Indira Gandhi National Open University
School of Sciences

BMTC-133
REAL ANALYSIS

Block

3

Infinite Series

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BLOCK INTRODUCTION

In the previous block you studied the notion of sequences and their limits. In this block we familiarize you to the notion of an infinite series. You will see that the concept of limit of a sequence in the foundation for the study of an inifinite series.

Infinite series occur naturally while considering the decimal expansion of numbers. For instance, the decimal expansion of a rational number, say $\frac{1}{3}$, was represented by

$$\frac{1}{3} = 0.333... = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + ...$$

even when the meaning of the sum of infinite number of terms was not established. Infinite series found a place in some the earlier works of mathematician Archimedes (288 BC – 212 BC). Later many other mathematicians worked on infinite series; B. Taylor formed Taylor series, Binomial series by L. Euler and so on. But they could not establish a precise meaning of an infinite sum. Later in 19th century with much rigorous definition of limit and continuity and differentiability, the mathematician A. Cauchy formulated the precise definition of sum of an infinite series that we use today.

In this block we define an infinite series which gives a precise meaning for an infinite sum. Then we explain its convergence and divergence. The whole content of this block is divided into 3 units. In Unit 7 we introduce the notion of infinite series and its convergence. Some general tests for convergence are also covered in this unit. There are many special tests for checking the convergence for the series with positive term (D' Alembert's ratio test, Cauchy root test, Raabe's test and Gauss's test) which are covered in Unit 8. Later in Unit 9 we discuss infinite series whose terms an alternatively positive and negative. We introduce two types of convergence for such series – absolute convergence and conditional convergence. Finally we discuss the rearrangement of terms of an alternating series. You will learn that the rearrangement can result in a series which entirely changes the nature of convergence or divergence of the original series.

Notations and Symbols (used in Block 3 apart from Block 1 and 2)

$a_1 + a_2 + a_3 + \ldots$	An infinite series in summation notation
$\sum_{n=1}^{\infty} a_n$	An infinite series in 'sigma' notation
a_n	nth term of an infinite series
$s_n = \sum_{k=1}^{n} a_k$	nth partial sum
(s_n)	sequence of partial sums

UNIT 7

CONVERGENCE OF SERIES

Structure

7.1 INTRODUCTION

We come across infinite series long before we have heard of them, formally. For instance, the fractions $\frac{1}{3}$ when represented in the decimal base 10 is

$$\frac{1}{3} = 0.333.... = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + ...$$

So is the following representation of $\sqrt{2}$:

$$\sqrt{2} = 1.414213... = 1 + \frac{4}{10} + \frac{1}{10^2} + \frac{4}{10^3} + \frac{2}{10^4} + ...$$

The end expressions on the L.H.S are examples of the so called "infinite series" which we shall define in this unit.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ...$$

Most functions such as $\sin x, \cos x, e^x$ can be represented as a series (see margin). This is an advantage in may physical applications where we may have to calculate very small values of such functions. Infinite series is very useful for finding solutions of differential equations which are models of many real-life problems.

The series which you might be most familiar is the geometric series. Taylor series which you have studied in the calculus course is an important class of series which has got lot of application in many areas of science. Binomial series is another type of series which you might be familiar from your school mathematics.

In Sec. 7.2 we shall first explain the meaning of an "infinite series". You will learn that an infinite series is nothing but a sequence each of whose term is sum of a finite number of its terms of a sequence in order. These sums are called partial sums. When we terminate the decimal somewhere as a matter of practical necessity, we drop all the terms in the sum on the right hand side from certain term onwards, the resulting approximation is an example of what is called the partial sum of the infinite series. This identification in terms of partial sums helps to apply many important theorems on sequences to the study of infinite series.

In the next section Sec. 7.3 we study the convergence of an infinite series by considering the corresponding results for the associated sequence of partial sums. We shall state and prove some general tests for convergence. In this connection we have discussed certain series such as harmonic series which are not convergent. We have termed them as "divergent" series. There are many examples and exercises which give enough practice to check if an infinite series is convergent or not. Lastly we have established the connection between infinite series and decimal expansion of fractions.

Objectives

After studying this unit you should be able to:

- define an infinite series as a sequence of partial sums of an infinite sequence associated with it;
- find the partial sum associated to a given infinite series;
- represent an infinite series in summation notation and sigma notation;
- define the convergence and divergence of an infinite series;
- state and use a necessary test for convergence of an infinite series;
- state and use a necessary and sufficient condition for the convergence of an infinite series whose terms are non-negative real numbers;
- state and use a sufficient condition for divergence of an infinite series;
- find the sum and difference of two convergent series and also the product under scalar multiplication of a convergent series;
- write the fraction corresponding to a decimal number whose decimal expansion is infinite and recurring using the notion of convergence of infinite series.

7.2 PRELIMINARIES TO INFINITE SERIES

In this section we shall introduce you to the notion of 'an infinite series'. You are already familiar with the term 'series' from your school mathematics. You might have studied the series known as geometric series, arithmetic series, and harmonic series.

Some of the examples of these series are as follows:

$1+2+3+....$ (An arithmatic series) ... (1)

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...$ (A geometric series) ... (2)

$1+\frac{1}{2}+\frac{1}{3}+....$ (A harmonic series) ... (3)

In all the examples above, you can notice that the numbers are written as sum, followed by, three dots where the dots indicate that the summation is to be extended. You might have noticed that the terms of the series above are respectively elements of the sequence given by

$$(1, 2, 3, ...,) \qquad ... (4)$$

$$\left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\right) \qquad ... (5)$$

$$\left(1, \frac{1}{2}, \frac{1}{3}, ...\right) \qquad ... (6)$$

We shall use these examples to illustrate the basic concept of an infinite series.

Let us consider a sequence $(a_n)_{n\in\mathbb{N}}$. We form an expression of the form

$$a_1 + a_2 + a_3 + ... + a_n + ... \qquad ... (7)$$

The expression in (7) is called an infinite series. The numbers $a_1, a_2, a_3....$are called the terms of the series. For example, the expressions in (1), (2) and (3) are examples of an infinite series.

You may note that it is not necessary that the terms of a series are all positive numbers as shown in (1) (2) and (3). We can have series generated by the sequence $(1, -1, 1, -1....)$ which is of the form
$1 + (-1) + 1 + (-1) + ... = 1 - 1 + 1 - 1 + 1....$

Since we know that it is impossible to add infinitely many terms, then the question arises what do these expressions given by (1), (2) or (3) mean? Can we attach any value to be called as the sum of these infinite numbers? As an attempt to do this we shall construct certain sequences corresponding to an infinite series.

Let us, for instance, construct the sequence corresponding to the series in (1). We put

$$s_1 = 1 = a_1$$
$$s_2 = 1 + 2 = a_1 + a_2 = 3$$
$$s_3 = 1 + 2 + 3 = a_1 + a_2 + a_3 = 6$$
$$s_4 = 1 + 2 + 3 + 4 = a_1 + a_2 + a_3 + a_4 = 10$$

That is, s_1 is the first term and s_2 is the sum of the first two terms, s_3 is the sum of the first three terms and so on. Then we get the sequence $(1, 3, 610, 15, ...)$. This sequence is called the sequence of 'partial sums' associated with the series in (1) and is denoted by sequence $(s_n)_n$.

Similarly you can find the sequence of partial sums of series in (2) and (3).

The above discussion leads to the following definition.

Definition 1: Let $(a_n)_n$ be a sequence of real numbers with the associated infinite series given by $a_1 + a_2 + a_3 + ...$

We define

$$
\begin{aligned}
&s_1 = a_1 \\
&s_2 = a_1 + a_2 = s_1 + a_2 \\
&s_3 = a_1 + a_2 + a_3 = s_2 + a_3 \\
&\vdots \\
&s_k = a_1 + a_2 + a_3 + \ldots + a_k = s_{k-1} + a_k
\end{aligned} \qquad \ldots (8)
$$

(like this we continue)

Then the sequence (s_n), so obtained is called as the **sequence of partial sums** of the series.

Then, s_1 is called the first partial sum, s_2 is the second partial sum and s_3 is the third partial sum and so on.

This means that given an infinite series we can construct a sequence called sequence of partial sums.

On the other hand, suppose we know that the sequence of partial sums of a series is the sequence (s_n). How do we find the corresponding series? Let us see. From Eqn. (8) we get that

$$
\begin{aligned}
&a_1 = s_1 \\
&a_2 = s_2 - s_1 \\
&a_3 = s_3 - s_2 \\
&\vdots \quad \vdots \quad \vdots \\
&a_k = s_{k+1} - s_k
\end{aligned} \qquad \ldots (9)
$$

This shows that the terms of the series are given by the difference of the terms of the sequence of partial sums in order. This shows that a series is completely determined by its sequence of partial sums.

We next make a formal definition.

Definition: Given a real sequence (a_n) construct a sequence (s_n) as follows:

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n, \ \forall n, \qquad \ldots (10)$$

i.e. s_n is the sum of the first n terms of (a_n). The sequence (s_n) which is the sequence of partial sums is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \ldots \qquad \ldots (11)$$

Before we go further we introduce some notations.

Notations: You are familiar with sigma "Σ" notation from semester 1 Calculus course when you studied Taylor series expansion, binomial expansion etc.

Recall that the sum $a_1 + a_2 + a_3 + ... + a_n$ is denoted by $\sum_{i=1}^{n} a_i$ which is read as *"sigma a_i, i varies from 1 to n"*. For instance $\sum_{n=10}^{20} \frac{1}{n}$ *which we read as "sigma $\frac{1}{n}$, n varies from 10 to 20"* is the sigma notation for the sum

$$\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+...+\frac{1}{20}$$

We extend this idea and denote an infinite series expressed by (11) in sigma notation by

$$\sum_{n=1}^{\infty} a_n \qquad \text{... (12)}$$

which is read as "*sigma a_i, i runs from to 1 to infinity*". The symbol infinity stands for the fact that the series continue futher indefinitely.

In some cases we simple write $\sum a_n$ without mentioning $n=1$to ∞. In such cases it is assumed the summation is from 1 to ∞.

Another point to be noted is that it is not always necessary to start with 1, it can be any other whole number. For example, we can have $\sum_{n=0}^{\infty} a_n$ or $\sum_{n=867}^{\infty} a_n$ etc.

We use the notation a_n for the general term of an infinite series, and call it nth **term**.

Thus there are two notations commonly used to denote an infinite series. The expression given by Eqn. (11) is known as "**summation notation**" and other given by Eqn. (12) is known as "**sigma** (Σ) **notation**". The $'a_i$ in Σ' notation is called the general term. Here you may note that to express in Σ notation we should know the general term of the series. For example, if we want to express the series in Eqn. (1), (2) and (3) in Σ notation we should know the general terms of the series given by Eqn. $(1),(2)$ and (3) ? You might have got that they are given respectively by $n, \frac{1}{2^n}$ and $\frac{1}{n}$ and therefore the $\Sigma-$notation of these three series is respectively given by

$$\sum_{n=1}^{\infty} n, \quad \sum_{n=1}^{\infty} \frac{1}{2^n}, \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

You must have got some practise for writing the nth term of a sequence from Block 2.

Let us see some examples now to familiarise with the notation.

Example 1: Consider the following series in summation notation. Write these series in $'\Sigma'$ notation.

i) $1-1+1-1+...$

ii) $1+\frac{1}{2^5}+\frac{1}{3^5}+...$

iii) $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}$

iii) $1-\frac{1}{2}+\frac{1}{3}+...$

Solution: Let us try one by one

i) We first rewrite the sum as $1+(-1)+1+(-1)+...$ The terms are alternatively $+1$ and (-1). Then the nth term of the series is $(-1)^{n+1}$ and the $'\Sigma'$ notation is $\sum_{n=1}^{\infty}(-1)^{n+1}$.

ii) The nth term is $\frac{1}{n^5}$. Therefore the $\Sigma-$notation is $\sum_{n=1}^{\infty}\frac{1}{n^5}$.

iii) By observing the pattern of the terms we get that the nth term is $\frac{1}{n(n+1)}$. Therefore, the Σ notation is given by

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

iv) The terms are alternatively positive and negative. Therefore the nth term $(-1)^{n+1}\frac{1}{n}$. Hence the notation is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$.

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Example 2: Write the following series in summation notation

$$\sum_{n=1}^{\infty}\frac{n+4}{2n^3-n+1}$$

Solution: The nth term is $a_n=\frac{n+4}{2n^3-n+1}$. To get a_1, we put $n=1$, then $a_1=\frac{5}{2}$. Similarly $a_2=\frac{6}{15}, a_3=\frac{7}{40}$.

Therefore the summation notation is

$$\frac{5}{2}+\frac{6}{15}+\frac{7}{40}+....$$

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To get more practise with the notations you can try the following exercises.

E1) Find the nth term of the following infinite series and write in $\Sigma-$notation.

i) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+...$

ii) $\frac{3}{1}+\frac{7}{3}+\frac{13}{7}+...$

E2) Write the following series in summation notation.

i) $\sum\frac{n^2\,2^n}{n!}$

ii) $\sum_{n=1}^{\infty} \frac{(n+1)^2 - n}{(n-1)^2 + n}$

Now that you have become familiar with the different notations of an infinite series, let us now go back to the definition of partial sums associated with an infinite series.

For example, let us consider the partial sums of the series given by

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

By definition the partial sums are

$$s_1 = 1$$
$$s_2 = 1 + \frac{1}{2} = \frac{3}{2}$$
$$s_3 = 1 + \frac{1}{2} + \frac{1}{2^2} = \frac{7}{4}$$

Then, $s_n = s_{n-1} + a_n = 1 + \frac{1}{2} + \frac{1}{4} + .. + \frac{1}{2^{n-1}}$. You may note that the series is a geometric series with the first term 1 and the common ratio $\frac{1}{2}$ which is less than 1. Also there are (n) terms. Therefore from the formula of the sum upto n terms of a geometric series, we have

$$s_n = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{2^n}}{\frac{1}{2}} = 2 - \frac{1}{2^{n-1}}$$

Therefore the sequence of partial sums is $\left(1, \frac{3}{2}, \frac{7}{4}, \ldots 2 - \frac{1}{2^{n-1}}, \ldots\right)$.

Let us now consider another series $1 - 1 + 1 - 1 + \ldots$ and find the partial sums.

Here you note $s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 0$ and so on.

Then we have $s_n = \begin{cases} 1, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even} \end{cases}$

Therefore the sequence of partial sums is $(1, 0, 1, 0, \ldots)$.

In the cases above, it was not difficult to calculate the partial sums. But this is not always the case.
Let us see some more examples.

Example 3: Show that the sequence of partial sums of the series whose nth term is $\frac{1}{n(n+1)}$ is $\left(\frac{n}{n+1}\right)$.

Solution: The given series is $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.

To find the sequence of partial sums we try two methods.

Method 1: We rewrite the nth term $a_n = \frac{1}{n(n+1)}$ as

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\therefore s_n = a_1 + a_2 + \ldots + a_n$$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$= 1-\frac{1}{n+1}$ (all the other terms get cancelled)

$$= \frac{n}{n+1}$$

Therefore the sequence of partial sums is $\left(\frac{n}{n+1}\right)_n$.

Method 2 (Using Mathematical Induction): We first note that the series can be written in the summation form as

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{4\times 3}+\ldots.$$

$$\therefore s_n = \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{4\times 3}+\ldots+\frac{1}{n(n+1)}$$

We have to show that $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{4}\times\frac{1}{3}+\ldots+\frac{1}{n(n+1)} = \frac{n}{(n+1)}$ ... (13)

To show this we use the principle of Mathematical Induction on the number of terms.

We first note that the result is true for $n=1$. L.H.S = R.H.S if we put $n=1$ in Eqn. (13).

Assuming that the result is true for $n=k$, we have to show that the result is true for $n=k+1$.

Since the result is true for $n=k,$ Eqn. (13) implies that

$$\frac{1}{1\times 2}+\ldots+\frac{1}{k(k+1)} = \frac{k}{k+1} \quad \ldots (14)$$

Adding $\frac{1}{(k+1)(k+2)}$ to both the sides of the Eqn. (14), we get

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} = \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)}{(k+2)}$$

This shows that the result is true for $k+1$. Therefore, by the principle of Mathematical Induciton, Eqn. (13) holds for all $n \in \mathbb{N}$.

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Remark 1: In the example above you must have noticed that there are different methods of proof and the Method 2 is much longer. Therefore we prefer Method 1 but in certain cases we may have to use the method of mathematical induction.

Example 4: Write each of the following infinite series in Σ-notation and find the partial sum s_n.

i) $2+4+8+\ldots$

ii) $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots$

iii) $1+3+5+\ldots$

Solution: Let us try one by one.

i) The given series is a geometric series with the first term 2 and common ratio 2. Therefore, the nth term is $a_n = 2^n$. The series is written in Σ-notation by $\sum_{n=1}^{\infty} 2^n$.

