

Solution: We can see that the first few terms of the sequence $(a_n)_{n\in\mathbb{N}}$ are as follows.

$$1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, 8, \frac{1}{9}, \ldots$$

i) Now if we look at the odd terms, we get a decreasing subsequence. That is the subsequence $(a_{2n-1})_{n\in\mathbb{N}} = \left(\frac{1}{2n-1}\right)_{n\in\mathbb{N}}$ is decreasing, and also bounded below by 0.

ii) The subsequence formed by even terms is $(a_{2n})_{n\in\mathbb{N}} = (2n)_{n\in\mathbb{N}}$ which is increasing, but not bounded above.

iii) Note that the subsequence $(a_{n^2})_{n\in\mathbb{N}} = \left(n^{2(-1)^{n^2}}\right)_{n\in\mathbb{N}}$. Its first five terms are as follows: $1, 4, \frac{1}{9}, 16, \frac{1}{25}$. Clearly neither the subsequence is increasing nor decreasing. Hence, it is not a monotone subsequence. Consequently, it is not strictly monotone.

***

You should try some exercises now.

E19) Is $(1,2,3,4,\ldots)$ a subsequence of $(1^2, 2^2, 3^2, 4^2, \ldots)$?

E20) Show that $(0,0,0,\ldots)$ is a subsequence of $(0,0,1,1,0,0,1,1,\ldots)$.

E21) Identify which of the following are subsequences of the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = \left(\left(1+\frac{1}{n}\right)^{1-\frac{1}{n}}\right)_{n\in\mathbb{N}}$?

i) $\left(\left(1+\frac{1}{k}\right)^{-\frac{1}{k}}\right)_{k\in\mathbb{N}}$ ii) $\left(\left(1+\frac{1}{k}\right)^{1-\frac{1}{k}}\right)_{k\in\mathbb{N}}$ iii) $\left(\left(1+\frac{1}{k!}\right)^{1-\frac{1}{k!}}\right)_{k\in\mathbb{N}}$

E22) Let $(b_n)_{n\in\mathbb{N}}$ be a subsequence of $(a_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$ a subsequence of $(b_n)_{n\in\mathbb{N}}$. What is the relation between $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$?

E23) Is every subsequence of a decreasing sequence decreasing? Prove or disprove.

E24) If $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence, what can you say about the boundedness of the subsequences of $(a_n)_{n\in\mathbb{N}}$?

Thus far you have only seen the basic notions related to sequences. In the next section we shall introduce the concept of convergence of a sequence.

5.6 CONVERGENT SEQUENCES

The notion of convergence of a sequence is crucial to understand the behaviour of real valued functions, and is helpful in the estimation of sums of series. (You will study about series in the next block.)

Consider the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = 1$ if n is odd, and $a_n = 0$ if n is even. So, every of $(a_n)_{n\in\mathbb{N}}$ is 1 or 0. What happens to a_n when n tends to infinity? The notion of convergence addresses precisely the question: Given a sequence $(a_n)_{n\in\mathbb{N}}$, what happens to a_n when n tends to infinity?

Let us consider another example of a sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. In this case, we can see that a_n comes closer and closer to 0 as n becomes larger and larger. That is a_n tends to 0 as n tends to ∞. In other words, we can say that 0 is the 'limit' of the sequence $(a_n)_{n\in\mathbb{N}}$.

Let us now have a formal definition of limit.

Definition: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Then a real number L is said to be a **limit** of $(a_n)_{n\in\mathbb{N}}$ if given any $\varepsilon > 0$, there exists a number $n_0 \in \mathbb{N}$ such that for all $n \ge n_0, |a_n - L| < \varepsilon$. This is also expressed as $\lim\limits_{n\to\infty} a_n = L.$

Read the definition above a few more times. It has mainly three numbers L, ε and n_0. The number L is a fixed number given to you. The number ε is also given, but not fixed. You are free to choose any positive value for ε. The last number is n_0, which is not given to you. Instead, you have to find n_0 in order to prove that L is such that $|a_n - L| < \varepsilon$ for every $n \ge n_0$, i.e., all the terms of $(a_n)_{n\in\mathbb{N}}$ starting from n_0 lie in the interval $]L-\varepsilon, L+\varepsilon[$. Since the choice of selecting ε lies with us, the interval $]L-\varepsilon, L+\varepsilon[$ can be made as small as we wish. However, for smaller ε, we may get larger n_0, that is n_0 depend on ε. Let us see geometrically what it means in Fig. 4. Consider the horizontal strip of width 2ε, generated by the lines $y = L - \varepsilon$ and $y = L + \varepsilon$. If L is the limit of the sequence $(a_n)_{n\in\mathbb{N}}$ then there must be some natural number n_0 on the x-axis such that for every n to the right of n_0, a_n lies in the strip.

Note that $|a_n - L|$ represents the absolute value of the difference of a_n and L.

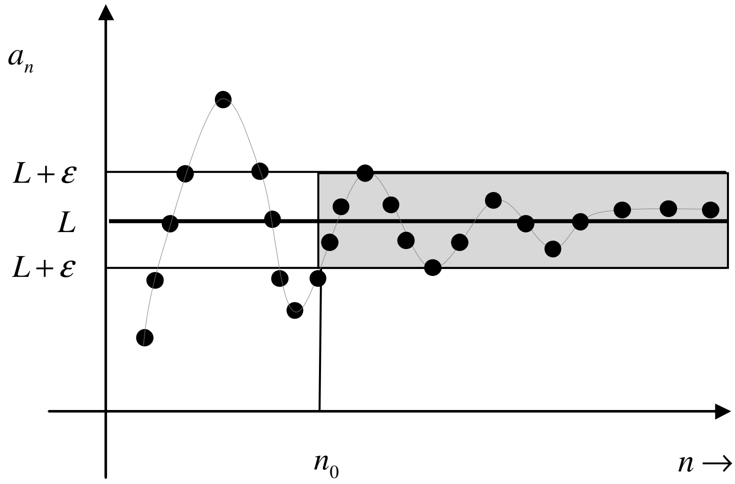


Fig.4: Geometrical illustration of limit

Now consider the following theorem, which says that the limit of a sequence is unique whenever it exists.

Theorem 2: If L_1 and L_2 are two limits of a sequence $(a_n)_{n\in\mathbb{N}}$ then $L_1 = L_2$.

Proof: Let $\varepsilon > 0$ be given. Then $\dfrac{\varepsilon}{2} > 0$. Since L_1 is a limit of $(a_n)_{n\in\mathbb{N}}$ there exists some $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies

$$|a_n - L_1| < \frac{\varepsilon}{2}. \qquad \text{... (1)}$$

Also L_2 is a limit of $(a_n)_{n\in\mathbb{N}}$. Hence there exists $m_0 \in \mathbb{N}$ such that $n \geq m_0$ implies

$$|a_n - L_2| < \frac{\varepsilon}{2}. \qquad \text{... (2)}$$

Now let $k_0 = \max\{m_0, n_0\}$. Then $n \geq k_0$ implies $n \geq m_0$ and $n \geq n_0$. Thus both Eq. (1) and (2) hold for all $n \geq k_0$. Therefore, for each $n \geq k_0$,

$$\begin{aligned} |L_1 - L_2| &= |L_1 - a_n + a_n - L_2| \\ &\leq |L_1 - a_n| + |a_n - L_2| \\ &= |a_n - L_1| + |a_n - L_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$|L_1 - L_2| \leq 0 \Rightarrow |L_1 - L_2| = 0 \Rightarrow L_1 - L_2 = 0 \Rightarrow L_1 = L_2. \qquad \blacksquare$$

Let us consider a few examples.

Example14: Using the definition of limit, show that 1 is the limit of the sequence $(1, 1, 1, 1, \ldots)$.

Solution: Here $a_n = 1$ and $L = 1$. Then

$$|a_n - L| = |1 - 1| = 0.$$

Let $\varepsilon > 0$ be given. Then we see that $|a_n - L| < \varepsilon, \forall n \geq 1$. So, we can take $n_0 = 1$. Thus for $n \geq 1$, we have $|a_n - L| < \varepsilon$. Hence 1 is the limit of the given sequence.

***

Example15: Using the definition of the limit, show that 0 is the limit of the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$.

Solution: Here $a_n = \frac{1}{n}, L = 0$. Let $\varepsilon > 0$ be given. Our task is to find an n_0, such that $n > n_0 \Rightarrow |a_n - L| < \varepsilon$. Now see that

$$|a_n - L| = \left|\frac{1}{n} - 0\right| = \frac{1}{n}$$

Then

$$|a_n - L| < \varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$$

Thus we see that for all $n > \frac{1}{\varepsilon}$

$$|a_n - L| < \varepsilon.$$

We choose $n_0 = \left\lceil \frac{1}{\varepsilon} \right\rceil$. Now we have

$$n \geq n_0 \Rightarrow n \geq \left\lceil \frac{1}{\varepsilon} \right\rceil \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow |a_n - L| < \varepsilon.$$

Hence 0 is the limit of the given sequence.
0.00000000000.....

***

Example 16: Using the definition of the limit, show that $\lim_{n\to\infty} \frac{2n+3}{3n-2} = \frac{2}{3}$.

Solution: Let $\varepsilon > 0$ be given. Here $a_n = \frac{2n+3}{3n-2}$ and $L = \frac{2}{3}$. Let us consider

$$|a_n - L| = \left|\frac{2n+3}{3n-2} - \frac{2}{3}\right| = \frac{13}{3(3n-2)}$$

So,

$$|a_n - L| < \varepsilon \Leftrightarrow \frac{13}{3(3n-2)} < \varepsilon \Leftrightarrow 3n - 2 > \frac{13}{3\varepsilon} \Leftrightarrow n > \frac{13+6\varepsilon}{9\varepsilon}$$

So if we choose $n_0 = \left\lceil \frac{13+6\varepsilon}{9\varepsilon} \right\rceil$, we have for all $n \geq n_0$, $|a_n - L| < \varepsilon$. Hence,

$$\lim_{n\to\infty} \frac{2n+3}{3n-2} = \frac{2}{3}.$$

***

Now let us look at some examples of sequences that have no limit.

Example 17: Show that the sequence $\left((-1)^n\right)_{n\in\mathbb{N}}$ has no limit.

Solution: Assume, if possible, a real number L is the limit of the given sequence. Then for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$

$$|(-1)^n - L| < \varepsilon.$$

Now you can split the statement above into two statements according to weather n is even or odd.

For odd $n \geq n_0$, we have

$$|-1-L| < \varepsilon \Leftrightarrow |L+1| < \varepsilon \Leftrightarrow -1-\varepsilon < \underline{L < \varepsilon - 1}$$

Whereas, for even $n \geq n_0$, we have

$$|1-L| < \varepsilon \Leftrightarrow |L-1| < \varepsilon \Leftrightarrow \underline{1-\varepsilon < L} < \varepsilon + 1$$

Now focus at the underlined inequalities above. Do they hold for all $\varepsilon > 0$? In particular, what happens when you choose $\varepsilon = 1$? You can see that one inequality gives you $L < 0$ and the other one $L \geq 0$. This is absurd. Hence, L cannot be the limit of the sequence. Since L is arbitrary, no real number is the limit of the sequence.

***

Example18: Show that the sequence $(a_n)_{n\in\mathbb{N}}$ has no limit, where

$$a_n = \begin{cases} 1, & \text{if } n = 2^k \text{ for some } k \in \mathbb{N} \\ n, & \text{otherwise} \end{cases}$$

Solution: First let us plot this sequence to understand the behaviour of the terms a_n as n increases. (See Fig. 5.)

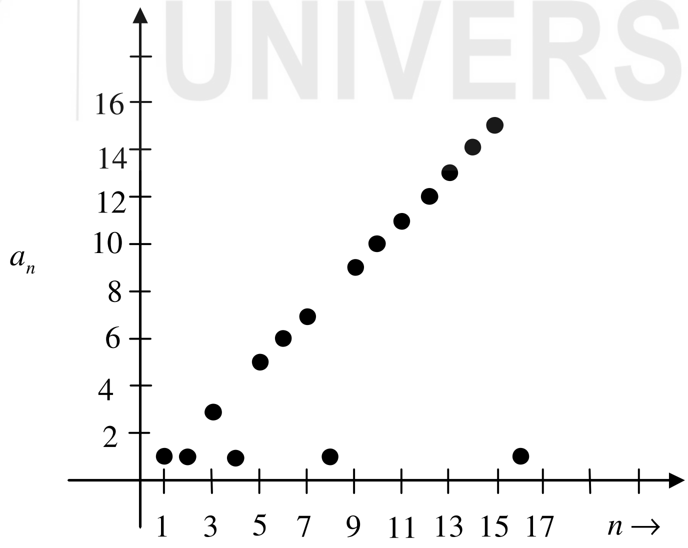


Fig. 5

Now assume, if possible, that L is the limit of this sequence. Then for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall n > n_0$

$$|a_n - L| < \varepsilon.$$

Looking at the definition of a_n, we get two cases. In case of $n = 2^k$ for some $k \in \mathbb{N},\ n \geq n_0$ we have

$$|a_n - L| < \varepsilon \Rightarrow |1 - L| < \varepsilon \Rightarrow 1 - \varepsilon < \underline{L < 1 + \varepsilon}$$

On the other hand, when $n \neq 2^k$ for any $k \in \mathrm{N},\ n \geq n_0,$ we have

$$|a_n - L| < \varepsilon \Rightarrow |n - L| < \varepsilon \Rightarrow \underline{n - \varepsilon < L} < n + \varepsilon$$

Now look at the underlined inequalities. Take $\varepsilon = 1$, for example. Then we have $L < 2$ as well as $n - 1 < L,\ \ \forall n > n_0$. This is impossible. Hence, the sequence has no limit.

***

We have seen the examples of the sequences that have limit and also of those that do not have. To distinguish between the two classes of sequences, you need the following definition.

Definition: A sequence is said to be **convergent** if it has a limit, and **divergent** otherwise.

When a sequence $(a_n)_{n\in N}$ has the limit L, we write $\lim_{n\to\infty} a_n = L.$ The same statement can also be written as "$(a_n)_{n\in N}$ converges to L", or as "$a_n \to L$ as $n \to \infty$" or just as "$a_n \to L$".

Let us look at the following example.

Example 19: Show that $\left(\dfrac{\sin n}{n}\right)_{n\in\mathbb{N}}$ converges to 0.

Solution: Let $a_n = \dfrac{\sin n}{n}$ and $L = 0.$ Take $\varepsilon > 0.$ We have

$$|a_n - L| = \left|\frac{\sin n}{n} - 0\right| \leq \frac{1}{n}$$

Thus

$$|a_n - L| < \varepsilon \text{ if } \frac{1}{n} < \varepsilon \ \left(\text{if } n > \frac{1}{\varepsilon}.\right)$$

Choose $n_0 = \left\lceil \dfrac{1}{\varepsilon} \right\rceil$. Now

$$n \geq n_0 \Rightarrow n \geq \left\lceil \frac{1}{\varepsilon} \right\rceil \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow |a_n - L| < \varepsilon.$$

Therefore, $\left(\dfrac{\sin n}{n}\right)_{n\in\mathbb{N}}$ converges to 0.

***

Now let us consider the following property of convergent sequences.

Theorem 3: Every convergent sequence is bounded.

Proof: Let us consider a convergent sequence $(a_n)_{n\in\mathbb{N}}$ which converges to $L.$ Then for a given $\varepsilon > 0,$ we have some $n_0 \in \mathbb{N}$ such that for all $n > n_0$ $|a_n - L| < \varepsilon.$

Then for all $n > n_0,$

$$|a_n| = |a_n - L + L| \le |a_n - L| + |L| < |L| + \varepsilon$$

Let $M =$ max $\{|a_1|, |a_2|, \ldots, |a_{n_0}|\}$, and $K =$ max $\{|L| + \varepsilon, M\}.$

Thus for all $n \ge 1$ we have $|a_n| \le K.$ This proves that $(a_n)_{n\in\mathbb{N}}$ is bounded. ■

What is the contrapositive of the statement of Theorem 3? It is "every unbounded sequence is divergent." You know that a statement is true if and only if its contrapositive is true. This can help us in proving the divergence of many sequences. See one such example, given below.

Example 20: Show that the sequence $(n^n)_{n\in\mathbb{N}}$ is divergent.

Solution: We know that for each $k \in \mathbb{N}$ there exists some $n \in \mathbb{N}$ such that $n > k.$ This implies $n^n > k^n > k.$ Thus for each $k \in \mathbb{N}$ some term of $(n^n)_{n\in\mathbb{N}}$ is larger than $k.$ Therefore, $(n^n)_{n\in\mathbb{N}}$ is not bounded above. Hence $(n^n)_{n\in\mathbb{N}}$ is unbounded. It, therefore, follows from Theorem 3, that $(n^n)_{n\in\mathbb{N}}$ is divergent.

***

Try now the following exercises.

E25) Using the definition of limit, prove the following limits.

i) $\lim\limits_{n\to\infty} \dfrac{4}{n^2} = 0$ ii) $\lim\limits_{n\to\infty} \dfrac{2(-1)^n}{n} = 0$

iii) $\lim\limits_{n\to\infty} \left(1 - \dfrac{(-1)^n}{n}\right) = 1$ iv) $\lim\limits_{n\to\infty} \dfrac{\sin n + \cos n}{n} = 0$

v) $\lim\limits_{n\to\infty} c = c$ vi) $\lim\limits_{n\to\infty} \sqrt{n+1} - \sqrt{n-1} = 0$

E26) Prove that if $a_n \to L,$ then (i) $a_n^2 \to L^2,$ and (ii) $|a_n| \to |L|.$

E27) Give an example of a divergent sequence that is not monotone.

E28) Is every sequence that is not monotone, divergent? Give reasons.

E29) Write the converse of the statement of Theorem 3. Is the converse true? Justify.

E30) Give an example of a sequence $(a_n)_{n\in\mathbb{N}}$ which is not constant, a_n is nonzero for all $n \in \mathbb{N},$ and both $(a_n)_{n\in\mathbb{N}}$ and $\left(\dfrac{1}{a_n}\right)_{n\in\mathbb{N}}$ converge to the same limit.

E31) Using the definition of limit, prove that the sequence $(\sqrt{n})_{n\in\mathbb{N}}$ has no limit.

The section above was meant to give you a basic understanding of convergent, and divergent sequences. In the forthcoming section we shall explore more about their characteristics.

5.7 CAUCHY SEQUENCES

In the last section, we saw that to prove that a sequence is convergent we, first need to find a number L which could be the limit of the sequence. This makes the task of proving convergence difficult in many cases. For example, by looking at the terms of a sequence, we might have an intuitive idea that the sequence approach to a number, but what that number is may not be easy to determine.

Augustin-Louis Cauchy (1789-1857)
Fig. 6

To resolve this **Augustin -Louis Cauchy** (1789-1857), an eminent mathematician, presented a criterion for convergence of sequences, which does not depend on L.

Let us first assume that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to L. Then for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$

$$|a_n - L| < \frac{\varepsilon}{2}$$

Now, let $m, n \geq n_0$. Then

$$\begin{aligned} |a_m - a_n| &= |a_m - L + L - a_n| \\ &\leq |a_m - L| + |L - a_n| \\ &= |a_m - L| + |a_n - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This shows that, no matter what L is, the terms a_m and a_n can be made as close as we wish after a certain term.

Let us consider, for example, the sequence $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$. Let $m > n.$ then

$$|a_m - a_n| = \left|\frac{1}{m} - \frac{1}{n}\right| < \frac{1}{m} + \frac{1}{n} < \frac{1}{2n}$$

This shows that as m and n increase, the terms a_m and a_n come close to each other. This leads to the following definition.

Definition: A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be a **Cauchy sequence** if for every $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ such that

$$|a_m - a_n| < \varepsilon, \quad \forall m, n \geq n_0$$

The sequence $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is Cauchy. To prove, let $\varepsilon > 0.$ Now choose $n_0 \in \mathbb{N}$ such that $\frac{1}{2n_0} < \varepsilon$ i.e. $n_0 > \frac{1}{2\varepsilon}.$

Now

$$m > n \geq n_0 \Rightarrow \left|\frac{1}{m} - \frac{1}{n}\right| < \frac{1}{2n} \leq \frac{1}{2n_0} < \varepsilon$$
$$\Rightarrow |a_m - a_n| < \varepsilon.$$

Hence $(a_n)_{n\in\mathbb{N}}$ is Cauchy.

We have seen above that **every convergent sequence is a Cauchy sequence**. Interestingly, the converse is also true. It is the content of the following theorem.

Theorem 4: Every Cauchy sequence (in $\mathbb{R}$) is convergent.

Proof: Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. We define a set S as below.

$$S = \{x \in \mathbb{R} \ : \exists k_0 \in \mathbb{N} \text{ such that } x < a_n, \forall n \geq k_0\}$$

The proof involves two steps.

Step 1: $S \neq \emptyset$ and S is bounded above.

Since $(a_n)_{n\in\mathbb{N}}$ is bounded, there exists some $M > 0$ such that

$$-M < a_n < M, \quad \forall n \in \mathbb{N}.$$

Hence, $-M \in S$. This shows that $S \neq \emptyset$. Let $x \in S$ be arbitrary. There exists some $n_0 \in \mathbb{N}$ such that $x < a_n, \ \forall n \geq n_0$. But

$$a_n < M, \qquad \forall n \in \mathbb{N}.$$

This shows that $x < M$. Since $x \in S$ is arbitrary, it follows that S is bounded above. The completeness property of $\mathbb{R}$ now implies that the supremum of S exists in $\mathbb{R}$.

Step 2: sup S is the limit of $(a_n)_{n\in\mathbb{N}}$

Let $u = \sup S$. Let $\varepsilon > 0$ be given. Since the sequence is Cauchy, there exists $n_0 \in \mathbb{N}$ such that

$$|a_n - a_m| < \frac{\varepsilon}{2}, \quad \forall m, n \geq n_0$$

In particular,

$$|a_n - a_{n_0}| < \frac{\varepsilon}{2}, \quad \forall n \geq n_0 \Rightarrow a_{n_0} - \frac{\varepsilon}{2} < a_n < a_{n_0} + \frac{\varepsilon}{2}, \forall n \geq n_0$$
$$\Rightarrow a_{n_0} - \frac{\varepsilon}{2} \in S \text{ and } a_{n_0} + \frac{\varepsilon}{2} \notin S.$$

$$\Rightarrow a_{n_0} - \frac{\varepsilon}{2} \le u \le a_{n_0} + \frac{\varepsilon}{2} \qquad (\because u = \sup S)$$

$$\Rightarrow -\frac{\varepsilon}{2} \le u - a_{n_0} \le \frac{\varepsilon}{2}$$

$$\Rightarrow |u - a_{n_0}| \le \frac{\varepsilon}{2}$$

Thus for all $n \ge n_0$, we have

$$|a_n - u| = |a_n - a_{n_0} + a_{n_0} - u|$$

$$\le |a_n - a_{n_0}| + |u - a_{n_0}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Consequently $(a_n)_{n\in\mathbb{N}}$ converges to u. This completes the proof. ■

Thus Cauchy sequences provide a characterisation of sequences as stated below.

