Indira Gandhi National Open University
School of Sciences

BMTC-133
REAL ANALYSIS

Volume-II

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BMTC-133
REAL ANALYSIS

Indira Gandhi National Open University
School of Sciences

Block
4

CONTINUITY AND DIFFERENTIABILITY OF FUNCTIONS

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BLOCK INTRODUCTION

In Volume 1, you were introduced to the system of real numbers and the limit point of a set of real numbers. Also, you were introduced to some real functions in this volume. We considered a special function called the sequence and began the study of limiting processes with the notion of convergence of infinite sequence and series. We also discussed a related concept known as 'infinite series' and study the notion of convergence of an infinite series.

In this block, we shall start with the limit concept as applied to arbitrary real functions. The limit of a function, in general, is an abstract notion in the sense that the function never attains its value at a point but tries to approach a value called the limiting value. The limit concept is fundamental to all further ideas in Real Analysis. Therefore, we shall develop it in this block and then use it to discuss the differentiability of function.

This block contains 4 units. In the first unit of the block i.e. in Unit 10, we have review the notion of the limit of a function to which you are already familiar from your study of Calculus. We illustrate certain basic facts about limits through a number of examples. The attempts are made to help you to appreciate the rigorous notion of epsilon-delta definition of the limit of a function and its geometrical meaning. Closely related to the limit of a function is the notion of sequential limits which also we shall introduce in this unit. Finally, we discuss the algebra of limits. In this unit, we also introduce the notion of the continuity of a function at a point and extend it to the continuity of a function on an interval or on a non-empty set of real numbers. Also, we discuss some continuous and discontinuous functions as well as the algebra of continuous functions. Finally we shall introduce the notion of uniform continuity of a function.

In Unit 11, we introduce the notion of the derivative of a function and give its geometrical interpretation. Also, we discuss its relationship with the continuity of a function and then we define the algebraic operations of addition, subtraction, multiplication and division on the differentiable functions.

Unit 12 deals with the important contributions made by Rolle, Lagrange and Cauchy in the form of mean-value theorems. We also discuss the generalized mean-value theorem, intermediate-value theorem and Darboux theorem.

In Unit 13, we confine our discussion to Taylor's and Maclaurin's theorem and discuss applications of differentiability to evaluate some intermediate forms of the functions as well as their extreme-values.

Notations and Symbols (used in Block 4)

(Also see the notations used in Volume I)

$\in \; (\notin)$	belongs to (does not belong to)
$\in$	epsilon
$\mathbb{R}\left(\mathbb{R}^{+}\right)\left(\mathbb{R}^{-}\right)$	the set of real numbers(the set of positive real numbers)(the set of negative real numbers)
$\forall$	for all
$\{x \mid x \text{ satisfies } P\}$	the set of all x such that x satisfies the property P
$\lvert x \rvert$	modulus of the real x
$\lim_{x \to a} f(x)$	limit of $f(x)$ as x tends to a
$x \to f(x)$	a functions f taking x to $f(x)$
$\approx$	is approximately equal to
$\max\{x, y\}$	the maximum of x and y
$\min\{x, y\}$	the minimum of x and y
w.r.t.	with respect to
$\frac{dy}{dx}, y^{(1)}, y', D(y)$	the first derivative of y w.r.t. x.
$\frac{d}{dx}(f(x), f'(x)$	the first derivative of $f(x)$ w.r.t x.
$\frac{d^2y}{dx^2}, y^{(2)}, f''(x)$	the second derivative of y or $f(x)$ w.r.t. x.
$\frac{d^ny}{dx^n}, y^{(n)}, f^{(n)}(x)$	the nth derivative of y or $f(x)$ w.r.t. x.
$\approx$	is approximately equal to

UNIT 10

LIMIT AND CONTINUITY

Structure **Page No.**

10.1 INTRODUCTION

You have studied the course on Calculus. In that course you have been introduced to functions, their types, their domains and ranges. You have also studied the concepts of limits and continuity of functions in that course. So, you already know how to calculate the limit of a function at a point, and to decide whether a function is continuous at a point or not. Here we are going to revisit these concepts. But this time we are going to look closely at the theory, rather than just calculate the limits.

In Section 10.2 we formally introduce the concept of limit by defining it in terms of ε and δ neighbourhoods. The uniqueness of limits is shown and one-sided limits are briefly discussed. We state and prove theorems which characterises the limit of a function in terms of convergence of a sequence thereby providing a link to the earlier units 5 and 6 of this course. Analogous results about limits are also proved such as basic limit properties, the squeeze theorem for limits and the fact that the inequalities are preserved in limit.

In Sec. 10.3 infinite limits are introduced as an extension of the concept of limit and proved some of the basic theorems which are analogs of the results of finite limit.

Continuity of functions is discussed in Sec. 10.4 formally using the concept of limits of functions. Basic properties of continuous functions are shown along with a short discussion on discontinuity.

We define uniform continuity in Sec. 10.5 and establish the relationship between continuity and uniform continuity. We prove an important theorem which states that a continuous function on a bounded closed interval $[a,b]$ is uniformly continuous. The theorem which establishes interface of Cauchy sequences with uniformly continuous functions is also proved.

There are many other important theorems for continuous function which are discussed in the next unit on differentiability such as Bolzano theorem, inverse function theorem etc. These theorems are important not only in proving theorems on differentiation but also to prove many other important theorems in analysis.

Objectives

After studying this Unit, you should be able to

- Define the limit of a function using ε and δ neighbourhoods;
- Calculate the limit, if it exists;
- Define continuity of a function at a point using the $\varepsilon-\delta$ definition of the limit;
- Check whether a function is continuous or not;
- Classify the type of discontinuities;
- Define uniform continuity and differentiate it from continuity.

10.2 LIMIT OF A FUNCTION

In Block 2, Unit 5 and 6 we have defined limit of a sequence in terms of ε and N and proved some important theorems on limits. In this unit we define limit of a function using the concept of neighbourhood.

You recall that when we talk of the limit of a function at a point, we are interested in the behavior of the function very close to that point. How do the function values change when we approach that point? If the function values also seem to be very close to a particular value, then we say, that value is the limit of the function at that point. That is the general idea. But for this idea to make precise, we have to define 'very close', 'approach', in the language of Mathematics. We shall now get down to that. But you would have that noted, we are interested in the behavior of the function **near** the point. However the function value at that point doesn't interest us!

Here is the definition.

It is very important to know the role of ε and δ when apply this definition to find the limits.

Definition 1: Suppose a real-valued function f is defined on a deleted neighbourhood, I, of a point $p \in \mathbb{R}$. We say that $l \in \mathbb{R}$ is a limit of f at p, if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$0 < |x-p| < \delta, x \in I \Rightarrow |f(x)-l| < \varepsilon \qquad \dots (1)$$

If $l \in \mathbb{R}$ is a limit of f at p, we write $\lim_{x \to p} f(x) = l$. We also express it by saying:

***f* approaches *l* as *x* approaches *p*.**

Sometimes we also write $f(x) \to l$ **as** $x \to p$.

Remark 1: Note that whenever we are discussing the limit of a function at a point p, we consider only the values of the function around that point. Certain terminologies will help us to simplify the discussion. An open interval containing a point p of the form $]p-\varepsilon, p+\varepsilon[$ for any $\varepsilon > 0$ is called an **ε-neighbourhood of** p. Also if I is a neighbourhood of p, then $I \setminus \{p\}$ is called a **deleted neighbourhood** of p. Based on these we can rewrite the definition as follows:

Definition 2: A function f defined on a deleted neighbourhood I of a point $p \in \mathbb{R}$, is said to have a limit $\ell \in \mathbb{R}$, if given an ε-neighbourhood of ℓ there exists a deleted δ-neighbourhood of p such that whenever x lies in the deleted δ-neighbourhood of p, $f(x)$ lies in ε-neighbourhood of ℓ. The following figure illustrates this

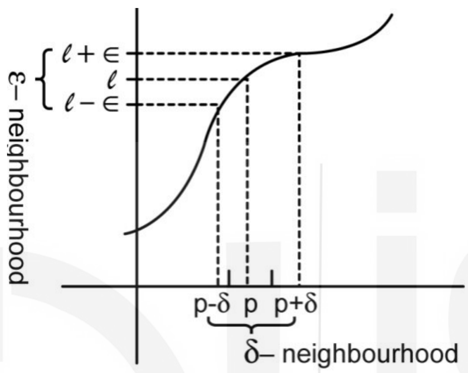

Fig. 1: Definition of Functional Limit

Sometimes an ε-neighbourhood of p is denoted $N_\varepsilon(p)$ and a deleted δ-neighbourhood of p is denoted by $N'_\delta(p)$. We can then say that the limit exists if

$$x \in N'_\delta(p) \cap I \Rightarrow f(x) \in N_\varepsilon(\ell).$$

Remark 2: i) The positive number δ, in the definition, depends on ε.

ii) The condition $0 < |x-p|$ tells us that $x \neq p$, making it clear, that for defining the limit of a function at a point, we do not consider the value of the function at that point. This means, even if a function is not defined at a point, the limit of the function at that point may exist. Recall that in Definitions 1 and 2 we say that f is defined on a **deleted neighbourhood**, $N'_\delta(p)$ of the point $p \in \mathbb{R}$.

iii) If the limit of the function f at p does not exist, then we say that f **diverges** at p.

We will now illustrate these points through some examples. For each of the functions in the examples below, the domain is taken as the largest set of real numbers for which the definition of the function makes sense. So, the domain for the first two functions is $\mathbb{R}$, while that for the third one is $\mathbb{R} \setminus \{2\}$.

Example 1: If $f(x) = K$ is a constant function, then show that $\lim_{x \to p} f(x) = K$.

Solution: For every positive number ε, we need to find a positive number δ, such that

$$0 < |x-p| < \delta, x \in \mathbb{R} \Rightarrow |f(x) - K| < \varepsilon. \qquad \ldots (2)$$

Now, $|f(x)-K| = |K-K| = 0 < \varepsilon$ for all $x \in \mathbb{R}$.

Hence, Eqn. (2) is true for any value of $x \in \mathbb{R}$.

Thus, $\lim_{x \to p} f(x) = K.$

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Example 2: If $f(x) = 2x+3$ and $p = 3,$ then using your knowledge of Calculus, you know that $\lim_{x \to 3} f(x) = 9$. Find a value for δ, when $\varepsilon = 1,$ and when $\varepsilon = 0.1.$ Also we find the limit using Definition 1.

Solution: Let us first take the case when $\varepsilon = 1.$

Now, $|f(x)-9| = |2x+3-9| = 2|x-3| < 1$ will be true, if $|x-3| < 0.5$. So, if we take $\delta = 0.5,$ then Eqn. (1) will be satisfied for $\varepsilon = 1$. Next, if $\varepsilon = 0.1$, you can easily work out that $\delta = 0.05$ will satisfy Eqn. (1).

Next we shall find the limit using Definition 1.

Let $\varepsilon > 0$ be given. We must find a δ so that if $0 < |x-3| < \delta,$ then

$|2x+3-9| < \varepsilon$

$|2x+3-9| = 2|x-3|$

So we set $\delta = \frac{\varepsilon}{2}$. Then if x satisfies $0 < |x-3| < \delta,$ then

$$|2x+3-9| = 2|x-3| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon$$

Hence the limit exists and it is 9.

***

Example 3: If $f(x) = \frac{x^2-4}{x-2}$, then f is not defined at 2. Show that the limit of f at 2 is 4.

Solution: Let $\varepsilon > 0$ be given. We must find a δ so that if $0 < |x-2| < \delta,$ then $|f(x)-4| < \varepsilon$. To find $\delta,$ we proceed as follows:

Now, 2 does not belong to the domain of this function. And when $x \neq 2, x-2 \neq 0.$ Therefore, we can divide by $x-2$.

Thus, $|f(x)-4| = \left|\frac{x^2-4}{x-2} - 4\right| = |x+2-4| = |x-2|$. This will be less than $\delta,$ whenever we choose $\delta = \varepsilon.$ Then if $|x-2| < \delta,$ then $|f(x)-4| = |x-2| < \delta = \varepsilon.$

This shows that the required limit is 4.

***

Note: From the above examples, you must have observed that in each case, we first find $|f(x)-l|$, and try to express it in terms of $|x-p|$, to arrive at a value of δ.

Next we shall show that the limit of a function, if it exists, is a unique number.

Theorem 1: Suppose a real-valued function f is defined on a deleted neighbourhood, I, of a point $p \in \mathbb{R}$. Then f can have only one limit at p.

Proof: On the contrary, suppose l and m are limits of f at p. Then by Definition 1 for any $\varepsilon > 0,$ there exists $\delta_1 > 0,$ such that $0 < |x - p| < \delta_1,\ x \in I \Rightarrow |f(x) - l| < \varepsilon/2,$ and there also exists $\delta_2 > 0,$ such that $0 < |x - p| < \delta_2, x \in I \Rightarrow |f(x) - m| < \varepsilon/2$.

Suppose $\delta = \min\{\delta_1, \delta_2\}$.

Then, if we choose $0 < |x - p| < \delta, x \in I,$ we get that
$|l - m| = |l - f(x) + f(x) - m| \leq |f(x) - l| + |f(x) - m| < \varepsilon$.

This means that we can make $|l - m|$ less than any positive number ε. This can only be true if $|l - m| = 0,$ that is, if $l = m.$

So, if the limit exists, it is unique. ■

We shall make some remarks now.

Remark 3: i) Definition 1 can also be expressed as follows:
$\lim\limits_{x \to p} f(x) = l,$ if $\forall \varepsilon > 0, \exists \delta > 0,$ such that whenever the distance between x and p is less than δ, and x belongs to I, then the distance between $f(x)$ and l is less than ε.

We can also say, whenever x belongs to the deleted δ neighbourhood I of p, the function value $f(x)$ belongs to the ε neighbourhood of l or,

$$x \in N'_\delta(p) \cap I \Rightarrow f(x) \in N_\varepsilon(\ell).$$

ii) Another interpretation of Definition 2 is that If $\lim\limits_{x \to p} f(x) \neq l,$ then the statement *"there exists* $\varepsilon > 0,$ *such that for every* $\delta > 0,$
$x \in N'_\delta(p) \cap I \Rightarrow f(x) \in N_\varepsilon(l)$" is not true. This means, there exists $x \in N'_\delta(p) \cap I,$ such that $f(x) \notin N_\varepsilon(l)$. Or, in other words, there is a member $x_0 \neq p$ of I, whose distance from p is less than δ, but the distance of $f(x)$ from l is not less than ε.