The nth partial sum is given by

$s_n = a_1 + a_2 + \ldots. + a_n$.

Here $a_1 = 2, a_2 = 2^2, a_3 = 2^3, \ldots a_n = 2^n$.

Therefore $s_n = 2 + 2^2 + \ldots + 2^n$

$= 2\frac{(2^n - 1)}{2-1}$ (the sum upto n term of a geometric series with the first 2 and common ratio 2.)

$= 2(2^n - 1)$

or we can directly calculate that and get that for $n=1, s_1 = 2$ and for $n=2, s_2 = 6$ and so on.

ii) The series is a geometric series with the first term $\frac{1}{3}$ and common ratio $\frac{1}{3}$.

Therefore the nth term is $a_n = \frac{1}{3^n}$.

The series in Σ-notation is given by $\sum_{n=1}^{\infty} \frac{1}{3^n}$.

The nth partial sum is given by $s_n = a_1 + a_2 + a_3 + ... + a_n$

Here $a_1 = \frac{1}{3}, a_2 = \frac{1}{3^2}, a_3 = \frac{1}{3^3} + ... + a_n = \frac{1}{3^n}$

$$\therefore s_n = \frac{1}{3} + \frac{1}{3^2} + ... + \frac{1}{3^n}$$

$$= \frac{1}{3} \frac{\left(1 - \left(\frac{1}{3}\right)^n\right)}{1 - \frac{1}{3}} = \frac{1}{2}\left[1 - \left(\frac{1}{3}\right)^n\right]$$

iii) The series is an arithmetic series with the first term 1 and common difference 2. The formula for the nth term is $a_n = a + (n-1)d$. Then the nth term of the given series is $a_n = 1 + (n-1)2 = 2n - 1$.

Therefore the series is $\sum_{n=1}^{\infty}(2n-1)$.

The nth partial sum s_n is given by $s_n = a_1 + a_2 + ... + a_n$.

Here $a_1 = 1, a_2 = 3, a_3 = 5, ..., a_n = 2n - 1$.

$$\therefore s_n = 1 + 3 + 5 + ... + (2n-1) = \frac{n(2n-1)}{2}.$$

Using the formula for summing an arithmetic progression with the first term 1 and common difference d is

$$s_n = \frac{n}{2}[2 + (n-1)2]$$
$$= \frac{n}{2}[2 + 2n - 2] = \frac{n \times 2n}{2} = n^2$$

i.e we have $s_1 = 1, s_2 = 4, s_3 = 9, ...$

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To get more practises, you can try some exercises.

E3) Find the n^{th} partial sums of the following series

i) $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$ ii) $1 + \frac{5}{8} + \left(\frac{5}{8}\right)^2 + ...$

E4) Can you find a series whose all terms are positive and the nth partial sum is $\frac{1}{n}$? Why? Why not?

So far we have learnt that an infinite series can be identified with a sequence of its partial sums. Does this help us to attach a particular value which can be considered as the sum of infinitely many terms? In the next section we address this question by definining the notion of convergence of an infinite series.

7.3 CONVERGENCE OF AN INFINITE SERIES

In this section we shall discuss the notion of convergence of an infinite series and discuss some general tests for convergence.

Let us start with the most familiar series called geometric series given by

$$1+a+a^2+...+a^{n-1}+...$$

The sum upto n terms of this series is given by

$$s_n = 1+a+a^2+...+a^{n-1} = \frac{1-a^n}{1-a}, |a| \neq 1$$

$$= \frac{1}{1-a} - \frac{a^n}{1-a}$$

We know that $\lim_{n\to\infty} a^n = 0,$ when $|a|<1.$

So, $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{1-a}\right) + \lim_{n\to\infty}\frac{a^n}{1-n} = \frac{1}{1-a}.$

Thus we write $\lim_{n\to\infty} s_n = \frac{1}{1-a}$ when $|a|<1$. This implies that the sequence of partial sums is convergent. This is the same as saying that

$1+a+a^2+...+a^{n-1} \to \frac{1}{1-a}$ as $n\to\infty$ if $|a|<1$.

We describe this limit by writing the series as an infinite series on the L.H.S and the limit on R.H.S.

$$1+a+a^2+... = \frac{1}{1-a} \text{ if } |a|<1.$$

The three dots are important, they indicate that the pattern continuous without end.

For any given $a, |a|<1,$ the value $\frac{1}{1-a}$ is called the sum of the infinite series.

For example if $a = \frac{1}{2}$, then we have

$$1+\frac{1}{2}+\frac{1}{2^2}.... = \frac{1}{1-\frac{1}{2}} = 2.$$

We write this as $\sum_{n=0}^{\infty}\frac{1}{2^n} = 2$. In this case we also say that the infinite series converges to the limit '2'.

The following figure explains that the geometric series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...$ converges to 1.

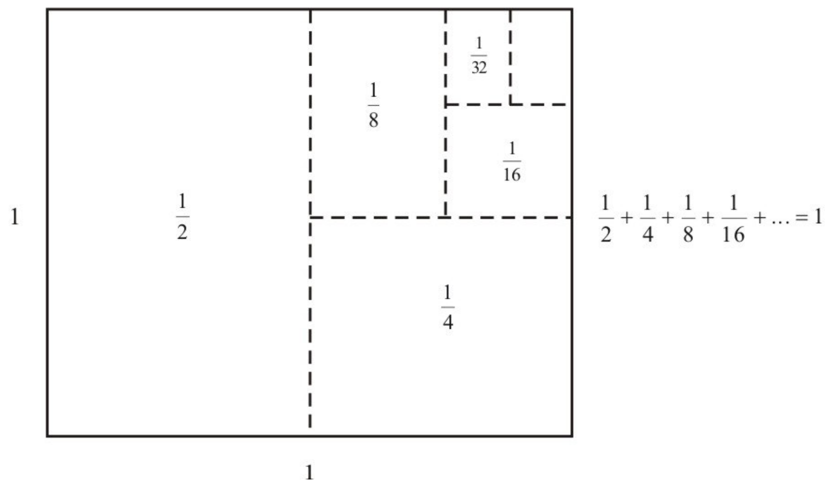

Fig. 1: Convergence of the geometric series

Let us consider another example. Consider the series whose nth term is $(-1)^{n+1}$. We note here that the sequence of the sequence $(s_n)_n$ of its partial sum is given by

$$s_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

i.e. we have $s_1 = 1, s_2 = 1-1 = 0, s_3 = 1-1+1 = 1$ etc.

Recall from Unit 5 of the Block 2 that the sequence (s_n) diverges. In this case we say that the series $\sum_{n=0}^{\infty}(-1)^n$ is divergent.

These examples show that the behaviour of an infinite series can be studied by the sequence of its partial sums.

We have the following definition.

Definition 2: An infinite series $\sum_{n=1}^{\infty} a_n$ is said to be **convergent** if its sequence of partial sums (s_n) converges to a limit S. The limit S is called the sum of the infinite series and we write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... = S \qquad ... (15)$$

In case the sequence of partial sums diverges, we say that the infinite series is **divergent**.

Remark: You may note that in many problems it is convenient to begin indexing an infinite series at 0 instead of 1. We then start with a_0 and write.

$$a_0 + a_1 + a_2 + ... + a_n + ... = \sum_{n=0}^{\infty} a_n$$

In this setting the partial sums are $s_0 = a_0 s_1 = a_0 + a_1, \ldots$ etc.

Before we consider some examples we shall discuss some important results.

Theorem 1: The infinite series $\sum a_n$ converges implies that $\lim\limits_{n\to\infty} a_n = 0$. That means, if an infinite series converges, then its nth term $a_n \to 0$ as $n \to \infty$.

Proof: To prove this, we first note that the series $\sum a_n$ converges means that the sequence of partial sums $(s_n)_n$ converges. Let the limit be S. We can write the nth term a_n as

$$\begin{aligned} a_n &= s_n - s_{n-1} \\ \therefore \lim_{n\to\infty} a_n &= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \\ &= S - S = 0 \end{aligned}$$

Thus $a_n \to 0$ as $n \to \infty$.

Hence the result. ■

You might be thinking whether the converse of the theorem above is true or not. Note that Theorem 1 says that the condition $a_n \to 0$ as $n \to \infty$ is necessary for the convergence of the series $\sum a_n$. Is the condition sufficient? We show that it is not. That means if $\lim\limits_{n\to\infty} a_n = 0$ for a series $\sum a_n$, then the series may not converge. The following example illustrates this.

Example 5: The harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, even though the nth term $\frac{1}{n} \to 0$ as $n \to \infty$.

Solution: The given series is

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

The nth term of the series is $a_n = \frac{1}{n} \to 0$ as $n \to \infty$.

We have to show that the series diverges. That is the sequence of partial sums diverges. To show that, we proceed as follows:

We first consider the sum of the first four terms of the series given by

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

Then we have

$$s_{2^2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{2}{4} = 2 \qquad \ldots (16)$$

Similarly we cosider that the sum of the next eight terms of the series and observe that

$$s_8 = \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)$$
$$> \left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)=2\frac{1}{2}$$

This shows that $s_8 > 2\frac{1}{2} = 1+3\left(\frac{1}{2}\right)$.

We continue like this and consider

$$s_{2^k} = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+....+\frac{1}{8}\right)+...+\left(\frac{1}{2^{k-1}+1}+...+\frac{1}{2^k}\right)$$
$$>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+...+\frac{1}{8}\right)+...+\left(\frac{1}{2^k}+...+\frac{1}{2^k}\right)$$

$$=1+\frac{1}{2}+2\left(\frac{1}{4}\right)+4\left(\frac{1}{8}\right)+...+2^{k-1}\left(\frac{1}{2^k}\right)$$
$$=1+\underbrace{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+...+\frac{1}{2}}_{k \text{ terms}}$$
$$=1+k\left(\frac{1}{2}\right),$$

This shows that sequence (s_n) is unbounded. This together with the fact that the sequence of partial sums is an increasing sequence shows that (s_n) is divergent (See E9 of Unit 6). Hence the series is divergent.

***

The Theorem 1 infact shows that the terms of a convergent series are ultimately very small.

Here is a corollary to Theorem 1 which is very useful in checking the divergence of a series.

Corollary 1(Sufficient Condition for Divergence): If the sequence $\{a_n\}$ either fails to have a limit or has a limit different from 0 as $n \to \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Let us see an example.

Example 6: Show that the following series are divergent.

i) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+...$

ii) $-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+...$

Solution: i) Here we note that $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots=\sum_{n=1}^{\infty}\frac{n}{n+1}$

We observe that $a_n=\frac{n}{n+1}$ and $a_n \to 1$ as $n\to\infty$

Therefore by corollary 1 the series diverges.

ii) Similarly we note that $-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^n n}{n+1}$

Here $a_n=\frac{(-1)^n n}{n+1}$. Then $a_{2n}\to 1$ and $a_{2n-1}\to -1$ as $n\to\infty$.

Therefore by corollary 1 the series diverges.

***

Now we shall state a necessary and sufficient condition for convergence of an infinite series.

Theorem 2: The series $\sum a_n$ converges if and only if for every $\varepsilon>0$ there exists $M\in\mathbb{N}$, depending on ε, such that if $m>n>M$, then

$$|s_m-s_n|=|a_{n+1}+a_{n+2}+\ldots+a_m|<\varepsilon. \quad \blacksquare$$

The proof of this theorem follows directly from the Cauchy criterion for convergence of a sequence. [SeeTheorem Unit]

Series with non-negative terms play an important role while considering convergence. Their partial sums are non-decreasing. In this case we have an important theorem.

Theorem 3: Let (a_n) be a sequence of non-negative real numbers. Then the series $\sum a_n$ converges if and only if the sequence of partial sums (s_k) is bounded above. In this case it follows that

$$\sum_{n=1}^{\infty}a_n=\lim_{k\to\infty}(s_k)=\sup\{s_k : k\in\mathbb{N}\}$$

Proof: If Part: Suppose that the series $\sum_{n=1}^{\infty}a_n$ converges. Then the sequence of partial sums (s_k) is convergent and is increasing since the terms of the series $\sum a_n$ are non-negative. Since every convergent sequence is bounded (by Theorem 2, unit 5, Block 2), the sequence (s_k) is bounded. Infact it is above and in this case

$$\sum_{n=1}^{\infty}a_n=\lim_{k\to\infty}s_k=\sup\{s_k : k\in\mathbb{N}\}$$ [Refer… of Unit 5 and 6 of Block 2]

Only if Part: Let the series $\sum_{n=1}^{\infty}a_n$ be a series such that all $a_i's$ are non-negative and such that the sequence of partial sums (s_k) is bounded above.

Since $s_{n+1} - s_n = \sum_{k=1}^{n+1} s_k - \sum_{k=1}^{n} s_k = a_{k+1} \geq 0$, we get that the sequence (s_k) of parital sums is increasing. Thus by Monotone convergence Theorem (Theorem 3, Unit 6, Block 2) the sequence (s_k) must converge and hence the series converge and we have

$$\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} s_k = \sup\{s_k : k \in \mathbb{N}\}$$ ■

Next we shall use this theorem to check the convergence of the following series.

Example 7: The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Solution: We first note that the sequence of partial sums of the given series is given by $s_1 = 1, s_2 = 1 + \frac{1}{2^2}, s_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2}, ..., s_k = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2}$.

It is monotonic. Therefore by Theorem 3 it sufficies to show that the sequence of partial sums is bounded above. "To show this" we write the given series as

$1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) + ...$ Then we have $\frac{1}{2^2} + \frac{1}{3^2} < \frac{1}{2}$

$\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} < \frac{1}{2^2}$ is $\frac{1}{1 + \frac{1}{2}} = 2.$

This shows that the sequence of partial sums (s_n) of the given series is bounded. This together with the fact that the sequences is monotonic shows that the series is convergent.

***

The argument used in proving the result in Example 7 can be used in proving a more general result.

Example 8: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$.

(We leave it as an exercise for you to try, see E5).

It is now interesting to ask what happens to $\sum \frac{1}{n^p}$ when $0 < p \leq 1$. The following example gives an answer to this question.

Example 9: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $0 < p \leq 1$.

Solution: To prove the result we will use the elementary inequality $n^p \leq n$ when $n \in \mathbb{N}$ and $0 < p \leq 1$. Then it follows that $\frac{1}{n} \leq \frac{1}{n^p}$ for $n \in \mathbb{N}$.

This shows that the sequence of partial sums (s_n) and (s_{n^p}) are such that

$s_n \le s_{n^p}$ where $s_n = 1+\frac{1}{2}+\frac{1}{3}+..n$ and $s_{n^p} = 1+\frac{1}{2^p}+\frac{1}{3^{p^{-1}}}+...+\frac{1}{n^p}$. Since the partial sums of the harmonic series $\left(\frac{1}{n}\right)$ diverges, (s_n) is not bounded, this together with the inequality shows that the partial sums of the series are not bounded when $0 < p \le 1$. Hence the series diverges for these values of p.

***

Example 10: Show that the nth term of the following series tends to 0 as $n \to \infty$ and yet the series diverges.

i) $\sum \ell n\left(1+\frac{1}{n}\right)$

ii) $\sum \frac{1}{\sqrt{n}}$

Solution: i) Let $a_n = \ell n(1+1/n)$. We observe that $1+\frac{1}{n} \to 1$ as $n \to \infty$. Thus $\ln\left(1+\frac{1}{n}\right) \to 0$ as $n \to \infty$.

We shall check that the series diverges.

We note that

$$\begin{aligned}
s_n &= a_1 + a_2 + ... + a_n \\
&= \ell n(1+1) + \ell n\left(1+\frac{1}{2}\right) + \ell n\left(1+\frac{1}{3}\right) + ...\ell n\left(1+\frac{1}{n}\right) \\
&= \ell n 2 + \ell n\,\frac{3}{2} + \ell n\,\frac{4}{3} + ... + \ell n\,\frac{n+1}{n} \\
&= \ell n\left(2 \times \frac{3}{2} \times \frac{4}{3} + ... + \frac{n+1}{n}\right) \\
&= \ell n\,(n+1) \quad \text{(by cancellation)}
\end{aligned}$$

Therefore $s_n = a_1 + a_2 + ... + a_n = \ln(n+1) \to \infty$ as $n \to \infty$.

Hence the series diverges.

ii) The given series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ...$$

The nth term $a_n = \frac{1}{\sqrt{n}}$ tends to 0 as $n \to \infty$.

We will check that the series diverges

$$s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > \frac{n}{\sqrt{n}} = \sqrt{n} \to \infty \text{ as } n \to \infty.$$

Hence the series diverges. [This can be concluded from Example 9 as well by taking $p = \frac{1}{2}$].