Theorem 5 (Cauchy's Criterion of Convergence): A real sequence is convergent iff it is Cauchy.

Proof: See Theorem 4 and the discussion above Theorem 4.
Let us consider an example.

Example 21: Show that $\left(\frac{(-1)^n}{2^n}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Solution: Let $a_n = \frac{(-1)^n}{2^n}$. For $n > m$, we have $2^n > 2^m$. This implies $\frac{1}{2^n} < \frac{1}{2^m}$.
Now we can see that, for $n > m$

$$|a_n - a_m| = \left|\frac{(-1)^n}{2^n} - \frac{(-1)^m}{2^m}\right|$$

$$< \frac{1}{2^n} + \frac{1}{2^m}$$

$$< \frac{1}{2^m} + \frac{1}{2^m} = \frac{1}{2^{m-1}}$$

Let $\varepsilon > 0$ be given. Then for all $n > m$ we have

$$|a_n - a_m| < \varepsilon \quad \text{if} \quad \frac{1}{2^{m-1}} < \varepsilon$$

i.e.

$$|a_n - a_m| < \varepsilon \quad \text{if} \quad 2^{m-1} > \frac{1}{\varepsilon}$$

The inequality $2^{m-1} > \frac{1}{\varepsilon}$ is true for all $m \in \mathbb{N}$, if $\varepsilon \ge 1$. For $0 < \varepsilon < 1$, we take logarithm with base 2 on both sides to get,

$$m - 1 > \log_2\left(\frac{1}{\varepsilon}\right), \text{ i.e. } m > 1 + \log_2\left(\frac{1}{\varepsilon}\right).$$

So let us take $n_0 = \max\left\{1,\ 1+\log_2\left(\frac{1}{\varepsilon}\right)\right\}$. Now we have

$$
\begin{aligned}
n > m \geq n_0 &\Rightarrow m > 1+\log_2\left(\frac{1}{\varepsilon}\right)\\
&\Rightarrow m-1 > \log_2\left(\frac{1}{\varepsilon}\right)\\
&\Rightarrow 2^{m-1} > \frac{1}{\varepsilon}\\
&\Rightarrow \frac{1}{2^{m-1}} < \varepsilon\\
&\Rightarrow |a_n - a_m| < \varepsilon.
\end{aligned}
$$

This proves that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

***

The sequence in Example 21 has been proved Cauchy and hence convergent. (See Theorem 4.) The convergence of the sequence in Example 21 could also be proved using definition. Indeed, observe that the terms of the sequence approach 0. We next show you a sequence whose limit is not predictable. In Section 5.2 we introduced the Fibonacci sequence $(f_n)_{n\in\mathbb{N}}$ defined as

$$f_n = f_{n-1} + f_{n-2}, \forall n \geq 3,$$

and $f_1 = f_2 = 1$.

From E14(vi), you know that $(f_n)_{n\in\mathbb{N}}$ is increasing. Now define the sequence $(a_n)_{n\in\mathbb{N}}$ by

$$a_n = \frac{f_n}{f_{n+1}}, \qquad \forall n \geq 1.$$

You can see that the first few terms of $(a_n)_{n\in\mathbb{N}}$ are

$$1,\ 0.\ 5,\ 0.667,\ 0.6,\ 0.625, \ldots$$

The terms do not seem to approach a familiar number. You can compute a few more terms to convince yourself. However, this sequence is Cauchy and hence convergent (by Theorem 4) as shown in the following example.

Example 22: Show that the sequence $(a_n)_{n\in\mathbb{N}}$ defined above is Cauchy.

Solution: First let us observe that for all $n \in \mathbb{N}$

$$a_n = \frac{f_n}{f_{n+1}} = \frac{f_n}{f_n + f_{n-1}} \geq \frac{f_n}{f_n + f_n} = \frac{1}{2},$$

since the sequence $(f_n)_{n\in\mathbb{N}}$ is increasing. Also observe that

$$a_{n+1} = \frac{f_{n+1}}{f_{n+2}} = \frac{f_{n+1}}{f_{n+1} + f_n} = \frac{1}{1+\left(\frac{f_n}{f_{n+1}}\right)} = \frac{1}{1+a_n}$$

Hence $a_{n+1} - a_n = \dfrac{1}{1+a_n} - \dfrac{1}{1+a_{n-1}} = \dfrac{a_{n-1} - a_n}{(1+a_n)(1+a_{n-1})}$

But $(1+a_n)(1+a_{n-1}) \geq \left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{2}\right) > 2, \quad \forall n \geq 2$.

Now using the Principle of Mathematical Induction, you should show that

$$|a_{n+1} - a_n| \leq \frac{1}{2}|a_n - a_{n-1}|, \qquad \forall n \geq 2.$$

Since the inequality above is true for all $n \geq 2$, we can replace n by $n-1$, to get

$$|a_n - a_{n-1}| \leq \frac{1}{2}|a_{n-1} - a_{n-2}|.$$

Thus we get

$$|a_{n+1} - a_n| \leq \frac{1}{2^2}|a_{n-1} - a_{n-2}|$$

Repeating the above process we get

$$|a_{n+1} - a_n| \leq \frac{1}{2^{n-2}}|a_2 - a_1| = \frac{1}{2^{n-1}}, \ \forall n \geq 1$$

Now let $n = m + k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned}(a_{m+1} - a_m) + (a_{m+2} - a_{m+1}) + (a_{m+3} - a_{m+2}) + \ldots + (a_{m+k} - a_{m+k-1})\\ = -a_m + a_{m+k} = a_n - a_m\end{aligned}$$

Hence

$$\begin{aligned}|a_n - a_m| &\leq |a_{m+1} - a_m| + |a_{m+2} - a_{m+1}| + |a_{m+3} - a_{m+2}| + \ldots + |a_{m+k} - a_{m+k-1}|\\ &\leq \frac{1}{2^{m-1}} + \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{m+k-2}}\\ &= \frac{1}{2^{m-1}}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{k-1}}\right]\\ &= \frac{\frac{1}{2^{m-1}}\left(1 - \frac{1}{2^k}\right)}{1 - \frac{1}{2}}\\ &< \frac{1}{2^m}\end{aligned}$$

Now for arbitrary $\varepsilon > 0$ use arguments similar to the preceding example to get

$$n_0 > \max\left\{2, 2 + \log_2\left(\frac{1}{\varepsilon}\right)\right\}.$$

Now we have

$$
\begin{aligned}
n > m \geq n_0 &\Rightarrow m > 2 + \log_2\left(\frac{1}{\varepsilon}\right) \\
&\Rightarrow m - 2 > \log_2\left(\frac{1}{\varepsilon}\right) \\
&\Rightarrow 2^{m-2} > \frac{1}{\varepsilon} \\
&\Rightarrow \frac{1}{2^{m-2}} < \varepsilon \\
&\Rightarrow |a_n - a_m| < \varepsilon
\end{aligned}
$$

Hence $(a_n)_{n\in\mathbb{N}}$ is Cauchy.

***

Now you should do some exercises.

E32) Show that the following sequences are Cauchy. Hence, conclude that they are convergent.

i) $\left(1, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{5}, \ldots\right)$ (ii) $\left(\frac{n}{2n^2 - n + 1}\right)_{n\in\mathbb{N}}$

iii) $(c^n)_{n\in\mathbb{N}}$, where $|c| < 1$

iv) $(a_n)_{n\in\mathbb{N}}$, where $a_n = \begin{cases} 1, & \text{if } n \text{ is a prime} \\ 1 - \frac{1}{n}, & \text{otherwise} \end{cases}$

E33) Show that the following sequences are not Cauchy. Hence, conclude that they do not converge.

i) $\left((-4)^n\right)_{n\in\mathbb{N}}$ ii) $(a_n)_{n\in\mathbb{N}}$, where $a_n = \sum_{k=1}^{n} \frac{1}{k}$

E34) Show that every Cauchy sequence is bounded. In particular, every convergent sequence is bounded.

By now you must have understood what convergence is, and how to use the definition of limit, and the Cauchy's convergence criterion to test the convergence of a sequence. In the next section we shall discuss some more tools to compute the limits of sequences.

5.8 CRITERIA FOR CONVERGENCE OR DIVERGENCE OF SEQUENCES

In this section we shall discuss some criteria that establish the convergence or divergence of sequences. We start with the following criterion.

Theorem 6: A sequence $(a_n)_{n\in\mathbb{N}}$ converges to L iff every subsequence of $(a_n)_{n\in\mathbb{N}}$ converges to L.

Proof: Let us first assume that $(a_n)_{n\in\mathbb{N}}$ converges to L. Let $\varepsilon > 0$ be given. Then, there exists some $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow |a_n - L| < \varepsilon.$$

Consider an arbitrary subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$. Let $k_0 \in \mathbb{N}$ be such that $n_{k_0} \geq n_0$. Now

$$k \geq k_0 \Rightarrow n_k \geq n_{k_0} \geq n_0 \Rightarrow |a_{n_k} - L| < \varepsilon$$

This implies $(a_{n_k})_{k\in\mathbb{N}}$ converges to L.

Conversely, let us assume that each subsequence of $(a_n)_{n\in\mathbb{N}}$ converges to L. Then $(a_n)_{n\in\mathbb{N}}$ converges to L because $(a_n)_{n\in\mathbb{N}}$ is a subsequence of itself. ■

The theorem above gives a characterisation of convergence of a sequence in terms of the convergence of its subsequences. In particular, it can be used to show the divergence of a sequence just by finding its two subsequences converging to two distinct limits. As an application consider the following example.

Example 23: Show that the sequence $(2^{2^{(-1)^n}})_{n\in\mathbb{N}}$ is divergent.

Solution: Let $a_n = 2^{2^{(-1)^n}}$. Consider the subsequence $(a_{2k+1})_{k\in\mathbb{N}}$ and $(a_{2k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$.

Here

$$a_{2k+1} = 2^{2^{(-1)^{2k+1}}} = 2^{2^{-1}} = \sqrt{2}, \forall k \in \mathbb{N}.$$

And

$$a_{2k} = 2^{2^{(-1)^{2k}}} = 2^{2^1} = 4, \forall k \in \mathbb{N}.$$

Thus $a_{2k+1} \to \sqrt{2}$ and $a_{2k} \to 4$. Hence $(a_n)_{n\in\mathbb{N}}$ is divergent. The above sequence is not even Cauchy. Indeed, $|a_{n+1} - a_n| = -\sqrt{2} > 2$ for all $n \in \mathbb{N}$.

***

Theorem 6 has a nice corollary which characterises the convergence of a sequence in terms of the convergence of its odd and even subsequences. It is given below.

Corollary 1: A sequence $(a_n)_{n\in\mathbb{N}}$ converges to L iff its subsequences $(a_{2n-1})_{n\in\mathbb{N}}$ and $(a_{2n})_{n\in\mathbb{N}}$ to L.

Proof: First let us assume that $(a_n)_{n\in\mathbb{N}}$ converges to L. Then by Theorem 6, both the sequences $(a_{2n-1})_{n\in\mathbb{N}}$ and $(a_{2n})_{n\in\mathbb{N}}$ converge to L.
Conversely, assume that $(a_{2n-1})_{n\in\mathbb{N}}$ and $(a_{2n})_{n\in\mathbb{N}}$ converge to L. We have to show that $(a_n)_{n\in\mathbb{N}}$ converges to L. Let $\varepsilon > 0$ be arbitrary. Then there exist

natural numbers n_0 and n_1 such that for all $n \geq n_0$

$$|a_{2n-1} - L| < \varepsilon$$

and for all $n \geq n_1$

$$|a_{2n} - L| < \varepsilon.$$

Take $n_2 = \max\{n_0, n_1\}$. Then for all $n \geq n_2$ we have $|a_n - L| < \varepsilon$. Hence $(a_n)_{n\in\mathbb{N}}$ converges to L. ■

Let us look at an example.

Example 24: Using Corollary 1, show that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to 1, where

$$a_n = \begin{cases} 1+\dfrac{1}{n}, & \text{if } n \text{ is odd} \\ 1-\dfrac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

Solution: The odd and even subsequences of $(a_n)_{n\in\mathbb{N}}$ are $\left(1+\dfrac{1}{2n-1}\right)_{n\in\mathbb{N}}$ and $\left(1-\dfrac{1}{2n}\right)_{n\in\mathbb{N}}$, respectively.

Let $\varepsilon > 0$ be given. Then

$$\left|\left(1+\frac{1}{2n-1}\right)-1\right| = \frac{1}{2n-1} < \varepsilon \text{ if } n > \frac{1}{2}\left(1+\frac{1}{\varepsilon}\right).$$

So let $n_0 = \left\lceil \dfrac{1}{2}\left(1+\dfrac{1}{\varepsilon}\right)\right\rceil$. Then for all $n \geq n_0$ we have

$$\left|\left(1+\frac{1}{2n-1}\right)-1\right| < \varepsilon.$$

Hence $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{2n-1}\right) = 1$. Similarly, $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{2n}\right) = 1$.

Therefore, $\lim\limits_{n\to\infty} a_n = 1$.

***

Divergence to ∞ or $-\infty$

When a sequence converges we know there is some real number to which it converges. Given a divergent sequence, will it be legitimate to ask where it diverges? To be specific, let us look at the sequence $(a_n)_{n\in\mathbb{N}} = \left(n+\dfrac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$.

Its first few terms are as follows:

$$0, \frac{5}{2}, \frac{2}{3}, \frac{17}{4}, \frac{24}{5}, \frac{37}{6}, \frac{48}{7}, \ldots$$

So, where is the n^{th} term a_n heading to? Have you observed that its numerator is becoming larger much faster than the denominator? Thus, it seems that "a_n is diverging to ∞". Let us look at the formal definition.

Definition: A sequence $(a_n)_{n\in\mathbb{N}}$ is said to **diverge to** ∞ if for every $M > 0$ there exists some $n_0 \in \mathbb{N}$ such that $a_n > M$ for all $n > n_0$.

The definition above says that a sequence diverges to ∞ if given any positive real number M the sequence has a term after which all the terms are greater than M.

Let us come back to the discussion of the sequence $(a_n)_{n\in\mathbb{N}} = \left(n + \frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$.

Pick any $M > 0$. Note that when n is even $a_n = n + \frac{1}{n}$ and when n is odd, $a_n = n - \frac{1}{n}$. Thus in any case

$$a_n \geq n - 1 > M \text{ if } n > M + 1$$

So, let $n_0 = \lceil M + 1 \rceil$. Then we can see that $n > n_0 \Rightarrow n > M + 1 \Rightarrow n - 1 > M \Rightarrow a_n > M$.

Hence $(a_n)_{n\in\mathbb{N}}$ diverges to ∞.

Let us look at one more example.

Example 25: Show that $\left(\left(\frac{3}{2}\right)^n\right)_{n\in\mathbb{N}}$ diverges to ∞.

Solution: We know that for all $n \in \mathbb{N}$, $a^n \geq na$ if $a > 1$. This implies for all $n \in \mathbb{N}$, $\left(\frac{3}{2}\right)^n \geq n \cdot \frac{3}{2} > n.$

Now let $M > 0$ be arbitrary. Then $a_n > M$ if $n > M$. So, let $n_0 = \lceil M \rceil$. Then we have for all $n > n_0, a_n > M$. Hence the given sequence diverges to ∞.

***

The notion of divergence to $-\infty$ is analogous to divergence to ∞, as you can observe from the definition given below.

Definition: A sequence $(a_n)_{n\in\mathbb{N}}$ is said to **diverge to** $-\infty$ if for every $M > 0$, there exists some $n_0 \in \mathbb{N}$ such that $a_n < -M$ for all $n > n_0$.
Let us have an example.

Example 26: Show that $(-\sqrt{n})_n$ diverges to $-\infty$.

Solution: Let $M > 0$ be arbitrary. Now

$$-\sqrt{n} < -M \text{ if } \sqrt{n} > M \text{ if } n > M^2.$$

So, let $n_0 = \left| M^2 \right|$. Then

$$n > n_0 \Rightarrow n > M^2 \Rightarrow \sqrt{n} > M \Rightarrow -\sqrt{n} < -M.$$

Therefore, the given sequence diverges to $-\infty$.

***

Now you should try the following exercises.

E35) Check whether or not the following sequences converges.

i) $\left(\dfrac{4n+(-1)^n}{2n+(-1)^n}\right)_{n\in\mathbb{N}}$ ii) $\left(\sin\dfrac{(2n+1)\pi}{2}\right)_{n\in\mathbb{N}}$

E36) Does the sequence $\left(\cos\left(\dfrac{n\pi}{n+1}\right)\right)_{n\in\mathbb{N}}$ have two subsequences converging to different limits? Justify.

E37) Show that $(n!)_{n\in\mathbb{N}}$ diverges to ∞.

E38) Does the sequence $(-n\,2^{-n})_{n\in\mathbb{N}}$ diverge to $-\infty$? Justify.

In the next we shall discuss some more tools for computation of limits convergent sequences.

5.9 ALGEBRA OF CONVERGENT SEQUENCES

Often complicated sequences can be expressed as addition, multiplication, or quotient of simple sequences. In this context, it is imperative to know how limits of sequences behave with respect to such operations. So, in this section, we shall discuss how to find the limits of sequences which are sum, product or quotient of sequences. Consider the sequences

$$(a_n)_{n\in\mathbb{N}} = \left(\frac{n+1}{2n+1}\right)_{n\in\mathbb{N}} \text{ and } (b_n)_{n\in\mathbb{N}} = \left(\frac{n-1}{n+1}\right)_{n\in\mathbb{N}}.$$

You can prove yourself that $(a_n)_{n\in\mathbb{N}}$ converges to $\frac{1}{2}$ and $(b_n)_{n\in\mathbb{N}}$ converges to 1. Let $(c_n)_{n\in\mathbb{N}}$ be the sequence $(a_n + b_n)_{n\in\mathbb{N}}$. That is,

$$c_n = \frac{n+1}{2n+1} + \frac{n-1}{n+1} = \frac{3n^2+n}{(2n+1)(n+1)}, \forall n \in \mathbb{N}.$$

We now ask you the question: does $(c_n)_{n\in\mathbb{N}}$ converge to $\frac{1}{2}+1$?

To see, consider

$$c_n - \frac{3}{2} = \frac{3n^2+n}{(2n+1)(n+1)} - \frac{3}{2} = \frac{-7n-3}{2(2n+1)(n+1)}$$

Now

$$\left|c_n - \frac{3}{2}\right| = \frac{7n+3}{2(2n+1)(n+1)} \le \frac{10n}{2(2n+1)(n+1)} < \frac{10n}{2(2n)(n)} = \frac{5}{2n}$$

Thus if we pick $n_0 \in \mathbb{N}$ in such a way that $\frac{5}{2n_0} < \varepsilon$, i.e. $n_0 > \frac{5}{2\varepsilon}$, then we have

$$\left|c_n - \frac{3}{2}\right| < \varepsilon, \qquad \forall n \ge n_0$$

Hence $(c_n)_{n\in\mathbb{N}}$ converges to $\frac{3}{2}$.

Thus, in this case, we find that

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

Does the equality above hold for all convergent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$?

To investigate let us assume that

$$\lim_{n\to\infty} a_n = L_1 \text{ and } \lim_{n\to\infty} b_n = L_2$$

Then for a given $\varepsilon > 0$, there exists $n_0, n_1 \in \mathbb{N}$ such that for all $n \ge n_0$

$$|a_n - L_1| < \frac{\varepsilon}{2},$$

and for all $n \ge n_1$

$$|b_n - L_2| < \frac{\varepsilon}{2}.$$

Now let $n_2 = \max\{n_0, n_1\}$. Then for all $n \ge n_2$, we have

$$|(a_n + b_n) - (L_1 + L_2)| \le |a_n - L_1| + |b_n - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, $\lim_{n\to\infty}(a_n + b_n) = L_1 + L_2$.

The arguments above prove the following theorem.

Theorem 7: If the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent then so is $(a_n + b_n)_{n\in\mathbb{N}}$. Further,

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Consider an example now.

Example 27: Find the limit of the sequence $\left(\frac{1}{n^2}+\frac{1-n}{n}\right)_{n\in\mathbb{N}}$

Solution: First you must prove that

$$\lim_{n\to\infty}\frac{1}{n^2}=0 \text{ and } \lim_{n\to\infty}\frac{1-n}{n}=-1.$$

From Theorem 7, it follows that

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{1-n}{n}\right)=0-1=-1.$$

***

As in the case of sum, we have a similar result about the product of two convergent sequences.

Theorem 8: If the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent sequences, then so is $(a_nb_n)_{n\in\mathbb{N}}$, and

$$\lim_{n\to\infty}(a_nb_n)=(\lim_{n\to\infty}a_n).(\lim_{n\to\infty}b_n).$$

Proof: Let us assume that

$$\lim_{n\to\infty}a_n=L_1 \text{ and } \lim_{n\to\infty}b_n=L_2.$$

Assume that $L_1\neq 0$. Let $\varepsilon>0$ be given. Let us consider the quantity

$$\begin{aligned}|a_nb_n-L_1L_2| &= |a_nb_n-L_1b_n+L_1b_n-L_1L_2|\\ &\le |a_nb_n-L_1b_n|+|L_1b_n-L_1L_2|\\ &= |a_n-L_1||b_n|+|L_1||b_n-L_2|\end{aligned}$$

Now for L_1L_2 to be the limit of $(a_nb_n)_{n\in\mathbb{N}}$, we have to find some $n_0\in\mathbb{N}$ such that for all $n\ge n_0$ the quantity $|a_n-L_1||b_n|+|L_1||b_n-L_2|$ is less than ε. Look at the term $|a_n-L_1||b_n|$. Since $(b_n)_{n\in\mathbb{N}}$ is convergent, $(b_n)_{n\in\mathbb{N}}$ must be bounded. (See E34.) That is, there exists some $M>0$ such that

$$|b_n|\le M, \text{ for all } n\ge 1.$$

Since $\lim_{n\to\infty}a_n=L_1$ there exists some $n_1\in\mathbb{N}$ such that for all $n\ge n_1$

$$|a_n-L_1|<\frac{\varepsilon}{2M}.$$

Also since $\lim_{n\to\infty}b_n=L_2$, there exists some $n_2\in\mathbb{N}$ such that for all $n\ge n_2$,

$$|b_n-L_2|<\frac{\varepsilon}{2|L_1|}.$$

Thus, if we choose $n_0=\max\{n_1,n_2\}$, we have for all $n\ge n_0$

$$|a_n b_n - L_1 L_2| < \frac{\varepsilon}{2M}.M + |L_1|.\frac{\varepsilon}{2|L_1|} = \varepsilon .$$

This proves that $(a_n b_n)_n$ is convergent and $\lim_{n\to\infty}(a_n b_n) = L_1 L_2$.