Let us see example.

Example 4: Show that $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{x}{|x|}$ does not exist.

Solution: Suppose that the limit does exist. Let us take $\varepsilon = 1$ and let $\delta > 0$ be arbitrary. Consider $0 < |x - 0| < \delta$ or $0 < |x| < \delta$. Then there are two possibilities: either $0 < x < \delta$ or $-\delta < x < 0.$

If $0 < x < \delta,$ then $f(x) = 1$ and if $-\delta < x < 0,$ then $f(x) = -1.$ Therefore the right hand limit is 1 and the left hand limit is $-1.$

Consequently the desired limit does not exist (refer theorem 1).

***

Next we establish the connection between the limit of a function and limit of a sequence that you have studied in Unit 5. Our next theorem brings out the connection between these two limits.

Theorem 2: Suppose a real-valued function f is defined on a deleted neighbourhood, I, of a point p. Then the following statements are equivalent:

i) $\lim_{x \to p} f(x) = l$

ii) If (x_n) is a sequence in I that converges to p, then the sequence $(f(x_n))$ converges to l.

Proof: We need to prove i) implies ii), and ii) implies i).

i) implies ii): Suppose $\lim_{x \to p} f(x) = l$. Further, suppose (x_n) is a sequence in I, which converges to p. We have to prove that $(f(x_n)) \to l$. Let $\varepsilon > 0$, be given. Then since $\lim_{x \to p} f(x) = l$, there exists a $\delta > 0$, such that

$$0 < |x - p| < \delta, x \in I \Rightarrow |f(x) - l| < \varepsilon \qquad (3)$$

Then, since $\{x_n\}$ converges to p, for this positive number δ, there exists a natural number n_0, such that

$$n \geq n_0 \Rightarrow |x_n - p| < \delta \qquad (4)$$

Combining (3) and (4), we get

$$n \geq n_0 \Rightarrow |x_n - p| < \delta \Rightarrow |f(x_n) - l| < \varepsilon.$$

This shows that $(f(x_n)) \to l$, and we have proved that i) implies ii).

ii) implies i): Here we are given that for every sequence (x_n) in I, which converges to p, the sequence $(f(x_n))$ converges to l. Then we have to prove that $\lim_{x \to p} f(x) = l$.

We assume that $\lim_{x \to p} f(x) \neq l$, then by Remark 2 (ii), there exists $\varepsilon > 0$, such that for every $\delta > 0$ there exists an element $x \in I$, such that $0 < |x - p| < \delta$, but $|f(x) - l| > \varepsilon$.

In particular, for every natural number n, (i.e. by taking $\delta = \frac{1}{n}$) there exists $x_n \in I$, such that

$$0 < |x_n - p| < \frac{1}{n}, \text{ but } |f(x_n) - l| > \varepsilon.$$

This shows that $(f(x_n))$ does not converges to p whereas the sequence (x_n)

converges to p. This is not possible (in view of the hypothesis).

Hence our assumption that $\lim_{x\to p} f(x) \neq \ell$ is not true. Thus ii)$\Rightarrow$i).

Hence the theorem. ■

Example 5: Show that the following functions do not have a limit at 0.

i) $f(x) = 1/x, x \neq 0,$

ii) $f(x) = \sin(1/x), x \neq 0$.

Solution: i) Consider the sequence $(x_n) = (1/n)$. Then $(f(x_n)) = (n)$. This sequence does not have a limit, as it is unbounded. So, we have a sequence $(x_n) = (1/n)$ which converges to 0, but the sequence $(f(x_n)) = (n)$ does not converge. Therefore by Theorem 2 we conclude that limit of $f(x) = 1/x$, $x \neq 0,$ as $x \to 0$ does not exist.

ii) In this case we are going to consider two sequences converging to 0.

$(x_n) = \left(\frac{1}{n\pi}\right)$, and $(y_n) = \left(\frac{1}{2n\pi + \frac{\pi}{2}}\right)$. Both these sequences converge to 0 as $n \to \infty$. [Please refer to Unit 5]

Now $(f(x_n)) = (0)$, the constant sequence, which converges to 0.

And $f(y_n) = \sin(2n\pi + \frac{\pi}{2}) = 1$. So, $(f(y_n)) = (1)$, the constant sequence, which converges to 1. Therefore, by Theorem 1 (on uniqueness of limits) $\lim_{x\to 0} \sin(1/x)$ does not exist.

***

Next we shall prove another theorem that relates existence of limit of a function and boundedness of f, in a neighbourhood.

Theorem 3: Suppose a real-valued function f is defined on a deleted neighbourhood, I, of a point $p \in \mathbb{R}$. If $\lim_{x\to p} f(x)$ exists, then f is bounded in some deleted neighbourhood of p.

Proof: Suppose $\lim_{x\to p} f(x) = l$. Then for $\varepsilon = 1$, there exists $\delta > 0$, such that

$0 < |x - p| < \delta, x \in I \Rightarrow |f(x) - l| < 1$.

Therefore, $0 < |x - p| < \delta \Rightarrow ||f(x)| - |l|| \leq |f(x) - l| < 1$.

Or, $0 < |x - p| < \delta, x \in I \Rightarrow |f(x)| < |l| + 1$. This means f is bounded on $N'_\delta(p) \cap I$, which is a deleted neighbourhood of p. ■

So far we discussed how to compute limits of some simple functions at some given points by directly applying the definition. Infact, using the definition to get the limit is little tedious. Here are two theorems, which will help us get the limits of many more functions, if they exist, more easily. The first one of these is called the **Sandwich or the Squeeze Theorem.**

Theorem 4 (Sandwich or squeeze theorem): Suppose the real-valued functions f, g and h are defined on a deleted neighbourhood, I, of a point $p \in \mathbb{R}$. Suppose

i) $f(x) \le g(x) \le h(x), \forall x \in I$

and

ii) $\lim_{x \to p} f(x) = l = \lim_{x \to p} h(x)$.

holds. Then $\lim_{x \to p} g(x)$ exists and is equal to l.

Proof: Since (ii) holds, for any given $\varepsilon > 0, \exists \delta_1 > 0, \delta_2 > 0$, such that

$0 < |x - p| < \delta_1, x \in I \Rightarrow |f(x) - l| < \varepsilon$, and

$0 < |x - p| < \delta_2, x \in I \Rightarrow |h(x) - l| < \varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - p| < \delta, x \in I \Rightarrow l - \varepsilon < f(x) < l + \varepsilon,$ and $l - \varepsilon < h(x) < l + \varepsilon$.

Therefore, $0 < |x - p| < \delta, x \in I \Rightarrow l - \varepsilon < f(x) \le g(x) \le h(x) < l + \varepsilon$.

This means $0 < |x - p| < \delta, x \in I \Rightarrow |g(x) - l| < \varepsilon$, that is, $\lim_{x \to p} g(x) = l$. ■

Theorem 5 (Algebra of Limits): Suppose the real-valued functions *f* and *g* are defined on a deleted neighbourhood, I, of a point $p \in \mathbb{R}$. If $\lim_{x \to p} f(x) = l$, and $\lim_{x \to p} g(x) = m$, then

i) $\lim_{x \to p} (f + g)(x) = l + m.$

ii) $\lim_{x \to p} (f - g)(x) = l - m.$

iii) $\lim_{x \to p} fg(x) = lm.$

iv) $\lim_{x \to p} kf(x) = kl,$ where $k \in \mathbb{R}$.

v) If $g(x) \ne 0$ for all $x \in I$, and if $\lim_{x \to p} g(x) = l \ne 0,$ then $\lim_{x \to p} (\frac{f}{g})(x) = l/m.$

vi) If $f(x) \ge 0$ for all $x \in I$, then $\lim_{x \to p} \sqrt{f(x)} = \sqrt{l}$.

This theorem can be proved using the definition of limit, or by using Theorem 2, which is the sequential criterion of limit. You have proved a similar theorem, about limits of sequences in Unit 7. Using that theorem, it becomes easier to use sequential criterion to prove this theorem. For illustration, we shall use the definition to prove i), and then use sequential criterion to prove the remaining. Let us start proving i) to vi) one by one.

Proof: We shall begin with (i).

i) Since $\lim_{x \to p} f(x) = l,$ and $\lim_{x \to p} g(x) = m$, for any given $\varepsilon > 0, \exists \delta_1 > 0, \delta_2 > 0,$ such that

$0 < |x - p| < \delta_1, x \in I \Rightarrow |f(x) - l| < \varepsilon/2,$ and

$0 < |x - p| < \delta_2, x \in I \Rightarrow |g(x) - m| < \varepsilon/2$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $0 < |x - p| < \delta, x \in I \Rightarrow |(f + g)(x) - (l + m)| = |f(x) + g(x) - l - m|$

$\le |f(x) - l| + |g(x) - m| < \varepsilon$.

Thus, we have proved that $\lim_{x\to p}(f+g)(x) = l+m$.

ii) Since $\lim_{x\to p} f(x) = l$, and $\lim_{x\to p} g(x) = m$, by Theorem 2 if (x_n) is a sequence in I, such that $(x_n) \to p$, then $(f(x_n)) \to l$ and $(g(x_n)) \to m$.

Therefore,
$((f-g)(x_n)) = (f(x_n) - g(x_n)) \to l - m$. [Recall the results from the Unit 5 on limits of sequence.]

Thus ii) is proved.

iii) Again, since $\lim_{x\to p} f(x) = l$, and $\lim_{x\to p} g(x) = m$, if (x_n) is a sequence in I, such that $(x_n) \to p$, then $(f(x_n)) \to l$ and $(g(x_n)) \to m$. Therefore, $((fg)(x_n)) = (f(x_n)g(x_n)) \to lm$.

Thus, iii) is proved.

iv) We prove this exactly in the same way. We leave the proof to you. See E1).

v) We first show that $\lim_{x\to p} \frac{1}{g(x)} = \frac{1}{m}$.

Since $\lim_{x\to p} g(x) = m$, for any given $\varepsilon > 0, \exists \delta_1 > 0,$ such that

$$0 < |x-p| < \delta_1, x \in I \Rightarrow |g(x) - m| < \left|\frac{m}{2}\right|.$$

$$\Rightarrow |m| - |g(x)| \le |g(x) - m| < \left|\frac{m}{2}\right|$$

$$\Rightarrow |g(x)| > \left|\frac{m}{2}\right|.$$

Again, since $\lim_{x\to p} g(x) = m$, for any given $\varepsilon > 0, \exists \delta_2 > 0,$ such that

$$0 < |x-p| < \delta_2, x \in I \Rightarrow |g(x) - m| < \frac{\varepsilon m^2}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $0 < |x-p| < \delta, x \in I \Rightarrow \left|\frac{1}{g(x)} - \frac{1}{m}\right| = \left|\frac{g(x)-m}{mg(x)}\right| < 2\left|\frac{g(x)-m}{m^2}\right| < \varepsilon$

So, we have proved that $\lim_{x\to p} \frac{1}{g(x)} = \frac{1}{m}$.

Now using the product rule, iii), we conclude that $\lim_{x\to p} \left(\frac{f}{g}\right)(x) = l/m$.

vi) Since the limit of f exists at p, f is bounded on some deleted neighbourhood of p. Further, $f(x) \ge 0 \forall x \Rightarrow l \ge 0$. (Can you prove this statement? We have left it as an exercise for you. See E3).

We first consider the case when, $l = 0$.

Since $\lim_{x\to p} f(x) = 0, \forall \varepsilon > 0, \exists \delta > 0,$ such that

$$0 < |x-p| < \delta, x \in I \Rightarrow |f(x)| < \varepsilon^2$$

Then $0 < |x - p| < \delta, x \in I \Rightarrow \left|\sqrt{f(x)}\right| < \varepsilon$.

Thus, $\lim\limits_{x \to p} \sqrt{f(x)} = 0$.

Now, suppose $l > 0$.

Since $\lim\limits_{x \to p} f(x) = l, \forall \varepsilon > 0, \exists \delta > 0,$ such that

$0 < |x - p| < \delta, x \in I \Rightarrow |f(x) - l| < \varepsilon\sqrt{l}$

Then $0 < |x - p| < \delta, x \in I \Rightarrow \left|\sqrt{f(x)} - \sqrt{l}\right| = \left|\dfrac{f(x) - l}{\sqrt{f(x)} + \sqrt{l}}\right|$

$$= \frac{|f(x) - l|}{\sqrt{f(x)} + \sqrt{l}} \leq \frac{f(x) - l}{\sqrt{l}} < \varepsilon.$$

Thus, $\lim\limits_{x \to p} \sqrt{f(x)} = \sqrt{l}$.

This result can also be proved by using the sequential criterion and Theorem in Unit 7.

Hence the theorem. ■

Using Theorem 5 we can easily find the limits of polynomial functions and also some rational functions. Our next example shows this.

Example 6: Find the limits of the following functions at the given points.

i) $f(x) = 4x^3 + 3x^2 - 7$, as $x \to -3$.

ii) $f(x) = \dfrac{2x^4 - 3x^3 - 6x + 15}{x^2 - 4}$, as $x \to 1$.

Solution: i) Now, $\lim\limits_{x \to -3} x = -3.$ Therefore, using Theorem 5, we get $\lim\limits_{x \to -3} x^2 = 9,$ and $\lim\limits_{x \to -3} x^3 = -27.$ Again using Theorem 5, we get

$\lim\limits_{x \to -3} (4x^3 + 3x^2 - 7) = 4(-27) + 3(9) - 7 = -88.$

ii) We shall first check whether the function satisfies the conditions stated in Theorem 5. The function f is a rational function. The polynomial in the denominator has two zeroes, 2 and -2. So, we can choose a neighbourhood of 1, on which this polynomial is non-zero. Further, $\lim\limits_{x \to 1} (x^2 - 4) = -3 \neq 0.$ Thus the function satisfies all the required conditions of Theorem 5 v). Therefore we have

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{2x^4 - 3x^3 - 6x + 15}{x^2 - 4} = \frac{\lim\limits_{x \to 1} (2x^4 - 3x^3 - 6x + 15)}{\lim\limits_{x \to 1} (x^2 - 4)} = \frac{8}{-3}.$$

***

As you have seen in Example 3, we can easily find the limits of polynomial and rational functions.