***

Example 11: Examine the convergence of the following infinite series:

i) $\sum_{m=1}^{\infty} n$ ii) $\sum_{n=1}^{\infty} [1 + (-1)^{n+1}]$.

Solution: i) Consider the series $1 + 2 + 3 + \cdots$. In this case,

$$s_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

The (s_n) is a sequence of positive terms, monotonic and unbounded and therefore diverges.

Hence the series $\sum_{n=1}^{\infty} n$ is divergent.

ii) Consider the series $\sum_{n=1}^{\infty} [1 + (-1)^{n+1}]$. As noted earlier a_n is 2 or 0 according as n is odd or even. Therefore, $(a_n) = (2, 0, 2, 0, \ldots)$ fail to have a limit as $n \to \infty$. Therefore by corollary 1, the series diverges.

***

Try these exercises now.

E5) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$.

E6) Show that the following series diverges.

$$\sum_{k=1}^{\infty} \frac{k^2 + k + 1}{k^2 - k + 1} = \frac{3}{1} + \frac{7}{3} + \frac{13}{7} + \ldots, a_n \to 1 \text{ as } n \to \infty$$

A convergent series can easily be added, subtracted and multiplied by real numbers. Multiplication of two series is complicated and therefore we do not consider it now.

We shall prove the following theorem.

Theorem 4: Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent seris of real numbers and let c be a real number. Then the following hold.

i) The series $\sum_{n=1}^{\infty} (x_n + y_n)$ converges and $\sum_{n=1}^{\infty} (x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$.

ii) The series $\sum_{n=1}^{\infty} (x_n - y_n)$ converges and $\sum_{n=1}^{\infty} (x_n - y_n) = \sum_{n=1}^{\infty} x_n - \sum_{n=1}^{\infty} y_n$.

iii) The series $\sum_{n=1}^{\infty} c\,x_n$ converges and $\sum_{n=1}^{\infty} c\,x_n = c\sum_{n=1}^{\infty} x_n$.

Proof: To prove part i), note that

$$\sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n = \lim_{n\to\infty}\sum_{n=1}^{n} x_n + \lim_{n\to\infty}\sum_{n=1}^{n} y_n = \lim_{n\to\infty}\left(\sum_{n=1}^{n} x_n + \sum_{n=1}^{n} y_n\right)$$

$$= \lim_{n\to\infty}\sum_{n=1}^{n}(x_n + y_n) = \sum_{n=1}^{\infty}(x_n + y_n)$$

where in each step the existence of the quantity on the left side of the equal sign implies the existence of the quantity on the right side. Hence, if $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge, then $\sum_{n=1}^{\infty}(x_n + y_n)$ converges and we have

$\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$.

The remaining parts are left as an exercise for you See E9).

Similar to the algebraic operations, comparability also holds as stated in the following theorem.

Theorem 5: Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent series so that $x_n \le y_n$ for all $n \in \mathbb{R}$. Then $\sum_{n=1}^{\infty} x_n \le \sum_{n=1}^{\infty} y_n$.

The proof is left as an exercise for you to do. ■

Example 12: Find the sum of the series $\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \ldots$

Solution: This is a geometric series $a + ax + ax^2 + \ldots$ in which $a = \left(\frac{1}{2}\right)^3$. We simply factor a out:

$$\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \ldots + = \left(\frac{1}{2}\right)^3\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots\right]$$

$$= \left(\frac{1}{2}\right)^3\left(\frac{1}{1-1/2}\right) = \frac{1}{4}$$

***

Example 13: Prove that the following series is convergent and calculate it sum $\sum\left(\frac{1}{2^n} + \frac{3}{n(n+1)}\right)$.

Solution: We know that $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is convergent since it is a geometric series with sum $\frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right)=1.$

Also $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ is convergent since the sequence of partial sum is the (s_n) where $s_n=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$$=1-\frac{1}{n}=\frac{n}{n+1}$$

and $\lim_{n\to\infty}s_n=1.$

Therefore by Theorem 4(iii), the series $\sum\frac{3}{n(n+1)}$ is convergent and it sum is 3. Hence by Theorem 4(i), we get that $\sum\frac{1}{2^n}+\frac{3}{n(n+1)}$ is convergent and the sum is 4.

***

The Theorems 4 and 5 state the results for convergent series. Can we have corresponding results for divergent series? The answer is obtained by applying Theorem 4(iii). We have the following theorem.

Theorem 6: If $\sum_{n=1}^{\infty}a_n$ diverges and $c\neq 0$, then $\sum_{n=1}^{\infty}c\,a_n$ diverges.

Proof: To prove that we assume the contrary. Let us assume that $\sum c\,a_n$ converges $c\neq 0.$ Then by Theorem 4 (iii) $c\sum a_n$ converges which is turn proves that $\frac{1}{c}\times c\sum a_n$ converges. That is $\sum a_n$ converges. This is not possible. Hence $\sum c\,a_n$ diverges. ■

For example $\sum_{n=1}^{\infty}\frac{5}{n}$ diverges because $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. Similarly $\sum_{n=1}^{\infty}\frac{3n}{n+1}$ diverges because $\sum_{n=1}^{\infty}\frac{n}{n+1}$ diverges.

Thus a known divergent series cannot be turned into a convergent series by multiplying each term by a constant, unless that constant is zero.

Here we make an important point. So far we have used only series starting with a_0 or a_1, but it is also sometimes necessary to use series that begin with a_k as given below:

$$\sum_{n=k}^{\infty}a_n=a_k+a_{k+1}+a_{k+2}+\ldots$$

Indeed, any convergent series can be written as the sum of an initial block of terms plus another convergent series. Thus we have,

$$\sum_{n=1}^{\infty} a_1 = a_1 + \sum_{n=2}^{\infty} a_n,$$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \sum_{n=3}^{\infty} a_n,$$

and, in general,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_k + \sum_{n=k+1}^{\infty} a_n$$

for each positive integer k. For example, we can write $\sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n$ if $|x| < 1$ and $\sum_{n=1}^{\infty} \frac{n}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \sum_{n=3}^{\infty} \frac{1}{n(n+1)}$.

We shall illustrate this with an example.

Example 14: Find the sum of the series $\sum_{k=3}^{\infty} \frac{1}{2^k}$.

Solution: Note that $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{2^2} + \sum_{k=3}^{\infty} \frac{1}{2^k}$;

also $\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1 - \frac{1}{2}} = 2.$

Therefore, $\sum_{k=3}^{\infty} \frac{1}{2^n} = \sum_{k=0}^{\infty} \frac{1}{2^k} - 1\frac{1}{2} - \frac{1}{2^2} = 2 - \frac{7}{4} = \frac{1}{4}.$

***

Whether a series converges or diverges has nothing to do with the starting index.

Specifically, if the series $\sum_{n=k}^{\infty} a_n$ converges for some starting index k, it will converge for any other starting index; and if $\sum_{n=k}^{\infty} a_n$ diverges for some starting Index k, it will diverge for any other starting index. To illustrate, since

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots$$

is a convergent series, so is the series

$$\sum_{n=100}^{\infty} \frac{1}{2^n} = \frac{1}{2^{100}} + \frac{1}{2^{101}} + \frac{1}{2^{102}} + \ldots;$$

and since the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ...$$

diverges, the series

$$\sum_{n=1}^{\infty} \frac{1}{n+5} = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + ...$$

is also divergent.

***

You can try some exercises now.

E7) Prove (ii) and (iii) in Theorem 4.

E8) Prove Theorem 5.

E9) Decide whether the following series converge or diverge. If a series diverges, state why. If it converges, find the sum.

i) $1 + e^{-2} + e^{-4} + e^{-6} + ...$

ii) $\sum_{n=0}^{\infty} (-1)^{n+2} (0.9)^n$

Now you will see that the geometric series is the foundation for the decimal expansion of rational numbers.

So far we have considered convergence and divergence of an infinite series. You might have noticed that among the convergent series, geometric series plays a vital role. We shall now show that the basic fundamental decimal expansion of real numbers can be expressed as a geometric series.

The connection arises from the observation that 0.9 means $9/10$ and 0.99 means $9/10 + 9/10^2$, and the meaning for the infinite decimal 0.999 ... is the infinite series given by

$$9/10 + 9/10^2 + 9/10^3 + ... + 9/10^n + ...$$

The sum of this series is, always the limit of its partial sums, of which the nth partial sum s_n is given by

$$s_n = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + ... + \frac{9}{10^n}$$
$$= \frac{9}{10}\left[1 + \frac{1}{10} + \frac{1}{10^2} + ... + \frac{1}{10^{n-1}}\right] = \frac{9}{10}\left[\frac{1 - 1/10^n}{1 - 1/10}\right] = 1 - (1/10^n)$$

Since $\lim_{n \to \infty} s_n = 1$, we conclude that

$$0.999... = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + ... = 1.$$

In general, if $(a_n)_{n=}^{\infty}$ is a sequence of integers taken from 0 through 9, then corresponding to the infinite decimal expansion, $0.a_1a_2a_3...$ we have the infinite series given by

$$\sum_{n=1}^{\infty}\frac{a_n}{10^n}=\frac{a_1}{10}+\frac{a_2}{10^2}+\frac{a_3}{10^3}+...+\frac{a_n}{10^n}+...$$

Thus we have another method of finding the fraction or rational number corresponding to a given recurring decimal by calculating the sum of the corresponding geometric series, associated with that.

Let us see some examples.

The notation $0.\bar{4}$ is the short form of $0.444...$

Example 15: Find the rational number represented by $0.444...$ or $0.\bar{4}$.

Solution: The given decimal $0.444....$ corresponds to the geometric series given by

$$\sum_{n=1}^{\infty}\frac{4}{10^n}=\frac{4}{10}+\frac{4}{10^2}+\frac{4}{10^3}+...$$

which has as its partial sum s_n given by

$$\begin{aligned} s_n &= \frac{4}{10}+\frac{4}{10^2}+...+\frac{4}{10^n} \\ &= \frac{4}{10}\left[1+\frac{1}{10}+...+\frac{1}{10^{n-1}}\right]=\frac{4}{10}\,\frac{1-1/10^n}{1-1/10}=\frac{4}{9}(1-1/10^n). \end{aligned}$$

Since $\lim\limits_{n\to\infty} s_n = 4/9$, it follows that $0.444...=4/9$.

***

Example 16: Convert each of the following infinite decimal exapansion $0.9\overline{67}$ into a fraction.

Solution: We first note that $0.9\overline{67}=0.96767....$

$$=\frac{9}{10}+\frac{67}{10^3}+\frac{67}{10^5}+\frac{67}{10^7}+...$$

$$=\frac{9}{10}+\frac{67}{10^3}\left[1+\frac{1}{10^3}+\frac{1}{10^4}+...\right]$$

$$=\frac{9}{10}+\frac{67}{10^3}\left[\frac{1}{1-\frac{1}{10^2}}\right]$$

$$=\frac{9}{10}+\frac{67}{10^3}\times\frac{10^2}{99}$$

$$=\frac{9}{10}+\frac{67}{990}$$

$$= \frac{9 \times 99 + 67}{990}$$

$$= \frac{891 + 67}{990}$$

$$= \frac{958}{990}$$

Therefore the fraction corresponding to the decimal expansion is $\frac{958}{990}$.

***

Here are some exercises for you.

E10) Find the rational number represented by the following

i) $0.3\overline{12}$ ii) $0.297297...$

E11) Find the infinite series corresponding to $\frac{11}{7}$.

With this we come to an end of this unit.

Let us briefly recall what we have done in this unit.

7.4 SUMMARY

In this unit we have covered the following:

1) We have explained the notion of infinite series. An infinite series is an expression of the form $a_1 + a_2 + a_3 + ...$ (extending indefinitely) where each a_n is an element of a sequence $(a_n)_{n \in \mathbb{N}}$.

2) An infinite series is expressed as

   i) $a_1 + a_2 + a_3 + ...$ (summation notation)

   ii) $\sum_{n=1}^{\infty} a_n$ (symbolic notation or Σ-notation).

3) We have defined the term "partial sums" associated with a series $\sum_{n=1}^{\infty} a_n$, and we have defined a sequence known as the sequence of partial sums (s_n) assicated with a series $\sum_{i=1}^{n} a_i$ where

   $$s_n = a_1 + a_2 + a_3 + ... + a_n.$$

4) We defined convergence and divergence of a series in terms of the convergence and divergence of the sequence of partial sums associated with the series. The limit of the sequence is called the sum of the series.

5) We discussed certain necessary and certain sufficient conditions for the convergence and divergence of the series.

6) We established the connection between an infinite series and infinite decimal expansion for numbers, specially rationals.

7.5 SOLUTIONS/ANSWERS

E1) i) $\frac{n}{n+1}$ ii) $\frac{n^2+n+1}{n^2-n+1}$

E2) i) $2+8+12+...$

ii) $\frac{3}{1}+\frac{7}{2}+\frac{13}{7}+...$

E3) i) We have $\frac{2}{n(n+2)}=\frac{1}{n}-\frac{1}{n+2}$

$$\therefore s_n = a_1+a_2+...+a_n = 1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+...+\frac{2}{n(n+2)}$$

$$=\frac{3}{2}+\frac{2}{n(n+2)}$$

ii) $$1+\frac{5}{8}+\left(\frac{5}{8}\right)^2+...+\left(\frac{5}{8}\right)^n = \frac{1-\left(\frac{5}{8}\right)^{n+1}}{1-\frac{5}{8}}$$

$$=\frac{1-\left(\frac{5}{8}\right)^{n+1}\times 8}{3}$$

E4) Suppose there exists a sequence $\{a_n\}$ of positive terms with $s_n=\frac{1}{n}$. Then by the definition of partial sum $a_n = s_{n+1}-s_n = \frac{1}{n+1}-\frac{1}{n}=\frac{-1}{n(n+1)}<0.$

This is a contradiction.

E5) Since the argument is very similar to the special case considered in part (a), we will leave some of the details. We have $\frac{1}{1^p}=1$ and

$$\frac{1}{2^p}+\frac{1}{3^p}<\frac{2}{2^p}=\frac{1}{2^{p-1}}$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{4}{4^p}=\left(\frac{1}{2^{p-1}}\right)^2$$

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This shows that the sequence of partial sums are bounded. This together with the fact that the sequence is monotonic shows that the series is convergent.

E6) Here $a_n=\frac{k^2+k+1}{k^2-k+1}$ and $a_n \to 1$ as $n\to\infty$.

Since a_n tends to a limit different from 0, the series diverges.

E7) We first prove (iii). The nth partial sum of the series $\sum_{n=1}^{\infty} \lambda a_n$ is

$$\sum_{k=1}^{\infty} \lambda a_k = \lambda a_1 + \lambda a_2 + ... + \lambda a_n$$
$$= \lambda(a_1 + a_2 + ... + a_n) = \lambda s_n$$

Since (s_n) is convergent, (λs_n) is convergent. Moreover $\lim_{n\to\infty}(\lambda s_n) = \lambda \lim_{n\to\infty} s_n = \lambda s$.

This shows that $\sum \lambda a_n = \lambda s.$

ii) This follows from Theorem 4(i) and (iii), using the choice $c = -1$:

$$\therefore \sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty}(a_n + (-b_n)) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty}(-b_n)$$
$$= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

E8) Consider the partial sums of $\sum a_n$ and $\sum b_n$ given by

$$s_n = a_1 + a_2 + ... + a_n$$
$$t_n = b_1 + b_2 + ... + b_n$$

We are given that $a_1 \le b_1, a_2 \le b_2, ..., a_n \le b_n$.

Therefore we have $s_n = \sum_{k=1}^{n} a_k \le t_n = \sum_{k=1}^{n} b_k$

$\therefore\ \forall_n,\ t_n - s_n \ge 0.$

Consequently $\lim_n (t_n - s_n) \ge 0.$ Hence the result.

E9) i) $1 + e^{-2} + e^{-4} + e^{-6} + ... = 1 + \frac{1}{e^2} + \frac{1}{e^4} + \frac{1}{e^6} + ...$

This is geometric series with the first term 1 and common ratio $\frac{1}{e^2} < 1.$ $\therefore$ the series converges and the sum is given by

$$\frac{1}{1 - \frac{1}{e^2}} = \frac{e^2}{e^2 - 1}.$$

ii) Here $a_0 = 1, a_1 = (-1)(0.9)^1, a_2 = (0.9)^2, a_3 = (-1)(0.9)^3$

This shows that the given series has two subseries, namely $\sum_{k=0}^{2n}(0.9)^k$ and $\sum_{k=1}^{2n+1}(0.9)^k$, one converges to a positive number and the other to its opposite number.