The proof is still not complete. We ask you to prove the case $L_1 = 0$ (See E 41). ■

Finally let us look at the following result.

Theorem 9: If for all $n \in \mathbb{N}$, $a_n \neq 0$, and $(a_n)_{n\in\mathbb{N}}$ converges to $L \neq 0$, then $\left(\frac{1}{a_n}\right)_{n\in\mathbb{N}}$ converges to $\frac{1}{L}$.

Proof: Let $\varepsilon > 0$ be given. Consider

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \left|\frac{L - a_n}{a_n L}\right| = \frac{1}{|a_n|}.\frac{1}{|L|}.|a_n - L|$$

Since L is the limit of $(a_n)_{n\in\mathbb{N}}$, the quantity $|a_n - L|$ can be made smaller than any positive number, by choosing sufficiently large n. Our concern is can the quantity $\frac{1}{|a_n|}$ be made smaller than a certain positive number by choosing sufficiently large n?

Since a_n converges to L, $|a_n|$ converges to $|L|$. (See E26). Thus, there are always infinitely many terms of $(|a_n|)_{n\in\mathbb{N}}$ that lie in $\left]|L| - \frac{|L|}{2}, |L| + \frac{|L|}{2}\right[$, that is, in $\left]\frac{|L|}{2}, 3\frac{|L|}{2}\right[$. (Why?)

That is, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\frac{|L|}{2} \leq |a_n| \leq \frac{3}{2}|L|.$$

This gives $\frac{1}{|a_n|} \leq \frac{2}{|L|}$. Now since $(a_n)_{n\in\mathbb{N}}$ converges to L, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$

$$|a_n - L| < \frac{\varepsilon}{2|L|^2}$$

Thus if we choose $n_2 = \max\{n_0, n_1\}$, we have for all $n \geq n_2$

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| < \frac{2}{|L|}.\frac{1}{|L|}.\frac{\varepsilon}{2|L|^2} = \varepsilon.$$

Hence, $\lim\limits_{n\to\infty}\frac{1}{a_n}=\frac{1}{L}$. ■

All the theorems above are of immense importance while evaluating the limits of sequences. Let us see some applications.

Example 28: Find the limit of the sequence $\left(\frac{2n^3+n}{n^3+2}\right)_{n\in\mathbb{N}}$.

Solution: We can rewrite n^{th} term of the given sequence as

$$\frac{2n^3+n}{n^3+2}=\frac{2+\frac{1}{n^2}}{1+\frac{2}{n^3}}.$$

Let $a_n=2+\frac{1}{n^2}$ and $b_n=1+\frac{2}{n^3}$.

Since $\frac{1}{n^2}\to 0,$ we have $a_n\to 2+0=2.$ Similarly we can see that $b_n\to 1$.

Thus

$$\frac{a_n}{b_n}\to\frac{2}{1}=2.$$

***

Now it is time that you do some exercises.

E39) If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent sequences then show that $(\alpha a_n+\beta b_n)_{n\in\mathbb{N}}$ is also convergent and $\lim\limits_{n\to\infty}(\alpha a_n+\beta b_n)=\alpha\lim\limits_{n\to\infty}a_n+\beta\lim\limits_{n\to\infty}b_n$, for all $\alpha,\beta\in\mathbb{R}$.

E40) Give an example of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ which are divergent, but their sum $(a_n+b_n)_{n\in\mathbb{N}}$ is convergent.

E41) Show that if $(a_n)_{n\in\mathbb{N}}$ converges to 0 and $(b_n)_{n\in\mathbb{N}}$ converges to L, then $(a_nb_n)_{n\in\mathbb{N}}$ converges to 0.

E42) Let $1\le k\le n.$ Show that $\lim\limits_{k\to\infty}\frac{\mathrm{k}}{\mathrm{n}^2}=0.$

E43) Show that if $(a_n)_{n\in\mathbb{N}}$ converges to L_1 and $(b_n)_{n\in\mathbb{N}}$ converges to $L_2\neq 0,$ then

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\frac{L_1}{L_2}.$$

E44) Find the limit of the sequence $\left(\frac{n^2+n+1}{n^2+1}\right)_{n\in\mathbb{N}}$.

In this unit we have addressed the fundamental issues related to sequences. Of paramount importance is the striking relationship between convergent sequences and Cauchy sequences. This relationship reveals that convergent sequences are nothing but Cauchy sequences. The last section is also no less important. In fact it is the one which empowers us with tools to compute the limit of complicated sequences in terms of elementary sequences.

5.10 SUMMARY

In this unit you have studied the following points:

i) definitions and examples of real sequences, and subsequences.

ii) definitions, examples, and properties of bounded, monotone, Cauchy, convergent, and divergent sequences.

iii) criteria of convergence and divergence of sequences.

iv) algebra of convergent sequences.

5.11 SOLUTIONS / ANSWERS

E1) i) We can observe that the sequence follows the pattern

$$\frac{1}{3}, \frac{1}{3\cdot 5}, \frac{1}{5\cdot 7}, \frac{1}{7\cdot 9}, \ldots, \frac{1}{(2n-1)(2n+1)}, \ldots\ldots$$

Hence, n^{th} term is $\frac{1}{(2n-1)(2n+1)}$.

ii) In this case the pattern is

$$-1, 1, -1, 2, -1, 3, \ldots, -1, n, \ldots$$

So, we have

$$a_n = \begin{cases} -1, & \text{if } n \text{ is odd} \\ n, & \text{otherwise} \end{cases}$$

iii) We can see that $a_n = n^{\frac{1}{n}}$.

E2) i) $2, \left(\frac{3}{2}\right)^{\frac{1}{2}}, \left(\frac{10}{3}\right)^{\frac{1}{3}}, \left(\frac{17}{4}\right)^{\frac{1}{4}}, \left(\frac{26}{5}\right)^{\frac{1}{5}}$

ii) $10, 50, \frac{500}{3}, \frac{1250}{3}, \frac{3125}{3}$

iii) $0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}$

iv) $1, \frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{1}{5}$

E3) No. We can see, for example, that the sequences differ at the first term.

E4) The cube of the given sequence is $\left(\frac{1}{n^3}\right)_{n\in\mathbb{N}}$ and cube root is $\left(\frac{1}{n^{\frac{1}{3}}}\right)_{n\in\mathbb{N}}$.

E5) The following plot represents the sequence.

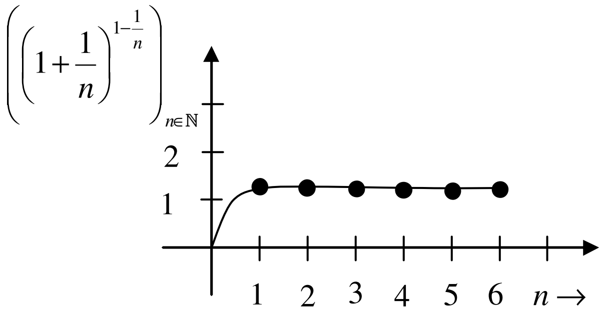


Fig. 7

E6) i) Let $a_n = \frac{n^2+n}{n^2+1}$. Then $a_n \le 2 \Leftrightarrow n \le n^2+2,$ which is true for all $n \in \mathbb{N}.$ Hence $(a_n)_{n\in\mathbb{N}}$ is bounded above.

Also note that $a_n = \frac{n^2+n}{n^2+1} \ge 1 \Leftrightarrow n \ge 1,$ which is true for all $n \in \mathbb{N}$. Thus $(a_n)_{n\in\mathbb{N}}$ is bounded below as well. Therefore $(a_n)_{n\in\mathbb{N}}$ is bounded.

ii) Let $a_n = \frac{(-1)^n n}{n+1}$. Then $|a_n| = \frac{n}{n+1} \le 1 \ \forall\, n \in \mathbb{N}.$

Hence for all $n \in \mathbb{N}, -1 \le a_n \le 1.$ Therefore, $(a_n)_{n\in\mathbb{N}}$ is bounded.

iii) We can see that

$$\sqrt{n} - \sqrt{n+1} = \frac{(\sqrt{n} - \sqrt{n+1})\,(\sqrt{n} + \sqrt{n+1})}{\sqrt{n} + \sqrt{n+1}} = \frac{-1}{\sqrt{n} + \sqrt{n+1}},$$

which lies between -1 and $0.$ Hence the sequence is bounded.

iv) We have $|1+(-1)^n| \le 1+1 = 2.$ So $-2 \le 1+(-1)^n \le 2$ for all $n \in \mathbb{N}.$ Hence, the sequence is bounded.

E7) Observe that $-K < a_n < K \Leftrightarrow |a_n| < K$. The result follows from here.

E8) i) Let $a_n = (-1)^n n^2.$ From E7 to conclude the boundedness of $(a_n)_{n\in\mathbb{N}}$ it is sufficient to check the boundedness of $(|a_n|)_{n\in\mathbb{N}}$ We have $|a_n| = n^2.$ If possible, assume for some $u \in \mathbb{R}$ that $|a_n| \le u$ for all $n \in \mathbb{N}.$ This means for all $n \in \mathbb{N}$ $n^2 \le u$ i.e. $n \le \sqrt{u},$ which is impossible. Hence $(|a_n|)_{n\in\mathbb{N}}$ is unbounded. Therefore, $(a_n)_{n\in\mathbb{N}}$ is unbounded.

ii) Observe that $2^n + (-1)^n \ge 2^n - 1 \ge n \quad \forall n \in \mathbb{N}.$ Therefore the sequence is unbounded.

iii) Since 2^k+1 is not of the form 2^m, we have for all $k \in \mathbb{N}$

$$a_{2^k+1} = -(2^k+1) \le -2^k \le -k.$$

Now, assume if possible, that for all $n \in \mathbb{N}, a_n \ge \ell$. Then $a_{2^k+1} \ge \ell$ for all $k \in \mathbb{N}$. This implies for all $k \in \mathbb{N}$, $-k \ge \ell$ that is, $k \le -\ell$. This is impossible. Hence $(a_n)_{n\in\mathbb{N}}$ is unbounded.

E9) Since $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded, their exist $L, M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$|a_n| \le L \text{ and } |b_n| \le M.$$

This implies, for all $n \in \mathbb{N}$

$$|a_n + b_n| \le |a_n| + |b_n| \le L + M$$

and

$$|a_n\, b_n| = |a_n||b_n| \le LM.$$

Hence $(a_n + b_n)_{n\in\mathbb{N}}$ and $(a_n\, b_n)_{n\in\mathbb{N}}$ are bounded.

E10) False. We know that $\left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$ is bounded, and $\dfrac{1}{n} \ne 0\ \forall n \in \mathbb{N}$, but $(n)_{n\in\mathbb{N}}$ is not bounded.

E11) True, by definition.

E12) True. Let $(a_n)_{n\in\mathbb{N}}$ be a strictly monotone sequence. If $(a_n)_{n\in\mathbb{N}}$ is strictly increasing then $m < n \Rightarrow a_m < a_n \Rightarrow a_m \le a_n$, and hence $(a_n)_{n\in\mathbb{N}}$ is increasing. Similarly, if $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing then $(a_n)_{n\in\mathbb{N}}$ is decreasing.

The converse statement is: every monotone sequence is strictly monotone, which is not true. A counter-example is $(1,1,2,3,4,\ldots)$.

E13) Let $(a_n)_{n\in\mathbb{N}}$ be increasing and as well as decreasing. Then

$$m < n \Rightarrow a_m \le a_n \text{ and } a_n \le a_m$$
$$\Rightarrow a_m = a_n.$$

Hence $(a_n)_{n\in\mathbb{N}}$ is constant.

E14) i) Let $a_n = 1 - \dfrac{1}{10^n}$. Then for all $n \ge 1$ we have

$$a_{n+1} = 1 - \frac{1}{10^{n+1}} = 1 - \frac{1}{10 \cdot 10^n} > 1 - \frac{1}{10^n} = a_n.$$

So, $(a_n)_{n\in\mathbb{N}}$ is increasing, and hence monotone.

ii) Let $a_n = \dfrac{n^3+n^2-n+2}{n^3+1}$. By factorising the numerator and the denominator, we get

$$a_n = \frac{(n+2)(n^2-n+1)}{(n+1)(n^2-n+1)} = \frac{n+2}{n+1} = 1+\frac{1}{n+1}$$

Now for all $n \geq 1$ we have

$$a_{n+1}-a_n = \frac{1}{n+2}-\frac{1}{n+1} = \frac{-1}{(n+1)(n+2)} < 0.$$

So, $(a_n)_{n\in\mathbb{N}}$ is decreasing, and hence monotone.

iii) The first three terms of the sequence are $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$. This shows that the sequence is neither increasing nor decreasing. Hence it is not monotone.

iv) Let $a_n = c^n$. Then for all $n \geq 1$ we have

$$a_{n+1} = c^{n+1} = c^n \cdot c < c^n = a_n.$$

Thus $(a_n)_{n\in\mathbb{N}}$ is decreasing, and hence monotone.

v) The first three terms of the sequence are $1, 2, \frac{3}{2}$. This shows that the sequence is not monotone.

vi) Let $P(n)$ be the statement $f_{n+1} \geq f_n$, for all $n \geq 1$. We shall use the Principle of Mathematical Induction to show that $P(n)$ is true for all $n \geq 1$. Note that $P(1)$ is $f_2 \geq f_1$, which is true. Let $P(k)$ be true for all $1 \leq k \leq n$. Consider $P(n+1)$. We have

$$\begin{aligned} f_{n+2} &= f_{n+1} + f_n \\ &\geq f_n + f_{n-1}, \text{ by Induction hypothesis} \\ &= f_{n+1} \end{aligned}$$

This proves that $P(n+1)$ is true. Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \geq 1$. That is, $f_{n+1} \geq f_n$ for all $n \geq 1$. This means $(f_n)_{n\in N}$ is increasing, and hence monotone.

E15) Let $(a_n)_{n\in\mathbb{N}}$ be increasing. Then for all $n \geq 1$. we have $a_{n+1} \geq a_n$, which implies $-a_{n+1} \leq -a_n$. Hence $(-a_n)_{n\in\mathbb{N}}$ is decreasing.

E16) i) One such sequence is $\left(1-\frac{1}{n}\right)_{n\in\mathbb{N}}$.

iii) One such sequence is $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$.

E17) One such sequence is $((-1)^n n)_{n\in\mathbb{N}}$.

E18) False. A counter-example is the sequence $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ which is increasing, and also has all positive terms, but $\left(\frac{n}{n+1}+\frac{n+1}{n}\right)_{n\in\mathbb{N}}$ is not increasing.

E19) No. Because, for example, 2 is a term of the first sequence, where as 2 never occurs in the second sequence.

E20) Let $(a_n)_{n\in\mathbb{N}} = (0,0,1,1,0,0,1,1,\ldots)$ and $(b_k)_{k\in\mathbb{N}} = (0,0,0,\ldots)$. Then we can write for each $k \in \mathbb{N}, b_k = a_{n_k}$, where $(n_k)_{k\in\mathbb{N}} = (1,2,5,6,9,10,\ldots)$. Here $(n_k)_{k\in\mathbb{N}}$ is increasing sequence. Thus $(b_k)_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

E21) We are given that $a_n = \left(1+\frac{1}{n}\right)^{1-\frac{1}{n}} = \left(\frac{n+1}{n}\right)^{\frac{n-1}{n}}$.

i) Since $\frac{n+1}{n} > 1$ and $\frac{n-1}{n} \geq 0$ for all n, we have $\left(\frac{n+1}{n}\right)^{\frac{n-1}{n}} > 1$ for all n. That is, $a_n > 1$ for all n. Let $b_k = \left(1+\frac{1}{k}\right)^{-\frac{1}{k}} = \left(\frac{k}{k+1}\right)^{\frac{1}{k}}$. Then for all k, $b_k < 1$. Thus $(b_k)_{k\in\mathbb{N}}$ is not a subsequence of $(a_n)_{n\in\mathbb{N}}$.

ii) Let $b_k = \left(1+\frac{1}{k^2}\right)^{1-\frac{1}{k^2}} = a_{k^2}$. We get $n_k = k^2$ such that $b_k = a_{n_k}$. Also $n_k < n_{k+1}$ for all k. Hence $(b_k)_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

iii) Let $b_k = \left(1+\frac{1}{k!}\right)^{1-\frac{1}{k!}} = a_{k!}$. Then we get $n_k = k!$ for all $k \in \mathbb{N}$ such that $b_k = a_{n_k}$. Also $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Thus $(b_k)_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

E22) Since $(b_n)_{n\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$. there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that $b_k = a_{n_k}$ for all $k \in \mathbb{N}$. Similarly, we have a strictly increasing sequence $(m_\ell)_{\ell\in\mathbb{N}}$ of natural numbers such that $c_\ell = b_{m_\ell}$ for all $\ell \in \mathbb{N}$. Then for all $\ell \in \mathbb{N}$, we have $c_\ell = b_{m_\ell} = a_{n_{m_\ell}}$. It remains to show that $(n_{m_\ell})_{\ell\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers. Clearly, for all $\ell \in \mathbb{N}$, $m_\ell \in \mathbb{N}$ and hence $n_{m_\ell} \in \mathbb{N}$. Now for all $\ell \in \mathbb{N}$ we have $m_\ell < m_{\ell+1} \Rightarrow n_{m_\ell} < n_{m_{\ell+1}}$.

So, $(n_{m_\ell})_{\ell\in\mathbb{N}}$ is strictly increasing sequence of natural numbers.

Therefore, $(c_n)_{n\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

E23) Yes, every subsequence of a decreasing sequence is decreasing. The proof is as follows. Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence, and $(a_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(a_n)_{n\in\mathbb{N}}$. Now

$$k < k+1 \Rightarrow n_k < n_{k+1} \qquad (\because (n_k)_{k\in\mathbb{N}} \text{ is increasing.})$$
$$\Rightarrow a_{n_k} \geq a_{n_{n+1}} \qquad (\because (n_k)_{k\in\mathbb{N}} \text{ is increasing.})$$

This proves that $(a_{n_k})_{k\in\mathbb{N}}$ is decreasing.

E24) Consider a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ Since $(a_n)_{n\in\mathbb{N}}$ is bounded, there exist some $\ell, u \in \mathbb{R}$ such that for all $n\in\mathbb{N}$, $\ell \leq a_n \leq u.$ This means for all $k\in\mathbb{N}$, $\ell \leq a_{n_k} \leq u.$ Consequently, $(a_{n_k})_{k\in\mathbb{N}}$ is bounded.

E25) i) Let $a_n = \dfrac{4}{n^2}.$ Let $\varepsilon > 0$ be given. We have to find an $n_0 \in \mathbb{N}$ such that $n > n_0$ implies $|a_n - 0| < \varepsilon.$ We know that

$$|a_n - 0| < \varepsilon \text{ if } \left|\frac{4}{n^2} - 0\right| < \varepsilon \text{ if } \frac{4}{n^2} < \varepsilon \text{ if } n > \frac{2}{\sqrt{\varepsilon}}$$

So, if we choose $n_0 = \left\lceil \dfrac{2}{\sqrt{\varepsilon}} \right\rceil$, we have

$$n > n_0 \Rightarrow n > \frac{2}{\sqrt{\varepsilon}}, \text{ i.e., } n^2 > \frac{4}{\varepsilon}.$$

Equivalently, $\dfrac{1}{n^2} < \dfrac{\varepsilon}{4} \Leftrightarrow \dfrac{4}{n^2} < \varepsilon.$

This proves that $\lim\limits_{n\to\infty} \dfrac{4}{n^2} = 0.$

ii) Let $a_n = \dfrac{2(-1)^n}{n}.$ Take $\varepsilon > 0$ arbitrary.

We can see that

$$|a_n - 0| < \varepsilon \text{ if } \left|\frac{2(-1)^n}{n} - 0\right| < \varepsilon \text{ if } \frac{2}{n} < \varepsilon \text{ if } n > \frac{2}{\varepsilon}$$

Choose $n_0 = \left\lceil \dfrac{2}{\varepsilon} \right\rceil$. Then for all $n > n_0$ we have $n > \dfrac{2}{\varepsilon} \Leftrightarrow \dfrac{2}{n} < \varepsilon$.