Next we discuss the limits of another important class of functions, namely, trigonometric functions.

You must have some idea of the limits of trigonometric functions in Calculus.

We start with the sine function, and prove that $\lim_{x\to p} \sin x = \sin p$. Then using the trigonometric identities, and Theorem 5, we shall find the limits of the remaining functions.

Example 7: Show that i) $\lim_{x\to 0} \sin x = 0.$

ii) $\lim_{x\to p} \sin x = \sin p.$

Solution: i) We are first going to show that

$$0 \le |\sin x| \le |x|, \forall x \in \mathbb{R}. \quad \dots (5)$$

Let us consider the case when $0 < x \le 1$. Look at Fig. 1 in which $OP = 1$. In triangle OPM, $\sin x = PM < (arc \text{ length } PA) = x.$ Therefore, (5) is true in this case.

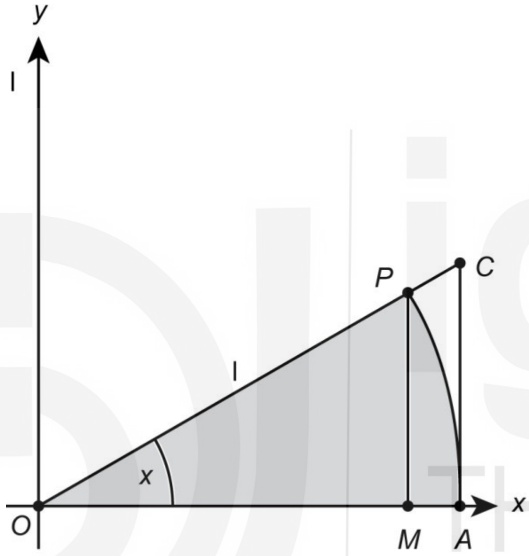

Fig. 2: Triangle OPM, Sector OPA

Now, suppose $-1 < x < 0.$ Let $y = -x.$ Then $0 < y < 1,$ and therefore, $0 \le |\sin y| \le |y|$.

This means, $0 \le |\sin(-x)| \le |-x| \Rightarrow 0 \le |-\sin x| \le |-x| \Rightarrow 0 \le |\sin x| \le |x|$.

Thus, we have shown that $0 \le |\sin x| \le |x|, \forall x \in \mathbb{R}.$

This means, $-x \le \sin x \le x, \forall x \in \mathbb{R}$.

We know that $\lim_{x\to 0} x = 0.$, and $\lim_{x\to 0}(-x) = -\lim_{x\to 0} x = 0.$

Using the Sandwich Theorem, we can conclude that $\lim_{x\to 0} \sin x = 0.$

ii) Here we have to show that $\lim_{x\to p} \sin x = \sin p.$ For that, we have to show that $\forall \varepsilon > 0, \exists \delta > 0,$ such that $0 < |x-p| < \delta \Rightarrow |\sin x - \sin p| < \varepsilon$. But,

$$|\sin x - \sin p| = \left|2\cos\left(\frac{x+p}{2}\right)\sin\left(\frac{x-p}{2}\right)\right| \le 2\left|\sin\left(\frac{x-p}{2}\right)\right| \le 2\left|\frac{x-p}{2}\right| = |x-p|$$

Therefore, if we choose $\delta = \varepsilon$, then $0 < |x - p| < \delta \Rightarrow |\sin x - \sin p| < \varepsilon$, and it follows that $\lim_{x \to p} \sin x = \sin p.$

***

Example 8: Show that $\lim_{x \to p} \cos x = \cos p$.

Solution: Now,

$$|\cos x - \cos p| = \left| -2\sin\frac{x+p}{2}\sin\frac{x-p}{2} \right| \le 2\left|\sin\frac{x-p}{2}\right| \le 2.\frac{1}{2}|x-p| = |x-p|.$$

So, if we choose $\delta = \varepsilon$, then $0 < |x - p| < \delta, \Rightarrow |\cos x - \cos p| < \varepsilon$.

Thus, $\lim_{x \to p} \cos x = \cos p.$

***

Now using Examples 4 and 5, and the theorem on Algebra of Limits, we can show that

i) $\lim_{x \to p} \tan x = \tan p$.

ii) $\lim_{x \to p} \cot x = \cot p$.

iii) $\lim_{x \to p} \sec x = \sec p$.

iv) $\lim_{x \to p} \operatorname{cosec} x = \operatorname{cosec} p$.

Example 9: Show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Solution: Consider the part of the unit circle in the first quadrant shown in Fig. 3.

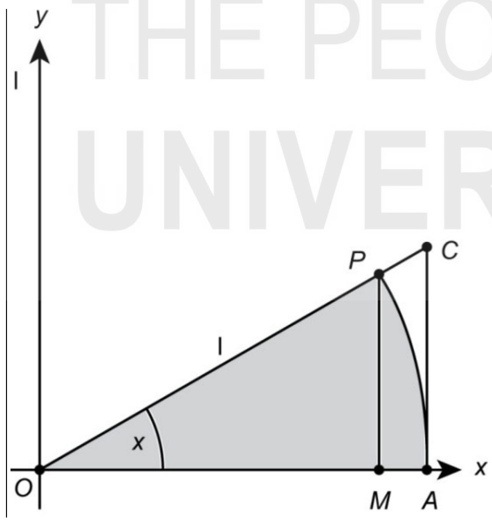

Fig. 3: Triangle OPM, Sector OPA, Traingle OPT

Then PT is perpendicular to OP and PM is perpendicular to OA. Since $(PM) < (arc\text{ length } PA) < (PT)$, we get

$$\sin x < x < \tan x \Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x} \Rightarrow \cos x < \frac{\sin x}{x} < 1.$$

Now taking limits as $x \to 0$, and using the Sandwich Theorem, we get

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

***

Example 10: Show that i) $\lim_{x\to 0} \sin(1/x)$ does not exist

ii) $\lim_{x\to 0} x\sin(1/x) = 0.$

Solution: i) Consider the sequence $\left(\frac{1}{2n\pi + \pi/2}\right)_{n\in\mathbb{N}}$. This sequence tends to zero as $n \to \infty$.

Now the sequence $\left(\sin\left(1/\frac{1}{2n\pi+\pi/2}\right)\right) = \left(\sin\left(2n\pi + \frac{\pi}{2}\right)\right) = (1) \to 1,$ as $n \to \infty$.

On the other hand, if we take the sequence $\left(\frac{1}{2n\pi}\right)_{n\in\mathbb{N}}$, which also has limit 0, we find that the sequence $\left(\sin\left(1/\frac{1}{2n\pi}\right)\right)_{n\in\mathbb{N}} = (\sin(2n\pi)) = (0) \to 0,$ as $n \to \infty$. Since these two sequences tending to zero have two different limits, we conclude that $\lim_{x\to o} \sin(1/x)$ does not exist.

ii) We know that $\sin(1/x) \le 1$. Therefore, $0 \le |x\sin(1/x)| \le |x|$.
Taking limits as $x \to 0$, and using the Sandwich theorem we get $\lim_{x\to 0} x\sin(1/x) = 0$. Fig. 4 shows the graphs of these two functions.

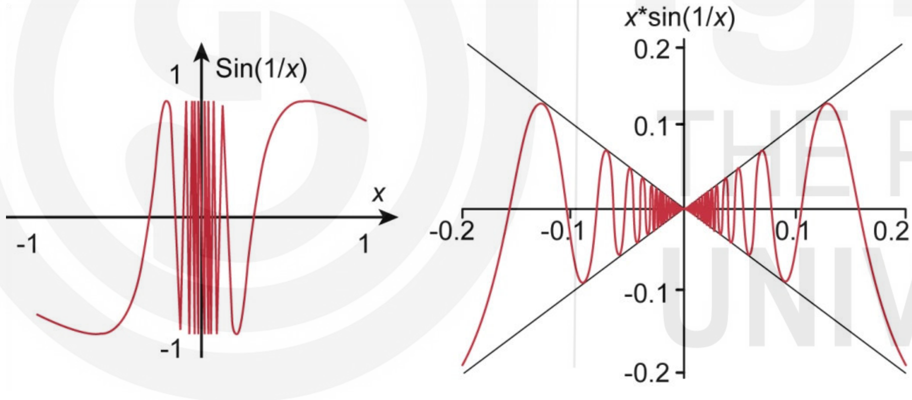

Fig. 4

You can see that the graph of $\sin(1/x)$ oscillates wildly between -1 and 1, as x approaches zero. The graph of $x\sin(1/x)$ also oscillates, but it is clear that the amplitude of the oscillations is decreasing as x tends to zero.

***

Before we proceed further we want you to try the following exercises and see if you have understood what you have learnt so far.

E1) Prove Theorem 5 iv).

E2) If $\delta, \varepsilon, l, p, f$ have the same meaning as in Definition 1, find δ in the following:

i) $f(x) = 5x - 11, p = 2, \varepsilon = 0.5$

ii) $f(x) = \sqrt{x}, p = 4, \varepsilon = 0.5$

iii) $f(x)=\dfrac{x}{1+x}, p=1, \varepsilon=1$.

E3) If $f(x)\geq 0$ for every x in a deleted neighbourhood, I, of p, and if $\lim\limits_{x\to p} f(x)$ exists, then $\lim\limits_{x\to p} f(x)\geq 0$.

E4) Using $\varepsilon-\delta$ definition, find $\lim\limits_{x\to 2}(x^2-7x)$.

In this section you have seen various examples of limits of functions. In the next one we are going to extend the concept of limits. We are going to see how a function behaves as $x\to\infty$.

10.3 SOME EXTENSIONS OF THE LIMIT CONCEPT

In this section we study the limit of a function as x tends to ∞ or $-\infty$ and the notion of one-sided limits.

Till now we have considered limits of functions as x tends to some real number p But in many cases for instance $\sqrt{a}, e^x, \sqrt{x}$ etc. We need to study the behavior of a function as x takes larger and larger values, that is, as $x\to\infty$. Here is the definition:

Definition 3: i) Let $f:(a,\infty)\to\mathbb{R},$ where $a\in\mathbb{R}$. We say that limit of f as $x\to\infty$ is l, if $\forall\varepsilon>0, \exists K\in\mathbb{R}, K>0,$ such that $x>K\Rightarrow |f(x)-l|<\varepsilon$.
ii) Let $f:(-\infty,b)\to\mathbb{R},$ where $b\in\mathbb{R}$. We say that limit of f as $x\to-\infty$ is $l,$ if $\forall\varepsilon>0, \exists K\in\mathbb{R}, K>0,$ such that $x<-K\Rightarrow |f(x)-l|<\varepsilon$.

The following example will help you understand this definition.

Example 11: Show that i) $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)=0$.

ii) $\lim\limits_{x\to\infty}\left(\dfrac{1}{x^n}\right)=0,$ where $n\in\mathbb{N}$.

Solution: i) Now, $|f(x)-l|=\left|\dfrac{1}{x}-0\right|=\left|\dfrac{1}{x}\right|$. This will be less than ε, if $|x|>\dfrac{1}{\varepsilon}$.

So, for a given $\varepsilon>0$, if we take $K=\dfrac{1}{\varepsilon}$, then $x>K\Rightarrow |f(x)-l|=\left|\dfrac{1}{x}\right|<\varepsilon$.

Therefore, $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)=0$.

ii) Here $|f(x)-l|=\left|\dfrac{1}{x^n}-0\right|=\left|\dfrac{1}{x^n}\right|$. This will be less than ε, if $|x^n|>\dfrac{1}{\varepsilon}$. So, for a given $\varepsilon>0,$ if we take $K=\dfrac{1}{\varepsilon^{1/n}}$, then $x>K\Rightarrow |f(x)-l|=\left|\dfrac{1}{x^n}\right|<\varepsilon$.

Therefore, $\lim\limits_{x\to\infty}\left(\dfrac{1}{x^n}\right)=0$.

***

The results in Theorem 5 on algebra of limits holds for infinite limits also. Here we shall state it without proof.

Theorem 6: Suppose real-valued functions f and g are defined on $(a,\infty), a \in \mathbb{R}$. If $\lim_{x\to\infty} f(x) = l$, and $\lim_{x\to\infty} g(x) = m$, then prove that

i) $\lim_{x\to\infty}(f+g)(x) = l+m$.

ii) $\lim_{x\to\infty}(f-g)(x) = l-m$.

iii) $\lim_{x\to\infty} fg(x) = lm$.

iv) $\lim_{x\to\infty} kf(x) = kl$, where $k \in \mathbb{R}$.

v) If $g(x) \neq 0$ for all $x \in I$, and if $\lim_{x\to p} g(x) = m \neq 0$, the $\lim_{x\to p} f(x) = l$.

vi) If $f(x) \geq 0$ for all $x \in I$, then $\lim_{x\to\infty} \sqrt{f(x)} = \sqrt{l}$. ■

Example 12: Show that $\lim_{x\to\infty}\left(\frac{3x^2-2x+5}{x^2+1}\right) = 3.$

Solution: We divide the numerator and denominator of f by the highest power of x.

Then we get $f(x) = \dfrac{3-\frac{2}{x}+\frac{5}{x^2}}{1+\frac{1}{x^2}}$. Now, the limit of the numerator is 3, and that of the denominator is 1. Therefore, the required limit is 3.

You must have noticed that we have obtained the limits obtained in i) and ii), using the algebra of limits. We have proved the algebra of limits for limits as $x \to p \in \mathbb{R}$. But it is also true for limits as $x \to \infty$. We ask you to prove this in E5).

***

You are already familiar with the evaluation of limits at infinity from your study of BMTC-131 Calculus course. Now we give you few exercises to check your knowledge.