Hence the given series is not convergent.

E10) i) We write $0.3121212... = 0.3 + \frac{12}{10^3} + \frac{12}{10^3}\left(\frac{1}{10^2}\right) + \frac{12}{10^3}\left(\frac{1}{10^2}\right)^2 + ...$

$$= \frac{3}{10} + \frac{12}{10^3}\left(\frac{1}{10^2} + \frac{1}{(10^2)^2} + ...\right)$$

$$= \frac{3}{10} + \frac{12}{10^3}\left(\frac{1}{10^2} + \frac{1}{(10^2)^2} + ...\right)$$

$$= \frac{3}{10} + \frac{12}{10^3} \times \frac{1}{1 - \frac{1}{10^2}}$$

$$= \frac{3}{10} + \frac{12}{10^3} \times \frac{1}{99/100}$$

$$= \frac{3}{10} + \frac{12}{10^3} \times \frac{100}{99}$$

$$= \frac{3 \times 99 + 12}{99 \times 10}$$

$$= \frac{99 + 4}{33 \times 10} = \frac{103}{330}$$

ii) $0.297297... = \frac{297}{1000} + \frac{297}{1000^2} + \frac{297}{1000^3}$

This is a geometric series with the first term $a = \frac{297}{1000}$ and

$r = \frac{1}{1000}$. So the series converges and the sum is given by

$$S = \frac{297}{1000} \times \frac{1}{1 - \frac{1}{1000}} = \frac{297}{1000} \times \frac{1000}{999} = \frac{9}{111}.$$

E11) We know that $\frac{11}{7} = 1.571428571428.....$

$$= 1 + 0.571428571428$$

$$= 1 + \frac{571428}{10^6} + \frac{571428}{(10^6)^2} + ...$$

$$= 1 + \frac{571428}{10^6}\left[1 + \frac{1}{10^2} + \frac{1}{10^3} + ...\right]$$

$$= 1 + \frac{571428}{10^6}\left[\frac{1}{1 - \frac{1}{10}}\right]$$

$$= 1 + \frac{571428}{10^6} \times \frac{10}{99}$$

$$= 1 + \frac{5772}{10^4}$$

$$= 1 + 0.5772$$

$$= 1.5772$$

UNIT 8

TESTS FOR CONVERGENCE

Structure

8.1 INTRODUCTION

In the last unit we discussed infinite series and its convergence. We defined the convergence of a series in terms of the convergence of its associated sequence of partial sums (s_n). However, as you have learnt from the previous unit, it is not always easy to find a simple formula for the nth partial sums.

Consequently, the technique of establishing convergence by directly finding the limit of (s_n) does not always work. However, the situation is better for series in which all the terms are positive. We call them positive term series. Recall that for such a series the sequence of partial sums is strictly increasing. Therefore if the partial sums are bounded, then the series must converge.

In this unit we shall discuss special tests for convergence of a positive term series. We shall avoid using the word positive as we are considering only such series in this unit. We discuss special tests for convergence of series, known as comparison test D' Alembert's ratio test, Cauchy root test, Raabe's test and Gauss's test respectively in Sec. 8.2 to Sec. 8.5. These tests enable us to deal with a fairly large number of positive term series.

Objectives

After studying this unit you should be able to:

- apply the following tests for convergence or divergence of a series:

  i) Comparison Test

ii) Cauchy Root Test

iii) D'Alembert's Ratio Test

iv) Raabe's Test and Gauss's Test

- decide which test is more appropriate for testing convergence depending upon nth term of the series.

8.2 COMPARISON TEST

The most common among the tests of convergence of a positive term series is the comparison test. In this test we compare the series $\sum a_n$ with another series $\sum b_n$ whose convergence or divergence is known. Accordingly we decide whether $\sum a_n$ converges or diverges. We shall illustrate this with the following series.

$$1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots+\frac{1}{n!}+\ldots \qquad \ldots (1)$$

Let $s_n = 1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots+\frac{1}{n!}$.

We observe that $2! \geq 2, 3! > 2^2, 4! > 2^3$ and, in general if $k > 1$, we have $k! \geq 2^{k-1}$, then

$$(k+1)! = k!(k+1) > 2^{k-1}(k+1) > 2^k \qquad \ldots (2)$$

Recall that $n! \geq 2^{n-1}$ for each positive integer (infact, equality holds only at $n = 1$ and 2). Hence $1/n! \leq 1/2^{n-1}$.

Therefore we have

$$s_n = 1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!} < 1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}} = \frac{1-1/2^n}{1-1/2} = 2-\frac{1}{2^{n-1}} < 2$$

for each positive integer n. The last inequality, above, shows that the partial sums of the series are bounded above by 2 and therefore by Theorem 10 in unit 7, the series converges.

The preceding argument shows that the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because its terms are less than or equal to the corresponding terms of the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$.

In general, given a series $\sum a_n$ if there exists $\sum b_n$ such that the following inequalities hold

$$0 < a_n \leq b_n$$

for all n, then we say that the **series $\sum b_n$ dominates the series $\sum a_n$**.

We are now ready to prove the following theorem.

Theorem 1(Comparison test): Let $\sum a_n$ and $\sum b_n$ be series of positive terms. Then the following holds:

Convergence Test (Case 1): If $0 \le a_n \le b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ is also convergent.

Divergence Test (Case 2): If $0 \le b_n \le a_n$ for all n and $\sum b_n$ diverges, then $\sum a_n$ is also divergent.

Proof: Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$.

We shall prove cases 1 and 2 one by one.

Case 1: We have, $s_n \le t_n$ for all n. Since $\sum b_n$ converges the sequence of parital sums (t_n) converges to the sum, say $T = \sum b_n$.

Therefore we have $s_n \le t_n \le T$. Thus partial sums (s_n) form an increasing and bounded sequence, that is,

$$s_1 \le s_2 \le s_3 \le \ldots \le s_n \le \ldots \le T .$$

Hence by Theorem 10 of Unit 7 converges $\sum a_n$.

Case 2: In the Divergence Test, the relation between s_n and t_n is reversed. Now, $t_n \le s_n$ because $b_n \le a_n$ for all n and it is clear that s_n must grow large if t_n does. But t_n tends to infinity because $\sum b_n$ is divergent; hence $s_n \to \infty$ as $n \to \infty$ and $\sum a_n$ diverges. ■

Let us see some examples.

Example 1: Use the comparison test to determine whether the following series is convergent: $\sum_{n=1}^{n} \frac{\sqrt{n}}{n^2+1}$.

Solution: Here $a_n = \frac{\sqrt{n}}{n^2+1}$. Then we have

$$0 \le \frac{\sqrt{n}}{n^2+1} = \frac{1}{n^{3/2} + \frac{1}{\sqrt{n}}} < \frac{1}{n^{3/2}}$$

we know by p-test (Example 8 of Unit 7) that $\sum_{n=1}^{n} \frac{1}{n^{3/2}}$ is convergent as $p = 3/2 > 1$.

Therefore by the comparison test (case 1), we get that the given series is convergent.

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Example 2: Applying the comparison test to the series $\sum_{n=1}^{n} \frac{n^{3/2}}{3n^2 - n}$, determine whether it converges or diverges.

Solution: Let $a_n = \frac{n^{3/2}}{3n^2 - n}$. We observe that the nth term of this series a_n behaves like $1/3\sqrt{n}$ for large n and that $\sum \frac{1}{3\sqrt{n}} = \frac{1}{3} \sum \frac{1}{\sqrt{n}}$ is a divergent series. This suggests we use the comparison test in its divergence form i.e. case 2. Specifically, we observe that the nth term a_n is such that

$$a_n = \frac{n^{3/2}}{3n^2 - n} = \frac{1}{3\sqrt{n} - \frac{1}{n}} > \frac{1}{3\sqrt{n}} > 0$$

If we take $b_n = \frac{1}{3\sqrt{n}}$, then we have $b_n \le a_n$. Again we apply p-test and note that $\sum b_n$ is divergent. Therefore by the comparison test (case 2), the series is divergent.

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Example 3: Prove that the series $\sum \frac{n+1}{n \times 2^n}$ converges.

Solution: We use the inequality $\frac{n+1}{n} \le 2$. Therefore $\frac{n+1}{n \times 2^n} \le \frac{2}{2^n}$.

Put $a_n = \frac{n+1}{n \times 2^n}, b_n = \frac{2}{2^n}$. Then $a_n \le b_n$. Also $\sum b_n = \sum_{n=1}^{\infty} \frac{2}{2^n}$ is a geometric series with common ratio $r = \frac{1}{2}$ and therefore converges. Hence by comparison test $\sum \frac{n+1}{n \times 2^n}$ is convergent.

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We have to be cautions when we apply the comparison test. For example, if $0 < a_n \le b_n$ and $\sum b_n$ is divergent, then we cannot conclude that $\sum a_n$ is divergent. For instance we have $0 < \frac{1}{n^2} < \frac{1}{n}$. But $\sum \frac{1}{n}$ is divergent whereas $\sum \frac{1}{n^2}$ is convergent. On the other hand we have $0 < \frac{1}{2^n} < \frac{1}{n}$. Here $\sum \frac{1}{n}$ is divergent and $\sum \frac{1}{2^n}$ is convergent. This shows that if the dominating series diverges, then the original series can either converge or diverge.

Next we consider another series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + 3\sqrt[4]{n} + 1}$.

We take $a_n = \frac{1}{\sqrt{n} + 3\sqrt[4]{n} + 1}$ and $b_n = \frac{1}{\sqrt{n}}$.

Then both a_n and b_n are positive. Since we can expess in the form,

$$\frac{1}{\sqrt{n}+3\sqrt[4]{n}+1}=\frac{1}{\sqrt{n}}\times\frac{1}{1+3n^{-1/4}+n^{-1/2}} \qquad \ldots (3)$$

We also have

$$\frac{a_n}{b_n}=\frac{\sqrt{n}}{\sqrt{n}(1+3n^{-1/4}+n^{-1/2})}$$
$$=\frac{1}{1+3n^{-1/4}+n^{-1/2}}$$

The R.H.S of the above equality tends to 1 as $n\to\infty$. We learn that the given expression is somewhat similar to $\frac{1}{\sqrt{n}}$ in the limiting case. We also know that $\frac{1}{\sqrt{n}}$ diverges and therefore it seems likely that in the limiting case the given series is divergent by applying case 2 of the comparison test which we state below.

We prove the following theorem.

Theorem 2 (Limit form of the comparison test): Suppose for all $n, a_n \geq 0$ and $b_n > 0$. Then

i) If $\lim\limits_{n\to\infty}\frac{a_n}{b_n}$ exists, then $\sum b_n$ is convergent implies that $\sum a_n$ is convergent.

ii) If either $\lim\limits_{n\to\infty}\frac{a_n}{b_n}$ is a positive number or if $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\infty$, then $\sum b_n$ is divergent implies that $\sum a_n$ is divergent.

Proof: i) Suppose $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\ell$. Then $\ell\geq 0,$ as $\frac{a_n}{b_n}\geq 0, \forall n$. Also, the sequence $\left(\frac{a_n}{b_n}\right)$ is bounded and therefore there exists a constant $k>0$ such that

$\forall n, \frac{a_n}{b_n}\leq k$ or $a_n\leq kb_n$. Hence, by the comparison test, if $\sum b_n$ is converges, then $\sum a_n$ converges.

ii) Here we first consider the case $\lim\frac{a_n}{b_n}=\ell>0.$ If $\ell>0,$ then for $\varepsilon=\frac{\ell}{2}$ there exists an integer r such that

$$n\geq r\Rightarrow \ell-\varepsilon<\frac{a_n}{b_n}<\ell+\varepsilon\Rightarrow\frac{a_n}{b_n}>\ell-\varepsilon=\frac{\ell}{2}\Rightarrow a_n>\frac{\ell}{2}b_n.$$

Hence, by the comparison test case 2, $\sum b_n$ is divergent implies that $\sum a_n$ is divergent.

Next we consider the case that $\lim_{n\to\infty}\frac{a_n}{b_n}=\infty$. Then by this definition infinite limits, there is an integer s such that $n \geq s \Rightarrow \frac{a_n}{b_n} \geq 1 \Rightarrow a_n \geq b_n$.

Hence, by comparison test case 2, $\sum b_n$ is divergent implies that $\sum a_n$ is divergent.

It follows from the limit form of the comparison test that the series $\sum \frac{1}{\sqrt{n}+3\sqrt[4]{n}+1}$ is divergent since $\frac{1}{\sqrt{n}}$ is divergent on taking $\ell = 1$. ■

Yet another form of comparison test is given by the following theorem.

Theorem 3 (Another form of Comparison test): Suppose $a_n, b_n > 0$, for all n and there exists an M such that $\frac{a_n}{a_{n+1}} \geq \frac{b_n}{b_{n+1}}$ for all $n \geq M$. Then $\sum a_n$ is convergent if $\sum b_n$ is convergent and $\sum b_n$ is divergent if $\sum a_n$ is divergent.

Proof: We first note that for any fixed positive interger M, we have

$$\begin{aligned}\frac{a_M}{a_n} &= \frac{a_M}{a_{M+1}}\frac{a_{M+1}}{a_{M+2}}\cdots\frac{a_{n-1}}{a_n}\\ &\geq \frac{b_M}{b_{M+1}}\frac{b_{M+1}}{b_{M+2}}\cdots\frac{\mathrm{b}_n-1}{b_n}=\frac{b_M}{b_n}.\end{aligned}$$

Thus, $\frac{a_M}{a_n} \geq \frac{b_M}{b_n}$ i.e. $\frac{a_M}{b_M} \geq \frac{a_n}{b_n}$. Since, M is a fixed positive integer, $\frac{a_M}{b_M}$ is a constant, say k. Thus, $a_n \leq kb_n$ for all $n \geq M$. Hence, by the comparison test, $\sum a_n$ is convergent if $\sum b_n$ is convergent and $\sum b_n$ is divergent if $\sum a_n$ is divergent. ■

Let us see some examples.

Example 4: Test the convergence of the following series.

i) $\sum \sqrt{\frac{n}{n^4+1}}$

ii) $\sum \frac{4^n+1}{5^n+3^n}$

iii) $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$

Solution: i) Let $a_n = \sqrt{\frac{n}{n^4+1}}$, $b_n = \frac{1}{n^{3/2}}$. Note that $\lim_{n\to\infty}\frac{a_n}{b_n}=1$. $\sum b_n$ is convergent as it is a $p-$series, $p = 3/2 > 1$. Hence, $\sum a_n$ is convergent.

ii) Now $a_n = \dfrac{4^n + 1}{5^n + 3^n}, b_n = \left(\dfrac{4}{5}\right)^n$.

Observe that $\sum b_n = \sum \left(\dfrac{4}{5}\right)^n$ being a geometric series with common ratio $\dfrac{4}{5} < 1,$ is convergent.

Note that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$, which is finite. Since, $\sum b_n$ is convergent, as note above, by case-1 $\sum a_n$ is convergent by the comparison test.

iii) Here $a_n = \dfrac{\tan^{-1} n}{1+n^2}$ and $b_n = \dfrac{1}{1+n^2}$. Now, $\lim \dfrac{a_n}{b_n} = \dfrac{\pi}{2}$, which is finite. Also note that $\dfrac{1}{1+n^2} < \dfrac{1}{n^2}$ and $\sum \dfrac{1}{n^2}$ is convergent by the p-test for $p = 2$.

Thus $\sum b_n = \sum \dfrac{1}{1+n^2}$ is convergent. Hence the comparison test $\sum a_n$ is convergent.

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Now you can try the following exercises.

E1) Test the convergence of the following series:

i) $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{5} + \ldots + \dfrac{1}{2^{n-1}+1} + \ldots$

ii) $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \ldots + \dfrac{1}{\sqrt{n}} + \ldots$

E2) Determine whether the following series are convergent.

i) $\dfrac{1}{4} + \dfrac{1}{7} + \dfrac{1}{10} + \cdots + \dfrac{1}{3n+1} + \cdots$.

ii) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n}}{n^2 - 4}$.

E3) Show that if the series $\sum\limits_{n=1}^{\infty} u_n$ of positive terms converges, then the series $\sum\limits_{n=1}^{\infty} u_n^2$ also converges. Hence, deduce that the series $\sum \dfrac{1}{\sqrt{n}}$ diverges.

Next we shall consider another special test.

8.3 D' Alembert's Ratio Test

In the previous section we discussed the comparison test. The comparison test needs another series with known behaviour for the purpose of comparison. Here we shall discuss another test known as Ratio Test. This test

J.D'Alembert

is due to the French mathematician J.D' Alembert. Therefore this test is called D'Alembert's Ratio Test or just Ratio Test as stated below.