Thus, $\left|\dfrac{2(-1)^n}{n} - 0\right| < \varepsilon$, that is, $|a_n - 0| < \varepsilon.$

Therefore, $\lim\limits_{n\to\infty} \dfrac{2(-1)^n}{n} = 0.$

iii) We can see that

$$\left|1 - \frac{(-1)^n}{n} - 1\right| = \frac{1}{n} < \varepsilon \text{ if } n > \frac{1}{\varepsilon}$$

So, we choose $n_0 = \left\lceil \dfrac{1}{\varepsilon} \right\rceil$. Then for all $n > n_0$ we have

$$n > \frac{1}{\varepsilon} \Leftrightarrow \frac{1}{n} < \varepsilon \text{. That is, } \left|1 - \frac{(-1)^n}{n} - 1\right| < \varepsilon.$$

Therefore, $\lim\limits_{n\to\infty} \left(1 - \dfrac{(-1)^n}{n}\right) = 1.$

iv) **Hint:** Use the fact that for all $n \in \mathbb{N}$,
$|\sin n + \cos n| \le |\sin n| + |\cos n| \le 2.$

v) Here $a_n = c.$ Take $\varepsilon > 0$ arbitrary. Then $|a_n - c| < \varepsilon$ is equivalent to $0 < \varepsilon.$ So $|a_n - c| < \varepsilon$ holds always regardless of what n is. Thus, any $n_0 \in \mathbb{N}$ would work.

vi) We can observe that

$$\sqrt{n+1} - \sqrt{n-1} = \frac{\left(\sqrt{n+1} - \sqrt{n-1}\right)\left(\sqrt{n+1} + \sqrt{n-1}\right)}{\sqrt{n+1} + \sqrt{n-1}} = \frac{2}{\sqrt{n+1} + \sqrt{n-1}}$$

This implies for all $n \ge 2$

$$\left|\sqrt{n+1} - \sqrt{n-1}\right| = \frac{2}{\sqrt{n+1} + \sqrt{n-1}} < \frac{1}{\sqrt{n-1}}$$

Now, let $a_n = \sqrt{n+1} - \sqrt{n-1}.$ Take $\varepsilon > 0$ arbitrary.
We have

$$|a_n - 0| < \varepsilon \text{ if } \frac{1}{\sqrt{n-1}} < \varepsilon \text{ if } n > \frac{1}{\varepsilon^2} + 1.$$

So, we choose

$$n \ge 2 \text{ and } n > \frac{1}{\varepsilon^2} + 1 \Rightarrow n \ge 2 \text{ and } \frac{1}{\sqrt{n-1}} < \varepsilon$$
$$\Rightarrow |a_n - 0| < \varepsilon$$

Hence, $\lim\limits_{n\to\infty} \sqrt{n+1} - \sqrt{n-1} = 0.$

E26) i) Since $(a_n)_{n\in\mathbb{N}}$ is bounded, hence there exists some $K > 0$ such that $|a_n| < K$ for all $n \ge 1.$ We have $a_n \to L.$ So, let $\varepsilon > 0$ be arbitrary. Then there exists some $n_0 \in \mathbb{N}$ such that $n > n_0$ implies

$$|a_n - L| < \frac{\varepsilon}{K+|L|}.$$

Now we observe that for all $n > n_0$

$$\begin{aligned} |a_n^2 - L^2| &= |(a_n + L)\,(a_n - L)| \\ &= |a_n + L|\,|a_n - L| \\ &\le (|a_n| + |L|)\,|a_n - L| \\ &< (K+|L|)\,\frac{\varepsilon}{K+|L|} = \varepsilon \end{aligned}$$

This shows that $a_n^{\,2} \to L^2.$

ii) Try yourself.

E27) One such sequence is $(a_n)_{n\in\mathbb{N}}$, where $a_n = (-1)^n, n = 1,2,3,...$

E28) No. For example, the sequence $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ is not monotone, and is convergent to 0.

E29) The converse is "every bounded sequence is convergent". It is false, because, for example, the sequence $((-1)^n)_{n\in\mathbb{N}}$ is bounded, but not convergent.

E30) One such sequence is $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$. We have

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 = \lim_{n\to\infty} \frac{n+1}{n}.$$

E31) Assume, if possible, that $(\sqrt{n})_{n\in\mathbb{N}}$ has the limit $L.$ Take $\varepsilon = 1.$ Then there must exist some $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$

$$\left|\sqrt{n} - L\right| < 1 \Leftrightarrow -1 + L < \underline{\sqrt{n} < 1 + L}$$

Then the underlined inequality above gives as $n < (1+L)^2$ for all $n \ge n_0,$ which is impossible. Thus $\left(\sqrt{n}\right)_{n\in\mathbb{N}}$ has no limit.

E32) i) Here $a_n = \frac{(-1)^n}{n}.$ Let $n > m.$ Then

$$\begin{aligned} |a_n - a_m| &= \left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \\ &\le \frac{1}{n} + \frac{1}{m} \\ &< \frac{1}{m} + \frac{1}{m} = \frac{2}{m} \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Then

$$|a_n - a_m| < \varepsilon \text{ if } \frac{2}{m} < \varepsilon \text{ if } m > \frac{2}{\varepsilon}.$$

So, we choose $n_0 = \left\lceil \frac{2}{\varepsilon} \right\rceil$. Then

$$n > m \geq n_0 \Rightarrow \frac{2}{m} < \varepsilon \Rightarrow |a_n - a_m| < \varepsilon.$$

Therefore, $(a_n)_{n\in\mathbb{N}}$ is Cauchy.

ii) Let $a_n = \dfrac{n}{2n^2 - n + 1}$. Let $n > m$. Then

$$\begin{aligned}|a_n - a_m| &= \left|\frac{n}{2n^2 - n + 1} - \frac{m}{2m^2 - m + 1}\right| \\ &\leq \frac{n}{2n^2 - n + 1} + \frac{m}{2m^2 - m + 1} \\ &< \frac{n}{2n^2 - n} + \frac{m}{2m^2 - m} \\ &= \frac{1}{2n-1} + \frac{1}{2m-1}\end{aligned}$$

Now $n > m \Rightarrow 2n - 1 > 2m - 1 \Rightarrow \dfrac{1}{2n-1} < \dfrac{1}{2m-1}$

Hence

$$|a_n - a_m| < \frac{1}{2m-1} + \frac{1}{2m-1} = \frac{2}{2m-1}$$

Now let $\varepsilon > 0$ be arbitrary. Then

$$|a_n - a_m| < \varepsilon \text{ if } \frac{2}{2m-1} < \varepsilon \text{ if } m > \frac{1}{2}\left(1 + \frac{2}{\varepsilon}\right).$$

So, we choose $n_0 = \left\lceil 1 + \frac{2}{\varepsilon} \right\rceil$. Then

$$n > m \geq n_0 \Rightarrow m > \frac{1}{2}\left(1 + \frac{2}{\varepsilon}\right) \Rightarrow \frac{2}{2m-1} < \varepsilon \Rightarrow |a_n - a_m| < \varepsilon$$

Therefore, $(a_n)_{n\in\mathbb{N}}$ is Cauchy.

iii) Let $a_n = c^n$, where $0 < c < 1$. For $n > m$, we can see that

$$|a_n - a_m| = |c^n - c^m| = |c^m(c^{n-m} - 1)| = c^m(1 - c^{n-m}) < c^m$$

Let $\varepsilon > 0$ be arbitrary. Then we have

$$|a_n - a_m| < \varepsilon \text{ iff } c^m < \varepsilon \text{ if } \frac{1}{\varepsilon} < \left(\frac{1}{c}\right)^m \text{ if } m > \log_{\frac{1}{c}}\left(\frac{1}{\varepsilon}\right)$$

So, let us choose $n_0 = \left\lceil \log_{\frac{1}{c}}\left(\frac{1}{\varepsilon}\right) \right\rceil$. Then

$$n > m \geq n_0 \Rightarrow m > \log_{\frac{1}{c}} \varepsilon \Rightarrow |a_n - a_m| < \varepsilon.$$

Hence, $(a_n)_{n\in\mathbb{N}}$ is Cauchy.

iv) For any $m, n \in \mathbb{N}$, we can see that

$$|a_n - a_m| = \begin{cases} 0, & \text{if } m, n \in \mathbb{N}_{prime} \\ \dfrac{1}{m}, & \text{if } m \notin \mathbb{N}_{prime}, n \in \mathbb{N}_{prime} \\ \dfrac{1}{n}, & \text{if } m \in \mathbb{N}_{prime}, n \notin \mathbb{N}_{prime} \\ \left|\dfrac{1}{n} - \dfrac{1}{n}\right|, & \text{if } m, n \notin \mathbb{N}_{prime} \end{cases}$$

Thus for $n > m$, we have

$$|a_n - a_m| < \frac{2}{m}.$$

So, for a given $\varepsilon > 0$, we find $n_0 = \left\lceil \frac{2}{\varepsilon} \right\rceil$ such that

$$n > m > n_0 \Rightarrow |a_n - a_m| < \varepsilon.$$

Hence, $(a_n)_{n\in\mathbb{N}}$ is Cauchy.

E33) i) If the sequence $((-4)^n)_n$ is Cauchy, then for each $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that

$$n > m \geq n_0 \Rightarrow |(-4)^n - (-4)^m| < \varepsilon.$$

Now fix $\varepsilon = 1$, and $m = n_0$. Then we have

$$\begin{aligned} n > n_0 &\Rightarrow |(-4)^n - (-4)^{n_0}| < 1 \\ &\Rightarrow -1 + (-4)^{n_0} < (-4)^n < 1 + (-4)^{n_0} \end{aligned}$$

This implies for each even $n > n_0$,

$$(-4)^n < 1 + (-4)^{n_0} \Rightarrow 4^n < 1 + (-4)^{n_0},$$

which is impossible. Hence the given sequence is not Cauchy.

ii) If $(a_n)_{n\in\mathbb{N}}$ is Cauchy, then for each $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ such that

$$n > m \geq n_0 \Rightarrow |a_n - a_m| < \varepsilon.$$

Now fix $\varepsilon = \frac{1}{2}$ and $m = n_0$. Then

$$n > n_0 \Rightarrow |a_n - a_{n_0}| < \frac{1}{2}$$

$$\Rightarrow \frac{1}{n_0+1} + \frac{1}{n_0+2} + \cdots + \frac{1}{n} < \frac{1}{2}$$

Take $n = 2n_0$. Then we have

$$\frac{1}{n_0+1} + \frac{1}{n_0+2} + \cdots + \frac{1}{n} > \underbrace{\frac{1}{2n_0} + \frac{1}{2n_0} + \cdots + \frac{1}{2n_0}}_{n_0 \text{ terms}}$$

$$= \frac{1}{2n_0} \cdot n_0 = \frac{1}{2}$$

This is a contradiction. Hence $(a_n)_{n\in\mathbb{N}}$ is not Cauchy.

E34) We know that every Cauchy sequence is convergent, and every convergent sequence is bounded. Therefore, every Cauchy sequence is bounded.

E35) i) Let $a_n = \frac{4n + (-1)^n}{2n + (-1)^n}$. Then

$$a_{2n-1} = \frac{8n-5}{4n-3} \text{ and } a_{2n} = \frac{8n+1}{4n+1}$$

Let $\varepsilon > 0$ be arbitrary. Then

$$|a_{2n-1} - 2| = \left|\frac{8n-5}{4n-3} - 2\right| = \frac{1}{4n-3} < \varepsilon$$

if $n > \frac{1}{4}\left(3 + \frac{1}{\varepsilon}\right)$.

So, let $n_0 = \left[\frac{1}{4}\left(3 + \frac{1}{\varepsilon}\right)\right]$. Then for all $n > n_0$

$|a_{2n-1} - 2| < \varepsilon$, which implies $\lim\limits_{n\to\infty} a_{2n-1} = 2.$

Now

$$|a_{2n} - 2| = \left|\frac{8n+1}{4n+1} - 2\right| = \frac{1}{4n+1} < \varepsilon \text{ if } n > \frac{1}{4}\left(\frac{1}{\varepsilon} - 1\right).$$

So, let $n_0 = \left[\frac{1}{4}\left(\frac{1}{\varepsilon} - 1\right)\right]$. Then for all $n > n_0$

$|a_{2n} - 2| < \varepsilon$, which implies $\lim\limits_{n\to\infty} a_{2n} = 2.$

Therefore, $\lim\limits_{n\to\infty} a_n = 2.$ Hence, the sequence is convergent.

ii) Let $a_n = \sin\left(\frac{(2n+1)\pi}{2}\right)$. Then

$$a_{2n-1} = \sin\left(\frac{4n-1}{2}\right)\pi = \sin\left(2n\pi - \frac{\pi}{2}\right) = -1$$

and

$$a_{2n} = \sin\left(\frac{4n+1}{2}\right)\pi = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$$

So we have $a_{2n-1} \to -1$ and $a_{2n} \to 1.$

Hence $(a_n)_{n\in\mathbb{N}}$ is not convergent.

E36) As $n \to \infty, \frac{n}{n+1} \to 1,$ and so it seems that $\cos\left(\frac{n\pi}{n+1}\right) \to \cos\pi.$ To prove it, let $\varepsilon > 0$ be given. Then we can write

$$\left|\cos\frac{n\pi}{n+1} - \cos\pi\right| = \left|-2\sin\left(\frac{\frac{n\pi}{n+1}+\pi}{2}\right)\sin\left(\frac{\frac{n\pi}{n+1}-\pi}{2}\right)\right|$$

$$= 2\left|\sin\left(\frac{2n+1}{2(n+1)}\pi\right)\sin\left(\frac{-\pi}{2(n+1)}\right)\right|$$

$$= 2\left|\sin\left(\pi - \frac{\pi}{2(n+1)}\right)\sin\left(\frac{-\pi}{2(n+1)}\right)\right|$$

$$= 2\left|\sin\left(\frac{\pi}{2(n+1)}\right)\sin\left(\frac{-\pi}{2(n+1)}\right)\right|$$

$$\le 2\left(\sin\frac{\pi}{2(n+1)}\right)^2$$

$$\le 2\cdot\left(\frac{\pi}{2(n+1)}\right)^2 \quad (\because |\sin x| \le |x| \text{ for small } x.)$$

$$\le \frac{\pi^2}{2n^2} \text{ if } n > \frac{\pi}{\sqrt{\varepsilon}}.$$

So, we choose $n_0 = \left[\frac{\pi}{\sqrt{\varepsilon}}\right]$. Then for all $n > n_0$

$$\left|\cos\frac{n\pi}{n+1} - \cos\pi\right| < \varepsilon. \text{ Hence } \cos\left(\frac{n\pi}{n+1}\right) \to \cos\pi.$$

Therefore, no two subsequences of $\left(\cos\left(\frac{n\pi}{n+1}\right)\right)_{n\in\mathbb{N}}$ have different limits.

E37) Let $a_n = n!$. Take $M \in \mathbb{N}.$ Then we know that $(M+1)! > M.$ So, for each $M \in \mathbb{N},$ we have found an $n_0 = M+1$ such that $a_{n_0} > M.$ Hence $(a_n)_{n\in\mathbb{N}}$ diverges to ∞.

E38) Let $a_n = -n2^{-n} = \dfrac{-n}{2^n}$. If $a_n \to -\infty$ then for $M = 1$, we must have an $n_0 \in \mathbb{N}$ such that

$$\frac{-n_0}{2^{n_0}} < -M \text{ , i.e., } n_0 > 2^{n_0},$$

which is not true. Hence $(a_n)_{n\in\mathbb{N}}$ does not converge to $-\infty$.

E39) Since $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent, we have

$$\begin{aligned}\lim_{n\to\infty} (\alpha a_n + \beta b_n) &= \lim_{n\to\infty} (\alpha a_n) + \lim_{n\to\infty} (\beta b_n)\\ &= \lim_{n\to\infty} \alpha \cdot \lim_{n\to\infty} a_n + \lim_{n\to\infty} \beta \cdot \lim_{n\to\infty} b_n\\ &= \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n\end{aligned}$$

E40) We can see that the sequences $(n)_{n\in\mathbb{N}}$ and $(-n)_{n\in\mathbb{N}}$ are both divergent, but their sum $(n + -n)_{n\in\mathbb{N}} = (0)_{n\in\mathbb{N}}$ is convergent.

E41) We have $a_n \to 0$ and $b_n \to L$. Let $\varepsilon > 0$ be arbitrary. If $L = 0$, then there exist $n_0, n_1 \in \mathbb{N}$ such that for all $n > n_0, |a_n| < \sqrt{\varepsilon}$, and for all $n > n_1, |b_n| < \sqrt{\varepsilon}$. Therefore, for all $n > \max\{n_0, n_1\}$

$$|a_n b_n| = |a_n||b_n| < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon.$$

Hence, $a_n b_n \to 0$.

On the other hand, when $L \neq 0$, we write

$$\begin{aligned}|a_n b_n| &= |a_n b_n - a_n L + a_n L|\\ &\le |a_n b_n - a_n L| + |a_n L|\\ &\le |a_n||b_n - L| + |a_n||L| \qquad \text{... (3)}\end{aligned}$$

Now we have to make the right hand side of the Eq. (3) smaller than ε, for sufficiently large values of n. We do this by making each of the terms $|a_n||b_n - L|$ and $|a_n||L|$ smaller than $\varepsilon/2$. The quantity $|a_n|$ occurs in each of these terms, we know that $a_n \to 0$ implies $|a_n| \to 0$. (See E26(ii)). Therefore, for the first occurrence of $|a_n|$, we choose $n_0 \in \mathbb{N}$ such that for all $n > n_0, |a_n| < \sqrt{\varepsilon}$.

For the second occurrence of $|a_n|$, we choose $n_1 \in \mathbb{N}$ such that for all $n > n_1, |a_n| < \varepsilon/(2|L|)$. Since $b_n \to L$, there exists some $n_2 \in \mathbb{N}$ such that for all $n > n_2, |b_n - L| < \dfrac{\sqrt{\varepsilon}}{2}$. Now choose $n_3 = \max\{n_0, n_1, n_2\}$. Then for all $n > n_3$ we have

$$|a_n b_n| \le |a_n||b_n - L| + |a_n||L|$$

$$< \sqrt{\varepsilon} \cdot \frac{\sqrt{\varepsilon}}{2} + \frac{\varepsilon}{2\,|L|} \cdot |L| = \varepsilon.$$

Therefore, $a_n b_n \to 0.$ This completes the proof.

E42) We are given that $1 \le k \le n.$ Let $\varepsilon > 0$ be arbitrary. Consider

$$\left|\frac{k}{n^2} - 0\right| = \frac{k}{n^2} \le \frac{1}{k} < \varepsilon \text{ if } k > \frac{1}{\varepsilon}.$$

So, choose $n_0 = \left[\frac{1}{\varepsilon}\right].$ Then for all $k > n_0$ we have

$$\left|\frac{k}{n^2} - 0\right| < \varepsilon.$$

Hence, $\lim\limits_{k\to\infty} \frac{k}{n^2} = 0.$

E43) We know that $\lim\limits_{n\to\infty} b_n = L_2 \ne 0.$ Hence

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{L_2}, \text{ provided } b_n \ne 0 \ \forall\ n \in \mathbb{N}.$$

Now

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n} = L_1 \cdot \frac{1}{L_2} = \frac{L_1}{L_2}.$$

E44) We can write

$$\frac{n^2+n+1}{n^2+1} = 1 + \frac{n}{n^2+1}$$

Now let $\varepsilon > 0$ be given. Then

$$\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \varepsilon \text{ if } n > \frac{1}{\varepsilon}$$

So by choosing $n_0 = \left[\frac{1}{\varepsilon}\right],$ we find that for all $n > n_0,$ $\left|\frac{n}{n^2+1} - 0\right| < \varepsilon.$

Therefore, $\lim\limits_{n\to\infty} \frac{n}{n^2+1} = 0.$

Hence,

$$\lim_{n\to\infty} \frac{n^2+n+1}{n^2+1} = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{n}{n^2+1} = 1 + 0 = 1$$

UNIT 6

LIMITS OF SEQUENCES

Structure

6.1 INTRODUCTION

In Unit 5 you studied different kinds of sequences and the criteria for the convergence of a sequence. You also studied that Cauchy sequences are convergent and convergent sequences are Cauchy. You have also seen some algebraic tools to compute the limit of convergent sequences. But as you will see, those tools are still inadequate to compute the limit, or prove divergence, of a wide variety of sequences.

Beginning with Section 6.2, you will study how the limits of sequences behave regarding order (on $\mathbb{R}$). Specifically, you will see how this behaviour gives us the squeeze theorem as a powerful tool to compute the limit of a sequence lying between two sequences converging to the same limit. In Section 6.3, we shall discuss the Monotone Convergence Theorem, which talks about the convergence of monotone sequences.

In Section 6.4, you will see Cauchy's First Theorem on Limits and its applications. There is also Cauchy's Second Theorem on Limits, which you will study in Section 6.5.

Objectives

After studying this unit, you should be able to:

- explain how limits of sequences behave with respect to order on $\mathbb{R}$;

- apply the Squeeze Theorem to compute the limits of sequences;
- explain, through monotone convergence theorem, under what conditions monotone sequences converge;
- state, prove and apply Cauchy's first and second theorem on limits;
- compute the limits of sequences whose n[th] terms involve a power of 1/n.

6.2 ORDER AND LIMITS

In Unit 3, the order properties on $\mathbb{R}$ were introduced. Here, we shall study the relationship between the order (on $\mathbb{R}$) and the limit of convergent sequences in $\mathbb{R}$. Once you understand this relationship it will be convenient for you to apply the Squeeze Theorem, for computation of limits, whenever applicable.

Let us consider the sequence $\left(1-\frac{1}{2^n}\right)_{n\in\mathbb{N}}$. You can see that all its terms are positive because $\frac{1}{2^n}<1, \forall n$. You can also show that its limit is 1, which also is positive. Now consider the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$. This sequence also has all positive terms. What is its limit? Is it positive? In fact, its limit is nonnegative. More generally, must the limit, if it exists, of a sequence with nonnegative terms be nonnegative? The answer lies in the following theorem.

Theorem 1: If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence with $a_n \geq 0$, for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \geq 0$.

Proof: We shall prove this by contradiction. Let $L = \lim_{n\to\infty} a_n$. If possible, take $L<0$. Then $-L>0$. Let $\varepsilon=-L$. Then there must be some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$|a_n - L| < \varepsilon \Rightarrow L-\varepsilon < a_n < L+\varepsilon \Rightarrow 2L < a_n < 0.$$

This means the terms $a_{n_0}, a_{n_0+1}, a_{n_0+2}, \cdots$ are all negative. Thus, we have arrived at a contradiction. Hence $L \geq 0$. ■

For example, the sequence $\left(\frac{1}{n^3+1}\right)_{n\in\mathbb{N}}$ is convergent and has all terms positive. Hence its limit is nonnegative.

There are many implications of Theorem 1. First, let us look at the following corollary.

Corollary 1: if $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence with $a_n \leq 0, \forall n \in \mathbb{N}$ then $\lim_{n\to\infty} a_n \leq 0$.

Proof: Since $a_n \leq 0 \ \forall n \in \mathbb{N}$, we have $-a_n \geq 0 \ \forall n \in \mathbb{N}$. Also $(-a_n)_{n\in\mathbb{N}}$ is convergent as $(a_n)_{n\in\mathbb{N}}$ is convergent. Now applying Theorem 1 on the sequence $(-a_n)_{n\in\mathbb{N}}$, we get

$$\lim_{n\to\infty}(-a_n)\geq 0 \Rightarrow -\lim_{n\to\infty} a_n \geq 0 \Rightarrow \lim_{n\to\infty} a_n \leq 0. \qquad \blacksquare$$

Let us consider an example.