E5) Evaluate the following limits using only the definition.

i) $\lim_{x\to\infty} \frac{3x^2}{x^2+2} = 3$

ii) $\lim_{x\to\infty} e^{-ax} = 0,$ where a is a positive real number

iii) $\lim_{x\to\infty} \frac{e^x}{e^x+1} = 1$

E6) Determine the following limits. Justify your answer.

i) $\lim_{x\to\infty} \frac{x^3-2x+5}{3x^3+6x^2+7}$

ii) $\lim_{x\to\infty} \frac{x+\cos x}{x+\sin x}$

Next we shall consider one-sided limits.

One-sided limits

You have seen that in finding the limit of a function at a point, we consider the behavior of the function as x, the independent variable approaches that point. Now, on a real line, a point can be approached in two ways: from the left, and from the right (see Fig. 5). Both these approaches are taken into account while finding the limit at a point.

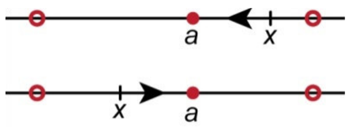

Fig.5

But, what if we consider only one approach at a time, say, we study the behavior of the function as x approaches the point from the left. If we can arrive at a limit, that limit will be called the **left hand limit of the function** at that point. Similarly, if we consider only the right hand approach, and arrive at a limit, that will be called the **right hand limit of the function** at that point. These are also called one-sided limits.

We formally give the definition.

Definition 4: Suppose a real-valued function f is defined on a deleted neighbourhood, I , of a point $p \in \mathbb{R}$. Then,

i) we say that $l \in \mathbb{R}$ is a right hand limit of f at p, if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$0 < x - p < \delta, x \in I \Rightarrow |f(x) - l| < \varepsilon \qquad \dots (6)$$

If $l \in \mathbb{R}$ is a right hand limit of f at p, we write $\lim_{x \to p^+} f(x) = l$.

ii) we say that $l \in \mathbb{R}$ is a left hand limit of f at p, if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$0 < p - x < \delta, x \in I \Rightarrow |f(x) - l| < \varepsilon. \qquad \dots (7)$$

If $l \in \mathbb{R}$ is a left hand limit of f at p, we write $\lim_{x \to p^-} f(x) = l$.

We have examples of functions, which do not have a limit at a given point, but the left hand limit of that function at that point exists and the right hand limit also exists. We discuss one such function below.

For example, let us consider the greatest integer function defined on $\mathbb{R}$. Its graph is shown in Fig. 6. Focus your attention on the graph around $x = 2$. You can see from the graph, that, for $1 \le x < 2, [x] = 1$, and for $2 \le x < 3, [x] = 2$.

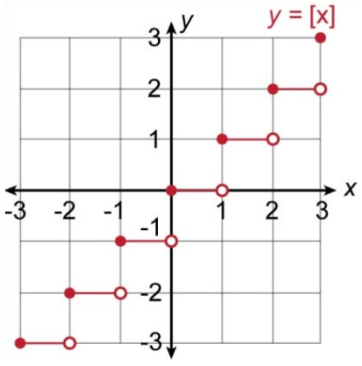

Fig. 6: Greatest Integer Function

So, as we approach 2 from the right side, $[x]$ approaches 2. That is,
$\lim_{x\to 2^+}[x]=2$.

It is also clear that as we approach 2 from the left side, $[x]$ approaches 1. That is, $\lim_{x\to 2^-}[x]=1$.

Thus, for the greatest interger function, both the one-sided limits exist, but are not equal at 2.

Do you realize that the limit of $[x]$ at 2 does not exist? Indeed the $\lim_{x\to 2^{\frac{1}{4}}}[x]=2$

and the $\lim_{x\to 2^-}[x]=1$ and they are unequal.

This is the situation for this function at every integer point. That is, at every integer, both the one-sided limits of $[x]$ exist and are unequal.

Let us see some more example.

Example 13: Let the function f be given by

$$f(x)=\begin{cases}\dfrac{x^3}{5x+12}, x<4\\ \sqrt{x}, x>4.\end{cases}$$

Evaluate $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$.

Solution: Using the algebra of limits, we get that

$$\lim_{x\to 4^-} f(x)=\lim_{x\to 4^-}\frac{x^3}{5x+12}=\frac{4^3}{20+12}=\frac{64}{32}=2.$$

By definition of f, $\lim_{x\to 4^+} f(x)=\lim_{x\to 4^+}\sqrt{x}=\sqrt{4}=2.$

Thus, in this case the left hand limit and right hand limit both exists and are equal.

***

The next theorem tells you how these different types of limits are related.

Theorem 7: Suppose a real-valued function f is defined on a deleted neighbourhood, I, of a point $p\in\mathbb{R}$. Then the following are equivalent:

i) $\lim_{x\to p} f(x)=l$

ii) $\lim_{x\to p^-} f(x)=\lim_{x\to p^+} f(x)=l$.

Proof: Suppose $\lim_{x\to p} f(x)=l$

Then for every $\varepsilon>0$, there exists a $\delta>0$, such that

$0<|x-p|<\delta, x\in I\Rightarrow |f(x)-l|<\varepsilon$. This means

$0 < x - p < \delta, x \in I \Rightarrow |f(x) - l| < \varepsilon$, and
$0 < p - x < \delta, x \in I \Rightarrow |f(x) - l| < \varepsilon$.

Therfore, $\lim_{x \to p^+} f(x) = l$ and $\lim_{x \to p^-} f(x) = l.$
Thus, i) implies ii).

Now, suppose the two one-sided limits exist and are equal to l. Then for every $\varepsilon > 0,$ there exists $\delta_1 > 0,$ such that

$$0 < x - p < \delta_1, x \in I \Rightarrow |f(x) - l| < \varepsilon$$

and there exists a $\delta_2 > 0,$ such that $0 < p - x < \delta_2, x \in I \Rightarrow |f(x) - l| < \varepsilon$
If we choose $\delta = \min\{\delta_1, \delta_2\}$, then $0 < |x - p| < \delta, x \in I \Rightarrow |f(x) - l| < \varepsilon$.
Therefore, $\lim_{x \to p} f(x) = l$.

Thus, ii) implies i). ■

Remark 4: Theorem 7 implies that the following holds

i) If any one of the one-sided limit of a function does not exists, then the limit of the function does not exist.
ii) If both the one-side limits exist and are unequal, then the limit does not exist.

We illustrate Remark (1) in the following example.

Example 14: Check whether the $\lim_{x \to 0^+}$ and $\lim_{x \to 0^-} f(x)$ exist for the following function

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

What can you conclude about $\lim_{x \to 0} f(x)$?

Solution: Since $f(x) = 1$ for all $x > 0,$ $\lim_{x \to 0+} f(x) = +1$. Similarly since $f(x) = -1$ for all $x < 0, \lim_{0^-} f(x) = -1.$

Since $\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x), \lim_{x \to 0} f(x)$ does not exist.

***

Next we shall a theorem which gives the connection between the limit of a function and limit of a sequence.

Theorem 8: Suppose a real-valued function f is defined on a deleted neighbourhood, I, of a point, $p \in \mathbb{R}$. Then the following are equivalent:

i) $\lim_{x \to p^+} f(x) = l$
ii) If $(x_n), x_n > p \forall n$ is a sequence in I that converges to p, then the sequence $(f(x_n))$ converges to l. ■

The proof of Theorem 2 can be modified to prove the above stated results.

Theorem 8 is also useful in checking the existence of a limit of a function. The following example shows this.

Example 15: Check whether the limit $f(x)$ exists for the function $f(x) = 2^{\frac{1}{(x-1)}}$ as $x \to 1$.

Solution: We shall check whether the right hand side limit and left hand side exist. Let us first consider the right hand side limit $\lim_{x\to 1^+} 2^{\frac{1}{(x-1)}}$. Let $\varepsilon > 0$ be given. Then for any $\delta > 0$, we choose a positive integer M_0 such that $\frac{1}{M_0} < \delta$. Then if $n \geq M_0$, we get $1+\frac{1}{n} \in]1, 1+\delta[$ and $\frac{1}{2^{1+\frac{1}{n}-1}} = 2^n \geq 2^{M_0}$. This shows that $\lim_{x\to 1^+} g(x)$ does not exist. Therefore by Remark 4 the $\lim_{x\to 1} f(x)$ does not exist.

***

To refresh your memory, we repeat the exercises on one-sided limits that you have done in BMTC-131.

E6) Find $\lim_{x\to\infty^+} f(x)$, where $f(x)$ is given by the following:

i) $\frac{\sqrt{x}-5}{\sqrt{x}+6}, x > 0$

ii) $\frac{\sqrt{x}+1}{x}$

E7) Prove that

i) $\lim_{x\to 4^+} (x - [x]) = 0$

ii) $\lim_{x\to 0^+} \frac{|x|}{x} = 1$

This discussion on limits of functions leads us to the definition of continuous functions, which we take up in the next section.

10.4 CONTINUOUS FUNCTIONS

Continuous functions form a very important class of real-valued functions. A continuous function can be thought of as one, whose graph is unbroken. So, the greatest integer function is not continuous, since its graph consists of a number of steps with gaps. But this characterization of continuity is not precise. In fact, later you will come across a function, whose graph appears unbroken, but which is not continuous. The precise definition of continuous functions emerged in the nineteenth century through the works of mathematicians Bolzano and Cauchy.

We shall now give the definition.

Definition 5: Let f be a function defined on a neighbourhood I of p.
Then f is said to be continuous at p, if $\forall \varepsilon > 0, \exists \delta > 0$, such that

$|x-p|<\delta, x\in I \Rightarrow |f(x)-f(p)|<\varepsilon$.

You must have noticed that this definition is similar to the definition of a limit. In fact a function is continuous at a point, if the limit of $f(x)$ as x approaches the point p is $f(p)$, the value of the function at that point. So, a function is continuous at p, if

i) f is defined at p, and
ii) $\lim\limits_{x\to p} f(x) = f(p)$.

If a function is not continuous at a point, then we say that it is **discontinuous at that point.**

If a function is continuous at every point of a set A, then it is said to be continuous on A.

Our study of limits in the earlier section helps us decide about the continuity of functions.

For example the function $f(x)=x$ is continuous on $\mathbb{R}$. The function $f(x)=1/x$ is not continuous at $x=0$, since it is not defined at 0. But it is continuous at every point of its domain. The function $f(x)=\sin x$ is continuous on $\mathbb{R}$. The greatest integer function $f(x)=[x]$ is not continuous at integers, since we have seen that this function does not have a limit at integer points. But it is continuous on $\mathbb{R}-\mathbb{Z}$.

The sequential criterion for limits also helps to formulate a similar criterion for continuity at a point:

We shall prove the following theorem.

Theorem 9 (Sequential definition for continuity): Let f be a function defined on a neighbourhood I of p. Then f is continuous at p if and only if for every sequence (x_n) of A, converging to p, the sequence $(f(x_n))$ converges to $f(p)$.

Proof: Let us suppose that f is continuous at p. Then $\lim\limits_{x\to p} f(x) = f(p)$. Given $\varepsilon>0$, there exists a $\delta>0$ such that $|x-p|<\delta \Rightarrow |f(x)-f(p)|<\varepsilon$.
If x_n is a sequence converging to 'a', then corresponding to $\delta>0$, there exists a positive integer M such that $|x_n-a|<\delta$ for $n\geq M$.
Thus, for $n\geq M$, we have $|x_n-p|<\delta$ which, in turn, implies that $|f(x_n)-f(p)|<\varepsilon$, proving thereby $f(x_n)$ converges to $f(p)$.

Conversely, let us suppose that whenever x_n converges to p, $f(x_n)$ converges to $f(p)$. Then we have to prove that f is continuous at p. For this, we have to show that corresponding to an $\varepsilon>0$, there exists some $\delta>0$ such that

$$|f(x)-f(p)|<\varepsilon, \text{ whenever } |x-p|<\delta.$$

If not, i.e., if f is not continuous at p, then there exists an $\varepsilon>0$ such that whatever $\delta > 0$ we take there exists an x_δ such that

$$|x_\delta - p| < \delta \text{ but } |f(x_\delta) - f(a)| \geq \varepsilon.$$

By taking $\delta = 1, 1/2, 1/3, \ldots$ in succession we get a sequence $\{x_n\}$, where $x_n = x_\delta$ for $\delta = 1/n$, such that $|f(x_n) - f(p)| \geq \varepsilon$. The sequence $\{x_n\}$ converges to p. For, if $m > 0$, these exists M such that $1/n < m$ for $n \geq M$ and therefore $|x_n - p| < m$ for $n \geq M$. But $f(x_n)$ does not converge to $f(p)$, a contradiction to our hypothesis. This completes the proof of the theorem. ■

That brings us to the next theorem about the **algebra of continuous functions.**

Theorem 10: Suppose f, g, h are functions defined on an interval I of a point p. If f, g, h are continuous at p, and if $k \in \mathbb{R}$, and $h(x) \neq 0$ for all $x \in I$, then $f + g, f - g, fg, f, f/h$ are all continuous at p.

Proof: We prove the continuity for $f + g$, and leave the rest to you.
Since f is continuous at p, $\forall \varepsilon > 0, \exists \delta_1 > 0$, such that

$$|x - p| < \delta_1 \Rightarrow |f(x) - f(p)| < \frac{\varepsilon}{2}.$$

Similarly, since g is continuous at p, for this same $\varepsilon > 0, \exists \delta_2 > 0$, such that

$$|x - p| < \delta_2 \Rightarrow |g(x) - g(p)| < \frac{\varepsilon}{2}.$$

If we choose $\delta = \min\{\delta_1, \delta_2\}$, then it follows that

$$|x - p| < \delta \Rightarrow |(f + g)(x) - (f + g)(p)|$$
$$= |f(x) + g(x) - f(p) - g(p)| \leq |f(x) - f(p)| + |g(x) - g(p)| < \varepsilon$$

Thus, $f + g$ is continuous at p.

In this proof we have used Definition 5. We can also prove it using the Algebra of Limits:

Now, since f and g are defined at p, $f + g$ is also defined at p.
Further, $\lim\limits_{x \to p}(f + g)(x) = \lim\limits_{x \to p}(f(x) + g(x)) = \lim\limits_{x \to p} f(x) + \lim\limits_{x \to p} g(x) = f(p) + g(p)$, since f and g are continuous at p.

Hence we have proved that $f + g$ is continuous at p. ■

Theorem 11: If f is continuous at p, then $|f|$ is also continuous at p.