Theorem 4 (Ratio Test): Let $a_n > 0, \forall n.$ Suppose for all $n,$

i) $\dfrac{a_{n+1}}{a_n} \le k < 1$, where k is a constant, then $\sum a_n$ is convergent.

ii) $\dfrac{a_{n+1}}{a_n} \ge 1,$ then $\sum a_n$ is divergent.

Proof: (i) We note that $\dfrac{a_n}{a_1} = \dfrac{a_n}{a_{n-1}} \dfrac{a_{n-1}}{a_{n-2}} \dots \dfrac{a_2}{a_1} < k^n$

Hence, $\forall n,$ we have $a_n \le k^{n-1} a_1.$ Also since $0 < k < 1,$ the geometric series $\sum k^n$ is convergent. Hence by the comparison test $\sum a_n$ is convergent.

ii) Since, $a_{n+1} \ge a_n \quad \forall n, a_n \ge a_1 > 0$ and so

$$s_n = a_1 + a_2 + \dots + a_n > na_1$$

Thus $\lim\limits_{n\to\infty} s_n = \infty$. Hence $\sum a_n$ is divergent. ■

Note that in the above theorem if $k = 1,$ then we cannot conclude whether the series converges. For example if $a_n = \dfrac{1}{n}$, then $\dfrac{a_{n+1}}{a_n} = \dfrac{n}{n+1} = \dfrac{1}{1+\dfrac{1}{n}} \to 1$ as $n \to \infty$ and therefore there does not exist any $k < 1$ for which $\dfrac{a_{n+1}}{a_n} \le k$ for all n and we have see that $\sum a_n = \sum \dfrac{1}{n}$ is divergent. On the other hand if $a_n = \dfrac{1}{n^2}$, then $\dfrac{a_{n+1}}{a_n} = \dfrac{n^2}{(n+1)^2} = \dfrac{1}{1+\dfrac{2}{n}+\dfrac{1}{n^2}} \to 1$ as $n \to \infty$ and therefore there does not exist any $k < 1$ for which $\dfrac{a_{n+1}}{a_n} \le k$ for all $n,$ but the series $\sum \dfrac{1}{n^2}$ converges. This shows that if $k = 1,$ then the test cannot be used.

Example 5: Applying the ratio test check whether the following series converges

$$\sum_{n=1}^{\infty} \frac{3n+1}{3n-1}.$$

Solution: Here $\dfrac{a_{n+1}}{a_n} = \dfrac{(3n+3)\,(3n+3)}{(3n+2)\,(3n+1)} = \dfrac{9n^2+9n-4}{9n^2+9n+2} < 1,$ for all n and so $\sum a_n$ is convergent by the ratio test.

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In the above example it was possible to find that the ratio is greater >1 and therefore the series diverges. Whereas the test for converges involves a number k which is no easy to find.

Here is another theorem which is generalised form of ratio test and more easy to apply.

Next we shall prove a more generalised form of the Ratio test.

Theorem 5 (Limit Form of Ratio Test): Suppose $a_n > 0,$ for all n and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell.$ Then $\sum a_n$ is convergent if $\ell < 1$ and divergent if $\ell > 1$. The test is inconclusive if $\ell = 1$.

Proof: Here $\ell \geq 0$.

i) Let $\ell < 1.$ Then $1-\ell > 0.$ Hence, for $\varepsilon = \frac{1}{2}(1-\ell),$ there is an integer r such that

$$n > r \Rightarrow \frac{a_{n+1}}{a_n} < \ell + \varepsilon = \frac{1}{2}(1+\ell) < 1.$$

Hence, letting $k = \frac{1}{2}(1+\ell),$ part (i) of the Theorem 4 shows that $\sum a_n$ is convergent.

ii) Let $\ell > 1.$ Then $\ell - 1 > 0$. Hence, for $\varepsilon = \frac{1}{2}(\ell - 1),$ there is an integer r such that

$$n > r \Rightarrow \frac{a_{n+1}}{a_n} > \ell - \varepsilon = \frac{1}{2}(1+\ell) > 1$$

Hence, letting $k = \frac{1}{2}(1+\ell),$ part ii) of Theorem 4 shows that $\sum a_n$ is divergent.

iii) If $\ell = 1,$ the series may or may not be convergent. For example, let $a_n = \frac{1}{n}, b_n = \frac{1}{n^2}$. Then,

$$\lim \frac{a_{n+1}}{a_n} = 1 \text{ and } \lim \frac{b_{n+1}}{b_n} = 1$$

Observe that $\sum a_n$ is divergent while $\sum b_n$ is convergent. ■

Example 6: Test the convergence of the series $\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \cdots + \frac{2n-1}{2^n} + \cdots$.

Solution: Here $a_n = \frac{2n-1}{2^n}$. So that $a_{n+1} = \frac{2n+1}{2^{n+1}}$.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(2n+1)2^n}{2^{n+1}(2n-1)}=\frac{1}{2}\lim_{n\to\infty}\frac{1+\frac{1}{2n}}{1-\frac{1}{2n}}=\frac{1}{2}.$$

Since $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{2}<1,$ the series converges.

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Example 7: Check the convergence of the following series.

i) $\sum\frac{n^3}{1+2^n}.$ ii) $\sum\frac{(n+2)^2}{n!}.$

Solution: i) $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^3}{1+2^{n+1}}\frac{1+2^n}{n^3}=\frac{1}{2}<1$

Hence by the limit form of the ratio test, $\sum a_n$ is convergent.

ii) $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+3)^2}{(n+1)!}\frac{n!}{(n+2)^2}=\lim_{n\to\infty}\frac{1}{n+1}\left(\frac{1+3/n}{1+2/n}\right)^2=0<1.$

Hence by the limit form of the ratio test, $\sum a_n$ is convergent.

***

Example 8: Let $x>0$ and r be constants and $a_n=n^r x^n$. Check the convergence of the series $\sum n^r x^n$.

Solution: We have $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^r x^{n+1}}{n^r x^n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^r\cdot x=x.$

Hence by the limit form of the ratio test, $\sum a_n$ is convergent if $0<x<1$ and divergent if $x>1$. If $x=1,$ then the Ratio Test fails. But then $a_n=n^r$ and we know that the series $\sum a_n=\sum n^r=\sum\frac{1}{n^{-r}}$ converges if $-r>1$ i.e. if $r<-1$ and divergent if $-r\le 1$ i.e. if $r\ge -1$.

***

Example 9: Check whether or not the Limit Form of the Ratio Test implies the convergence of the series $\sum\frac{(n!)^3 27^n}{(3n)!}$. Check whether the series $\sum\frac{(n!)^3 27^n}{(3n)!}$ satisfies the conditions of the limit form of the ratio test.

Solution: We have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{[(n+1)!]^3 27^{n+1}}{(3n+3)!}\cdot\frac{(3n)!}{(n!)^3 27^n}$$

$$=\lim_{n\to\infty}\frac{27(n+1)^3}{(3n+3)(3n+2)(3n+1)}=1.$$

Hence, the Ratio Test (limit form) fails.

***

Now try the following exercise.

E4) Show, using the Ratio Test, that the series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$ converges.

E5) For what positive values of x does the series $\sum_{n=1}^{\infty}\frac{x^n}{n}$ converge?

E6) Test for convergence the series $\sum_{n=1}^{\infty}\frac{5^n}{(2n+1)!}$.

You must have observed by now that in the ratio test we concentrate on the series $\sum a_n$ itself and do not need to think of some other series $\sum b_n$ as compared to the comparison test. But ratio test also has limitations especially when the limit tends to 1.

Next we shall consider another special test.

8.4 CAUCHY ROOT TEST

You have seen that D'Alembert's Ratio Test fails to give any definite information about the convergence or divergence of series in some situations. Here we shall study another test for determining the convergence or divergence of series whose nth term involves a power of $n.$ In the Root Test, the convergence of a given series is based on the behaviour of the sequence formed by taking the nth root of the terms of the given series.

The root test was developed first by Augustin-Louis Cauchy who published it in his textbook Cours d'analyse (1821). Thus, it is sometimes known as the Cauchy root test or Cauchy's radical test.

We shall now state and prove this test.

Theorem 6 (Root Test): Let $a_n > 0,$ for all n.

i) Suppose $\forall n, \sqrt[n]{a_n} \le k < 1,$ where k is a constant. Then $\sum a_n$ is convergent.

ii) Suppose $\forall n, \sqrt[n]{a_n} \ge 1$. Then $\sum a_n$ is divergent.

Proof: (i) Let us first consider (i). Then $\forall n$ we have $a_n \le k^n$.
Also since $0 < k < 1,$ the geometric series $\sum k^n$ is convergent. Hence the comparison test shows that $\sum a_n$ is convergent.

ii) Next we consider (ii). Here we have $\sqrt[n]{a_n} \ge 1 \forall n,$ then $a_n \ge 1$ and so
$s_n = a_1 + a_2 + ... + a_n > 1+1+...+1 = n,$
so that $\lim s_n = \infty$. Hence $\sum a_n$ is divergent. ■

As we discussed in the case of earlier tests, we state and prove the limit form of the test.

Theorem 7 (Limit form of root test): Suppose $a_n > 0, \forall\, n$. Suppose $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ exists and is l. Then $\sum a_n$ is convergent if $l < 1$ and divergent if $l > 1$. The test is inconclusive if $l = 1$.

Proof: We first consider (i). Then $l \geq 0$ and $l < 1$. Then for $\varepsilon = \frac{1}{2}(1-l)$, there is an integer r such that

$$n > r \Rightarrow \sqrt[n]{a_n} < l + \varepsilon = \frac{1}{2}(1+l) < 1.$$

Hence, letting $k = \frac{1}{2}(1+l)$, part (i) of the Theorem 6 shows that $\sum a_n$ is convergent.

ii) Let $l > 1$. Then for $\varepsilon = \frac{1}{2}(l-1)$, there is an integer r such that

$$n > r \Rightarrow \sqrt[n]{a_n} > l - \varepsilon = \frac{1}{2}(1+l) > 1.$$

Hence, letting $k = \frac{1}{2}(1+l)$, part (ii) of Theorem 6 shows that $\sum a_n$ is convergent.

iii) If $l = 1$, the series may or may not be convergent therefore the test can be used. For example, let $a_n = \frac{1}{n}, b_n = \frac{1}{n^2} \cdot$ Then $\lim\limits_{n\to\infty}\left(\frac{1}{n}\right)^{1/n} = 1$ and $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}\right)^{1/n} = 1$. But $\sum a_n$ is divergent while $\sum b_n$ is convergent.

Let us see some examples.

Example 10: Test for convergence the series

$$\frac{2}{1} + \left(\frac{3}{3}\right)^2 + \left(\frac{4}{5}\right)^3 + \cdots + \left(\frac{n+1}{2n-1}\right)^n + \cdots.$$

Solution: Here $a_n = \left(\frac{n+1}{2n-1}\right)^n$.

Since n occurs in the exponent of a_n, so we apply the Cauchy's Root Test. Here

$$a_n^{1/n} = \frac{n+1}{2n-1}$$

Therefore,

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2} < 1.$$

Hence the series converges.

***

In the above example, if you wish to apply Ratio Test, you will have to evaluate $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}$. That is, for $a_{n+1}=\left(\frac{n+1+1}{2(n+1)-1}\right)^{n+1}=\left(\frac{n+2}{2n+1}\right)^{n+1}$ and $a_n=\left(\frac{n+1}{2n-1}\right)^n$, we have to evaluate $\lim\limits_{n\to\infty}\left[\frac{n+2}{2n+1}\times\left(\frac{n+2}{n+1}\right)^n\times\left(\frac{2n-1}{n+1}\right)^n\right]$. Which is equal to $\frac{1}{2}$.

This is certainly more lengthy. Therefore here the root test is better.

Now we take an example where the Ratio Test fails, but the Root Test gives a definite answer.

Example 11: Test the convergence of the series $\sum\frac{x^n}{n^n}$, where $x>0$ is a constant.

Solution: Let $a_n=\frac{x^n}{n^n}$. Then $\sqrt[n]{a_n}=\frac{x}{n}\to 0<1$, as $n\to\infty$. Hence, by the root test the series is convergent.

***

Example 12: Check whether the series $\sum\left(1+\frac{1}{n}\right)^{-n^2}x^n$, where $x>0$ is a constant, converge.

Solution: Here $a_n=\left(1+\frac{1}{n}\right)^{-n^2}x^n=\frac{x^n}{\left(1+\frac{1}{n}\right)^{n^2}}$.

Then $\sqrt[n]{a_n}=\frac{x}{\left(1+\frac{1}{n}\right)^n}\to\frac{x}{e}$, as $n\to\infty$.

Hence, the series is convergent if $\frac{x}{e}<1$, i.e., $x<e$ and the series is divergent if $x>e$, by the Root Test. If $x=e$ then the root test (limit form) fails.

Now if $x=e$, then we use some other argument.

We note that if $x=e$, then we have $a_n=\left(1+\frac{1}{n}\right)^{-n}e.$

We know that $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$. This shows that $\sup\left(1+\frac{1}{n}\right)^n=e$ since $\left(1+\frac{1}{n}\right)^n$ is an increasing sequence. Thus we have

$$\sup\left(1+\frac{1}{n}\right)^n=e\Rightarrow\left(1+\frac{1}{n}\right)^n<e\Rightarrow\sqrt[n]{a_n}=e\left(1+\frac{1}{n}\right)^{-n}>1$$

Hence, by the root test, the series is divergent.

***

Remark: You may note that the test fails does not imply that the series is neither convergent nor divergent. We can always use some other argument. We shall state and prove a simple and important theorem, known as Abel's Theorem. This theorem gives a sufficient test for divergence only and not for convergence.

Theorem 8 (Abel's Test or Pringsheim's): If $\sum_{n=0}^{\infty} u_n$ is a convergent series of positive and decreasing terms, then $\lim_{n\to\infty} nu_n = 0$.

Proof: We shall prove this by contrapositive method. We assume that nu_n does not tend to zero. Then it is possible to find a positive number δ such that $nu_n \geq \delta$ for an infinity of values of n. Let n_1 be the first such value of n; n_2 the next such value of n which is more than twice as large as n_1; n_3 the next such value of n which is more than twice as large as n_2; and so on. Then we have a sequence numbers $n_1, n_2, n_3 ...$ such that $n_2 > 2n_1, n_3 > 2n_2, ...$ and so $n_2 - n_1 > \frac{1}{2}n_2$, since $\frac{1}{2}n_2 > n_1$. Similarly, $n_3 - n_2 > \frac{1}{2}n_3, ...$ and also $n_1 u_{n_1} \geq \delta$, $n_2 u_{n_2} \geq \delta, ...$ But, since u_n decreases as n increases, we have

$$u_0 + u_1 + ... + u_{n_1 - 1} \geq n_1 u_{n_1} \geq \delta,$$

$$u_{n_1} + ... + u_{n_2 - 1} \geq (n_2 - n_1) u_{n_2} > \frac{1}{2} n_2 u_{n_2} \geq \frac{1}{2}\delta,$$

$$u_{n_2} + ... + u_{n_3 - 1} \geq (n_3 - n_2) u_{n_3} > \frac{1}{2} n_3 u_{n_3} \geq \frac{1}{2}\delta,$$

and so on. Thus we can bracket the terms of the series $\sum u_n$ so as to obtain a new series whose terms are greater than those of the divergent series

$$\delta + \frac{1}{2}\delta + \frac{1}{2}\delta + ...$$

and therefore $\sum u_n$ is divergent. ■

Remark: The Theorem above gives a sufficient condition for the divergence since $\lim_{n\to\infty} n u_n \neq 0$ implies that $\sum u_n$ is divergent when u_n's are positive and decreasing.

Example 13: Use Abel's theorem to show that $\sum \left(\frac{1}{an+b} \right)$ is divergent where $a \neq 0, b$ are constants.

Proof: We note that $\sum \frac{1}{n}$ is a series whose terms are positive, and decreasing and $\lim_{n\to\infty} \frac{n}{an+b} = \lim_{n\to\infty} \frac{n}{an+b} = \frac{1}{a} \neq 0$. Hence by Theorem 8, the series is divergent.

***

Example 14: Show that Abel's theorem is not true if we omit the condition that u_n decreses as n increases.

Proof: The series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9}+\frac{1}{10^2}+...$$

in which $u_n = 1/n$ or $1/n^2$, according as n is or is not a perfect square, is convergent, since it may be rearranged in the form

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{10^2}+...+\left(1+\frac{1}{4}+\frac{1}{9}+...\right),$$

and each of these series is convergent. But, since $nu_n = 1$ whenever n is a perfect square, it is clearly not true that $nu_n \to 0$].

***

Remark: The converse of Abel's theorem is not true, i.e., it is not true that, if u_n decreases with n and $\lim nu_n = 0$, then $\sum u_n$ is convergent.