Example 1: Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent sequences such that $a_n \leq b_n \forall n\in\mathbb{N}$. Then show that

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

Solution: Define a sequence $(c_n)_{n\in\mathbb{N}}$ by $c_n = b_n - a_n, \forall n\in\mathbb{N}$. We have $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent, so is $(c_n)_{n\in\mathbb{N}}$ (See E39 of Unit 5.) Also,

$$a_n \leq b_n \Rightarrow b_n - a_n \geq 0 \Rightarrow c_n \geq 0.$$

Therefore, by Theorem 1,

$$\begin{aligned}\lim_{n\to\infty} c_n \geq 0 &\Rightarrow \lim_{n\to\infty}(b_n - a_n)\geq 0\\ &\Rightarrow \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n \geq 0 \text{ (See E39 of Unit 5.)}\\ &\Rightarrow \lim_{n\to\infty} b_n \geq \lim_{n\to\infty} a_n.\end{aligned}$$

This completes the proof.

***

Now consider the sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ where

$$a_n = 1 + \frac{1}{n^2+1}, \qquad b_n = 1 - \frac{1}{n}, \qquad c_n = 1 + \frac{1}{n}$$

You know that $-\frac{1}{n} < \frac{1}{n^2+1}$, so that $b_n < a_n$. Also, you can check that $\frac{1}{n^2+1} < \frac{1}{n}$. Hence $a_n < c_n$. Thus, we have $b_n < a_n < c_n \ \forall n\in\mathbb{N}$. Now verify that both the sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ converge to 1. Is this information sufficient to conclude about the convergence of $(a_n)_{n\in\mathbb{N}}$? Intuitively, you can see that as n tends to ∞ both b_n and c_n approach to 1 and a_n lies between b_n and c_n for all $n\in\mathbb{N}$. So it seems natural to expect that $a_n \to 1$ as n tends to ∞, also. In fact, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right) = 1 = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$$

What is essentially happening here is that the terms b_n and c_n "squeeze" (or force) a_n to approach 1, the common limit of $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$. In general, we have the following theorem.

Theorem 2 (Squeeze Theorem): Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be three sequences such that $b_n \le a_n \le c_n, \forall n \in \mathbb{N}$. If $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ converge to the same limit L, then $(a_n)_{n\in\mathbb{N}}$ also converges to L.

We say that a sequence $(a_n)_{n\in\mathbb{N}}$ lies **between** two sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ if for all $n \in \mathbb{N}$ either $b_n \le a_n \le c_n$ or $c_n \le a_n \le b_n$. In this terminology Theorem 2 can be reworded as follows:

"If $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are sequences converging to the same limit L, and $(a_n)_{n\in\mathbb{N}}$ is any sequence lying between $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$, then $(a_n)_{n\in\mathbb{N}}$ must converge to L."

Theorem 2 is sometimes also called **Sandwich Theorem**.

Proof: Let $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ both converge to L. Then, for given $\varepsilon > 0$ we have $n_0, n_1 \in \mathbb{N}$ such that

$$L - \varepsilon < b_n < L + \varepsilon, \qquad \forall n > n_0$$

And

$$L - \varepsilon < c_n < L + \varepsilon, \qquad \forall n > n_1$$

Now let $n_2 = \max\{n_0, n_1\}$. Then for all $n > n_2$

$$L - \varepsilon < b_n \le a_n \le c_n < L + \varepsilon.$$

This implies,

$$L - \varepsilon < a_n < L + \varepsilon, \qquad \forall n > n_2$$

Hence $\lim\limits_{n\to\infty} a_n = L.$ ■

Let us think about whether its converse is true or not. That is, whether for any sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n$, any one of them lies between the other two? Let us consider, for example, the sequences $\left(-\frac{1}{n^2}\right)_{n\in\mathbb{N}}, \left(\frac{1}{n^2}\right)_{n\in\mathbb{N}}$ and $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$. We can see that $-\frac{1}{n^2} < \frac{1}{n^2}$, but $\frac{(-1)^n}{n}$ does not lie between $-\frac{1}{n^2}$ and $\frac{1}{n^2}$.

The Squeeze Theorem is a powerful tool as it empowers you to compute the limit of a sequence by comparing it with two "suitably chosen" sequences. But, to use it effectively, some knowledge of inequalities is required. Let us consider some examples of its applications, now.

Example 2: Using the Squeeze Theorem, show that $\lim\limits_{n\to\infty} \frac{\sin n}{n} = 0.$

Solution: Let $a_n = \frac{\sin n}{n}$. Then we know that $-1 \le \sin n \le 1$, for all $n \in \mathbb{N}$.

This implies

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}.$$

So we can take $b_n = -\dfrac{1}{n}$, and $c_n = \dfrac{1}{n}$. Now $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = 0.$ Since $b_n \le a_n \le c_n$, by the Squeeze Theorem $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0.$

***

Example 3: Let $(a_n)_{n\in\mathbb{N}}$ be the sequence as defined below:

$$a_n = \sum_{k=1}^{n} \frac{1}{(n+k)^2}, \forall n \in \mathbb{N}$$

Find $\lim\limits_{n\to\infty} a_n$, if it exists.

Solution: Before we use the Squeeze Theorem, we need to find two sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ so that $b_n \le a_n \le c_n$ holds for all $n \ge 1$. To get such sequences, let us consider the quantity $\dfrac{1}{(n+k)^2}$ for $1 \le k \le n$. We know that for all $k \ge 1$

$$n+k > n \Rightarrow (n+k)^2 > n^2 \Rightarrow \frac{1}{(n+k)^2} < \frac{1}{n^2}$$

Also, for all $1 \le k \le n$

$$n+n \ge n+k \Rightarrow (2n)^2 \ge (n+k)^2 \Rightarrow \frac{1}{(2n)^2} \le \frac{1}{(n+k)^2}$$

Thus for all $k = 1, 2, \ldots, n$ we have

$$\frac{1}{(2n)^2} \le \frac{1}{(n+k)^2} < \frac{1}{n^2}.$$

This implies

$$\sum_{k=1}^{n} \frac{1}{(2n)^2} \le \sum_{k=1}^{n} \frac{1}{(n+k)^2} < \sum_{k=1}^{n} \frac{1}{n^2} \Rightarrow \frac{1}{4n^2}\sum_{k=1}^{n} 1 < a_n < \frac{1}{n^2}\sum_{k=1}^{n} 1$$

$$\Rightarrow \frac{1}{4n} < a_n < \frac{1}{n}, \qquad \left(\because \sum_{k=1}^{n} 1 = n\right)$$

Take $b_n = \dfrac{1}{4n}$ and $c_n = \dfrac{1}{n}$. Then $\lim\limits_{n\to\infty} b_n = 0 = \lim\limits_{n\to\infty} c_n$. Hence by the Squeeze Theorem, $\lim\limits_{n\to\infty} a_n = 0.$

***

Example 4: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative terms. If $a_n \to \ell$, then show that $\sqrt{a_n} \to \sqrt{\ell}$.

Solution: Since $a_n \ge 0,\ \forall n \in \mathbb{N}$ we have $\sqrt{a_n} \ge 0,\ \forall n \in \mathbb{N}$. Now

$$0 \le \left|\sqrt{a_n} - \sqrt{\ell}\right| = \frac{\left|\sqrt{a_n} - \sqrt{\ell}\right|}{\sqrt{a_n} + \sqrt{\ell}}(\sqrt{a_n} + \sqrt{\ell})$$

$$= \frac{|a_n - \ell|}{\sqrt{a_n} + \sqrt{\ell}}$$

$$\le \frac{1}{\sqrt{\ell}}|a_n - \ell| \qquad (\because \sqrt{a_n} + \sqrt{\ell} \ge \sqrt{\ell})$$

Since $a_n \to \ell$, we have $(a_n - \ell) \to 0$. This implies $\frac{1}{\sqrt{\ell}}|a_n - \ell| \to 0$. So, let $b_n = 0$ and $c_n = \frac{1}{\sqrt{\ell}} | a_n - \ell |$. Then $b_n \le | \sqrt{a_n} - \sqrt{\ell} | \le c_n$ for all $n \ge 1$. Hence by the Squeeze Theorem, $\left|\sqrt{a_n} - \sqrt{\ell}\right| \to 0$, which implies $\sqrt{a_n} \to \sqrt{\ell}$.

***

Example 5: Find $\lim\limits_{n\to\infty} \frac{\sin n \, \cos 2n}{n^2}$, if it exists.

Solution: Let $a_n = \frac{\sin n \, \cos 2n}{n^2}$. Then,

$$0 \le |a_n| = \left|\frac{\sin n \cos 2n}{n^2}\right| \le \frac{1}{n^2} \qquad \left(\because |\sin n| \le 1 \text{ and } |\cos 2n| \le 1\right)$$

Take $b_n = 0$ and $c_n = \frac{1}{n^2}$. Then $\lim\limits_{n\to\infty} b_n = 0 = \lim\limits_{n\to\infty} c_n$.

Hence, by the squeeze Theorem,

$$\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0.$$

***

Example 6: Show that $\lim\limits_{n\to\infty} 2^{\frac{1}{n}} = 1$.

Solution: We know that $2 > 1$, so $2^{\frac{1}{n}} > 1$ for all $n \in \mathbb{N}$. This means, we can write $2^{\frac{1}{n}} = 1 + a_n$ for some sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n > 0$ for all $n \in \mathbb{N}$. Then $2 = (1 + a_n)^n \ge 1 + n a_n$, which implies $1 \ge n a_n$, i.e., $a_n \le \frac{1}{n}$. Thus $0 < a_n \le \frac{1}{n}$ for all $n \in \mathbb{N}$. We know that $\frac{1}{n} \to 0$. So, applying the Squeeze Theorem, we get $\lim\limits_{n\to\infty} a_n = 0$. Therefore, $\lim\limits_{n\to\infty} 2^{\frac{1}{n}} = \lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} a_n = 1$.

***

You should do the following exercises now.

E1) Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence with $\alpha \le a_n \le \beta$, where $\alpha, \beta \in \mathbb{R}$, for all $n \in \mathbb{N}$. Then show that $\alpha \le \lim\limits_{n\to\infty} a_n \le \beta$.

E2) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $a_n < \frac{1}{n} \; \forall n \in \mathbb{N}$. Must $(a_n)_{n\in\mathbb{N}}$ be convergent? Why, or why not?

E3) Use the Squeeze Theorem to compute the limits of the following sequences:

i) $\left(\left(2+\frac{1}{n}\right)^2\right)_{n\in\mathbb{N}}$ ii) $\left(\frac{3n^2+1}{3n^2-1}\right)_{n\in\mathbb{N}}$

iii) $\left(\frac{2^n}{(n+2)!}\right)_{n\in\mathbb{N}}$ iv) $\left(\frac{1}{n}\sin\frac{n\pi}{2}\right)_{n\in\mathbb{N}}$

E4) Find the following limits.

i) $\lim\limits_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$ ii) $\lim\limits_{n\to\infty} n^{\frac{1}{n}}$

iii) $\lim\limits_{n\to\infty}\left(\prod_{i=0}^{n-1}\frac{n-i}{n}\right)$ iv) $\lim\limits_{n\to\infty}(n!)^{\frac{1}{n^2}}$

Sometimes it is not easy to find convergent sequences squeezing a given sequence. So, we need to look for other properties of a sequence to decide whether it converges or not. In the next section we shall discuss one such property.

6.3 MONOTONE CONVERGENCE THEOREM

From Unit 5, you know that monotone sequences either increase consistently or decrease consistently. This property makes their convergence or divergence predictable. In this section you will study an important theorem about the sequences that are monotone. Such sequences play a crucial role in real analysis. For example, consider the sequence $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$.

You can easily prove that it is increasing and bounded above. This means its terms increase successively. But since it is bounded above, they will never cross the least upper bound. What does this imply? Does the sequence converge to its least upper bound, namely 1? This is indeed the case, as the following theorem shows.

Theorem 3(Monotone Convergence Theorem): Every monotone bounded sequence is convergent.

Proof: Let $(a_n)_{n\in\mathbb{N}}$ be a monotone bounded sequence. There are two cases.

Case 1: Assume that $(a_n)_{n\in\mathbb{N}}$ is increasing . Since $(a_n)_{n\in\mathbb{N}}$ is bounded, it follows that $(a_n)_{n\in\mathbb{N}}$ is bounded above. Let

$$L=\sup\{a_n \mid n\in\mathbb{N}\}.$$

Then by Theorem 10 of Unit 3, for each $\varepsilon>0$ there exists a natural number n_0 such that $L-\varepsilon<a_{n_0}$. But $(a_n)_{n\in\mathbb{N}}$ is increasing, hence

$$a_{n_0} \le a_n, \qquad \forall\, n \ge n_0.$$

Thus, for all $n \ge n_0$

$$L - \varepsilon < a_n \Rightarrow L - a_n < \varepsilon$$
$$\Rightarrow |L - a_n| < \varepsilon, \text{since } L - a_n \ge 0.$$

This proves $\lim\limits_{n\to\infty} a_n = L$. Hence, $(a_n)_{n\in\mathbb{N}}$ is convergent.

Case 2: Assume that $(a_n)_{n\in\mathbb{N}}$ is decreasing. The proof of this case requires similar reasoning, and we leave it for you to work out (see E7). ■

Let us consider an example.

Example 7: Show that the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)_{n\in\mathbb{N}}$ is convergent. What is its limit?

Solution: Let $a_n = \left(1+\frac{1}{n}\right)^n$. First let us show that $(a_n)_{n\in\mathbb{N}}$ is increasing. Using the Binomial Theorem we can write

$$a_n = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{n^r} \text{ and } a_{n+1} = \sum_{r=0}^{n+1} \binom{n+1}{r} \frac{1}{(n+1)^r}$$

Now we write for $0 \le r \le n$

$$\binom{n}{r}\frac{1}{n^r} = \frac{1}{r!}\, n(n-1)\,(n-2)\ldots(n-(r-1))\cdot\frac{1}{n^r}$$
$$= \frac{1}{r!}\,\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{(n-(r-1))}{n} \qquad \text{... (1)}$$

Similarly for $0 \le r \le n,$ we write

$$\binom{n+1}{r}\frac{1}{(n+1)^r} = \frac{1}{r!}\,\frac{n+1}{n+1}\cdot\frac{n}{n+1}\cdot\frac{n-1}{n+1}\cdots\frac{(n+1-(r-1))}{n+1} \qquad \text{... (2)}$$

Comparing the right hand sides of Eqs. (1) and (2) we have

$$\frac{1}{r!} = \frac{1}{r!},\quad \frac{n}{n} = \frac{n+1}{n+1},\quad \frac{n-1}{n} \le \frac{n}{n+1},\quad \ldots,\quad \frac{n-(r-1)}{n} \le \frac{(n+1-(r-1))}{n+1}$$

Thus, for all $0 \le r \le n$

$$\binom{n}{r}\frac{1}{n^r} \le \binom{n+1}{r}\frac{1}{(n+1)^r}$$

Hence

$$\sum_{r=0}^{n}\binom{n}{r}\frac{1}{n^r} \le \sum_{r=0}^{n}\binom{n+1}{r}\frac{1}{(n+1)^r} + \binom{n+1}{n+1}\frac{1}{(n+1)^{n+1}}$$

Thus for all $n \in \mathbb{N}$ we have proved that $a_n \le a_{n+1}$. Therefore, $(a_n)_{n\in\mathbb{N}}$ is increasing. To prove that $(a_n)_{n\in\mathbb{N}}$ is bounded observe that $a_n > 0$ for all $n \in \mathbb{N}$ and for $0 \le r \le n$

$$\binom{n}{r}\frac{1}{n^r} = \frac{1}{r!}\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{(n-(r-1))}{n}$$
$$\le \frac{1}{r!}1.1.1......1 = \frac{1}{r!} \le \frac{1}{2^{r-1}}$$

Then

$$a_n = \sum_{r=0}^{n}\binom{n}{r}\frac{1}{n^r} = 1+\sum_{r=1}^{n}\binom{n}{r}\frac{1}{n^r} \le 1+\sum_{r=1}^{n}\frac{1}{2^{r-1}} < 3 \text{ (Why?)}$$

Thus $(a_n)_{n\in\mathbb{N}}$ is bounded above. Now the Monotone Convergence Theorem implies that $(a_n)_{n\in\mathbb{N}}$ is convergent.

The limit of this sequence is known as the **Euler number** e, that is,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Leonhard Euler (1707-1783)

Unfortunately, this sequence converges very slowly. That is, you need to take sufficiently large values of n to get a desired approximation to e. For instance, when you take $n = 1000$, you get the number

$$2.7169239322358924573830881219476$$

of which only the first three digits match in the exact value of e. Note that the decimal expansion of e goes as

$$e = 2.71828182845904523536..$$

***

It can be proved that e is an **irrational** number, however we shall not prove it here.

Let us consider some more examples, where Monotone Convergence Theorem can be applied.

Example 8: Show that the sequence $(a_n)_{n\in\mathbb{N}}$, where

$$a_n = \frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+n},$$

is monotone and bounded. Hence conclude that it is convergent.

Solution: Let us first show that the sequence is monotone. So, consider

$$a_{n+1}-a_n = \frac{1}{(n+1)+1}+\frac{1}{(n+1)+2}+\ldots+\frac{1}{(n+1)+(n-1)}+\frac{1}{(n+1)+n}$$

$$
\begin{aligned}
&+\frac{1}{(n+1)+(n+1)} \\
&-\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+n}\right) \\
&=\frac{1}{(n+1)+n}+\frac{1}{(n+1)+(n+1)}-\frac{1}{n+1} \\
&=\frac{1}{2(n+1)(2n+1)} \quad \forall n \in \mathbb{N}.
\end{aligned}
$$

This implies $(a_n)_{n\in\mathbb{N}}$ is increasing. Now note that $0 < a_n$, and

$$a_n < \underbrace{\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}}_{n \text{ terms}} = 1$$

Consequently, $(a_n)_{n\in\mathbb{N}}$ is bounded, and hence convergent.

***

Example 9: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence, defined by $a_1 = 2,$ and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \text{ for all } n \geq 1.$$

Show that $(a_n)_{n\in\mathbb{N}}$ is convergent. Also find the limit.

Solution: Let us compute the first few terms of this sequence. These are

$$
\begin{aligned}
a_1 &= 2, \\
a_2 &= \frac{3}{2} = 1.5, \\
a_3 &= \frac{17}{12} \approx 1.416667, \\
a_4 &= \frac{577}{408} \approx 1.414216
\end{aligned}
$$

This shows that the first 4 terms are decreasing. Soon you shall see that this behaviour continues for all the terms. Using the Principle of Mathematical Induction you can see that all the terms are positive. That is, 0 is a lower bound of the sequence. In fact, we can get a tighter lower bound, as shown below.

$$
\begin{aligned}
a_n^{\,2} - 2 &= \frac{1}{4}\left(a_{n-1}^2 + \frac{4}{a_{n-1}^2} + 4\right) - 2 \\
&= \frac{1}{4}\left(a_{n-1}^2 + \frac{4}{a_{n-1}^2} - 4\right) \\
&= \frac{1}{4}\left(a_{n-1} - \frac{2}{a_{n-1}}\right)^2 > 0 \qquad \forall n \geq 2.
\end{aligned}
$$

This implies

$$\left(a_n - \sqrt{2}\right)\left(a_n + \sqrt{2}\right) > 0.$$

Since you already have seen that $a_n > 0,$ we have $a_n + \sqrt{2} > 0$. This implies $a_n - \sqrt{2} > 0,$ i.e., $a_n > \sqrt{2}$.

Thus the sequence is bounded below. Now we prove that it is decreasing. Note that

$$a_n - a_{n+1} = a_n - \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{1}{2}\left(a_n - \frac{2}{a_n}\right) = \frac{a_n^{\,2} - 2}{2a_n} > 0 \left(\because a_n > \sqrt{2}\right).$$

This implies $a_n > a_{n+1}$ $\forall n \geq 1$.

Thus the sequence is decreasing as well. Hence, by the Monotone Convergence Theorem, it is convergent. Now if $\lim\limits_{n\to\infty} a_n = L$ then $\lim\limits_{n\to\infty} a_{n+1} = L$.

Hence we have

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right) \Rightarrow L = \sqrt{2}.$$

***

Now you should try the following exercises.

E5) Evaluate the following limits

i) $\lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n+100}$ ii) $\lim\limits_{n\to\infty} e^{-n^2}$

iii) $\lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^{100n}$ iv) $\lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}$

E6) Check, for convergence, the sequence $\left(a_n\right)_{n\in\mathbb{N}}$, defined by

$$a_{n+1} = \sqrt{1 + a_n} \;\; \forall n \geq 1 \text{ and } a_1 = 1.$$

E7) Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence that is bounded below. Show that $\lim\limits_{n\to\infty} a_n = \inf\{a_n \mid n \in \mathbb{N}\}$.

E8) Check whether or not the sequence $(a_n)_{n\in\mathbb{N}}$, defined by

$$a_{n+1} = 2 - \frac{1}{a_n + 2}, \qquad \forall n \geq 1 \text{ and } a_1 = 2$$

is convergent.

E9) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive terms that is increasing, but not bounded above. Show that $(a_n)_{n\in\mathbb{N}}$ diverges to ∞.

E10) Use E9 to show that the following sequences diverge to ∞.

i) $(n!)_{n\in\mathbb{N}}$ ii) $\left(\dfrac{n^2+n+1}{n+1}\right)_{n\in\mathbb{N}}$

E11) Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence, and $(a_{n_k})_{k\in\mathbb{N}}$ a decreasing subsequence of $(a_n)_{n\in\mathbb{N}}$. Must $(a_{n_k})_{k\in\mathbb{N}}$ converge? Justify.

E12) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence which is bounded but not monotone. For some $k \ge 1$, let $(a_{n+k})_{n\in\mathbb{N}}$ be a monotone subsequence of $(a_n)_{n\in\mathbb{N}}$. What can you say about the convergence of $(a_n)_{n\in\mathbb{N}}$? Justify.

In the next section we shall look at how to find the limit of sequences whose terms represent the arithmetic or geometric means of the terms of other convergent sequences.