Proof: Since f is continuous at p, $\forall \varepsilon > 0, \exists \delta > 0$, such that

$$|x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon.$$

Now, $||f(x)| - |f(p)|| \leq |f(x) - f(p)|$. Therefore,

$$|x - p| < \delta \Rightarrow ||f(x)| - |f(p)|| \leq |f(x) - f(p)| < \varepsilon.$$

Thus, we have shown that $|f|$ is continuous at p. ■

We have seen earlier, that f is continuous at p, if

a) f is defined at p, and
b) $\lim_{x \to p} f(x) = f(p)$.

So, even if one of a) and b) is not true, f cannot be continuous at p. Sometimes, f is not defined at p, but its limit exists at p. In such a case, we can define a new function, which is equal to $f(x)$ at all points except p, and which takes the value $\lim_{x \to p} f(x)$ at p. This new function is then continuous at p. But if $\lim_{x \to p} f(x)$ does not exist, then we cannot assign any value to $f(p)$ to make the function continuous. Our next example will illustrate this point.

Example 16: Check the functions, a) $\sin\left(\frac{1}{x}\right)$ and b) $x\sin\left(\frac{1}{x}\right)$ for continuity at point 0.

Solution: a) This function is not defined at $x = 0$, and $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist, as we have shown in Example 10. Therefore, the function is not continuous at 0.

[Look at the graph of this function shown in Fig. 4. It does not seem to have a break at $x = 0$. Looks are deceptive! The function is discontinuous at $x = 0$.]

b) This function is also not defined at 0. But since $0 \le \left| x\sin\left(\frac{1}{x}\right) \right| \le |x|$, by applying the Sandwich Theorem we can say that $\lim_{x \to 0} x\sin\left(\frac{1}{x}\right) = 0$, So, we define a function h as

$$h(x) = x\sin\left(\frac{1}{x}\right), x \ne 0,$$
$$= 0, x = 0$$

Then h is continuous at $x = 0$.

Note that it is also continuous at every other point in $\mathbb{R}$.

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Both the functions in this example are discontinuous at $x = 0$. But the discontinuity in the second one is removable in the sense that we can redefine the function at $x = 0$ to have continuity. We shall discuss about the types of discontinuities little later.

Given two functions f and g, we can define their composite function $f \circ g$ if the range of g is a contained in the domain of f.

The next theorem discusses the continuity of composite functions.

Theorem 12: Let f be a function defined on a neighbourhood, I, of $p \in \mathbb{R}$. Let f be a function defined on a neighbourhood, J, of $f(p)$, such that J is a subset of the range of f. If f is continuous at p, and if g is continuous at $f(p)$, then the composite function $f \circ g$ is continuous at p.

Proof: Since g is continuous at $f(p)$, $\forall \varepsilon > 0, \exists \eta > 0,$ such that
$|x - f(p)| < \eta, x \in J \Rightarrow |g(x) - g(f(p))| < \varepsilon.$

Now, since f is continuous at p, for this $\eta > 0, \exists \delta > 0,$ such that
$|x - p| < \delta, x \in I \Rightarrow |f(x) - f(p)| < \eta.$

Combining the two we can say that $\forall \varepsilon > 0, \exists \delta > 0,$ such that
$|x - p| < \delta, x \in I \Rightarrow |f(x) - f(p)| < \eta, f(x) \in J \Rightarrow |g(f(x)) - g(f(p))| < \varepsilon$.

Thus, $f \circ g$ is continuous at p. ■

Alternate Proof: Let (x_n) be a sequence in I, such that $\lim_{x \to p} x_n = p$. Then the sequence $f((x_n))$ converges to $f(p)$ since f is continuous at p. The sequence $(f(x_n))$ is in J. Now, since g is continuous at $f(p)$, the sequence $(g(f(x_n)))$ will converge to $g(f(p))$.

Therefore, we conclude that $f \circ g$ is continuous at p, using sequential criterion. ■

Algebra of Continuous Functions tells us that if f and g are functions continuous at a point p, then $f + g$ and fg are also continuous at p. The converse of this is not true. That is, if $f + g$ is continuous at p then f and g need not be continuous at p. Same is the case with fg. We now give you some examples to support our argument.

Suppose $f : \mathbb{R} \to \mathbb{R}, f(x) = 0,$ if $x \in [0,1]$, and $f(x) = 1,$ if $x \notin [0,1]$.
And suppose $g : \mathbb{R} \to \mathbb{R}, g(x) = 1,$ if $x \in [0,1]$, and $g(x) = 0,$, if $x \notin [0,1]$.

Both these functions are discontinuous at 0 and 1. But

$f + g : \mathbb{R} \to \mathbb{R}, (f + g)(x) = 1,$ which is continuous on $\mathbb{R}$.
$fg : \mathbb{R} \to \mathbb{R}, (fg)(x) = 0,$ which is also continuous on $\mathbb{R}$.

Using Theorems 8 and 9, we can decide on the continuity of many more functions. Take the example of $f(x) = \sqrt{\sin x}$. This function is continuous on its domain, because we can write it as a composite of two functions, g and h, that is, $f = g \circ h, h(x) = \sin x, g(x) = \sqrt{x}$. Since both g and h are continuous functions, f is also continuous. Here note that f will be defined only for those x, for which $\sin x \geq 0$.

So far you have seen that a function f is discontinuous at a point if either the limit of f exists at c and is not equal to $f(x)$ or the limit does not exist at the point c. Accordingly we categorise the types of discontinuities in the following way.

1. **Removable discontinuity:** A function f has removable discontinuity at a point c if $\lim_{x \to c} f(x)$ exists and is not equal to $f(c)$. Such discontinuity can be removed by assigning a suitable value to the function at $x = c$.

2. **Discontinuity of the first kind:** When $\lim_{x \to c-} f(x)$ and $\lim_{f(x) \to c+} f(x)$ exists and are unequal, then we say that f has **discontinuity of the first kind**.
3. **Discontinuity of the second kind:** When neither $\lim_{x \to c-} f(x)$ nor $\lim_{f(x) \to c+} f(x)$ exists, we say that f has **discontinuity of the second kind.**

Let us see some examples.

Example 16: Examine the types of Discontinuity of the functions at $x = 0$.

i) $$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & \text{if } x \neq 0 \\ 3, & \text{if } x = 0 \end{cases}$$

ii) $$f(x) = \begin{cases} x - |x|, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$$

Solution: Let us try one by one.

i) Note that here

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{2x} \times 2 = 2.$$

But $f(0) = 3.$ Therefore f is not continuous.

Since the $\lim_{x \to 0} f(x)$ exists and is not equal to $f(0),$ if has a removable discontinuity at 0. Here the discontinuity can be removed by redefining the function f at 0 by 2 instead of 3.

ii) We have $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x + x}{x} = 2.$

and

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x - x}{x} = 0.$$

Since both the left and right limits exists and are unequal, the function has discontinuity of the first kind.

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Example 17: Let f be the function defined on $]-1,1[$ by

$$f(x) = \begin{cases} x, & \text{if } x = \dfrac{1}{n} \text{ for } n \in 0 \\ 0, & \text{otherwise} \end{cases}$$

Show that f is continuous at 0 and discontinuous at all other points.

Solution: To determine where in the interval $]-1,1[\, f$ is continuous, we first take any point $c, 0 < c < 1$ and not of the form $1/n$ for $n \in \mathbb{N}$. Then exists a unique interger n_0 such that $1/(n_0 + 1) < c < 1/n_0$. On the neighbourhood

$(1/(n_0+1), 1/n_0)$ of c, $f(x)=0$, and so $\lim_{x\to c} f(x) = 0 = f(c)$. Thus f is continuous at c; and similarly f is continuous at c for any $c<0$ and not equal to $1/n$ for $n \in \mathbb{Z}$.

On the other hand, if $c = 1/n$ for some $n \in \mathbb{N}$, $f(x) = 0$ on the deleted neighborhood $\left(\frac{1}{(n+1)}, \frac{1}{(n-1)}\right) - \left\{\frac{1}{n}\right\}$ of $\frac{1}{n}$, so $\lim_{x\to 1/n} f(x) = 0 \neq 1/n = f(1/n)$; and similarly for $c = -1/n$ for some $n \in \mathbb{N}$. The function is discontinuous at the points $x = 1/n, n \in \mathbb{Z} - \{0\}$.

Consider now the point 0. Any neighborhood of 0 contains points for which $f(x) = 0$ and points for which $f(x) \neq 0$. Nonetheless, for all x we have $0 \leq |f(x)| \leq |x|$, so it follows that $\lim_{x\to 0} f(x) = 0 = f(0)$. Hence, f is continuous at 0.

Try these exercises now. You have already done similar ones in the course Calculus BMTC-131.

E10) If $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x) = \begin{cases} 2, x \in Z \\ 3, x \notin Z \end{cases}$, is f continuous at

i) $x = 1$, ii) $x = 1.5$?

E11) Let f be defined for all $x \in \mathbb{R}$ by $f(x) = \frac{x^2 - 6x + 9}{x - 3}, x \neq 3, f(3) = 5$. Is f continuous at $x = 3$? Is f continuous on $\mathbb{R}$? If f has a discontinuity, is it a removable discontinuity?

If so, how do you redefine f to make it continuous?

E12) Use the sequential definition of continuity to show that the function $f(x) = |x|$, is continuous on $\mathbb{R}$?

E13) Suppose f is defined by $f(x) = \begin{cases} x-1, & x \leq 0 \\ 2x, & 0 < x \leq 4 \\ 3x-4, & x > 4 \end{cases}$. Is f continuous on $\mathbb{R}$? What are the types of discontinuities? Justify your answer.

E14) Examine the following functions for continuity on $\mathbb{R}$. Also, draw the graph in each case and see if it is broken or not:

a) $f(x) = \begin{cases} x+1, x < 1 \\ x^2, 1 \leq x \leq 2 \\ x+2, x > 2 \end{cases}$

b) $f(x) = \begin{cases} x-1, x \leq 0 \\ x^2 - 1, 0 < x \leq 2 \\ \dfrac{3x^2}{4(x-1)}, x > 2 \end{cases}$

c) $f(x) = x - [x]$.

In the next section we shall discuss uniform continuity.

10.5 UNIFORM CONTINUITY

In this section we introduce a stronger form of continuity known as "uniform continuity". You will learn that this type of continuity has a number of important properties.

Let us recall the definition of a continuous function at a point p.

We say that a function f, is continuous at p, if $\forall \varepsilon > 0, \exists \delta > 0,$ such that $|x-p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$.

Here, a positive number ε is given to us and we find δ. This δ depends on ε, and it can also depend on p. Let us consider two functions, $f(x) = 3x$ and $g(x) = x^2$. Both these functions are continuous at 1 and at 10.

Suppose $\varepsilon = 0.1$ is given to us. Let us find the δ for f at 1 and 10.

At $p = 1$, we need to find δ_1 such that $|x-1| < \delta_1 \Rightarrow |3x-3| < 0.1$. Now $|3x-3| < 0.1 \Leftrightarrow 3|x-1| < 0.1 \Leftrightarrow |x-1| < 0.1/3$.

Thus, we can take $\delta_1 = 0.1/3$.

Similarly, at $p = 10$, we need to find δ_2 such that $|x-10| < \delta_2 \Rightarrow |3x-30| < 0.1$. Now $|3x-30| < 0.1 \Leftrightarrow 3|x-10| < 0.1 \Leftrightarrow |x-10| < 0.1/3$.

Thus, we can take $\delta_2 = 0.1/3$.

Here we find that we get the same δ at both the points. In fact, you can see that we will get the same value of δ at all points of $\mathbb{R}$. So, for this function, the value of δ depends only on the value of ε, and not on p.
Now let us take the next function, $g(x) = x^2$. Again, at $p = 1$ we need to find δ_1 such that $|x-1| < \delta_1 \Rightarrow |x^2 - 1| < 0.1$.

Suppose $\delta_1 < 1$.

Then, $|x-1| < \delta_1 < 1 \Rightarrow 0 < x < 2$

Now $|x^2 - 1| < 0.1$ if $|x+1||x-1| < 0.1$

If $|x-1| < \dfrac{0.1}{|x+1|} < \dfrac{0.1}{3}$

Thus, we can take $\delta_1 = \dfrac{0.1}{3}$.

Similarly, at $p = 10$, we need to find δ_2 such that

$|x-10|<\delta_2 \Rightarrow |x^2-100|<0.1.$

Again, if we take $\delta_2<1$
then $|x-10|<\delta_2<1 \Rightarrow 9<x<11$

Now $|x^2-100|<0.1$ if $|x+10||x-10|<0.1$

That is, if $|x-10|<\dfrac{0.1}{|x+10|}<\dfrac{0.1}{21}$

Thus, we can take $\delta_2=\dfrac{0.1}{21}$.

In this case δ_1 and δ_2 are different, and depend on $p=1$ or 10.
You can see this in Fig. 9.7.

Actually, $\delta=\dfrac{0.1}{21}$
will work for both points. But then, it does not work for $p=20$.

So, it is not possible to find a δ, which works for all points in the domain. Here we say that the continuity is not uniform.

This leads us to define uniform continuity of functions.

Definition 6: Let $f:S\to\mathbb{R}$, where $S\subseteq\mathbb{R}$. Then f is said to be uniformly continuous on S, if for every $\varepsilon>0$, there exists a $\delta>0$, such that, if x and y are any two points in S, with $|x-y|<\delta$, then $|f(x)-f(y)|<\varepsilon$.
The function that we considered earlier, $f(x)=3x$ is uniformly continuous on $\mathbb{R}$, whereas $f(x)=x^2$ is not uniformly continuous on $\mathbb{R}$.

Note that we talk of uniform continuity on a set, whereas we talk of continuity at a point also. Can you see from the definition that a function which is uniformly continuous on a set is continuous at every point of the set? The converse of this statement is not true as we observed in the case of the function $f(x)=x^2$, discussed above. We now give another example to support our statement.

Example 18: Show that the function, $f:]0,\infty[\to\mathbb{R}$, $f(x)=\dfrac{1}{x}$. is continuous on its domain $]0,\infty[$, but not uniformly continuous there. Whereas the function is uniformly continuous on $[c,\infty[$ for any fixed $c>0$.