For example let us take the series $\sum(1/n)$ and multiply the first term by 1, the second by $\frac{1}{2}$, the next two by $\frac{1}{3}$, the next four $\frac{1}{4}$, the next eight by $\frac{1}{5}$, and so on. On grouping in brackets the terms of the new series thus formed we obtain

$$1+\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{3}\left(\frac{1}{3}+\frac{1}{4}\right)+\frac{1}{4}\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+....;$$

and this series is divergent, since its terms are greater than those of

$$1+\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{3}\frac{1}{2}+\frac{1}{4}\frac{1}{2}+....,$$

which is divergent by Abel's theorem since $\lim\limits_{n\to\infty} n u_n = \frac{1}{2} \neq 0.$ But it is easy to see that the terms of the series

$$1+\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{3}\left(\frac{1}{3}\right)+\frac{1}{3}\left(\frac{1}{4}\right)+\frac{1}{4}\left(\frac{1}{5}\right)+\frac{1}{4}\left(\frac{1}{6}\right)+...$$

satisfy the condition that $nu_n \to 0$. In fact $nu_n = 1/v$ if $2^{v-2} < n \le 2^{v-1}$, and $v \to \infty$ as $n \to \infty$].

***

Here are some exercises for you.

E7) Check the convergence of the series $\sum 3^{-n-(-1)^n}$.

E8) Test the convergence or divergence of the series whose general term is

$$\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}.$$

So far we have discussed three important tests to determine whether a positive terms series is convergent. You recall that most of these tests have been derived in some way from one of the forms of the comparison test. In the next section we shall consider two more tests.

8.7 RAABE'S TEST AND GAUSS'S TEST

In this section we shall discuss two more tests which are much stronger than the earlier tests. These are Raabe's test and Gauss's test. We shall first consider Raabe's test.

Raabe (1801-1859) was Professor at Zurich. He made a lot of important contributions to geometry and analysis. He gave a test for the convergence of a series of positive terms, which is often decisive when the D'Alembert's Test fails. We state the test, without proof and discuss examples to illustrate its use.

Theorem 9 (Raabe's test): Let $\sum a_n$ be a series of positive real numbers. Suppose $\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=1$, and $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\ell$, then the series is convergent if $\ell>1$ and divergent if $\ell<1$.

Let us look at an example.

Example 15: Test the convergence of the series

$$\frac{2.4}{3.5}+\frac{2.4.6}{3.5.7}+\frac{2.4.6.8}{3.5.7.9}+\cdots$$

Solution: Here $a_n=\dfrac{2.4.6\ldots(2n+2)}{3.5.7\ldots(2n+3)}$

Hence $\dfrac{a_{n+1}}{a_n}=\dfrac{2n+4}{2n+5}$ and $\lim_{n\to\infty}\dfrac{a_{n+1}}{a_n}=\lim_{n\to\infty}\dfrac{2n+4}{2n+5}=1$.

Thus, the ratio test fails. Now we apply Raabe's test.

Note that $\lim_{n\to\infty}\dfrac{a_n}{a_{n+1}}=\dfrac{2n+5}{2n+4}=1$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\lim_{n\to\infty}\frac{n}{2n+4}=\frac{1}{2}<1.$$

Hence, by Raabe's Test $\sum_{n=1}^{\infty} a_n$ diverges.

We end this section by discussing Gauss' Test.

Gauss (1777-1855) an eminent German mathematician, gave a very powerful test for convergence which is applicable if Raabe's Test fails. It is not essential that first we apply Ratio Test, then Raabe's Test (if Ratio Test fails) and finally Gauss's Test (if Raabe's Test also fails). We can straightaway apply Gauss Test. We only state this test and then illustrate it by an example.

The proof is omitted as it is too technical and beyond the scope of this course.

Theorem 10: Gauss's Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. Suppose

$$\frac{a_n}{a_{n+1}} = a + \frac{b}{n} + \frac{r_n}{n^p}$$

where $a, b, p \in \boldsymbol{R}, a > 0, p > 1$ and (r_n) is a bounded sequence.
Then:

i) $\sum_{n=1}^{\infty} a_n$ converges, if $a > 1$

ii) $\sum_{n=1}^{\infty} a_n$ diverges, if $a < 1$

iii) $\sum_{n=1}^{\infty} a_n$ converges, if $a = 1, b > 1$

iv) $\sum_{n=1}^{\infty} a_n$ diverges, if $a = 1, b \le 1$

Example 16: Test the convergence of the series

$$\frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \cdots$$

Solution: In this case,

$$a_n = \frac{2^2.4^2.6^2 \ldots (2n)^2}{3^2.5^2.7^2 \ldots (2n+1)^2}.$$

Thus

$$\frac{a_n}{a_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2}$$

$$= \left(1 + \frac{3}{2n}\right)^2 \Big/ \left(1 + \frac{1}{n}\right)^2$$

$$= \left(1 + \frac{3}{2n}\right)^2 \left(1 + \frac{1}{n}\right)^{-2}$$

$$= \left(1 + \frac{9}{4n^2} + \frac{3}{n}\right)\left(1 - \frac{2}{n} + \frac{3}{n^2} - \frac{4}{n^3} + \cdots\right)$$ (By using binomial theorem for negative exponents)

$$= 1 + \frac{1}{n} - \frac{3}{4n^2} + \ldots \text{ higher powers of } \frac{1}{n}$$

$$= 1 + \frac{1}{n} + \frac{1}{n^2}\left(\frac{-3}{4}\right) + \ldots \text{ higher powers of } \frac{1}{n}$$

We put $a=1, b=1, r=-1$ and $r_n = \frac{3}{4} +$ powers of $\frac{1}{n}$. Therefore, (r_n) is a bounded sequence. Hence by Gauss's Test, the given series is divergent.

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Example 17: Test for convergence the series, by first applying Raabe's Test, and then Gauss's Test

$$\sum \frac{1^2 \cdot 3^2 ... (2n-1)^2}{2^2 \cdot 4^2 ... (2n)^2} x^{n-1}, x > 0.$$

Solution: Here $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+1)^2}{(2n+2)^2} \times x = x$

Hence by Ratio Test, the series converges if $x < 1$, and diverges if $x > 1$.

Now for $x = 1$,

$$\frac{a_n}{a_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{(2n+2)^2 - (2n+1)^2}{(2n+1)^2}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{4n^2 + 8n + 4 - 4n^2 - 4n - 1}{(2n+1)^2}\right]$$

$$= \lim_{n\to\infty} \frac{4n^2 + 3n}{(2n+1)^2} = 1.$$

Hence Raabe's Test fails.

Let us now apply Gauss's Test.

$$\frac{a_n}{a_{n+1}} = \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{2n}\right)^{-2}$$

$$= \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(1 - \frac{1}{n} + \frac{3}{4n^2} + ...\right)$$

$$= 1 + \frac{1}{n} - \frac{1}{4n^2} + ... \text{higher powers of } \frac{1}{n}$$

So that by Gauss's Test, the series diverges.

***

Note: We could get the result directly by Gauss's Test, for

$$\frac{a_n}{a_{n+1}} = \frac{1}{x}\left(\frac{2n+2}{2n+1}\right)^2 = \frac{1}{x} + \frac{1/x}{n} - \frac{1/4x}{n^2} + ...$$

where $\alpha = 1/x, \beta = 1/x$.

***

E9) Test the convergence of the series $\frac{1}{2}x+\frac{1.3}{2.4}x+\frac{1.3.5}{2.4.6}x^2+\cdots(x>0)$.

E10) Test the convergence of the series

$$\frac{1^2}{2^2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}+...$$

With this we come to an end of this unit.

Let us now briefly recall the main points discussed in this unit.

8.8 SUMMARY

In this unit we have discussed some special tests for checking the convergence of a positive term series.

1. **Comparison Test:**

   Convergence Test (Case-1): If $0 \le a_n \le b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ is also convergent.

   Divergence Test (Case-2): If $0 \le b_n \le a_n$ for all n and $\sum b_n$ diverges, then $\sum a_n$ is also divergent.

2. **D'Alembert Ratio Test:**

   If $\frac{a_{n+1}}{a_n} \le k < 1$, where k is a constant, then $\sum a_n$ is convergent.

   If $\frac{a_{n+1}}{a_n} \ge 1$, then $\sum a_n$ is divergent.

3. **Cauchy Root Test:**

   If $\forall n,\ \sqrt[n]{a_n} \le k < 1,$ where k is a constant, then $\sum a_n$ is convergent.

   If $\forall n,\ \sqrt[n]{a_n} \ge 1$. Then $\sum a_n$ is divergent.

4. **Raabe's Test:**

   If $\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=1,$ and $\lim\limits_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\ell,$ then the series is convergent if $\ell>1$ and divergent if $\ell<1$.

5. **Gauss's Test:**

   If $\frac{a_n}{a_{n+1}}=a+\frac{b}{n}+\frac{r_n}{n^p}$ where $a,\ b,\ p\in\mathbb{R},\ a>0,\ p>1$ and (r_n) is a bounded sequence. Then

   i) $\sum\limits_{n=1}^{\infty} a_n$ converges, if $a>1$

ii) $\sum_{n=1}^{\infty} a_n$ diverges, if $a < 1$

iii) $\sum_{n=1}^{\infty} a_n$ converges, if $a = 1,\ b > 1$

iv) $\sum_{n=1}^{\infty} a_n$ diverges, if $a = 1,\ b \le 1$

8.8 SOLUTIONS/ANSWERS

E1) i) Consider the series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2^{n-1}+1}+\ldots$$

Compare the series with the convergent geometric series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}+\ldots$$

Clearly $\frac{1}{2^{n-1}+1}<\frac{1}{2^{n-1}}$ for each n. That is, each term of the first series is less than the corresponding term of the second series. Hence, by the Comparison Test, (I) the given series converges.

ii) Consider the series $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$

Let us compare this series with the Harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

You have seen earlier that the Harmonic Series

Now, for each $n, \frac{1}{\sqrt{n}} \ge \frac{1}{n}$. In other words, each terms of the given series is greater than the corresponding term of the harmonic series. Hence, by the Comparison Test (I) the series diverges.

E2) i) Here $a_n = \frac{1}{3n+1}$. Take $b_n = \frac{1}{n}$.

Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{3n+1} = \frac{1}{3}$.

Also $\sum_{n-1}^{\infty} b_n = \sum_{n-1}^{\infty}\frac{1}{n}$ diverges.

Hence, $\sum_{n-1}^{\infty}\frac{1}{3n+1}$ diverges.

ii) Here $a_n = \frac{\sqrt{n}}{n^2-4}$. Take $b_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2}{n^2-4} = 1$$

Also $\sum_{n-1}^{\infty}\frac{1}{n^{3/2}}$ is the p -series will $p=3/2>1$, and hence converges.

Thus $\sum_{n-1}^{\infty}\frac{\sqrt{n}}{n^2-4}$ converges.

E3) Since $\sum_{n=1}^{\infty}u_n$ converges $\lim_{n\to\infty}u_n=0$.

Hence, there is an $M\in\mathbb{N}$ such that $u_n<1$ for $n\geq M$.
So, for $n\geq M$, $u_n^2<u_n$.

Hence, by the comparison Test I, $\sum_{n-1}^{\infty}u_n^2$ converges.

By putting $u_n=\frac{1}{\sqrt{n}}$ and noting that $\sum\frac{1}{n}$ diverges, we get that $\sum\frac{1}{\sqrt{n}}$ diverges.

E4) Here $a_n=\frac{1}{n!}$ and $a_{n+1}=\frac{1}{(n+1)!}$

Hence, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{n!}{(n+1)!}=\lim_{n\to\infty}\frac{1}{n+1}=0<1$.

Hence, $\sum_{n=1}^{\infty}\frac{1}{n!}$ converges.

E5) Let us first use the ratio test. Here

$a_n=\frac{x^n}{n}, a_{n+1}=\frac{x^{n+1}}{n+1}$.

Hence $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(\frac{x^{n+1}n}{(n+1)x^n}\right)=x\lim_{n\to\infty}\frac{n}{n+1}=x$

Thus, when $x<1$, $\sum_{n=1}^{\infty}\frac{x^n}{n}$ converges, and

when $x>1$, $\sum_{n=1}^{\infty}\frac{x^n}{n}$ diverges.

It remains to consider the case $x=1$ because the ratio test fails.

When $x=1$, the series $\sum_{n=1}^{\infty}\frac{1}{n}$, is the harmonic series, which diverges.

Thus, finally, $\sum_{n=1}^{\infty}\frac{x^n}{n}$ converges if $0<x<1$ and diverges if $x\geq 1$.

E6) Here $a_2=\frac{5^n}{(2n+1)!}a_{n+1}=\frac{5^{n+1}}{(2n+3)!}$

Hence $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{5^{n+1}}{2n+3)!},\frac{(2n+1)!}{5^n}$

$$=5\lim_{n\to\infty}\frac{1}{(2n+3)\ (2n+2)}=0<1.$$

Hence the series converges.

E7) Let $a_n = 3^{-n-(-1)^n}$.

Then $b_n = \dfrac{a_{n+1}}{a_n} = \begin{cases} 3, & \text{if } n \text{ is even} \\ \dfrac{1}{27}, & \text{if } n \text{ is odd} \end{cases}$

Hence, $\lim\limits_{n\to\infty} b_n$ does not exist and so the ratio test fails. But

$\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim 3^{-1-\frac{1}{n}(-1)^n} = \dfrac{1}{3} < 1$ and so $\sum a_n$ is convergent by root test.

E8) Here $a_n = \left(\dfrac{1}{(1+1/\sqrt{n})^{n^{3/2}}} \right)^{1/n}$

$$\lim_{n\to\infty} {a_n}^{1/n} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{1+\sqrt{n}}\right)^{\sqrt{n}}}$$

E9) Here $a_n = \dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} x^n$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x$$

By Ratio Test, the series converges if $x < 1$ and diverges if $x > 1$.
If $x = 1$, Ratio Test fails.

When $x = 1$, $\dfrac{a_n}{a_{n+1}} = \dfrac{2n+2}{2n+1}$ so that

$\lim\limits_{n\to\infty} n\left(\dfrac{a_n}{a_{n+1}} - 1\right) = \dfrac{1}{2}$ and so the series diverges by Raabe's Test.

Hence the series is convergent for $x < 1$ and divergent for $x \geq 1$.

E10)
$$\begin{aligned}
\frac{a_n}{a_{n+1}} = \frac{(2n+2)^2}{(2n+3)^2} &= (2n+2)^2 (2n+3)^{-2} \\
&= \left(1+\frac{1}{n}\right)^2 (2n+3)^{-2} \\
&= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1+\frac{3}{2n}\right)^{-2} \\
&= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{6}{2n}+\frac{9}{4n^2}+\ldots\right) \\
&= \left(1-\frac{1}{n}+\frac{1}{n^2}\right)\left(1+\frac{9}{4}\right) + \text{power of } \frac{1}{n} \\
&= \left(1-\frac{1}{n}+\frac{1}{n^2}\right)\left(\frac{13}{4}\right) + \ldots
\end{aligned}$$

Here $a = 1$ and $b \leq 1$ and $\{r_n\}$ is a bounded sequence. Hence the sequence diverges by Gauss test.

UNIT 9

ALTERNATING SERIES

Structure **Page Nos.**

9.1 INTRODUCTION

In the previous unit we discussed some special tests for checking the convergence or divergence of positive term series. You know that an infinite series need not always be a positive term series. In fact, infinite series, in general, can have both an infinite number of negative terms as well as an infinite number of positiver terms. The series which have both negative and positive terms may be classified into two major categories. The first category consists of those infinite series whose terms are alternately positive and negative. Such series are called **Alternating Series**. The other category is one in which terms need not necessarily be alternatively positive and negative that is to say, the infinite series whose terms are mixed and do not follow any specific pattern.

The question, now arises: **How to test the convergence of an infinite series which has both positive and negative terms?** The convergence tests discussed in Unit 8 are not suitable for this purpose because these tests in their present form cannot be applied to these series. Accordingly, the whole content of the unit is divided into three sections.

In Sec. 9.2, we discuss alternating series and its convergence. We shall discuss an important test known as **Leibniz Test** which gives sufficient conditions for convergence of an alternating series.

In Sec. 9.3, we introduce two types of convergence for an alternating series; namely **absolute** and **conditional.** You will see that all the tests for checking the convergence of positive term series are useful for checking the absolute convergence and the conditional convergence may hold when the absolute convergence fails.

Finally, in Sec. 9.4 we briefly discuss the method of **rearrangement** of series.