6.4 CAUCHY'S FIRST THEOREM ON LIMITS

Recall that in Unit 5, you studied Cauchy sequences and Cauchy's criterion of convergence. Here we shall discuss a theorem due to Cauchy, namely, Cauchy's First Theorem on Limits, applicable to certain types of sequences. As you must know, the arithmetic mean of n numbers $a_1, a_2, ..., a_n$ is the number

$$b_n = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

Let us consider

$$a_n = \frac{n-1}{n}, \qquad \forall n \ge 1.$$

Then the arithmetic mean of the first 20 terms of $(a_n)_{n\in\mathbb{N}}$ is

$$b_{20} = \frac{a_1 + a_2 + \ldots + a_{20}}{20} = \frac{0 + \frac{1}{2} + \frac{2}{3} + \ldots + \frac{19}{20}}{20} \approx 0.820113$$

and $a_{20} = \dfrac{19}{20} = 0.95.$ The difference between a_{20} and b_{20} is $|b_{20} - a_{20}| = 0.129887.$

With the help of a calculator you can check that $|b_{30} - a_{30}| = 0.099833.$

As we take larger and larger n, we find that a_n and b_n come closer and closer to each other. Practically this means that we can estimate the value of b_n by a_n, for large n. Can we always do so, without regard to what a_n is? Take for instance, $a_n = n$ and see what happens. You will find that in this case, b_n and a_n do not come closer, but move apart as n tends to ∞. Thus, it is not always possible to estimate b_n by a_n. So, what forces b_n to come closer to a_n in the first case then? Could it be related to the convergence of $(a_n)_{n\in\mathbb{N}}$? This was observed by the famous mathematician Cauchy, whom you have already met in Unit 5.

Let us see what he stated in this context.

Theorem 4 (Cauchy's First Theorem on Limits): Let $(a_n)_{n\in\mathbb{N}}$ be a sequence converging to L. Then the sequence

$$\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)_{n\in\mathbb{N}}$$

converges to L as well.

Proof: Since $(a_n)_{n\in\mathbb{N}}$ converges to L, for a given $\varepsilon > 0$, there exists a number $n_0 \in \mathbb{N}$ such that

$$|a_n - L| < \frac{\varepsilon}{2} \qquad \forall n > n_0.$$

Also, from Unit 5, you know that the convergent sequences are bounded. Hence there exists an $M > 0$ such that

$$|a_n| < M \qquad \forall n \geq 1.$$

This implies

$$|a_n - L| \leq |a_n| + |L| < M + |L|, \qquad \forall n \geq 1.$$

Now for all $n > n_0$, let us consider

$$\begin{aligned}
\left|\frac{a_1 + a_2 + .. + a_n}{n} - L\right| &= \left|\frac{a_1 + a_2 + \ldots + a_n - nL}{n}\right| \\
&= \left|\frac{(a_1 - L) + (a_2 - L) + \ldots + (a_n - L)}{n}\right| \\
&= \left|\frac{(a_1 - L) + \ldots + (a_{n_0} - L) + (a_{n_0+1} - L) + \ldots + (a_n - L)}{n}\right| \\
&\leq \frac{|(a_1 - L) + \ldots + (a_{n_0} - L)|}{n} + \frac{|(a_{n_0+1} - L) + \ldots + (a_n - L)|}{n} \\
&< \frac{n_0(M + |L|)}{n} + \frac{n - n_0}{n}\left(\frac{\varepsilon}{2}\right) \\
&< \frac{n_0(M + |L|)}{n} + \frac{\varepsilon}{2} \qquad \left(\because \frac{n - n_0}{n} < 1\right)
\end{aligned}$$

Now to finish the proof we must choose a sufficiently large n so that $\frac{n_0(M + |L|)}{n} < \frac{\varepsilon}{2}$. This can be done if we choose $n > \frac{2n_0(M + |L|)}{\varepsilon}$.

So, let

$$n_1 = \left\lceil \frac{2n_0(M+|L|)}{\varepsilon} \right\rceil.$$

Then, for all $n > \max\{n_0, n_1\}$ we have

$$\left| \frac{a_1 + a_2 + \ldots + a_n}{n} - L \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\lim\limits_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = L.$ ■

Let us now consider some examples.

Example 10: Find $\lim\limits_{n\to\infty}\left[\frac{1}{n}\left(1 + 2^{\frac{1}{2}} + 3^{\frac{1}{3}} + \ldots + n^{\frac{1}{n}}\right)\right]$ using Cauchy's First Theorem on Limits.

Solution: Note that $a_n = n^{\frac{1}{n}}$. From E4 you know that $\lim\limits_{n\to\infty} a_n = 1$. Hence by Cauchy's First Theorem on Limits,

$$\lim_{n\to\infty}\left(\frac{1 + 2^{\frac{1}{2}} + 3^{\frac{1}{3}} + \ldots + n^{\frac{1}{n}}}{n}\right) = 1.$$

***

Remark 1: Note that the **converse** of Theorem 4 **is not true**. For a counterexample, let us consider $a_n = (-1)^n$. Then

$$\begin{aligned}\left| \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \right| &= \left| \frac{(-1) + 1 + (-1) + \ldots + (-1)^n}{n} \right| \\ &= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{1}{n}, & \text{if } n \text{ is odd} \end{cases} \\ &\le \frac{1}{n}\end{aligned}$$

Thus, $0 \le \left| \frac{a_1 + a_2 + \cdots + a_n}{n} \right| \le \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence, by the Squeeze

Theorem

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = 0.$$

But you know that $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$ is not convergent. (See Example 17 of Unit 5).

Example 11: Find $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right]$.

Solution: We can rewrite the limit given above as

$$L=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{\sqrt{1+\frac{1}{n^2}}}+\frac{1}{\sqrt{1+\frac{2}{n^2}}}+\cdots+\frac{1}{\sqrt{1+\frac{n}{n^2}}}\right]$$

Now let,

$$a_k=\frac{1}{\sqrt{1+\frac{k}{n^2}}},\ k=1,2,\cdots,n.$$

Since $\lim_{k\to\infty}\frac{k}{n^2}=0$ (See E42 of Unit 5), it follows that $\lim_{k\to\infty}a_k=1$. Thus we have

$$L=\lim_{n\to\infty}\frac{1}{n}\left(a_1+a_2+\cdots+a_n\right)=1 \qquad \text{(by Theorem 4)}$$

***

There is a nice corollary to Theorem 4. It is about the limit of the sequence of geometric means of the first n terms of a convergent sequence.

Corollary 3: Let the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n>0, \forall n\in\mathbb{N}$, converge to the limit L. Then the sequence

$$\left(\left(a_1a_2\ldots a_n\right)^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$$

also converges to L.

Recall that the geometric mean of two positive numbers a and b is $\sqrt{ab}$. The geometric mean of three positive numbers a,b,c is $(abc)^{\frac{1}{3}}$. More generally, the geometric mean of n positive numbers $a_1,a_2,\ldots,a_n$ is $(a_1\,a_2\ldots a_n)^{\frac{1}{n}}$.

Proof: To prove this corollary we shall use a result that you will study in Unit 10 of Block 4. This result states that **if the sequence $(a_n)_{n\in\mathbb{N}}$ converges to L and f is a continuous function defined on the range of $(a_n)_{n\in\mathbb{N}}$, then the sequence $(f(a_n))_{n\in\mathbb{N}}$ converges to $f(L)$.**

Symbolically it means,

$$\lim_{n\to\infty}f(a_n)=f\left(\lim_{n\to\infty}a_n\right)=f(L),$$

where f is continuous. In the same unit you will also study that the natural logarithm function $\ln$, defined on $\mathbb{R}^+$, is continuous. Thus we have

$$\lim_{n\to\infty}\ln a_n=\ln\left(\lim_{n\to\infty}a_n\right)=\ln L$$

Therefore, by Theorem 4 we have

$$\lim_{n\to\infty}\left(\frac{\ln a_1 + \ln a_2 + \ldots + \ln a_n}{n}\right) = \ln L \quad \Rightarrow \quad \lim_{n\to\infty}\frac{\ln(a_1 a_2 \ldots a_n)}{n} = \ln L$$

$$\Rightarrow \quad \lim_{n\to\infty} \ln(a_1 a_2 \ldots a_n)^{\frac{1}{n}} = \ln L$$

$$\Rightarrow \quad \ln\left[\lim_{n\to\infty}(a_{1,} a_{2,} \ldots a_n)^{\frac{1}{n}}\right] = \ln L$$

$$\Rightarrow \quad \lim_{n\to\infty}(a_1 a_2 \ldots a_n)^{\frac{1}{n}} = L. \qquad \blacksquare$$

Corollary 3 truly provides us with a powerful tool for computation of limits. To appreciate its worth, let us consider a few examples.

Example 12: Find $\lim_{n\to\infty}(1+n)^{\frac{1}{n}}$.

Solution: Let us write

$$b_n = (n+1)^{\frac{1}{n}}$$

$$= \left(\frac{2}{1}.\frac{3}{2}.\frac{4}{3}\ldots\frac{n+1}{n}\right)^{\frac{1}{n}}$$

$$= (a_1.a_2.a_{3.}\ldots a_n)^{\frac{1}{n}}, \text{ where } a_n = \frac{n+1}{n}.$$

Thus b_n is the geometrical mean of $a_1, a_2, \ldots, a_n$. You already know that $\lim_{n\to\infty} a_n = 1$. Hence by Corollary 3, $\lim_{n\to\infty} b_n = 1$.

***

Example 13: Let $a_n = \left(\frac{n+1}{n}\right)^n$ and $b_n = \frac{n+1}{(n!)^{\frac{1}{n}}}$, for all $n \in \mathbb{N}$. Deduce the limit of $(b_n)_{n\in\mathbb{N}}$ by expressing $b_n = (a_1.a_{2.}\ldots a_n)^{\frac{1}{n}}$.

Solution: Note that

$$(a_1.a_2.\ldots.a_n)^{\frac{1}{n}} = \left[\frac{2}{1}.\left(\frac{3}{2}\right)^2.\left(\frac{4}{3}\right)^3\ldots.\left(\frac{n}{n-1}\right)^{n-1}.\left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}}$$

$$= \left[\frac{1}{1}.\frac{1}{2}.\frac{1}{3}\ldots.\frac{1}{n-1}\frac{(n+1)^n}{n}\right]^{\frac{1}{n}}$$

$$= \left[\frac{(n+1)^n}{n!}\right]^{\frac{1}{n}}$$

$$= \frac{n+1}{(n!)^{\frac{1}{n}}} = b_n$$

You know that $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

Hence by Corollary 3, $\lim\limits_{n\to\infty} b_n = e$.

***

A question that is often asked is -- given a sequence how does one know whether Cauchy's First Theorem on Limits or its corollary can be applied? A thumb rule is that when the n^th^ term of the sequence is expressed as a sum of a finite number of terms, you should try Cauchy's First Theorem on Limits.

And, when the n^th^ term involves the power of $\frac{1}{n}$, try its corollary.

Now try the following exercises to assess how much you have grasped.

E13) Show that

$$\lim_{n\to\infty}\left[\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(2n)^2}\right]=0.$$

E14) Does the sequence $\left(1-\frac{1}{n}\sum\limits_{k=1}^{n}\frac{1}{k^2}\right)_{n\in\mathbb{N}}$ converge to 1? Justify.

E15) Using Corollary 3, find the limits

i) $\lim\limits_{n\to\infty}\dfrac{n+2}{[(n+1)!]^{\frac{1}{n}}}$ ii) $\lim\limits_{n\to\infty}\left(\dfrac{1.3.5.7\ldots.(2n-1)}{n!}\right)^{\frac{1}{n}}$ iii) $\lim\limits_{n\to\infty}\dfrac{(n!)^{\frac{1}{n}}}{2^{\frac{n+1}{2}}}$

If you have gone through the exercises above, you must have understood how to apply Cauchy's First Theorem on Limits. In the next section we shall discuss one more theorem due to Cauchy.

6.5 CAUCHY'S SECOND THEOREM ON LIMITS

Consider the sequence $(a_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$. You know that it is not convergent.

But $\left(\dfrac{a_{n+1}}{a_n}\right)_{n\in\mathbb{N}}$ is convergent, and $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\dfrac{n+1}{n}=1.$

Now look at the sequence $\left(a_n^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$, where a_n is the same as above. Does it converge? Of course, it converges to the same limit as $\left(\dfrac{a_{n+1}}{a_n}\right)_{n\in\mathbb{N}}$ does. Now take $(a_n)_{n\in\mathbb{N}} = \left(\dfrac{c^n}{n^2}\right)_{n\in\mathbb{N}}$, where $c\in\mathbb{R}$. Without worrying about the convergence of $(a_n)_{n\in\mathbb{N}}$, consider $\dfrac{a_{n+1}}{a_n}=\dfrac{c^{n+1}}{(n+1)^2}.\dfrac{n^2}{c^n}=c\left(\dfrac{n}{n+1}\right)^2$. So,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=c.\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2=c.$$

And,

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{c}{n^{\frac{2}{n}}} = \frac{\lim_{n\to\infty} c}{\left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)^2} = \frac{c}{1^2} = c.$$

What do you observe from the examples above? Can we say, if $\left(\frac{a_{n+1}}{a_n}\right)_{n\in\mathbb{N}}$ converges to L, then so does $\left(a_n^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$? Note that the sequences above are of positive terms. It is exactly this condition, namely, the terms of the sequence $(a_n)_{n\in\mathbb{N}}$ are positive that ensures the limits of the sequences $(a_n^{\frac{1}{n}})_{n\in\mathbb{N}}$ and $\left(\frac{a_{n+1}}{a_n}\right)_{n\in\mathbb{N}}$ are equal, as you can see from the following theorem due to Cauchy.

Theorem 5 (Cauchy's Second Theorem on Limits): Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of **positive** terms. If $\left(\frac{a_{n+1}}{a_n}\right)_{n\in\mathbb{N}}$ is convergent, then

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Proof: Let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. Then, for any $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\left|\frac{a_{n+1}}{a_n} - L\right| < \varepsilon \Rightarrow L - \varepsilon < \frac{a_{n+1}}{a_n} < L + \varepsilon \qquad \text{... (3)}$$

Since $n \geq n_0$, let $n - n_0 = k$ for some $k \geq 0$. From Eq. (3), we get k inequalities for $n = n_0, n_0+1, \ldots, n_0+k-1$. Multiplying these k inequalities we get

$$(L-\varepsilon)^k < \frac{a_{n_0+1}}{a_{n_0}} . \frac{a_{n_0+2}}{a_{n_0+1}} \ldots \frac{a_{n_0+k}}{a_{n_0+k-1}} < (L+\varepsilon)^k$$

This implies,

$$(L-\varepsilon)^k < \frac{a_{n_0+k}}{a_{n_0}} < (L+\varepsilon)^k \qquad \Rightarrow (L-\varepsilon)^{n-n_0} < \frac{a_n}{a_{n_0}} < (L+\varepsilon)^{n-n_0}$$

$$\Rightarrow (L-\varepsilon)^{1-\frac{n_0}{n}} a_{n_0}^{\frac{1}{n}} < a_n^{\frac{1}{n}} < (L+\varepsilon)^{1-\frac{n_0}{n}} a_{n_0}^{\frac{1}{n}}$$

$$\Rightarrow b_n < a_n^{\frac{1}{n}} < c_n$$

where $b_n = (L-\varepsilon)^{1-\frac{n_0}{n}} . a_{n_0}^{\frac{1}{n}}$ and $c_n = (L+\varepsilon)^{1-\frac{n_0}{n}} . a_{a_{n_0}}^{\frac{1}{n}}$. Note that $\lim_{n\to\infty} b_n = L - \varepsilon$, and $\lim_{n\to\infty} c_n = L+\varepsilon$ (Why?) Hence by the Squeeze Theorem,

$$L-\varepsilon \le \lim_{n\to\infty} a_n^{\frac{1}{n}} \le L+\varepsilon.$$

But ε is arbitrary, hence $\lim_{n\to\infty} a_n^{\frac{1}{n}} = L.$ ■

Let us consider some applications of Theorem 5.

Example 14: Find the limit of the sequence

$$\left(\frac{n}{[(n+1)(n+2)\dots(2n)]^{\frac{1}{n}}}\right)_{n\in\mathbb{N}}.$$

Solution: Let $a_n = \frac{n^n}{(n+1)(n+2)\dots(2n)}$. Then $a_n > 0$ for all $n\in\mathbb{N}$. Now consider

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+2)(n+3)\dots(2n+1)(2n+2)}\frac{(n+1)(n+2)\dots(2n)}{n^n}$$

$$= \frac{(n+1)}{2(2n+1)}.\left(\frac{n+1}{n}\right)^n$$

$$= \frac{\left(1+\frac{1}{n}\right)}{2\left(2+\frac{1}{n}\right)}.\left(1+\frac{1}{n}\right)^n$$

Hence, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{2.2}.e = \frac{e}{4}$. So,

$$\lim_{n\to\infty}\frac{n}{[(n+1)(n+2)\dots(2n)]^{\frac{1}{n}}} = \lim_{n\to\infty}\left(\frac{n^n}{(n+1)(n+2)\dots(2n)}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} a_n^{\frac{1}{n}}$$

$$= \frac{e}{4} \quad \text{(by Theorem 5).}$$

***

Example 15: Find $\lim_{n\to\infty}\frac{1}{n}(n!)^{\frac{1}{n}}$.

Solution: Note that we can write $\frac{1}{n}(n!)^{\frac{1}{n}} = \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$ So, let $a_n = \frac{n!}{n^n}$. Hence

$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$. Now

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

So, $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}.$

Hence, by Cauchy's Second Theorem on Limits,

$$\lim_{n\to\infty} \frac{1}{n}(n!)^{\frac{1}{n}} = \lim_{n\to\infty} a_n^{\frac{1}{n}} = \frac{1}{e}.$$

***

You should do the following exercises now.

E16) Find $\lim\limits_{n\to\infty} \frac{[(k+1)(k+2)\ldots(k+n)]^{\frac{1}{n}}}{n}$, where $k \in \mathbb{N}$.

E17) Find

i) $\lim\limits_{n\to\infty}\left[\frac{(2n)!}{n!^2}\right]^{\frac{1}{n}}$ ii) $\lim\limits_{n\to\infty}\left[\frac{(kn)!}{n!^k}\right]^{\frac{1}{n}}$, where $k \in \mathbb{N}$.

With this we come to the end of this unit. Let us now see what we have covered in this unit.

6.6 SUMMARY

In this unit, we have discussed the following points:

1. How limits behave with respect to order;
2. How to find the limit of a sequence by the Squeeze Theorem;
3. The Convergence and divergence criteria of monotone sequences through Monotone Convergence Theorem;
4. How to apply the Cauchy's First Theorem on Limits;
5. How to apply the Cauchy's Second Theorem on Limits.

6.7 SOLUTIONS/ANSWERS

E1) Since $(a_n)_{n\in\mathbb{N}}$ is convergent, the sequence $(a_n - \alpha)_{n\in\mathbb{N}}$ is also convergent. Also $\alpha \le a_n$ implies $a_n - \alpha \ge 0$ for all $n \in \mathbb{N}$. Therefore, by Theorem 1, we have

$$\lim_{n\to\infty} (a_n - \alpha) \ge 0 \Rightarrow \lim_{n\to\infty} a_n - \lim_{n\to\infty} \alpha \ge 0$$
$$\Rightarrow \lim_{n\to\infty} a_n - \alpha \ge 0 \Rightarrow \lim_{n\to\infty} a_n \ge \alpha.$$

Similarly, show that $\lim\limits_{n\to\infty} a_n \le \beta$.

E2) No. For example, the sequence $(-n)_{n\in\mathbb{N}}$ satisfies the inequality $-n < \frac{1}{n}$ for all $n \in \mathbb{N}$, but $(-n)_{n\in\mathbb{N}}$ is divergent.

E3) i) Let $a_n = \left(2+\frac{1}{n}\right)^2$. Then we know that $2+\frac{1}{n} > 2$ for all $n \in \mathbb{N}$. This implies, $a_n > 2^2$ for all $n \in \mathbb{N}$. Now for all $n \in \mathbb{N}$

$$a_n = \left(2+\frac{1}{n}\right)^2 = 4+\frac{4}{n}+\frac{1}{n^2} < 4+\frac{4}{n}+\frac{1}{n} = 4+\frac{5}{n}.$$

So, take $b_n = 4$ and $c_n = 4+\frac{5}{n}$. We know that

$\lim\limits_{n\to\infty} c_n = 4+\lim\limits_{n\to\infty}\frac{5}{n} = 4 = \lim\limits_{n\to\infty} b_n.$

Therefore, by the Squeeze Theorem, $\lim\limits_{n\to\infty} a_n = 4.$

ii) Let $a_n = \frac{3n^2+1}{3n^2-1}$. Observe that for all $n \in \mathbb{N}$

$$1 < \frac{3n^2+1}{3n^2-1} < \frac{3n^2+1}{3n^2} < 1+\frac{1}{3n^2}.$$

Take $b_n = 1$ and $c_n = 1+\frac{1}{3n^2}$. We can see that $\lim\limits_{n\to\infty} c_n = 1$. Hence, by the Squeeze Theorem, $\lim\limits_{n\to\infty} a_n = 1.$

iii) Observe that for all $n \in \mathbb{N}$

$$0 < \frac{2^n}{(n+2)!} < \frac{2^n}{n!}$$

It can be proved that $\lim\limits_{n\to\infty}\frac{2^n}{n!} = 0$. Therefore, by the Squeeze Theorem, $\lim\limits_{n\to\infty}\frac{2^n}{(n+2)!} = 0.$

iv) Observe the inequality

$$-\frac{1}{n} \le \left|\frac{1}{n}\sin\frac{n\pi}{2}\right| \le \frac{1}{n}$$

for all $n \in \mathbb{N}$, and apply the Squeeze Theorem to get

$\lim\limits_{n\to\infty}\left(\frac{1}{n}\sin\frac{n\pi}{2}\right) = 0.$

E4) i) We can write

$$\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

Now we know that $\lim\limits_{n\to\infty} 1+\frac{1}{n}=1,$ and $1+\frac{1}{n}>0$ for all $n\in\mathbb{N}.$

Therefore, from Example 4, we have $\lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n}}=\sqrt{1}=1.$ Thus

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n}}+\lim\limits_{n\to\infty}1}=\frac{1}{1+1}=\frac{1}{2}$$

ii) We know that $n\geq 1$ implies $n^{\frac{1}{n}}\geq 1.$ We can, therefore, write $n^{\frac{1}{n}}=1+a_n,$ for some sequence $(a_n)_{n\in\mathbb{N}},$ where $a_n\geq 0$ for all $n\geq 1.$

Therefore,

$$n=(1+a_n)^n=1+n\,a_n+\frac{n(n-1)}{2}\,a_n^2+\cdots$$

This gives

$$n\geq 1+\frac{n(n-1)}{2}\,a_n^2\Rightarrow a_n\leq\sqrt{\frac{2}{n}}.$$

Now $1\leq n^{\frac{1}{n}}\leq 1+\sqrt{\frac{2}{n}}$ for all $n\in\mathbb{N}.$ Since $\lim\limits_{n\to\infty}\sqrt{\frac{2}{n}}=0$, it follows by the Squeeze Theorem, $\lim\limits_{n\to\infty}n^{\frac{1}{n}}=1.$

iii) Let $a_n=\prod\limits_{i=0}^{n-1}\frac{n-i}{n}=\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{1}{n}=\frac{n!}{n^n}$