Solution: Using the Algebra of Continuous Functions we can say that f is continuous on $]0,\infty[$. But, if we take $\varepsilon=0.5$, we cannot find a suitable δ, which works at all points of its domain. Because, for any $\delta>0$, we can find $n\in N$, such that $\left|\dfrac{1}{n}-\dfrac{1}{n+1}\right|=\left|\dfrac{1}{n(n+1)}\right|<\delta$, but then

$$\left|f\left(\frac{1}{n}\right)-f\left(\frac{1}{n+1}\right)\right|=|n+1-n|=1>0.5$$

Therefore, f is not uniformly continuous on $]0,\infty[$.

Next we shall check whether the function $f(x)=\frac{1}{x}$ is uniformly continuous on $[c,\infty[$ for any fixed $c>0.$ To show this, take any $\varepsilon>0,$ and consider

$$|f(x)-f(x')|=\left|\frac{1}{x}-\frac{1}{x'}\right|=\left|\frac{x'-x}{xx'}\right|.$$

For x and $x'\geq c>0,$ we have $0<1/x<1/c, 0<1/x'<1/c,$ and

$$|f(x)-f(x')|=\left|\frac{x'-x}{xx'}\right|\leq\frac{1}{c^2}|x'-x|<\varepsilon \text{ iff } |x-x'|<c^2\varepsilon.$$

Thus, given $\varepsilon>0,$ we can use $\delta=c^2\varepsilon.$ This is true for any fixed $c>0.$ Hence f is uniformly continuous on $[c,\infty],$ for any fixed $c>0.$

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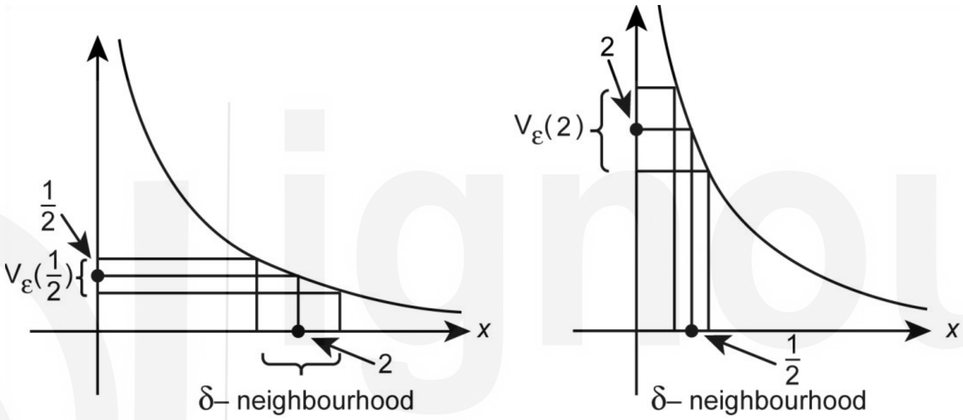
Remark 6: We can explain the results of uniform continuity in the earlier example geometrically as given in Fig. 7 below:



(a) $g(x)=1/x \quad (x>0)$ **(b)** $g(x) \quad (x>0)$

Fig. 7

You may note that for any subinterval I of fixed length $\delta>0$ in $[c,\infty[,$ the length on the y -axis of the interval $f(I)$ is less than or equal to the length of the interval $f([c,c+\delta])$ (see Fig. a), by the continuity of f at c we can make the length of $f([c,c+\delta])$ small by restricting $\delta.$

However, this argument does not extend to all of $(0,\infty).$ For $\delta>0,$ the length of $f((0,\delta])$ is not finite , and in particular, for any $\varepsilon>0$ and $\delta>0$ we can find a $\delta', 0<\delta'<\delta,$ such that the length of the interval $f([\delta',\delta])>\varepsilon.$ (see Fig. 7(b)). So, continuity does not guarantee uniform continuity, in general. The following theorem tells us that under certain extra conditions a continuous function is also uniformly continuous.

Theorem 13: If a real-valued function f is continuous on a closed and bounded interval I in $R,$ then it is uniformly continuous on $I.$

Proof: We prove this by assuming the contrary. If f is not uniformly continuous on $I,$ then taking the negation of Definition 5, there exists some $\varepsilon>0,$ for which no δ works. That is, for every $\delta>0,$ we can find x and y belonging to $I,$ such that $|x-y|<\delta,$ but $|f(x)-f(y)|>\varepsilon.$ In particular, for every $n\in\mathbb{N}$, we get x_n and y_n, such that $|x_n-y_n|<1/n,$ but $|f(x_n)-f(y_n)|>\varepsilon$.

Now, (x_n) and (y_n) are sequences in I, and therefore, are bounded sequences. Therefore, by Bolzano Weierstrass theorem, they have convergent subsequences, say, (x_{n_k}) and (y_{n_k}). Both these sequences will converge to the same limit, say, x_0.

Since the function, f, is continuous, the sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ should converge to the same limit. But this is not possible, since $|f(x_{n_k}) - f(y_{n_k})| > \varepsilon$. Hence f is uniformly continuous on I. ■

The condition in Theorem 10 is not necessary for a function f to be uniformly continuous as shown in the second part of Example 12. Note that the function which is continuous on the interval $[c, \infty), c > 0$ is uniformly continuous. But the domain of this function is neither bounded nor closed.
So the condition in Theorem 10 that the domain to be a closed and bounded interval is not a necessary one condition for a continuous function to be uniformly continuous. It is only sufficient.

Let us now take a further look on the function, $f :]0, \infty[\to \mathbb{R}$, $f(x) = 1/x$. We observe that the sequence, $\left(\frac{1}{n}\right)$ in $]0, \infty[$ is a Cauchy sequence, its image sequence. But under the continuous function $f(x) = \frac{1}{x}$ given by $\left(f\left(\frac{1}{n}\right)\right) = (n)$ is not a Cauchy sequence. But this never happen with a uniformly continuous function. ■

In the next theorem we prove that the image of a Cauchy sequence under a uniformly continuous function is Cauchy.

Theorem 14: If a function, f, is uniformly continuous on a subset S of $\mathbb{R}$, and if (x_n) is a Cauchy sequence in S, then the sequence $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

Proof: Suppose $\varepsilon > 0$ is given. Then, since f is uniformly continuous on S, there exists $\delta > 0$, such that, if x and y are any two points in S, with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Now since the given sequence is Cauchy, for this $\delta > 0$, there exists $n_0 \in \mathbb{N}$, such that $n, m \geq n_0 \Rightarrow |x_n - x_m| < \delta$. Thus,
$n, m \geq n_0 \Rightarrow |x_n - x_m| < \delta \Rightarrow |f(x_n) - f(x_m)| < \varepsilon$, which means that the sequence, $(f(x_n))$ is Cauchy. ■

See if you can solve these exercises now.

E16) Show that the following functions are not uniformly continuous on their domains:

i) $f(x) = x^2$ on $[0, \infty[$

ii) $g(x) = \sin(1/x)$ on $[0, \infty[$.

E17) Show that $f(x) = \dfrac{1}{1+x^2}$ is uniformly continuous on $[1, 2]$.

That brings us to the end of this unit.

10.6 SUMMARY

In this unit we have covered the following theory behind the concepts of limit and continuity, that you were acquainted.

A new concept, that of uniform continuity was introduced.

1. Discussed the theory behind the existence of limit and continuity that you were acquainted with in the course calculus.
2. Explained the logic behind the rules for finding the sum product and quotient of limit.
3. Some other forms of limits – one sided limits, infinite limits are also discussed.
4. We have proved the sequential criterion for continuity and other rules for finding sum, product of limits.
5. We introduced a new concept, that of uniform continuity. We have discussed the three different types of discontinuity and explained how this concept is different from continuity.
6. You have seen that every uniformly continuous function is continuous, but the converse is not true. But, if a function is continuous on a closed and bounded interval, then it is uniformly continuous there.

10.7 SOLUTION AND ANSWERS

E1) Let $k \in \mathbb{R}$ and $\lim\limits_{x \to p} f(x) = l$. We have to show that $\lim\limits_{x \to p} kf(x) = kl$.

Let (x_n) be a sequence such that $x_m \to x$ as $n \to p$. Then since the limit exists, $f(x_n) \to l$ and $n \to \infty$. Therefore by Theorem ? Unit 5 $kf(x_n) \to kl$ as $n \to \infty$. This is true for all (x_n) such that $x_n \to x$. Therefore by Theorem 2 stated earlier we get that $\lim\limits_{x \to p} f(x) = kl$.

Hence the result.

E2) i) $\ell = \lim\limits_{x \to 2} (5x - 11) = 1$

$$|5x - 11 - 1| < 0.5$$

$$\Rightarrow |5x - 12| < 0.5$$

$$\Rightarrow 5\left|x - \frac{12}{5}\right| < 0.5$$

$$\Rightarrow \left|x - \frac{12}{5}\right| < 0.1$$

$$\Rightarrow \left|x - 2 - \frac{2}{5}\right| < 0.1$$

$\Rightarrow |x-2| - \left|\frac{2}{5}\right| < \left|x-2-\frac{2}{5}\right| < 0.1$

$\Rightarrow |x-2| - \left|\frac{2}{5}\right| < 0.1$

$\Rightarrow |x-2| - 0.4 < 0.1$

$\Rightarrow |x-2| < 0.5$

$\Rightarrow |x-p| < \delta$

$\Rightarrow \delta \geq 0.5$

ii) $\lim_{x \to 4} \sqrt{x} = 2$

$\varepsilon = 0.5$

Given $\varepsilon > 0,$ we need to find $\delta > 0,$ such that

$0 < |x-4| < \delta \Rightarrow |\sqrt{x} - 2| < \varepsilon = 0.5$

$|\sqrt{x} - 2| < \varepsilon$

$\Rightarrow -\in < (\sqrt{x} - 2) < \varepsilon$

$\Rightarrow (2-\varepsilon)^2 < x < (2+\varepsilon)^2$

$\Rightarrow (2-e)^2 \leq 4-\delta$ and $4+\delta \leq (2+\varepsilon)^2$

$\Rightarrow \delta \leq 4-(2-\varepsilon)^2$ and $\delta \leq (2+\varepsilon)^2 - 4$

$\Rightarrow \delta = \min\{4-(2-\varepsilon)^2, (2+\varepsilon)^2 - 4\}$

$\varepsilon = 0.5$

The maximum value of $\delta = \min\{4-2.25,\ 6.25-4\}$

$= \min\{1.75, 2.25\}$

$= 1.75$

Therefore $\delta \leq 1.75.$

iii) $\lim_{x \to 1} \frac{x}{1+x} = \frac{1}{2},\ \varepsilon = 1$

By definition, $\varepsilon > 0, \delta > 0,\ 0 < |x-1| < \delta \Rightarrow \left|\frac{x}{1+x} - \frac{1}{2}\right| < \varepsilon$

$\Rightarrow \left|\frac{x}{1+x} - \frac{1}{2}\right| < \in$

$\Rightarrow -\in < \frac{x}{1+x} - \frac{1}{2} < \varepsilon$

$\Rightarrow -\varepsilon < \frac{2x-1-x}{2(1+x)} < \varepsilon$

$\Rightarrow -\varepsilon < \frac{x-1}{2(1+x)} < \varepsilon$

$\Rightarrow -2\in(1+x) < x-1 < 2\in(1+x)$

$\Rightarrow -2\in(1+x)+1 < x < 2\in(1+x)+1$

$\Rightarrow -2\in(1+x)+1 \leq 1-\delta, 1+\delta \leq 2\in(1+x)+1$

$\Rightarrow \delta \leq 2\in(1+x)$ and $\delta \leq 2\in(1+x)$

$\Rightarrow \delta \leq 2\in(1+x)$

$\Rightarrow \delta \leq 2.$ $\in = 1,\ x \to 1$

E3) On the contrary, we assume that $L < 0$ where $L = \lim_{x \to p} f(x)$. Let $\varepsilon > 0$ be such that $L + \varepsilon = 0$. Then given $\varepsilon > 0 \; \exists \, \delta > 0$ such that

$0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon$

i.e. $L - \varepsilon < f(x) < L + \varepsilon = 0$.

This is a contradiction to the assumption that $f(x) \geq 0$ in a added neighbourhood of p.

E4) We'll take these one by one.

i) Let $\varepsilon > 0$ be given. Then

$$\left| \frac{3x^2}{x^2+2} - 3 \right| = \left| \frac{6}{x^2+2} \right| < \frac{6}{x^2} < \varepsilon \text{ if } x > \sqrt{\frac{6}{\varepsilon}}.$$

Thus, given $\varepsilon > 0$, we take $G = \sqrt{\frac{6}{\varepsilon}}$ so that

$$x > G \Rightarrow \left| \frac{3x^2}{x^2+2} - 3 \right| < \varepsilon$$

This shows that $\lim_{x \to \infty} \frac{3x^2}{x^2+2} = 3.$

ii) Let ε be such that $0 < \varepsilon < 1$. Then

$|e^{-ax}| < \varepsilon$ if and only if $\frac{1}{\varepsilon} < e^{ax}$. This means that $|e^{-ax}| < \varepsilon$ if and only if $x > \frac{1}{a} \ln \frac{1}{\varepsilon}$

Thus, $\lim_{x \to \infty} e^{-ax} = 0$ when $a > 0$.

iii) Let ε be such that $0 < \varepsilon < 1$. Then

$$\left| \frac{e^x}{e^x+1} - 1 \right| = \frac{1}{e^x+1} < \frac{1}{e^x} < \varepsilon, \text{whenever } x > \ln \frac{1}{\varepsilon}$$

Thus, $x > G = \ln \frac{1}{\varepsilon} \Rightarrow \left| \frac{e^x}{e^x+1} - 1 \right| < \varepsilon$

This shows that $\lim_{x \to \infty} \frac{e^x}{e^x+1} = 1.$

E5) i) We shall prove that $\lim_{x \to \infty} \frac{1}{x} = 0.$

To find $\lim_{x \to -\infty} \left(\frac{1}{x} \right)$, we put $x = -t$. Then as $x \to -\infty, t \to \infty$. Then

$\left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| = \left| \frac{1}{-t} \right| = \left| \frac{1}{t} \right| = \frac{1}{t}$, if we take $t > 0$. Now $\frac{1}{t} < \varepsilon$, if $t > \frac{1}{\varepsilon}$.