Objectives

Therefore, after studying this unit, you should be able to

- recognize an Alternating Series;
- apply the Leibniz Test to check the convergence of an Alternating series;
- identify an absolutely convergent series and a conditionally convergent series;
- generate new series by rearranging terms of an infinite series and check the nature of the convergence or divergence of the new series.

9.2 ALTERNATING SERIES

In this section we turn our attention to series that have positive as well as negative terms. Of these, the simplest are those whose terms are alternatively positive and negative.

Some of the examples of an alternating series are

$$\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{9}}+\ldots,$$

$$1-\frac{1\cdot 2}{1\cdot 3}+\frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}-\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7}+\ldots,$$

and $1-1+1-1+\ldots$

Formally, we define an alternating series in the following way:

Defintion 1: An infinite series $\sum_{n=1}^{\infty} a_n$ is called an **alternating series** if any two consecutive terms of the series are of opposite sign.

An alternating series can be written as

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \ldots., \text{where each } a_n > 0. \qquad \ldots (1)$$

If the first term is negative, it can be written as

$$\sum (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 + \ldots. \qquad \ldots (2)$$

The second series can be obtained from the first if you multiply each term of the first series by (-1). Therefore, it is enough to discuss the convergence of the first series.

Note that the terms of an alternating series are alternatively positive and negative. The terms with positive signs are called positive terms and the terms with negative sign are called negative terms.

There is a very simple test for the convergence of an Alternating series. This test is known as Leibniz Test after the name of Leibniz, the eminent German mathematician.

Theorem 1 (Leibniz Test): Let $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ be an alternating series such that

i) $a_n > 0 \ \forall\, n = 1,\, 2,\, 3,\, \ldots$

ii) $a_1 \geq a_2 \geq a_3 \geq \ldots,$ i.e., $(a_n)_n$ is a monotonically decreasing sequence.

iii) $\lim_{n\to\infty} a_n = 0$

Gottfried Leibniz (1646-1716)

Leibniz Test is also Known as **Alternating Series Test.**

Then the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent. Moreover, if $s = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$ and s_n is the nth partial sum, then $|s - s_n| < a_{n+1}$.

Proof: Let us first look at the even partial sums $s_2, s_4, s_6, \ldots$ We have

$$s_2 = a_1 - a_2 \geq 0,$$
$$s_4 = s_2 + (a_3 - a_4) \geq s_2 \text{ because } a_3 \geq a_4,$$
$$s_6 = s_4 + (a_5 - a_6) \geq s_4 \text{ because } a_5 \geq a_6,$$

and, in general,

$$s_{2n+2} = s_{2n} + (a_{2n+1} - a_{2n+2}) \geq s_{2n} \text{ because } a_{2n+1} \geq a_{2n+2}$$

Then $s_2 \leq s_4 \leq s_6 \leq \ldots\ldots$ … (3)

Also $s_{2n} = a_1 - (a_2 - a_3)\ldots(a_{2n-2} - a_{2n-1}) - a_{2n}$.

It follows that $s_{2n} \leq a_1 \ \forall\, n \in \mathbb{N}$. Monotonic convergence then implies that the subsequence (s_{2n}) converges to some s. We next show that (s_n) converges to s. Indeed, if $\varepsilon > 0 \ \exists K$ such that $n \geq K \Rightarrow |s_{2n} - s| \leq \frac{1}{2}\varepsilon$, and $a_{2n+1} \leq \frac{1}{2}\varepsilon$.

Therefore $|s_{2n+1} - s| \leq |s_{2n} - s| + |a_{2n+1}| \leq \varepsilon.$

Thus every partial sum of odd terms is within ε of s. Since $\varepsilon > 0$ is arbitrary (s_n) converges to s and hence $\sum(-1)^{n+1}\cdot a_n$ is convergent and $\sum(-1)^{n+1}a_n = s.$

Now we check the inequality $|s - s_n| < a_{n+1}$.

we note that $\left|(-1)^{n+2}a_{n+1} + \ldots + (-1)^{n+p+1}a_{n+p}\right| = \left|a_{n+1} - a_{n+2} + \ldots + a^{p-1}a_{n+9}\right|$

The sum between absolute value signs can be expressed in the form

$$(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \ldots + \begin{cases} (a_{n+p-1} - a_{n+p}) & \text{of } p \text{ is even} \\ a_{n+p} & \text{if } p \text{ is odd} \end{cases}$$

Since (a_n) is decreasing this shows that the sum is ≥ 0 and therefore the absolute value sign on the right above can be removed. The sum is can be written as

$$a_{n+1} - (a_{n+2} - a_{n-3}) + \ldots \begin{cases} a_{n+p} & \text{if } p \text{ is even} \\ (a_{n+p-1} - a_{n+p)} & \text{if } p \text{ is odd} \end{cases}$$

This shows that the sum is $\leq a_{n+1}$.

Hence the inequality $|\mathrm{s} - \mathrm{s_n}| < \mathrm{a_{n+1}}$ holds.

Hence the theorem. ■

From Unit 7, you know that the condition $\lim_{n\to\infty} a_n = 0$ is necessary for the convergence of the series $\sum_{n=1}^{\infty} a_n$. But according to Leibniz Test, if the given series is an alternating series where the absolute value of the terms form a monotonically decreasing sequence, then the condition $\lim_{n\to\infty} a_n = 0$ is also sufficient for the convergence.

Let us do some examples and exercises:

Example 1: Using the Leibniz Test show that the following series converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots\ldots$$

Solution: This series is known as the Alternating Harmonic Series.

Let us check whether the conditions of the Leibniz Test are satisfied.

We can write the series as $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, where $a_n = \frac{1}{n}$.

Here

i) $a_n > 0$ for all n.

ii) $a_1 > a_2 > a_3 > a_4 > \ldots\ldots.$

iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Thus, the series satisfies all the conditions of the Leibniz Test.

Hence, the series is convergent.

Can you guess what the sum of this series is?

It was proved as a well known result that the limit or **sum of this series is equal to $\ln 2$.**

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Example 2: Check whether the following series satisfies all the conditions of the Leibniz Test.

$$3 - 3^{\frac{1}{2}} + 3^{\frac{1}{3}} - 3^{\frac{1}{4}} + 3^{\frac{1}{5}} \cdots$$

Solution: Let $a_n = 3^{\frac{1}{n}}$. Then the series can be written as $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$. Since $3 > 3^{\frac{1}{2}} > 3^{\frac{1}{3}} > \dots$, therefore the first and second conditons of the Leibniz test are satisfied.

However, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3^{\frac{1}{n}} = 1 \neq 0$. (Recall from Unit 6)

That means, the third condition of the Leibniz test is not satisfied.

***

Example 3: Show that the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n(n+1)} = \frac{3}{2} - \frac{5}{2.3} + \frac{7}{3.4} - \frac{9}{4.5} + \dots$$

is convergent.

Solution: For the given series we have, $a_n = \dfrac{2n+1}{n(n+1)}$. Clearly, $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$. Now

$$\frac{a_{n+1}}{a_n} = \frac{2n+3}{(n+1)(n+2)} \cdot \frac{n(n+1)}{2n+1} = \frac{2n^2+3n}{2n^2+5n+2} < 1$$

Hence $a_n > a_{n+1}$, and therefore the series converges.

***

Now see if you can solve these exercises.

E1) Show that the series $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots$ is convergent.

E2) Show that the series $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$ converges for $p > 0$.

Here we make an important point. The first step in applying the Alternating Series Test is to check that the terms actually alternate in sign. Clearly, the series

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} - \dots$$

has positive and negative terms; yet, it is not an alternating series and therefore the Alternating Series Test cannot be applied. The series diverges. Nor does the alternating series

$$e - e^{1/2} + e^{1/3} - e^{1/4} + \dots$$

meet the requirements of the Alternating Series Test, even though $a_n = e^{1/n}$ is positive and decrease with n. The difficulty is that $\lim_{n\to\infty} a_n \neq 0$ (Since as $n \to \infty$ $e^{1/n} = 1$.) Thus the series diverges [Refer Theorem 1 in Unit 7].

Let us now consider once again the Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

You know that this series diverges. But the Alternating Harmonic Series (as discussed in Example 1) namely

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges. Thus, we have a series that converges because it is an alternating series. If all the negative terms are replaced by the corresponding positive terms, then the convergence is lost. To study this phenomenon in a more general way, we introduce the notions of absolute convergence and conditional convergence in the next section.

9.3 ABSOLUTE AND CONDITIONAL CONVERGENCE

In this section we shall discuss two types of convergence of an altenating series namely absolute convergence and conditional convergence.

Let us consider the following two series:

i) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \cdots$

and

ii) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$

By the Leibniz Test, both the series converge.

Again consider the two series

iii) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \cdots$ and

iv) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots$

obtained from (i) and (ii) by replacing each negative term by its absolute value (i.e $-k$ to $+k$). Then you know from Unit 7 that the series (iii) converges, while the series (iv) diverges. This shows that if you replace an alternating series with a series with all terms having positive signs, then in some cases we get convergence whereas in other cases divergence. This leads us to divide the convergent series into two classes, namely, the 'absolutely convergent series' and the 'conditionally convergent series', which we define as follows:

Definition 2 (Absolute convergence): A series $\sum_{n=1}^{\infty} a_n$ of real numbers is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Definition 3 (Conditional Convergence): A series $\sum_{n=1}^{\infty} a_n$ of real numbers is said to be conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ does not converge i.e. the given series is convergent but not absolutely convergent.

For example the series in (i) in the paragraph at the beginning converges absolutely, while the series (ii) there converges conditonally.

Note: We have defined $\sum_{n=1}^{\infty} a_n$ to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent, but we have not said anything about the behaviour of $\sum_{n=1}^{\infty} a_n$ itself. Whereas we have defined $\sum_{n=1}^{\infty} a_n$ to be conditonally convergent if $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

Next we shall prove another theorem.

Theorem 2: If an infinite series is absolutely convergent, then it is convergent.

Proof: Let $\sum a_n$ be an absolutely convergent series i.e., $\sum |a_n|$ is convergent. We have to prove that $\sum a_n$ is convergent.
Let (s_n) be the sequence of partial sums of $\sum a_n$. Then

$$s_n = a_1 + a_2 + \cdots + a_n.$$

It is enough to show that $(s_n)_n$ is a Cauchy sequence.

Let $(t_n)_n$ be the sequence of partial sums associated with the series $\sum |a_n|$. Since $\sum |a_n|$ is convergent, therefore $(t_n)_n$ is convergent. Thus $(t_n)_n$ is a Cauchy sequence. In other words, for an $\varepsilon > 0$, there exists a positive integer m such that

$$|t_n - t_k| < \varepsilon \text{ for } n > m,\ k > m.$$

Suppose $n > k$. Then

$$\begin{aligned} |s_n - s_k| &= |a_{k+1} + a_{k+2} + \cdots + a_n| \\ &\le |a_{k+1}| + |a_{k+2}| + \cdots + |a_n| \text{ (Recall from Unit 3).} \\ &= |t_n - t_k| < \varepsilon. \end{aligned}$$

Which shows that $(s_n)_n$ is a Cauchy sequence. This completes the proof. ■

Thus every absolutely convergent series is convergent. The converse, however, is not true. That is to say if a series is convergent, then it may not be absolutely convergent. The following example shows this.

Example 4: Test the absolute and conditional convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}.$$

Solution: Here $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{2n+1}$

Let $b_n = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$ Then the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Also

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}.$$

Hence by the Comparison Test, it follows that $\sum_{n=1}^{\infty} |a_n|$ is divergent. Thus $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}}$ is not absolutely convergent.

However, since $\frac{1}{2n+1} < \frac{1}{2n-1} \ \forall\ n$, and $\lim_{n\to\infty} \frac{1}{2n+1} = 0$, therefore by the Leibniz Test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ is convergent. Hence the series is conditionally convergent.

***

All the tests of convergence of infinite series discussed in Unit 8 for series of positive terms can be used to decide absolute convergence of general series.

Example 5: Test the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$, $x \in \mathbb{R}$, for convergence.

Solution: Here $a_n = \frac{\cos nx}{n^2}$

Since $|\cos nx| \le 1$, we get that $|a_n| \le \frac{1}{n^2}$. But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, (Why ?)

This means that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Hence, the series $\sum_{n=1}^{\infty} a_n$, i.e., $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is convergent. [See Theorem 2].

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Now try the following exerices.

E3) Test the conditional convergence of the series $\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{n^3} + \frac{1}{n^5}\right)$.

E4) Test the absolute convergence of the series $\sum \frac{(-1)^{n+1}}{2^n}$.

E5) Determine the values of p for which the series

$1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\frac{1}{5^p}\cdots$ converges conditionally.

In the next section we shall see that we can generate new series from an infinite series with entirely different nature.

9.4 REARRANGEMENT OF SERIES

In this section we shall discuss another aspect of an alternating series. This involves rearrangement of the terms of the series. We shall explain this through an example.

Consider the following series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \qquad \text{... (4)}$$

Suppose we now rearrange the terms of the series by

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots \qquad \text{... (5)}$$

This series can be written in Σ- notation by

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2^n}\right)$$

Then the series in (8) is known as a rearrangement (changing the order of the terms) of the series in (7). Another rearrangement can be obtained by rearranging the terms of the series (7) so that each positive term is followed by two negative terms which gives the series

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\ldots$$

In this way we can have different permutation of the terms and get different rearranged series. You may note that this is very peculiar about an alternating series. This does not happen with positive term series as all the terms are of positive sign only.

We formally make a definition now.

Definition 4: Let $\sum_{n=1}^{\infty} u_n$ be series. Let σ be a one-to-one function from $\mathbb{N}$ onto $\mathbb{N}$. Then $\sum_{n=1}^{\infty} \sigma_{\pi(n)}$ is said to be a rearrangment of $\sum_{n=1}^{\infty} u_n$.

For example $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots$ is a rearrangement of the series

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\cdots$.

Next we shall check whether the rearrangement of a series will preserve the convergence or divergence nature or it will alter the sum of the series.

For that we shall again look into the series in (4) given by

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \qquad \text{... (6)}$$

We have seen in Sec. 9.2 that this series converges by Leibniz test. Infact this series converges to $\ln 2$. The proof of this is beyond the scope of this course.

Let us now consider the rearrangement of the series given in (5). That is

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots \qquad \text{... (7)}$$

Let $(s_n)_n$ denote the sequence of the partial sum of the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots,$$

and $(t_n)_n$ denote the sequence of the partial sum of the rearrangement of the series

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots.$$

We set $r_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln$. Then (r_n) is convergent. We leave this as an exercise for you to try (see E8).

We now have

$$\begin{aligned}
s_{2n} &= 1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{2n-1}-\frac{1}{2n}\\
&= 1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)\\
&= \left(1+\frac{1}{2}+\ldots+\frac{1}{2n}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)\\
&= \left(1+\frac{1}{2}+\cdots+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)\\
&= \left(1+\frac{1}{2}+\cdots+\frac{1}{2n}\right)-\left[1+\frac{1}{2}+\cdots+\frac{1}{n}\right]\\
&= [r_{2n}+\ln 2n]-[r_n+\ln n]\\
&= [r_{2n}-r_n]+\ln 2n-\ln n\\
&= [r_{2n}-r_n]+\ln\frac{2n}{n}\\
&= [r_{2n}-r_n]+\ln 2
\end{aligned}$$

Since $(r_n)_n$ is convergent, therefore $(r_n)_n$ is a Cauchy Sequence.
Consequently, there exists $m \in \mathbb{N}$ such that $| r_{2n} - r_n | < \varepsilon$ for $n \geq m$ where $\varepsilon > 0$ is any number.

This implies that $\lim\limits_{n\to\infty} s_{2n} = \ln 2$.

Now we consider the sequence (s_{2n+1}). Since $\dfrac{(-1)^{n+1}}{n} \to 0$ as $n \to \infty$, we have $\left|s_{2n+1} - \log 2\right| \leq \left|s_{2n} - \log 2\right| + \left|a_{2n} + 1\right| \leq \varepsilon$ for $n \geq m$.

Thus every partial sum of odd terms is within ε of $\log 2$.
We have thus shown that $\lim s_n = \ln 2$.

For the sequence (t_n), we have

$$t_{3n} = \left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}+\frac{1}{4}\right)+\cdots+\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right)$$

$$= \left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right)$$

$$= \left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{4n-1}+\frac{1}{4n}\right]$$

$$-\frac{1}{2}\left[1+\frac{1}{2}+\cdots+\frac{1}{2n}\right]-\frac{1}{2}\left[1+\frac{1}{2}+\cdots+\frac{1}{n}\right]$$

Thus,

$$t_{3n} = (r_{4n} + \ln 4n) - \frac{1}{2}(r_{2n} + \ln 2n) - \frac{1}{2}(r_n + \ln n)$$

$$= \left(r_{4n} - \frac{1}{2}r_{2n} - \frac{1}{2}r_n\right) + \frac{3}{2}\ln 2$$

Again since (r_n) is a Cauchy Sequence, therefore,

$$\lim_{n\to\infty} t_{3n} = \frac{3}{2}\ln 2 .$$

Since $t_{3n+1} = t_{3n} + \dfrac{1}{4n+1}$ and $t_{3n+2} = t_{3n} + \dfrac{1}{4n+1} + \dfrac{1}{4n+3}$.