To apply Squeeze Theorem, we must find an upper bound of $\left(\frac{n!}{n^n}\right)_{n\in\mathbb{N}}$. We know that $n-i\leq n$ for all $i=0,1,2,\ldots,n-2.$ This implies

$$\prod_{i=0}^{n-2}(n-i)\leq n^{n-1}\Rightarrow n!\leq n^{n-1}$$
$$\Rightarrow\frac{n!}{n^{n-1}}\leq 1$$
$$\Rightarrow\frac{n!}{n^n}\leq\frac{1}{n}$$

So $0<a_n\leq\frac{1}{n}$ for all $n\in\mathbb{N}.$ Now the Squeeze Theorem implies that $\lim\limits_{n\to\infty}a_n=0.$

iv) Observe that $n!\geq 1$ which implies $(n!)^{\frac{1}{n^2}}\geq 1.$ Also observe that

$$n!\leq n^{n-1}\leq n^n\Rightarrow(n!)^{\frac{1}{n^2}}\leq n^{\frac{1}{n}}$$

In part ii) we have proved that $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1.$ Therefore by the Squeeze Theorem, $\lim\limits_{n\to\infty} (n!)^{\frac{1}{n^2}} = 1.$

E5) i) We can see that

$$\left(1+\frac{1}{n}\right)^{n+100} = \left(1+\frac{1}{n}\right)^{n} \cdot \left(1+\frac{1}{n}\right)^{100}$$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+100} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{100} = e,$$

because $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{100} = \left[\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)\right]^{100} = 1^{100} = 1.$

ii) Let $a_n = \bar{e}^{n^2}$. We know that $a_n > 0,$ for all $n \in \mathbb{N}.$ Now $a_n = \dfrac{1}{e^{n^2}} < \dfrac{1}{n^2},$ because $e^{n^2} > n^2$ for all $n \in \mathbb{N}.$ Thus $0 < a_n < \dfrac{1}{n^2}.$ Applying the Squeeze Theorem, now we get $\lim\limits_{n\to\infty} a_n = 0.$

iii) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{100n} = \left[\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n}\right]^{100} = e^{100}.$

iv) $\lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^{n} = \lim\limits_{n\to\infty}\left(\dfrac{1}{1+\dfrac{1}{n}}\right)^{n} = \lim\limits_{n\to\infty}\dfrac{1}{\left(1+\dfrac{1}{n}\right)^{n}} = \dfrac{1}{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n}} = \dfrac{1}{e}.$

E6) It is easy to see that $a_n > 0$ for all $n \geq 1.$ (Why?) Let us prove that the sequence is increasing. We have $a_1 = 1,$ and $a_2 = \sqrt{2}.$ So $a_2 > a_1.$ Let $a_n > a_{n-1}$ for some $n > 1.$ Since $a_{n+1}^2 = 1 + a_n$ and $a_n^2 = 1 + a_{n-1}$, it follows that

$$a_{n+1}^2 - a_n^2 = a_n - a_{n-1} \Rightarrow (a_{n+1} - a_n)\,(a_{n+1} + a_n) = a_n - a_{n-1}$$

$$\Rightarrow a_{n+1} - a_n = \frac{a_n - a_{n-1}}{a_{n+1} + a_n} > 0$$

$$\Rightarrow a_{n+1} > a_n.$$

Thus, by the Principle of Mathematical Induction, $a_{n+1} > a_n$ for all $n \geq 1.$ Hence $(a_n)_{n\in\mathbb{N}}$ is increasing. Using the Principle of Mathematical Induction, now let us show that $(a_n)_{n\in\mathbb{N}}$ is bounded above by 2. We have $a_1 < 2$. Assume that $a_n < 2$ for some $n > 1$. Then

$$a_{n+1} = \sqrt{1 + a_n} < \sqrt{1+2} = \sqrt{3} < 2.$$

It follows that $(a_n)_{n\in\mathbb{N}}$ is bounded above by 2. Hence, by the Monotone Convergence Theorem, $(a_n)_{n\in\mathbb{N}}$ is convergent.

E7) Since $(a_n)_{n\in\mathbb{N}}$ is decreasing, $(-a_n)_{n\in\mathbb{N}}$ is increasing. Since $(a_n)_{n\in\mathbb{N}}$ is bounded below, $(-a_n)_{n\in\mathbb{N}}$ is bounded above. Thus $(-a_n)_{n\in\mathbb{N}}$ is increasing and bounded above. Therefore, by Case 1 of the proof of the Monotone Convergence Theorem, $(-a_n)_{n\in\mathbb{N}}$ is convergent, and hence $(a_n)_{n\in\mathbb{N}}$ is convergent. Now

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left[-(-a_n)\right]\\ &= -\lim_{n\to\infty}(-a_n)\\ &= -\sup\{-a_n \mid n\in\mathbb{N}\}\\ &= -(-\inf\{a_n \mid n\in\mathbb{N}\}) \qquad (\because \sup(-S) = -\inf S)\\ &= \inf\{a_n \mid n\in\mathbb{N}\}.\end{aligned}$$

E8) Let us compute the first few terms of the sequence.
$a_1 = 2,\ a_2 = 1\cdot 75,\ a_3 = 1\cdot 733,\ a_4 = 1\cdot 732.$

It appears that the sequence is decreasing. So, consider

$$a_{n+1} - a_n = 2 - \frac{1}{a_n+2} - a_n = \frac{3-a_n^2}{a_n+2}$$

But to make $a_{n+1} - a_n < 0$ we must show that $3 - a_n^2 < 0$ i.e. $a_n > \sqrt{3}$ for all $n \geq 1$. We already have $a_1 = 2 > \sqrt{3}$. So, let us assume that $a_n > \sqrt{3}$ for some $n > 1$. Then

$$\begin{aligned}a_n + 2 > \sqrt{3} + 2 &\Rightarrow \frac{1}{a_n+2} < \frac{1}{\sqrt{3}+2}\\ &\Rightarrow -\frac{1}{a_n+2} > -\frac{1}{\sqrt{3}+2}\\ &\Rightarrow 2 - \frac{1}{a_n+2} > 2 - \frac{1}{\sqrt{3}+2} = \frac{2(\sqrt{3}+2)-1}{\sqrt{3}+2} = \frac{2\sqrt{3}+3}{\sqrt{3}+2} = \sqrt{3}.\end{aligned}$$

It follows that $a_{n+1} = 2 - \dfrac{1}{a_n+2} > \sqrt{3}$. Thus, $a_n > \sqrt{3}$ for all $n \geq 1$, and hence $(a_n)_{n\in\mathbb{N}}$ is bounded below.

Now we have $a_{n+1} - a_n < 0$, i.e., $a_{n+1} < a_n$ for all $n \geq 1$. This implies $(a_n)_{n\in\mathbb{N}}$ is decreasing. Therefore, by the Monotone Convergence Theorem, $(a_n)_{n\in\mathbb{N}}$ is convergent.

E9) Let $M > 0$ be any real number. Since $(a_n)_{n\in\mathbb{N}}$ is not bounded above, there is some $n_0 \in \mathbb{N}$ such that $a_{n_0} > M$. But $(a_n)_{n\in\mathbb{N}}$ is increasing. So

$$n > n_0 \Rightarrow a_n > a_{n_0} > M.$$

Therefore, $(a_n)_{n\in\mathbb{N}}$ diverges to ∞.

E10) i) We know that $(n+1)! > n!$ for all $n \geq 1$. This means $(n!)_{n\in\mathbb{N}}$ is increasing. Now if $n! < K$ for some $K > 0$, then $n < K$ (why?), which is not possible. Hence $(n!)_{n\in\mathbb{N}}$ is not bounded above. Therefore, by E9, $(n!)_{n\in\mathbb{N}}$ diverges to ∞.

ii) Let $a_n = \dfrac{n^2+n+1}{n+1} = 1 + \dfrac{n^2}{n+1}$. Then $a_{n+1} = 1 + \dfrac{(n+1)^2}{n+2}$.

So, $a_{n+1} > a_n \Leftrightarrow \dfrac{(n+1)^2}{n+2} > \dfrac{n^2}{n+1}$

$$\Leftrightarrow (n+1)^3 > n^2(n+2)$$

$$\Leftrightarrow 1 + 3n + n^2 > 0$$

Thus $a_{n+1} > a_n$ for all $n \geq 1$, which implies $(a_n)_{n\in\mathbb{N}}$ is increasing.

Now, if possible, assume that $K > 0$ is an upper bound of $(a_n)_{n\in\mathbb{N}}$. Then for all $n \geq 1$,

$$1 + \frac{n^2}{n+1} < K \Rightarrow \frac{n^2}{n+1} < K - 1.$$

We can write

$$\frac{n^2}{n+1} = \frac{n(n+1)-n}{n+1} = n - \frac{n}{n+1} > n - 1.$$

This implies $n < K$ for all $n \geq 1$. This is a contradiction to the fact that $\mathbb{N}$ is unbounded. Hence $(a_n)_{n\in\mathbb{N}}$ is not bounded above.

Therefore, by E9, $(a_n)_{n\in\mathbb{N}}$ diverges to ∞.

E11) Yes. Since $(a_n)_{n\in\mathbb{N}}$ is bounded, by E24 of Unit 5, $(a_{n_k})_{k\in\mathbb{N}}$ is bounded. Also $(a_{n_k})_{k\in\mathbb{N}}$ is decreasing. Hence by the Monotone Convergence Theorem $(a_{n_k})_{k\in\mathbb{N}}$ is convergent.

E12) Since $(a_n)_{n\in\mathbb{N}}$ is bounded, $(a_{n+k})_{n\in\mathbb{N}}$ is bounded. Also $(a_{n+k})_{n\in\mathbb{N}}$ is monotone. Hence by the Monotone Convergence Theorem $(a_{n+k})_{n\in\mathbb{N}}$ is convergent. Now, let $\lim\limits_{n\to\infty} a_{n+k} = L$. Then $\lim\limits_{n\to\infty} a_n = L$. Hence $(a_n)_{n\in\mathbb{N}}$ is convergent.

E13) Let us write

$$b_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n^2)} = \frac{1}{n}\left[\frac{1}{n\left(1+\frac{1}{n}\right)^2} + \frac{1}{n\left(1+\frac{2}{n}\right)^2} + \cdots + \frac{1}{n(2)^2}\right]$$

Now take $a_k = \dfrac{1}{n\left(1+\frac{k}{n}\right)^2}$, for $1 \le k \le n.$ Then

$$\lim_{k\to\infty} a_k = \lim_{n\to\infty} \frac{1}{n\left(1+\frac{k}{n}\right)^2} = 0.$$

Therefore, by the Cauchy's First Theorem on Limits, we have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n}(a_1 + a_2 + \cdots + a_n) = 0.$$

E14) Let $a_n = \dfrac{1}{n^2}$. Then $\lim\limits_{n\to\infty} a_n = 0.$ Now

$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k^2}\right) = \lim_{n\to\infty}\left[1 - \frac{1}{n}\left(\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right)\right]$$
$$= 1 - \lim_{n\to\infty}\left(\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}\right)$$
$$= 1 - 0 = 1.$$

E15) i) Let $b_n = \dfrac{n+2}{[(n+1)!]^{\frac{1}{n}}}$. Then $b_n^n = \dfrac{(n+2)^n}{(n+1)!}$. We have to express

$b_n = (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$ for a suitable sequence $(a_n)_{n\in\mathbb{N}}$ of positive terms.

If we take $a_n = \left(\dfrac{n+2}{n+1}\right)^n$, then we have

$$a_1.a_2.a_3 \ldots a_{n-1}.a_n = \left(\frac{3}{2}\right).\left(\frac{4}{3}\right)^2.\left(\frac{5}{4}\right)^3 \ldots .\left(\frac{n+1}{n}\right)^{n-1}.\left(\frac{n+2}{n+1}\right)^n$$
$$= \frac{3}{2}.\frac{4^2}{3^2}.\frac{5^3}{4^3} \ldots .\frac{(n+1)^{n-1}}{n^{n-1}}.\frac{(n+2)^n}{(n+1)^n}$$
$$= \frac{(n+2)^n}{(n+1)!}$$

Hence $b_n = (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$. Now, we write

$$a_n = \left(1 + \frac{1}{n+1}\right)^{n+1-1} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)}$$

Hence, $\lim\limits_{n\to\infty} a_n = \dfrac{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n+1}\right)^{n+1}}{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n+1}\right)} = \dfrac{e}{1} = e.$

Therefore, by Corollary 3, $\lim\limits_{n\to\infty} b_n = e.$

ii) Let $b_n = \left(\dfrac{1.3.5.7\ldots(2n-1)}{n!}\right)^{\frac{1}{n}} \Rightarrow b_n^n = \dfrac{1.3.5.7\ldots(2n-1)}{1.2.3\ldots n}$

$$\Rightarrow b_n^n = a_1.a_2\ldots a_n$$

$$\Rightarrow b_n = (a_1.a_2\ldots a_n)^{\frac{1}{n}},$$

where $a_n = \dfrac{2n-1}{n} = 2-\dfrac{1}{n}$. Since $\lim\limits_{n\to\infty} a_n = 1,$ by Corollary 3 $\lim\limits_{n\to\infty} b_n = 1.$

iii) Let

$$b_n = \frac{(n!)^{\frac{1}{n}}}{2^{\frac{n+1}{2}}} \Rightarrow b_n^n = \frac{n!}{2^{\frac{n(n+1)}{2}}}$$

$$\Rightarrow b_n^n = \frac{n!}{2^{1+2+\cdots+n}}$$

$$\Rightarrow b_n^n = \frac{1}{2}.\frac{2}{2^2}.\frac{3}{2^3}\cdots\frac{n}{2^n}$$

$$\Rightarrow b_n^n = a_1.a_2.a_3\ldots a_n$$

$$\Rightarrow b_n = (a_1.a_2.a_3\ldots a_n)^{\frac{1}{n}},$$

where $a_n = \dfrac{n}{2^n}$. Since $\lim\limits_{n\to\infty} a_n = 0,$ by Corollary 3 $\lim\limits_{n\to\infty} b_n = 0.$

E16) Let $a_n = \dfrac{(k+1)(k+2)\ldots(k+n)}{n^n}$. Then we have $a_n > 0$ for all $n \in \mathbb{N}$.

Now

$$\frac{a_{n+1}}{a_n} = \frac{(k+1)(k+2)\ldots(k+n)(k+n+1)}{(n+1)^{n+1}} \cdot \frac{n^n}{(k+1)(k+2)\ldots(k+n)}$$

$$= \left(\frac{k+n+1}{n+1}\right)\cdot\left(\frac{n}{n+1}\right)^n$$

$$= \left(1+\frac{k}{n+1}\right)\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

So, $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = 1\cdot\dfrac{1}{e} = \dfrac{1}{e}$. Therefore, by Theorem 5, $\lim\limits_{n\to\infty} a_n^{\frac{1}{n}} = \dfrac{1}{e}.$

E17) i) Let $a_n = \dfrac{(2n)!}{n!^2}$. Then $a_n > 0$ for all $n \geq 1$. Now

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!^2} \cdot \frac{n!^2}{(2n)!}$$

$$= \frac{(2n+2)(2n+1)(2n)!}{(n+1)\,n!(n+1)\,n!} \cdot \frac{n!^2}{(2n)!}$$

$$= 2\left(2 - \frac{1}{n+1}\right)$$

Thus $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 4.$ Therefore, by Theorem 5, $\lim\limits_{n\to\infty} a_n^{\frac{1}{n}} = 4.$

ii) Let $a_n = \frac{(kn)!}{(n!)^k}$. Then $a_n > 0$ for all $n \geq 1$. Now

$$\frac{a_{n+1}}{a_n} = \frac{(kn+k)!}{((n+1)!)^k} \cdot \frac{(n!)^k}{(kn)!}$$

$$= \frac{(kn+k)(kn+k-1)\ldots(kn+1)(kn)!}{(n+1)^k\ (n!)^k} \cdot \frac{(n!)^k}{(kn)!}$$

$$= k\left(k - \frac{1}{n+1}\right)\cdots\left(k - \frac{k-1}{n+1}\right)$$

Thus

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \underbrace{k \cdot k \cdots k}_{k \text{ factors}} = k^k.$$

Therefore by Theorem 5, $\lim\limits_{n\to\infty} a_n^{\frac{1}{n}} = k^k$.

MISCELLANEOUS EXAMPLES AND EXERCISES

The examples and exercises given below cover the concepts and processes you have studied in this block. Doing them will give you a better understanding of the concepts concerned, as well as practice in solving such problems.

Miscellaneous Examples

Example 1: Find a lower bound of the sequence $\left(\frac{6}{3n+7}\right)_{n\in\mathbb{N}}$. Also find its infimum. Is it bounded above?

Solution: Let $a_n = \frac{6}{3n+7}$, for $n\in\mathbb{N}$. Since $a_n > 0$ for all $n\in\mathbb{N}$, it follows that 0 is a lower bound of $(a_n)_{n\in\mathbb{N}}$. Now observe that as n gets larger and larger, a_n gets smaller and smaller. It seems that 0 is the infimum of $(a_n)_{n\in\mathbb{N}}$. To prove it, let as take any $\varepsilon > 0$. We have to find an $n\in\mathbb{N}$ such that $\varepsilon > \frac{6}{3n+7}$. (See Theorem 7(i) of Unit 3.) Simplifying this inequality we get $n > \frac{6-7\varepsilon}{3\varepsilon}$, and such an n exists by the Archimedean Property of $\mathbb{R}$. Therefore, 0 is the infimum of $(a_n)_{n\in\mathbb{N}}$. To see that the sequence is bounded above, observe that $\frac{6}{3n+7} \le \frac{6}{10}$ for all $n\in\mathbb{N}$. This means, $\frac{6}{10}$ is an upper bound of $(a_n)_{n\in\mathbb{N}}$.

***

Example 2: Identify which of the following sequences are monotone, and which are not.

i) $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$

ii) $(n-n^2)_{n\in\mathbb{N}}$

iii) $\left((-1)^{n^2}\right)_{n\in\mathbb{N}}$

iv) $\left(\frac{3n^2+2n+1}{2n^2+3n+4}\right)_{n\in\mathbb{N}}$

v) $(a_n)_{n\in\mathbb{N}}$, where $a_{n+1} = \frac{2a_n+3}{4}$ for all $n \ge 1$, and $a_1 = 1$.

Solution: i) The first three terms of the given sequence are $-1, \frac{1}{2}, -\frac{1}{3}$. We can now see that the sequence is neither increasing nor decreasing. Therefore, the sequence is not monotone.

ii) Let $a_n = n - n^2$. Then $a_{n+1} = n+1-(n+1)^2$. Now $a_{n+1} - a_n = n+1-(n+1)^2 - n + n^2 = -2n$, which is negative for all $n \ge 1$. This means, $a_{n+1} < a_n$ for all $n \ge 1$. Therefore, $(a_n)_{n\in\mathbb{N}}$ is decreasing.

iii) We know that when n is odd, n^2 is also odd. And, when n is even, n^2 is even. Thus the sequence expands as $(-1, 1, -1, 1, \ldots)$, which is neither increasing nor decreasing. Therefore, the sequence is not monotone.

iv) Let $a_n = \dfrac{3n^2+2n+1}{2n^2+3n+4}$. Then

$$a_{n+1} = \frac{3(n+1)^2+2(n+1)+1}{2(n+1)^2+3(n+1)+4} = \frac{3n^2+8n+6}{2n^2+7n+9}$$

Now, $a_{n+1}-a_n = \dfrac{3n^2+8n+6}{2n^2+7n+9} - \dfrac{3n^2+2n+1}{2n^2+3n+4}$

$$= \frac{5(n^2+5n+3)}{(2n^2+7n+9)(2n^2+3n+4)} > 0 \text{ for all } n \geq 1.$$

This implies, $a_{n+1} > a_n$ for all $n \geq 1$. Therefore, $(a_n)_{n\in\mathbb{N}}$ is increasing, and hence monotone.

v) We are given $a_1 = 1$. So $a_2 = \frac{5}{4} > a_1$. Now assume that $a_{n+1} > a_n$ for some $n \in \mathbb{N}$. Then

$a_{n+2} > a_{n+1} \Leftrightarrow \dfrac{2a_{n+1}+3}{4} > \dfrac{2a_n+3}{4} \Leftrightarrow a_{n+1} > a_n$, which is true. Therefore, by the Principle of Mathematical Induction, $(a_n)_{n\in\mathbb{N}}$ is increasing, and hence monotone.

***

Example 3: If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two Cauchy sequences, then so is the sequence $(a_n+b_n)_{n\in\mathbb{N}}$. Examine the validity of the statement.

Solution: Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be Cauchy sequences. Take $\varepsilon > 0$. Then there exist $n_0, n_1 \in \mathbb{N}$ such that $n > m \geq n_0 \Rightarrow |a_n - a_m| < \dfrac{\varepsilon}{2}$ and

$n > m \geq n_1 \Rightarrow |b_n - b_m| < \dfrac{\varepsilon}{2}$.

Now, let $n_2 = \max\{n_0, n_1\}$. Then for all $n > m \geq n_2$ we have

$$\begin{aligned} |(a_n+b_n)-(a_m+b_m)| &= |(a_n-a_m)+(b_n-b_m)| \\ &\leq |a_n-a_m| + |b_n-b_m| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $(a_n+b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence the given statement is valid.

***

Example 4: Assume that $0 < r < 1$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence satisfying $|a_{n+1}-a_n| < r^n$ for all $n \geq 1$. Show that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Solution: Let $\varepsilon > 0$ be arbitrary. Take any $m, n \in \mathbb{N}$ such that $n > m$. Then $|a_n - a_m| = |a_n - a_{n-1} + a_{n-1} - a_m| \leq |a_n - a_{n-1}| + |a_{n-1} - a_m|$.