So, if we choose $K = \frac{1}{\varepsilon}$, then $x > K \Rightarrow |f(x) - l| = \left| \frac{1}{x} \right| = \frac{1}{t} < \varepsilon$. This prove that $\lim\limits_{x\to-\infty}\left(\frac{1}{x}\right)=0.$

Alternatively, we write the given expression as

$$\frac{x^3-2x+5}{3x^3+6x^2+7}=\frac{1-\frac{2}{x^2}+\frac{5}{x^3}}{3+\frac{6}{x}+\frac{7}{x^3}}.$$

Since $\lim\limits_{x\to\infty}\frac{1}{x}=0$, we get that $\lim\limits_{x\to\infty}\frac{1}{x^n}=0$ for $n>1$ also and

$$\lim_{x\to\infty}\frac{x^3-2x+5}{3x^3+6x^2+7}=\frac{1}{3}.$$

ii) The given expression can be written as

$$\lim_{x\to\infty}\frac{x+\cos x}{x+\sin x}=\lim_{x\to\infty}\frac{1+\frac{\cos x}{x}}{1+\frac{\sin x}{x}}.$$

We shall prove that $\lim\limits_{x\to\infty}\frac{\sin x}{x}=0, \left|\frac{\sin x}{x}\right|\le\frac{1}{|x|}$

$$\therefore \lim_{x\to\infty}\frac{\sin x}{x}=0.$$

Similarly $\lim\limits_{x\to\infty}\frac{\cos x}{x}=0.$

Hence $\lim\limits_{x\to\infty}\frac{x+\cos x}{x+\sin x}=1.$

E6) We first note that for $f(x)=\sqrt{x}, x>0, \lim\limits_{x\to\infty}\frac{1}{\sqrt{x}}=0.$

This follows from the algebra of limits.

i) $$\frac{\sqrt{x}-5}{\sqrt{x}+6}=\frac{1-\frac{5}{\sqrt{x}}}{1+\frac{6}{\sqrt{x}}}, x>0$$

$\therefore \lim\limits_{x\to\infty}\frac{\sqrt{x}+5}{\sqrt{x}+6}=1$ for $x>0.$

ii) $$\frac{\sqrt{x+1}}{x}, x>0$$

$$\lim_{x\to\infty}\frac{\sqrt{x}+1}{x}=\frac{1+\frac{1}{\sqrt{x}}}{\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{x}}\lim_{x\to\infty}\left(1+\frac{1}{\sqrt{x}}\right)$$

$$=0$$

E7) i) We first note that as $x\to 4$ from the right, $[x]\to 4.$

$\therefore$ Using the algebra limits

$$\lim_{x\to 4^+}[x-[x]] = \lim_{x\to 4^+} x - \lim_{x\to 4^+}[x]$$
$$= 4-4$$
$$= 0.$$

ii) We first note that for $x > 0, \frac{|x|}{x} = 1.$

Therefore as $x \to 0$ from the right, then $\frac{|x|}{x}$ approaches 1.

$\therefore \lim_{x\to 0+} \frac{|x|}{x} = 1.$

E14) i) To prove this, assume the contrary. Then, for $\varepsilon = 1$ (again, an arbitrary choice; any fixed $\varepsilon > 0$ would work here also) there exists $\delta > 0$ such that $x, x' \in (0,\infty), |x - x'| < \delta$ implies

$$|f(x) - f(x')| = |x^2 - x'^2| = |x + x'||x - x'| < 1 \qquad \dots (1)$$

To find an x and x' that violates (1), we should consider x and x' large. In particular, let $x = 1/\delta$ and $x' = 1/\delta + \delta/2.$ Then $x, x' \in [0,\infty), |x - x'| = \delta/2,$ and

$$|f(x) - f(x')| = |x + x'||x - x'| = \left(\frac{2}{\delta} + \frac{\delta}{2}\right)\frac{\delta}{2} > \frac{2}{\delta}.\frac{\delta}{2} = 1,$$

contradicting (1).

ii) We will now prove that f is not uniformly continuous function on $]0,1[.$

Let $0 < \varepsilon < 1$ and $\delta > 0$ be any positive number.

Take $x = \frac{1}{n\pi}, y = \frac{1}{(n\pi + \pi/2)}.$ Then $|f(x) - f(y)| = 1.$

So, for $\delta > 0,$ choose n so that $x < \delta.$ Then, $x, y \in]0,\delta[,$ implies that $|x - y| < \delta$ but $|f(x) - f(y)| = 1 > \varepsilon.$

Hence f is not uniformly continuous.

E15) The function is a continuous on the bounded closed interval and therefore uniformly continuous.

UNIT 11

DIFFERENTIATION

Structure **Page No.**

11.1 INTRODUCTION

We are sure you are quite familiar with differentiation. In Calculus you have done extensive practice of differentiating functions. You also know that this branch of Mathematics was discovered while trying to solve the problem of finding the tangent to a given curve. Differentiation also helps us in the study of moving objects. In fact, the derivative measures the rate of change of the function under consideration. As a result, it has proved to be an excellent tool to study our ever changing world.

In this unit we shall concentrate more on the theoretical aspects differentiation. We shall also approach the proofs of some theorems in a slightly different manner, as compared to the proofs in Calculus. We hope that this new approach better prepares you to understand advanced level mathematics.

In Sec. 11.2 we define the derivative of a function and establish the connections between the concepts of differentiability and continuity both theoretically and graphically. The geometrical interpretation of a derivative is also covered. In Sec. 11.3 we prove some basic results concerning differentiability of sum, product, quotient and chain rule of functions. Afterwards in Sec. 11.4 we focus on inverse function theorems for differentiability.

As the name suggests, these theorems consider whether a property owned by a function is carried over to its inverse function. Before we prove the inverse function theorem for differentiability, we prove the inverse function theorems for continuity. Some more theorems on continuity which are used for proving theorems on differentiability are also discussed here. These theorems are important in their own right.

Objectives

After reading this unit, you should be able to

- Define the derivative of a function at a given point, and find it, if it exists;
- Show that a function differentiable at a point is also continuous at that point;
- Write and prove the formulas for the derivatives of sum, difference, product, quotient, scalar multiples of differentiable functions;
- State and prove the rule for the derivative of the composite of two differentiable functions;
- State and prove Bolzano Weierstrass Theorem, Intermediate Value Theorem and Inverse Function Theorems for continuity and differentiability.

11.2 PRELIMINARIES

We recall the definition of derivative of a function with which you are familiar from the Calculus course.

Definition 1: Let f be a real-valued function defined in an interval, I, and $p \in I$. If $h \in \mathbb{R}$ and if

$$\lim_{h\to 0} \frac{f(p+h)-f(p)}{h} \qquad \dots (1)$$

exists, then we say that f **is differentiable at** $\boldsymbol{p}$, and $\lim_{h\to 0} \frac{f(p+h)-f(p)}{h}$ is called the **derivative of** $\boldsymbol{f}$ at $\boldsymbol{p}$.

On replacing $p+h$ by x, the Eqn. (1) becomes

$$\lim_{x\to p} \frac{f(x)-f(p)}{x-p}. \qquad \dots (2)$$

Note that we will use either Eqn. (1) or Eqn. (2) while considering the derivative of a function.

The derivative of f at p is denoted by $f'(p)$, or $\frac{df}{dx}(p)$.

If the point p is the left end point of I, then the limit in the above definition will be the **right-hand limit**. If p is the right end point of I, then the limit will be the **left-hand limit**.

Thus, $f'(p) = \lim_{h\to 0} \frac{f(p+h)-f(p)}{h}$, if the limit exists. This f' then defines a new function at all those points at which f is differentiable. It is called the **derived function of** f.

You may recall from the calculus course that the motivation for this definition comes from geometry. You might have some intuitive idea that a tangent line is a line which intersects a curve only at one point. To define such a line we should know the slope of the line. To estimate this, we consider the family of lines called secant lines which passes through the points $(x, f(x))$ and $(p, f(p))$ for x near p. (See Fig. 1).

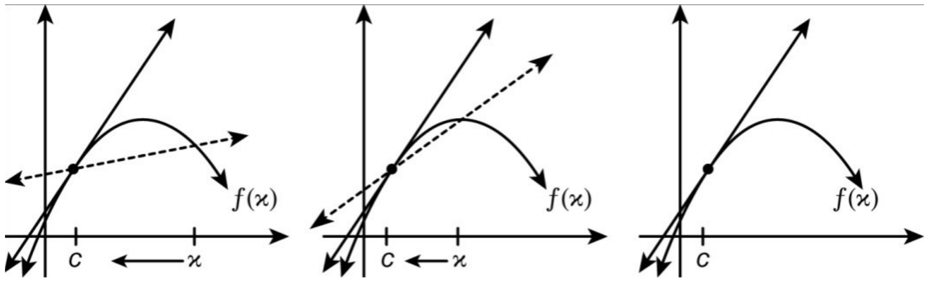


Fig. 1: Secant lines approaching the tangent line.

The slope of line through $(x, f(x))$ and $(p, f(p))$ is $\dfrac{f(x)-f(p)}{x-p}$. As x approaches p, in the limiting case, the line becomes a tangent line and then the slope of the tangent line should therefore be $\lim\limits_{x\to p}\dfrac{f(x)-f(p)}{x-p}$ which is $f'(p)$ if the derivative exists. Thus we have the definition of the tangent line as given by the following:

Definition 2: If the derivative $f'(p)$ exists at some point p, then the line passing through $(p, f(p))$ and having the slope $f'(p)$ is defined to be the tangent line to the curve $y = f(x)$ whose equation is given by

$$y - f(p) = f'(p)(x-p).$$

You are already familiar with calculating the derivative of a function using Definition 1 from the calculus course. Here we shall give some examples to familiarize you with the definition.

Example 1: Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by $f(x) = x^2$. Find the derivative of f at a point $p\in\mathbb{R}$.

Solution: Let $p\in\mathbb{R}$ be fixed. We first note that the function f is defined an open interval containing p. Using Eqn. (1) of Definition 1, we consider the limit

$$\begin{aligned}\lim_{h\to 0}\frac{f(p+h)-f(p)}{h} &= \lim_{h\to 0}\frac{(p+h)^2-p^2}{h^2}\\ &= \lim_{h\to 0}\frac{p^2+2ph+h^2-p^2}{h}\\ &= \lim_{h\to 0}(2p+h)\\ &= 2p.\end{aligned}$$

This shows that the limit exists and the derivative is $f'(p) = 2p$.
Alternatively using the Eqn. (2), of Definition 1 we get that

$$\begin{aligned}\lim_{x\to p}\frac{f(x)-f(p)}{x-p} &= \lim_{x\to p}\frac{x^2-p^2}{x-p}\\ &= \lim_{x\to p} x+p = 2p.\end{aligned}$$

This shows that the limit exists and the derivative is $f'(p) = 2p$.

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Replication of the method used above, will make the calculation of the derivative of $f(x) = x^n$ easier.

Example 2: Find the derivatives of the following functions, wherever they exist.

i) $f(x) = \sqrt{3x-5}, x > \frac{5}{3}$,

ii) $f(x) = \frac{1}{\sqrt{x}}, x > 0$,

iii) $f(x) = x^{1/4}, x \geq 0$

Solution: i) Now, $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{3(x+h)-5}-\sqrt{3x-5}}{h}$

$$= \frac{\left(\sqrt{3(x+h)-5}-\sqrt{3x-5}\right)\left(\sqrt{3(x+h)-5}+\sqrt{3x-5}\right)}{h\left(\sqrt{3(x+h-5)}+\sqrt{3x-5}\right)}$$

$$= \frac{(3(x+h)-5)-(3x-5)}{h\left(\sqrt{3(x+h)-5}+\sqrt{3x-5}\right)}$$

$$= \frac{3}{\left(\sqrt{3(x+h)-5}+\sqrt{3x-5}\right)}$$

Therefore, $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\frac{3}{\left(\sqrt{3(x+h)-5}+\sqrt{3x-5}\right)}$

$$= \frac{3}{2\sqrt{3x-5}}$$

Thus, the derivative of this function exists at all points of its domain.

ii) Here, $\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left(\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right)$

$$= \frac{1}{h}\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right)$$

$$= \frac{1}{h}\left(\frac{x-(x+h)}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}\right)$$

$$= \left(\frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}\right)$$

Therefore, $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\left(\frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}\right)$

$$= \frac{-1}{2x\sqrt{x}}$$

$$= \frac{-1}{2x^{2/3}}$$

Thus, $f'(x) = \frac{-1}{2x^{3/2}}$ for all $x > 0$.

iii) When $x = 0$, $\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left((0+h)^{1/4} - 0\right)$

$$= \frac{1}{h^{3/4}},$$

and the limit of this does not exist as $h \to 0$.

The derivative of the given function does not exist at $x = 0$.

When $x > 0$, $\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left((x+h)^{1/4} - x^{1/4}\right)$

$$= \frac{\left((x+h)^{1/4} - x^{1/4}\right)\left((x+h)^{1/4} + x^{1/4}\right)}{h\left((x+h)^{1/4} + x^{1/4}\right)}$$

$$= \frac{\sqrt{x+h} - \sqrt{x}}{h\left((x+h)^{1/4} + x^{1/4}\right)}$$

$$= \frac{(x+h) - x}{h\left((x+h)^{1/4} + x^{1/4}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{1}{\left((x+h)^{1/4} + x^{1/4}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}$$

Therefore, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0}\left(\frac{1}{\left((x+h)^{1/4} + x^{1/4}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}\right)$

$$= \frac{1}{4x^{1/4}\sqrt{x}} = \frac{1}{4x^{3/4}}.$$

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The next example illustrates how to compute the equation of a tangent to a given curve.

Example 3: Find the equation of the tangent line to the graph of the curve $y = f(x) = x^3 + 6x - 2$ at $(3, f(3))$.

Solution: We first find the slope of the tangent line. It is given by

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{x^3 + 6x - 2 - 43}{x - 3}$$

$$= \lim_{x \to 3} \frac{x^3 + 6x - 45}{x - 3} = \lim_{x \to 3} \frac{(x^2 + 3x + 15)(x - 3)}{x - 3} = 33.$$

Thus the equation of the tangent line is $y - 43 = 33(x - 3)$.