Therefore $\lim\limits_{n\to\infty} t_n = \dfrac{3}{2}\ln 2$.

That means the series in (7) which is the rearrangement of the series in (6) converges to a different limit $\dfrac{3}{2}\ln 2$.

This shows that the rearrangement of conditionaly convergent series may change its sum or limit.

Another rearrangement gives the series (with sign changed) is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...+\frac{1}{2^n}-1+\frac{1}{2^8+2}+...+\frac{1}{2^{16}}-\frac{1}{3}+...$$

We state here without proof that this series is divergent. The proof is so cumbursome that it is beyond the level of this course.

Hence the rearrangement not only changes the sum of the series but may change its nature also. *Now we ask under what conditions may we rearrange the terms of the series without altering its value?* An answer to this is given by Riemann Rearragement Theorem which says that if an infinite series is conditionally convergent, then its terms can be rearranged so that the new series converges or diverges. The proof of the theorem is not easy and therefore omitted.

We state two theorems (without proof) which will indicate the effect on the convergence of a series if we rearrange the terms of the series.

Theorem 3: If $\sum_{n=1}^{\infty} u_n$ is an absolutely convergent series converging to s, then every rearrangement of $\sum_{n=1}^{\infty} u_n$ also converges to s. ■

Thus, the order in which the terms occur is immaterial in absolutely convergent series.

Is this result true for a conditionally convergent series? Towards an answer to this question, we state the following result without proving it.

Bernhard Riemann (1826-1866)

Theorem 4(Riemann Rearrangement Theorem): Let $\sum_{n=1}^{\infty} u_n$ be a conditionally convergent series. Given any $\alpha \in \mathbb{R}$, there is a rearrangement of the series $\sum_{n=1}^{\infty} u_n$ which converges to α.

You may note that we have already explained how the nature of the series changes with rearrangement for the series.

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots .$$

You should be able to do the following exercise now.

E6) Show that $(r_n)_{n \in r\mathbb{N}}$ where $r_n = 1+\frac{1}{2}+...+\frac{1}{n}-\ell n$ is convergent,

E7) Suppose $\sum_{n=1}^{\infty} u_n$ is a series of positive terms diverging to $+\infty$. Show that every rearrangement of $\sum_{n=1}^{\infty} u_n$ also diverges to $+\infty$.

Let us briefly recall what we have covered in this unit.

9.5 SUMMARY

In this unit we have covered the following:

1) We have introduced a special class of infinite series known as alternating series – a series whose terms are alternatively positive and negative.

2) Stated and proved Leibniz test for checking the convergence of an alternating series.

3) We have discussed the two notions of convergence of an alternating series.

   i) Absolute convergence

   ii) Conditional convergence

4) We have explained how the rearrangement of the terms of an alternating series can change the nature of the original series.

9.6 SOLUTIONS/ANSWERS

E1) Here $a_n = \frac{1}{n!}$. Also $a_n > a_{n+1} > 0$.

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n!} \to 0$ as $n \to \infty$.

$\therefore$ Therefore all the condition of Leibniz test are satisfying.

$\therefore$ the series converges.

E2) Here $p > 0$. Then

$\frac{1}{1^p} > \frac{1}{2^p} > \frac{1}{3^p} > .....$and $\lim_{n\to\infty} \frac{1}{n^p} = 0.$

Hence the series converges by the Leibniz test.

E3) Since $\frac{1}{n^3} + \frac{1}{n^5} > \frac{1}{(n+1)^3} + \frac{1}{(n+1)^5}$, and $\lim_{n\to\infty}\left(\frac{1}{n^3} + \frac{1}{n^5}\right) = 0.$

Therefore by Leibniz test, $\sum_{n=2}^{\infty} (-1)^{n-1}\left(\frac{1}{n^3} + \frac{1}{n^5}\right)$ converges.

But $\sum_{n=1}^{\infty}\left(\frac{1}{n^3} + \frac{1}{n^5}\right) = \sum_{n=1}^{\infty}\frac{1}{n^3} + \sum_{n=1}^{\infty}\frac{1}{n^5}$ also converges.

This means that $\sum_{n=2}^{\infty}(-1)^{n+1}\left(\frac{1}{n^3} + \frac{1}{n^5}\right)$ is absolutely convergent, but not conditionally convergent.

E4) The series $\sum_{n=0}^{\infty}\frac{1}{2^n}$ is a convergent geometric series, so the series

$\sum \frac{(-1)^{n+1}}{2^n}$ is absolutely convergent.

E5) We have shown in E2 that the series converges for $p > 0$. But it is absolutely convergent only for $p \geq 2$. Hence this is conditionally convergent for $p \geq 2$.

E6) We shall show that (r_n) is convergent when

$$r_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln$$
$$= \sum \frac{1}{k} \int_1^n \frac{dx}{x}$$

$$r_n = \sum_{n=1}^{n} \frac{1}{k} - \int_1^n \frac{dx}{x}$$

Observe that $\frac{1}{k+1} \leq \int_k^{k+1} \frac{dx}{x} \leq \frac{1}{k}$.

$$r_{n+1} - r_n = \frac{1}{n+1} - \left[\int_1^{n+1} \frac{dx}{x} - \int_1^n \frac{dx}{x}\right]$$
$$= \frac{1}{n+1} - \int_n^{n+1} \frac{dx}{x}$$
$$\{r_n\}_{n \geq 1} \leq \frac{1}{n+1} - \frac{1}{n+1} \leq 0$$

Thus (r_n) is decreasing.

Also $r_n = \sum_{k=1}^{n} - \left[\int_k^{k+1} \frac{dx}{x}\right.$

$$\geq \sum_{n=1}^{n} \frac{1}{k} - \sum_{k=1}^{n-1} \frac{1}{k} = \frac{1}{n} \geq 0$$

$$r_n \geq 0 \quad \forall n$$

E7) Suppose a rearrangement $\sum_{n=1}^{\infty} u_{\pi(n)}$ of $\sum_{n=1}^{\infty} u_n$ converges.

Note that $\sum_{n=1}^{\infty} u_n$ is itself a rearrangement of $\sum_{n=1}^{\infty} u_{\pi(n)}$.

Hence, $\sum_{n=1}^{\infty} u_n$ would converge, contradicting the hypothesis that $\sum_{n=1}^{\infty} u_n$ diverges.

Hence, every rearrangement of $\sum_{n=1}^{\infty} u_n$ also diverges.

MISCELLANEOUS EXAMPLES AND EXERCISES

The examples and exercises given below cover the concepts and processes you have studied in this block. Doing them will give you a better understanding of the concepts concerned, as well as practice in solving such problems.

We shall first consider miscellaneous examples. You can solve the exercises in a similar way. We advise you not to look at the solutions of the exercises given at the end unless you have tried to solve them on your own.

Miscellaneous Examples

Example 1: Check whether the following statements are true or false. Give reasons for your answers.

a) The sequence of the partial sums of the series $1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}+...$is not a bounded sequence.

b) The series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...$is convergent.

c) The series $\sum a_n$ is convergent if $\lim\limits_{n\to\infty} a_n = 0.$

Solution: a) Let $s_n = 1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}$. Then (s_n) is the sequence of partial sums which is monotonically increasing and is of positive terms. Therefore it cannot be bounded. Thus the statement is true. We get that the statement is true.

b) The series is an alternating series and $a_{n+1} \le a_n \ \forall\ n$ and $\lim\limits_{n\to\infty} a_n = \frac{1}{2n+1} \to 0$ as $n\to\infty.$ Hence by the Leibnitz test, the series is convergent. Therefore the statement is true.

c) Let us consider the series $\sum\limits_{n=1}^{\infty}\frac{1}{n}.$ Then $a_n = \frac{1}{n} \to 0$ as $n\to\infty.$ But $\sum\frac{1}{n}$ is divergent. Hence the statement is false.

***

Example 2: Show that the series $1-\frac{1}{3\times2^2}+\frac{1}{5\times3^2}-\frac{1}{7\times4^2}+...$is convergent.

Solution: Let $\sum a_n = 1-\frac{1}{3\times2^2}+\frac{1}{5\times3^2}-....$ Here $a_n = \frac{1}{2n-1}\times\frac{(-1)^{n-1}}{n^2}.$ Then the series $\sum a_n$ is a series is an alternate terms and each of the terms is positive and monotonically decreasing and $\lim\limits_{n\to\infty}|a_n| = 0.$ Therefore by Liebniz rule, the given series is convergent.

Since $\sum u_n$ converges and b_n is positive and monotonically decreasing, therefore by Abel's test, the given series converge.

***

Example 3: Using the comparison test or the limit comparison test determine the convergence of the following series

i) $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ ii) $\sum_{n=1}^{\infty} \frac{\cos^2(2n)}{n^3}$

Solution: i) Let $a_n = \frac{1}{n+\sqrt{n}}$ and $b_n = \frac{1}{n}$, for $n = 1,2,...,$ so that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n}}$$
$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}}$$
$$= 1 \neq 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, we deduce, from the Limit Comparison Test, that $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ is divergent.

ii) We use the Comparison Test. Since $0 \le \cos^2(2n) \le 1,$ for $n = 1,2,...,$ we have $0 \le \frac{\cos^2(2n)}{n^3} \le \frac{1}{n^3}$, for $n = 1,2,....$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent, we deduce, from the Comparison Test, that $\sum_{n=1}^{\infty} \frac{\cos^2(2n)}{n^3}$ is convergent.

***

Example 4: Use Ratio test to determine whether the following series are convergent.

i) $\sum_{n=1}^{\infty} \frac{n^3}{n!}$ ii) $\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$

Solution: i) Let $a_n = \frac{n^3}{n!}$, for $n = 1,2,...,$ so that

$$\frac{a_{n+1}}{a_n} = \left(\frac{(n+1)^3}{(n+1)!}\right) \times \left(\frac{n!}{n^3}\right)$$
$$= \frac{(n+1)^2}{n^3}$$
$$= \frac{n^3 + 2n + 1}{n^3} = \frac{1}{n} + \frac{2}{n^2} + \frac{1}{n^3}.$$

Also $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0.$ Hence it follows, from the Ratio Test that $\sum_{n=1}^{\infty} \frac{n^3}{n!}$ is convergent.

ii) Let $a_n = \dfrac{n^2 2^n}{n!}$, for $n = 1,2,...,$ so that

$$\frac{a_{n+1}}{a_n} = \left(\frac{(n+1)^2 2^{n+1}}{(n+1)!}\right) \times \left(\frac{n!}{n^2 2^n}\right)$$
$$= \frac{2(n+1)}{n^2}$$
$$= 2\left(\frac{1}{n} + \frac{1}{n^2}\right).$$

Since $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 0$, it follows, from the Ratio Test that $\sum\limits_{n=1}^{\infty} \dfrac{n^2 2^n}{n!}$ is convergent.

***

Example 5: Prove that the following series is convergent and calculate its sum $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{2^n} + \dfrac{3}{n(n+1)}\right)$.

Solution: We know that $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n}$ is convergent, with sum 1, and that $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+1)}$ is convergent, with sum 1.

Hence, by the Sum Rule and the Multiple Rule the series $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{2^n} + \dfrac{3}{n(n+1)}\right)$ is convergent, with sum $1 + (3 \times 1) = 4$.

***

Example 6: Show that $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{n}{3n+2}$ diverges.

Solution: The series is an alternating series. Here $a_n = (-1)^{n+1} \dfrac{n}{3n+2}$.

Therefore $|a_n| = \dfrac{n}{3n+2}$. Also $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \dfrac{n}{3n+2} = \dfrac{1}{3}$.

Thus $\lim\limits_{n\to\infty} a_n$ is not equal 0, therefore the series is divergent.

***

Example 7: Give an example of a series $\sum\limits_{n=1}^{\infty} u_n$ of positive terms such that $\dfrac{u_{n+1}}{u_n} < 1$ for each n, but the series diverges. Does this example contradict the Ratio Test? Justify.

Solution: Let $\sum u_n = \sum \dfrac{1}{n}$, then $\dfrac{u_{n+1}}{u_n} = \dfrac{n}{n+1} < 1 \; \forall \; n$.

But the series $\sum \dfrac{1}{n}$ is divergent.

This does not contradict the Ratio Test because when we take the limits as $n \to \infty$, we find that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1.$

***

Example 8: Test the convergence of the series whose general term is $\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

Solution: Let $u_n = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}$

Then $\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$

$$= \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$= \frac{1}{e} < 1.$$

Hence the series converges.

***

Example 9: Determine whether or not $\sum_n \frac{(-1)n}{2^n}$ diverges, converges conditionally, or converges absolutely.

Solution: Let $a_n = \frac{(-1)^n n}{2^n}$. Then $|a_n| = \frac{n}{2^n}$

$$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} \right| = \frac{1}{2}\frac{n+1}{n}$$

$$\therefore \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \to \frac{1}{2} < 1$$

$\therefore$ The series is absolutely convergent and therefore convergent. Alternatively, the lebeniz rule can also be applied to show that the series is convergent.

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Example 10: Determine the convergence and divergence of the following series $\sum_{n=1}^{\infty} \frac{n^2-3n+4}{5n^4-n}$.

Solution: Let $a_n = \frac{n^2-3n+4}{5n^4-n}$ and $b_n = \frac{1}{n^2}$. Then both a_n and b_n are positive and $\frac{a_n}{b_n} = \frac{n^2 - 3n + 4}{5n^4 - n} \times \frac{n^2}{1}$

$$= \frac{n^4 - 3n^3 + 4n^2}{5n^4 - n}$$

$$= \frac{1 - 3n^{-1} + 4n^{-2}}{5 - n^{-3}}$$

Then $\lim\limits_{n\to\infty} \frac{a_n}{a_n} \to \frac{1}{5}$ as $n \to \infty$. Since $\frac{1}{5} \neq 0$, and the series $\sum \frac{1}{n^2}$ is convergent, the given series is convergent.

***

Miscellaneous Exercises

E1) Find the values of x for which the series $\sum \frac{n^2 - 1}{n^2 + 1} x^n$ converges.

E2) Prove that the following series is convergent and calculate its sum $\sum_{n=1}^{\infty} \left(\left(\frac{3}{4}\right)^n - \frac{2}{n(n+1)} \right)$.

E3) Interpret the decimal $0.999....$ as infinite series, and hence represent them as a fractions.

E4) Prove that the following series are convergent:

i) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^3 + 1}$ ii) $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$

E5) Determine which of the following series are convergent:

i) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ ii) $\sum \frac{n^2}{2n^2 + 1}$

– x –

SOLUTIONS/ANSWERS

E1) **Hint:** Apply D'Alembert's Ratio test. Then the series converges for $x < 1$ and diverges if $x > 1$.

E2) The series $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ is a geometric series, with $a = r = \frac{3}{4}$. Hence, it is convergent, with sum $\frac{\frac{3}{4}}{1 - \frac{3}{4}} = 3$.

The series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ is convergent, with sum 1 (cf. Sub-section 3.1.3).

Hence, by the combination Rules $\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^n-\frac{2}{n(n+1)}\right)$ is convergent,

with sum $3-(2\times 1)=1$.

E3) We interpret $0.999...$ as

$$\frac{9}{10^1}+\frac{9}{10^2}+\frac{9}{10^3}+....$$

This is a geometric series with $a=\frac{9}{10}$ and $r=\frac{1}{10}$. Since $\frac{1}{10}<1$, this series is convergent with sum $\frac{a}{1-r}=\frac{\frac{9}{10}}{1-\frac{1}{10}}=1$; hence $0.999...=1$.

E4) i) **Hint:** $a_n=\frac{(-1)^{n+1}n}{n^3+1}, n=1,2...$

$$|a_n|=\frac{n}{n^3+1}, n=1,2...$$

Apply leibniz test. Then the series is absolutely convergent. Hence it is convergent.

ii) **Hint:** Here $a_n=\frac{\cos n}{2^n}$.

$$|a_n|\le\frac{1}{2^n} \text{ since } |\cos n|\le 1.$$

The series is absolutely convergent and hence convergent.

E5) i) **Hint:** Apply Leibniz test.

ii) Let $a_n=\frac{n^2}{2n^2+1}$

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{2+\frac{1}{n^2}}=\frac{1}{2}$$

$\therefore$ the series is divergent.