Similarly, if $n-1>m,$ then we can write

$$|a_{n-1}-a_m|\le|a_{n-1}-a_{n-2}|+|a_{n-2}-a_m|$$

Thus we have for all $n>m,$

$$\begin{aligned}|a_n-a_m| &\le |a_n-a_{n-1}|+|a_{n-1}-a_{n-2}|+\cdots+|a_{m+1}-a_m|\\ &\le r^{n-1}+r^{n-2}+\cdots+r^m\\ &= r^m[1+r+r^2+\cdots+r^{n-m-1}]\\ &= r^m\frac{(1-r^{n-m})}{1-r}\\ &< \frac{r^m}{1-r}\end{aligned}$$

Now if we choose m such that $\frac{r^m}{1-r}<\varepsilon,$ we are done. That is, we have to choose m such that

$$r^m<(1-r)\varepsilon \Leftrightarrow m\ln r<\ln((1-r)\varepsilon)\Leftrightarrow m>\frac{\ln((1-r)\varepsilon)}{\ln r}.$$

Such an m always exists by the Archimedean Property of $\mathbb{R}.$ So, let $n_0=\left\lceil\frac{\ln((1-r)\varepsilon)}{\ln r}\right\rceil.$ Then $n>m>n_0\Rightarrow|a_n-a_m|<\frac{r^m}{1-r}\Rightarrow|a_n-a_m|<\varepsilon.$

Therefore, $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

***

Example 5: Let a sequence $(a_n)_{n\in\mathbb{N}}$ be defined by $a_1=1,$ and $a_{n+1}=\sqrt{2+a_n}$ for all $n\ge1.$ Show that $(a_n)_{n\in\mathbb{N}}$ is convergent. Also, find $\lim\limits_{n\to\infty}a_n.$

Solution: We are given that

$$\begin{aligned}a_{n+1}=\sqrt{2+a_n} &\Rightarrow a_{n+1}^2=2+a_n\\ &\Rightarrow a_{n+1}^2-a_n^2=2+a_n-a_n^2\\ &\Rightarrow (a_{n+1}-a_n)(a_{n+1}+a_n)=(2-a_n)(1+a_n)\end{aligned}$$

It is easy to see that 0 is a lower bound of $(a_n)_{n\in\mathbb{N}}.$ Therefore, $a_{n+1}+a_n>0$ for all $n\in\mathbb{N}.$ Hence $a_{n+1}-a_n=\frac{(2-a_n)(1+a_n)}{a_{n+1}+a_n}.$

Now $a_{n+1}-a_n>0$ if and only if $2-a_n>0,$ i.e., $a_n<2.$

We have $a_1=1.$ So, assume that $a_n<2$ for some $n\in\mathbb{N}.$ Then $a_{n+1}=\sqrt{2+a_n}<\sqrt{2+2}=2.$ Therefore, by the Principle of Mathematical Induction, $(a_n)_{n\in\mathbb{N}}$ is bounded above by 2. This implies $a_{n+1}-a_n>0$ for all $n\in\mathbb{N}.$ That is, $(a_n)_{n\in\mathbb{N}}$ is increasing. Consequently, by the Monotone Convergence Theorem, $(a_n)_{n\in\mathbb{N}}$ is convergent.

Now, assume that $\lim\limits_{n\to\infty}a_n=L.$ Then $\lim\limits_{n\to\infty}a_{n+1}=L.$ Hence, $L=\sqrt{2+L}.$ Solving this, we get $L=-1$ or $L=2$. But $a_n>0$ for all $n\in\mathbb{N}$. Therefore, by Theorem 1 of Unit 6, $L\geq 0$. Thus, it follows that $L=2$.

***

Example 6: Show that

$$\lim_{n\to\infty} n^{\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)}=\infty.$$

Solution: Observe that for all $n\in\mathbb{N}$, $\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)>\frac{1}{3}$.

Therefore, for all $n\in\mathbb{N}$,

$$n^{\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)}>n^{\frac{1}{3}}.$$

Now, given any $M>0$, there exists some $n\in\mathbb{N}$ such that $n>M^3$. This implies $n^{\frac{1}{3}}>M$, and hence

$$n^{\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)}>M.$$

This proves that the given sequence diverges to ∞. That is,

$$\lim_{n\to\infty} n^{\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)}=\infty.$$

***

Example 7: Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence, and $(a_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(a_n)_{n\in\mathbb{N}}$. Show

i) $(a_{n_k})_{k\in\mathbb{N}}$ is increasing,

ii) $(a_{n_k})_{k\in\mathbb{N}}$ is unbounded if $(a_n)_{n\in\mathbb{N}}$ is unbounded.

Solution: i) The proof of this is similar to the proof of E23 of Unit 5.

ii) Assume that $(a_n)_{n\in\mathbb{N}}$ is unbounded. If possible, let $(a_{n_k})_{k\in\mathbb{N}}$ be bounded. Then there exists some $M\in\mathbb{N}$ such that $a_{n_k}\leq M$ for all $k\in\mathbb{N}$. We know that $k\leq n_k$ for all $k\in\mathbb{N}$. Since $(a_n)_{n\in\mathbb{N}}$ is increasing, this implies that $a_k\leq a_{n_k}$ for all $k\in\mathbb{N}$. Consequently, $a_k\leq M$ for all $k\in\mathbb{N}$. This is a contradiction to the assumption that $(a_n)_{n\in\mathbb{N}}$ is bounded. Therefore, $(a_{n_k})_{k\in\mathbb{N}}$ is unbounded.

***

Miscellaneous Exercises

E1) For the sequence $(a_n)_{n\in\mathbb{N}}=\left(\cos\frac{n\pi}{2}\right)_{n\in\mathbb{N}}$ answer the following:

i) Does $(a_n)_{n\in\mathbb{N}}$ have a constant subsequence? If so, find one such.

ii) Does $(a_n)_{n\in\mathbb{N}}$ have any increasing subsequence?

iii) Is there any unbounded subsequence of $(a_n)_{n\in\mathbb{N}}$?

E2) Let $a_n = n^{\frac{1}{n}} \sin\frac{(2n-1)\pi}{2}$ for all $n\in\mathbb{N}$. Find two subsequences of $(a_n)_{n\in\mathbb{N}}$ that converge to distinct limits.

E3) Does the sequence in E1 above converge? If yes, what is its limit? If no, for each $L\in\mathbb{R}$, find an $\varepsilon>0$ such that infinitely many terms of $(a_n)_{n\in\mathbb{N}}$ satisfy $|a_n - L| \geq \varepsilon.$

E4) Check whether the following sequences are subsequences of $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ or not.

i) $\left(\frac{2}{k(k+1)}\right)_{k\in\mathbb{N}}$ ii) $\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right)_{k\in\mathbb{N}}$

iii) $\left(\frac{k!+1}{(k+1)!}\right)_{k\in\mathbb{N}}$ iv) $(b_k)_{k\in\mathbb{N}}$, where $b_k = \begin{cases}\frac{1}{k}, & \text{if } k \text{ is prime}\\ \frac{1}{k+1}, & \text{else}\end{cases}$

E5) Let $a_n = \left(1+\frac{1}{n}\right)^n$, where $n\in\mathbb{N}$. Show that

i) $(a_n)_{n\in\mathbb{N}}$ is increasing. ii) $(a_n)_{n\in\mathbb{N}}$ is bounded.

E6) Show that if $(a_n)_{n\in\mathbb{N}}$ is increasing, then $(1-a_n)_{n\in\mathbb{N}}$ is decreasing. Also show that if $(a_n)_{n\in\mathbb{N}}$ is bounded then so is $(1-a_n)_{n\in\mathbb{N}}$.

E7) If $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive terms, then $\left(\frac{a_{n+1}}{a_n}\right)_{n\in\mathbb{N}}$ is a monotone sequence. True or false? Justify your answer.

E8) Let $a_n = \sqrt{n} + (-1)^n n$ for all $n\in\mathbb{N}$.

i) Find a subsequence of $(a_n)_{n\in\mathbb{N}}$ that is bounded above.

ii) Find a subsequence of $(a_n)_{n\in\mathbb{N}}$ that is increasing.

iii) Does there exist a subsequence of $(a_n)_{n\in\mathbb{N}}$ that is increasing and bounded above both?

iv) Is $(a_n)_{n\in\mathbb{N}}$ convergent?

E9) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $|a_{n+1}-a_n| < \frac{1}{2^n}$ for all $n\geq 1$. Show that $(a_n)_{n\in\mathbb{N}}$ is convergent.

E10) Give an example of a sequence $(a_n)_{n\in\mathbb{N}}$ such that $|a_{n+1}-a_n| \leq |a_n - a_{n-1}|$ for all $n\geq 2$. Does such a sequence necessarily converge? Justify.

E11) Let α and β be two real numbers such that $0 < \alpha < \beta$. Show that if $a_n = (\alpha^n + \beta^n)^{\frac{1}{n}}$, then $\lim\limits_{n\to\infty} a_n = \beta$.

E12) Find $\lim\limits_{n\to\infty}\left(\dfrac{n+1}{n!}\right)^{\frac{1}{n}}$.

SOLUTIONS/ANSWERS

E1) i) Observe that if we put $n = 4k+1$, where $k = 0,1,2,\ldots$ in the sequence $(a_n)_{n\in\mathbb{N}}$ then we get the subsequence $(a_{4k+1})_{k\geq 0}$. This subsequence expands as

$$\left(\cos\frac{\pi}{2}, \cos\left(2\pi+\frac{\pi}{2}\right), \cos\left(4\pi+\frac{\pi}{2}\right), \ldots\right) = (0,0,0,\ldots)$$

which is a constant subsequence.

ii) Yes, the constant subsequence we have found in (i) is increasing.

iii) We know that $\left|\cos\frac{n\pi}{2}\right| \leq 1$ for all $n \in \mathbb{N}$. This means $(a_n)_{n\in\mathbb{N}}$ is bounded. Hence, there is no unbounded subsequence of $(a_n)_{n\in\mathbb{N}}$. (Why?)

E2) Note that

$$a_n = n^{\frac{1}{n}} \sin\left(n\pi - \frac{\pi}{2}\right) = n^{\frac{1}{n}}(-\cos n\pi) = (-1)^{n+1} n^{\frac{1}{n}}.$$

Taking n odd or even gives us two subsequences, namely $\left((2k-1)^{\frac{1}{2k-1}}\right)_{k\in\mathbb{N}}$ and $\left(-(2k)^{\frac{1}{2k}}\right)_{k\in\mathbb{N}}$. Since $\left(n^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ converges to 1, the above subsequences converge to 1 and -1, respectively.

E3) The sequence $(a_n)_{n\in\mathbb{N}} = \left(\cos\frac{n\pi}{2}\right)_{n\in\mathbb{N}}$ does not converge because its subsequences $(0,0,0,\ldots)$ and $(1,1,1,\ldots)$ converge to two distinct limits. Now, let $L \in \mathbb{R}$ be arbitrary. If $L \neq 0$, and if n is odd then,

$$|a_n - L| = \left|\cos\frac{n\pi}{2} - L\right| = |L|.$$

So, take $\varepsilon = |L|$ in this case. This gives $|a_{2n-1} - L| \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$. Now assume that $L = 0$. Then take $n = 4k$ for $k \in \mathbb{N}$. So, in this case

$$|a_n - L| = |\cos 2k\pi - L| = |1 - L| = 1.$$

This gives us $\varepsilon = 1$, such that $|a_{4n} - L| \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$.

E4) Let $a_n = \dfrac{1}{n}, \forall n \in \mathbb{N}$

i) We have $b_k = \dfrac{2}{k(k+1)} = \dfrac{1}{k(k+1)/2}$. This gives for each

$k \in \mathbb{N}, n_k = \dfrac{k(k+1)}{2} \in \mathbb{N}$ such that $b_k = a_{n_k}$.

Further, whenever $\mathrm{k} < \ell$, we have

$$\frac{\mathrm{k}(\mathrm{k}+1)}{2} < \frac{\ell(\ell+1)}{2}, \text{ i.e., } n_k < n_\ell.$$

Hence $(b_k)_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

ii) Let

$$b_k = \frac{1}{k!} - \frac{1}{(k+1)!} = \frac{(k+1)! - k!}{k!(k+1)!} = \frac{[(k+1)-1]k!}{k!(k+1)k!}$$
$$= \frac{1}{(k-1)!(k+1)}.$$

Thus, for each $k \in \mathbb{N}$, we have $n_k = (k-1)!(k+1)$. Now let us check whether or not $k < \ell$ implies $n_k < n_\ell$. Let $k < \ell$. Then $(k-1)! < (\ell-1)!$ and $(k+1) < (\ell+1)$. Hence $(k-1)!(k+1) < (\ell-1)!(\ell+1)$, which means $n_k < n_\ell$.

Thus $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers. Consequently, $(b_k)_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

iii) Let $b_k = \dfrac{k!+1}{(k+1)!}$. If we expand this we get

$$\left(\frac{2}{2}, \frac{3}{6}, \frac{7}{4!}, \cdots\right), \text{ i.e., } \left(1, \frac{1}{2}, \frac{7}{24}, \cdots\right)$$

Note that 1 is not a prime

You can observe that the term $\dfrac{7}{24}$ belongs to $(b_k)_{k\in\mathbb{N}}$. But there is no such term in $(a_n)_{n\in\mathbb{N}}$. Hence $(b_k)_{k\in\mathbb{N}}$ is not a subsequence of $(a_n)_{n\in\mathbb{N}}$.

iv) If we expand $(b_k)_{k\in\mathbb{N}}$, we get $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{5}, \dfrac{1}{7}, \cdots\right)$. This shows that the terms $\dfrac{1}{2}$ is repeated in $(b_k)_{k\in\mathbb{N}}$, whereas it appears only once in $(a_n)_{n\in\mathbb{N}}$. Hence $(b_k)_{k\in\mathbb{N}}$ is not a subsequence of $(a_n)_{n\in\mathbb{N}}$.

E5) i) Using the Binomial Theorem, we get

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \text{ and } a_{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{(n+1)^k}.$$

Now

$$a_{n+1} - a_n = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{(n+1)^k} - \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

$$= \sum_{k=0}^{n}\left[\binom{n+1}{k}\frac{1}{(n+1)^k} - \binom{n}{k}\frac{1}{n^k}\right] + \binom{n+1}{n+1}\frac{1}{(n+1)^{n+1}}$$

$$> \sum_{k=0}^{n}\left[\binom{n+1}{k}\frac{1}{(n+1)^k} - \binom{n}{k}\frac{1}{n^k}\right]$$

Now you can see that

$$\binom{n+1}{k}\frac{1}{(n+1)^k} = \frac{(n+1)(n+1-1)\cdots(n+1-(k-1))}{k!(n+1)^k}$$

$$= \frac{1}{k!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right) \qquad \text{... (1)}$$

And

$$\binom{n}{k}\frac{1}{n^k} = \frac{n.(n-1)(n-2)\cdots(n-(k-1))}{k!n^k}$$

$$= \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \qquad \text{... (2)}$$

Since you know that

$$1-\frac{i}{n+1} > 1-\frac{i}{n}, \text{ for } 1 \le i \le k-1,$$

the right hand side of Eq. (1) is greater than the right hand side of Eq. (2). Hence

$$\binom{n+1}{k}\frac{1}{(n+1)^k} > \binom{n}{k}\frac{1}{n^k},$$

for all $0 \le k \le n$. This implies $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. That is, $(a_n)_{n\in\mathbb{N}}$ is increasing.

ii) To see the boundedness, note that

$1-\frac{i}{n} < 1, \ \forall\ 1 \le i \le n.$

Hence

$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) < \frac{1}{k!}$$

Then $a_n < \sum_{k=0}^{n}\frac{1}{k!} = 1+\sum_{k=1}^{n}\frac{1}{k!}$

$$< 1+\sum_{k=1}^{n}\frac{1}{2^{k-1}} \qquad (\because k! > 2^{k-1}, \forall k \in \mathbb{N})$$

$$= 1+\frac{1}{1-\frac{1}{2}} = 3, \ \forall\ n \in \mathbb{N}.$$

Thus, $(a_n)_{n\in\mathbb{N}}$ is bounded.

E6) If $(a_n)_{n\in\mathbb{N}}$ is increasing then $a_n \le a_{n+1}, \ \forall n \in \mathbb{N}$. This implies $1-a_n \ge 1-a_{n+1}, \forall n \in \mathbb{N}$. Thus $(1-a_n)_{n\in\mathbb{N}}$ is decreasing. Now $(a_n)_{n\in\mathbb{N}}$ is bounded above means there exists some $u \in \mathbb{R}$ such that $a_n \le u,\ \forall n \in \mathbb{N}$. This implies $1-a_n \ge 1-u,\ \forall n \in \mathbb{N}$. Hence $(1-a_n)_{n\in\mathbb{N}}$ is bounded below.

E7) The statement is false. As a counter-example, consider the sequence $(a_n)_{n\in\mathbb{N}} = (1,2,1,2,1,2,\ldots)$ which has all the terms positive. However, $\left(\frac{a_{n+1}}{a_n}\right)_{n\in\mathbb{N}} = \left(2, \frac{1}{2}, 2, \frac{1}{2}, \cdots\right)$ which is not monotone.

E8) i) Let $b_k = \sqrt{(2k-1)} - (2k-1)$ where $k \in \mathbb{N}$. Then we have $b_k = a_{2k-1}$ for each $k \in \mathbb{N}$. Also, since $(2k-1)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, it follows that $(b_k)_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$. In order to show that $(b_k)_{k\in\mathbb{N}}$ is bounded above, note that for all $k \in \mathbb{N}$, $2k-1 \le (2k-1)^2$, which implies $\sqrt{2k-1} \le 2k-1$. That is, $b_k \le 0$ for all $k \in \mathbb{N}$.

Therefore, $(b_k)_{k\in\mathbb{N}}$ is bounded above.

ii) Let $b_k = \sqrt{2k} + 2k$ for all $k \in \mathbb{N}$. It is easy to see that $b_k = a_{2k}$ for all $k \in \mathbb{N}$. Since $(2k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, it follows that $(b_k)_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$. Now you can show that $(b_k)_{k\in\mathbb{N}}$ is increasing.

iii) You can prove that $(a_{2n-1})_{n\in\mathbb{N}}$ is decreasing, whereas $(a_{2n})_{n\in\mathbb{N}}$ is increasing. Thus any increasing subsequence of $(a_n)_{n\in\mathbb{N}}$ must be a subsequence of $(a_{2n})_{n\in\mathbb{N}}$. Note that $(a_{2n})_{n\in\mathbb{N}}$ is unbounded. Therefore, every subsequence of $(a_{2n})_{n\in\mathbb{N}}$ is unbounded. (See Example 7.) Consequently, $(a_n)_{n\in\mathbb{N}}$ has no subsequence that is increasing and bounded above both.

iv) The fact that $(a_{2n})_{n\in\mathbb{N}}$ is unbounded implies that $(a_n)_{n\in\mathbb{N}}$ is also unbounded. Therefore, $(a_n)_{n\in\mathbb{N}}$ is not convergent.

E9) See Example 4.

E10) Define a sequence $(a_n)_{n\in\mathbb{N}}$ by

$$a_n = \begin{cases} 2+\dfrac{1}{n}, & \text{when } n \text{ is odd} \\ 1-\dfrac{1}{n}, & \text{when } n \text{ is even.} \end{cases}$$

First let us assume that n is odd. Then

$$a_{n+1} - a_n = \left(1-\frac{1}{n+1}\right) - \left(2+\frac{1}{n}\right) = -1-\frac{1}{n}-\frac{1}{n+1}$$

and

$$a_n - a_{n-1} = \left(2+\frac{1}{n}\right) - \left(1-\frac{1}{n-1}\right) = 1+\frac{1}{n}+\frac{1}{n-1}.$$

So,

$$|a_{n+1} - a_n| = 1 + \frac{1}{n} + \frac{1}{n+1} < 1 + \frac{1}{n} + \frac{1}{n-1} = |a_n - a_{n-1}|.$$

Therefore, $|a_{n+1} - a_n| \le |a_n - a_{n-1}|$.

Now let us take n to be even. Then

$$a_{n+1} - a_n = \left(2 + \frac{1}{n+1}\right) - \left(1 - \frac{1}{n}\right) = 1 + \frac{1}{n+1} + \frac{1}{n}$$

and

$$a_n - a_{n-1} = \left(1 - \frac{1}{n}\right) - \left(2 + \frac{1}{n-1}\right) = -1 - \frac{1}{n} - \frac{1}{n-1}.$$

So, in this case we get

$$|a_{n+1} - a_n| = \left|1 + \frac{1}{n+1} + \frac{1}{n}\right| = 1 + \frac{1}{n+1} + \frac{1}{n} < 1 + \frac{1}{n} + \frac{1}{n-1}$$
$$= |a_n - a_{n-1}|.$$

This shows that $|a_{n+1} - a_n| \le |a_n - a_{n-1}|$ for all $n \in \mathbb{N}$. Now let us check whether $(a_n)_{n\in\mathbb{N}}$ is convergent or not. Note that the odd and even subsequence of $(a_n)_{n\in\mathbb{N}}$ are $\left(2 + \frac{1}{2n-1}\right)_{n\in\mathbb{N}}$ and $\left(1 - \frac{1}{2n}\right)_{n\in\mathbb{N}}$, respectively. You can see that both these subsequences do not converge to the same limit. Therefore, by Theorem 6 of Unit 5 $(a_n)_{n\in\mathbb{N}}$ is not convergent.

Thus we conclude that any sequence $(a_n)_{n\in\mathbb{N}}$ that satisfies $|a_{n+1} - a_n| \le |a_n - a_{n-1}|$ for all $n \in \mathbb{N}$, need not be convergent.

E11) Since $0 < \alpha < \beta$, we have $\alpha^n < \beta^n$. Now $a_n^n = \alpha^n + \beta^n < 2\beta^n$, which implies $a_n < 2^{\frac{1}{n}} \beta$ for all $n \in \mathbb{N}$. Also $\alpha > 0$ implies $\alpha^n > 0$. Therefore, $a_n^n = \alpha^n + \beta^n > \beta^n$, which means $a_n > \beta$. Thus we have $\beta < a_n < 2^{\frac{1}{n}} \beta$ for all $n \in \mathbb{N}$. Since $2^{\frac{1}{n}} \beta \to \beta$ as $n \to \infty$, by the Squeeze Theorem, we get $\lim\limits_{n\to\infty} a_n = \beta$.

E12) Let $a_n = \dfrac{n+1}{n!}$ for all $n \in \mathbb{N}$. Then $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive terms. Now

$$\frac{a_{n+1}}{a_n} = \frac{n+2}{(n+1)!} \cdot \frac{n!}{n+1} = \frac{n+2}{(n+1)^2}.$$

It can be shown that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 0$. Therefore, by the Cauchy's Second Theorem on Limits, we get $\lim\limits_{n\to\infty} a_n^{\frac{1}{n}} = \lim\limits_{n\to\infty} \left(\dfrac{n+1}{n!}\right)^{\frac{1}{n}} = 0$.