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We shall now prove a theorem which shows that continuity of a function at a point is a necessary condition for the existence of its derivative at that point.

Theorem 1: Let $f : I \to \mathbb{R}$, where I be an open interval in $\mathbb{R}$, and $p \in I$. If f is differentiable at p, then f is also continuous at p.

Proof: If $x \in I, x \neq p,$ then we have

$$f(x) - f(p) = \left(\frac{f(x) - f(p)}{x - p}\right)(x - p).$$

Therefore, using the product rule of limits we can say that

$$\lim_{x \to p}\big(f(x) - f(p)\big) = \lim_{x \to p}\left(\frac{f(x) - f(p)}{x - p}\right)\lim_{x \to p}(x - p)$$
$$= f'(p).0$$
$$= 0$$

Thus, $\lim_{x \to p} f(x) = f(p)$. Therefore, we conclude that f is continuous at p. ■

Thus, we have proved that the continuity is a necessary condition for the existence of the derivative at a point. But it is not a sufficient one. That is, if a function is continuous at a point, it does not follow that it is also differentiable there. An example of this is the function $f(x) = |x|$ at $x = 0$. If you have solved E10) of Unit 10, you would have realized that this function is continuous at 0. To see whether it is differentiable, we need to find whether the limit of $\frac{f(x) - f(0)}{x - 0} = \frac{|x| - 0}{x} = \frac{|x|}{x}$ exists as $x \to 0$, or not.

Now, if $x < 0,$ then $\frac{|x|}{x} = \frac{-x}{x} = -1$. Hence the left hand limit is $\lim_{x \to 0^-} \frac{|x|}{x} = -1$.

On the other hand, if $x > 0,$ then $\frac{|x|}{x} = \frac{x}{x} = 1$. Hence the right hand limit is

$\lim_{x \to 0^+} \frac{|x|}{x} = 1$.

Thus the left hand limit is not the same as the right hand limit. Therefore, $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. Hence $f(x) = |x|$ is not differentiable at $x = 0$.

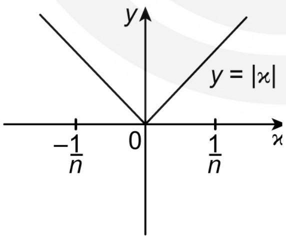

Fig. 2: Graph of $f(x) = |x|$.

If you look at the graph of $f(x) = |x|$ in Fig. 2 in the margin, you see that the graph is unbroken at $x = 0$, implying the continuity at that point. But note that the graph is not smooth there. It has a corner at that point. This geometrical feature indicates that the function is not differentiable there.

More precisely, we have the following:

Example 4: The absolute value function, $f(x) = |x|$ is not differentiable at $x = 0$. Show that it is differentiable at all other points in $\mathbb{R}$. In fact, for all $x < 0,$ the derivative is -1, and for all $x > 0,$ the derivative of the function is 1.

In 1872, K. Weierstrass remarkably gave an example of a function that is continuous everywhere and nowhere differentiable. The function is

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}(\cos 3^n x).$$

Solution: Suppose $x > 0$. If we choose $h > 0,$ then $x + h > 0$, and

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} 1 = 1.$$

On the other hand, if $x < 0,$ we choose $h < 0,$ $x + h < 0$ and we get

$$f'(x) = \lim_{h\to 0} \frac{|x+h|-|x|}{h} = \lim_{h\to 0} \frac{-(x+h)-(-x)}{h} = \lim_{h\to 0}(-1) = -1\,.$$

In Fig.2 you can see that the graph on the right hand side of the origin is a line with slope,1, and the graph on the left of the origin is a line with slope, -1.

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Now try this exercise on your own.

E1) Does $f(x) = x^{2/3}$ have a tangent line at $x=0$? Justify your answer.

In all these cases we have used the definition to find the derivative. But the process is tedious. And, as you know, there is a way out. We find the derivatives of some standard functions, and then use Algebra of derivatives to get the derivatives of combinations of these functions. You have already done this in your course on Calculus.

We shall prove some theorems on algebra of derivatives and some more theorems in the next section.

11.3 BASIC THEOREMS ON DERIVATIVES

There are many basic properties of various combination of functions which you must have used while computing derivatives in the calculus course. Here we give justifications of some of these properties in the form of proofs of the theorems.

We start with the Algebra of derivatives.

Theorem 2 (Algebra of Derivatives): Suppose f and g are two real-valued functions defined on an interval $I \subseteq \mathbb{R}$. Suppose f and g are differentiable at $p \in I$. Then

i) kf is differentiable at p, where $k \in \mathbb{R}$, and $(kf)'(p) = kf'(p)$

ii) $f+g$ is differentiable at p, and $(f+g)'(p) = f'(p) + g'(p)$

iii) fg is differentiable at p, and $(fg)'(p) = f'(p)g(p) + f(p)g'(p)$

iv) If $g(p) \neq 0$, then f/g is differentiable at p, and

$$\left(\frac{f}{g}\right)'(p) = \frac{f'(p)g(p) - f(p)g'(p)}{(g(p))^2}.$$

Proof: We prove these one by one.

i) $$\frac{kf(p+h)-kf(p)}{h} = k\,\frac{f(p+h)-f(p)}{h}.$$

Therefore, taking limits on both sides, as $h \to 0$, we get the required result.

ii) $$\frac{(f+g)(p+h)-(f+g)(p)}{h}=\frac{f(p+h)+g(p+h)-f(p)-g(p)}{h}$$
$$=\frac{f(p+h)-f(p)}{h}+\frac{g(p+h)-g(p)}{h}$$

Again, taking limits on both sides, as $h \to 0$, we get the required result.

iii) $$\frac{fg(p+h)-fg(p)}{h}=\frac{f(p+h)g(p+h)-f(p)g(p)}{h}$$
$$=\frac{f(p+h)g(p+h)-f(p+h)g(p)+f(p+h)g(p)-f(p)g(p)}{h}$$
$$=\frac{f(p+h)\left(g(p+h)-g(p)\right)+\left(f(p+h)-f(p)\right)g(p)}{h}$$
$$=f(p+h)\frac{g(p+h)-g(p)}{h}+\frac{f(p+h)-f(p)}{h}g(p)$$

We take limits on both sides, as $h \to 0$. Note that since f is differentiable at p, it is also continuous at p, and therefore, $\lim_{h\to 0} f(p+h)=f(p)$. This along with the fact that f and g are differentiable at p gives the required result.

iv) Now, $$\frac{(f/g)(p+h)-(f/g)(p)}{h}=\frac{1}{h}\left(\frac{f(p+h)}{g(p+h)}-\frac{f(p)}{g(p)}\right)$$
$$=\frac{1}{h}\left(\frac{f(p+h)g(p)-f(p)g(p+h)}{g(p)g(p+h)}\right)$$

$$=\frac{1}{h}\left(\frac{f(p+h)g(p)-f(p)g(p)+f(p)g(p)-f(p)g(p+h)}{g(p)g(p+h)}\right)$$
$$=\frac{1}{g(p)g(p+h)}\left(\frac{f(p+h)-f(p)}{h}g(p)-f(p)\frac{g(p+h)-g(p)}{h}\right)$$

Note that f and g are both differentiable and, therefore, continuous. So, taking limit as $h \to 0$ on both sides, we get the required formula for the derivative of the quotient of two functions.

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Remark 1: i) The statement (ii) in Theorem 2 can extended to any number of functions by using induction. So, if for $n \in N, f_1, f_2, ..., f_n$ are all differentiable at p, we get

$$(f_1+f_2+...+f_n)'(p)=f_1'(p)+f_2'(p)+...+f_n'(p).$$

ii) Similarly, extending Statement iii) of the theorem, we get

$$(f_1f_2....f_n)'(p)=f_1'(p)f_2(p)f_3(p)...f_n(p)+f_1(p)f_2'(p)f_3(p)...f_n(p)$$
$$+...+f_1(p)f_2(p)....f_{n=1}(p)f_n'(p)$$

Further, if $f_1=f_2=....=f_n$, we can write

$(f^n)'(p) = n(f(p))^{n-1} f'(p)$.

iii) Using the statements (i) and (ii) in Theorem 2, and taking $k = -1$, we get
$(f - g)'(p) = f'(p) - g'(p)$.

Let us see how this theorem can be used to get more insight into derivatives. Before stating the theorem we recall some definitions.

Definition 3: Recall that a function is called an **even function**, if $f(-x) = f(x)$ for all x in its domain, and a function is called an **odd function**, if $f(-x) = -f(x)$ for all x in its domain.

Theorem 3: Show that, if f is odd, then its derived function is even, and vice versa.

Proof: Suppose f is an odd function. Then $f(-x) = -f(x)$. Differentiating both sides, we get $f'(-x) = -f'x(-1)$, and hence f' is an even function. Similarly if f is an even function. Then $f(-x) = f(x)$ i.e. $f'(-x) = -f'(x)$. Thus, $-f'(-x) = f'(x)$, and therefore, f' is an odd function. ■

Let us see an example.

Example 5: A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |x| + |x+5|$. Find the points at which this function is differentiable.

Solution: The given function is the sum of two functions, $g(x) = |x|$, and $h(x) = |x+5|$. We know that g is differentiable everywhere, except at $x = 0$, and h is differentiable everywhere, except at $x = -5$. So, applying the formula for the derivative of the sum of two differentiable functions, we get that f is differentiable at all points, except at $x = 0$ and $x = -5$. Further, for all $x < -5$, $g'(x) = -1$, and $h'(x) = -1$. Therefore, $f'(x) = -2$. For all $-5 < x < 0$, $g'(x) = -1$, and $h'(x) = 1$. Therefore, $f'(x) = 0$. For all $x > 0$, $g'(x) = 1$, and $h'(x) = 1$. Therefore, $f'(x) = 2$.
f is not differentiable at $x = 0$ and at $x = -5$.

***

Along with addition, subtraction, multiplication and division of functions, we have another important operation on functions: composition of functions. We shall now see how to get the derivative of the composite of two differentiable functions. That is, we shall derive the chain rule. But first we prove a very useful result.

Theorem 4 (Caratheodory theorem): Let $f : I \to \mathbb{R}$, be a function defined on interval I. Let $p \in I$. Then f is differentiable at p if and only if there exists a function $\varphi : I \to \mathbb{R}$, such that φ is continuous at p, and for all $x \in I$, we have

$$f(x) - f(p) = \varphi(x)(x - p) \qquad \ldots (3)$$

Further, in that case, we have $\varphi(p) = f'(p)$.

Proof: We first prove the 'only if ' part. For that we assume that f is differentiable at p. If f is differentiable at p, then we define $\varphi : I \to R$ by

$$\varphi(x) = \begin{cases} \dfrac{f(x)-f(p)}{x-p}, & x \in I, x \neq p \\ f'(p), & x = p \end{cases}$$

Then $\lim\limits_{x \to p} \varphi(x) = \lim\limits_{x \to p} \dfrac{f(x)-f(p)}{x-p} = f'(p) = \varphi(p)$.

Therefore, φ is continuous at p. Also, by definition,
$f(x)-f(p) = \varphi(x)(x-p)$ for all $x \in I$.

Note that if $x = p$, then both sides of Eqn. 3 become zero.

Thus we have shown that there exists a continuous function φ satisfying Eqn. (3). Now we prove the 'if ' part. suppose there exists a function φ, satisfying

$$f(x)-f(p) = \varphi(x)(x-p) \qquad \dots (4)$$

for all $x \in I$, and that the function $\varphi(x)$ is continuous at $x = p$. We have to show that f is differentiable at p.

If $x \neq p$, we can divide both sides of Eqn. (4) by $x - p$. So, if $x \neq p$, we get $\dfrac{f(x)-f(p)}{x-p} = \varphi(x)$. Therefore, $\lim\limits_{x \to p} \dfrac{f(x)-f(p)}{x-p} = \lim\limits_{x \to p} \varphi(x)$. This limit is equal to $\varphi(p)$, since φ is continuous at p. Thus, $\lim\limits_{x \to p} \dfrac{f(x)-f(p)}{x-p}$ exists, and is equal to $\varphi(p)$. This shows that f is differentiable at p, and $f'(p) = \varphi(p)$. ■

This theorem above helps us prove the chain rule, which tells us how to differentiate a composite function.

Theorem 5 (The Chain Rule): Suppose $f : J \to \mathbb{R}$ and $g : I \to \mathbb{R}$, where I and J are intervals in $\mathbb{R}$, and $f(J) \subseteq I$. Let $p \in J$, such that f is differentiable at p, and let g be differentiable at $f(p)$. Then the composite function, $g \circ f : J \to R$ is differentiable at p, and
$(g \circ f)'(p) = g'(f(p))f'(p)$.

Proof: Since f is differentiable at p, by Caratheodory Theorem, there exists a function $\varphi : J \to R$, such that
$f(x)-f(p) = \varphi(x)(x-p)$, and $\varphi(p) = f'(p)$. … (5)

Similarly, since g is differentiable at $q = f(p)$, by Caratheodory Theorem, there exists a function $\psi : I \to R$, such that

$g(y)-g(q) = \psi(y)(y-q)$, and $\psi(q) = g'(q)$.

When we put $y = f(x)$, and $q = f(p)$, we get
$g(f(x))-g(f(p)) = \psi(f(x))(f(x)-f(p))$, and $\psi(f(p)) = g'(f(p))$ … (6)

Now, using Eqn. (4) we get

$$g(f(x)) - g(f(p)) = \psi(f(x))\varphi(x)(x-p)$$
$$= [\psi(f(x)).\varphi(x)](x-p) = [(\psi \circ f((x)).\ \varphi(x))]\ (x-p)$$

This is true for all $x \in J$. Since φ is continuous at p , and ψ is continuous at $f(p)$, the function, $(\psi \circ f).\varphi$ is continuous at p , and using Eqn. (4) and Eqn. (5), we get

$$\lim_{x\to p} \frac{g(f(x)) - g(f(p))}{x-p} = \lim_{x\to p}[(\psi \circ f(x)).\varphi(x)] = (\psi \circ f(p))\ .\ \varphi(p)$$
$$= \psi(f(p)).\varphi(p) = g'(f(p)).\ f'(p).$$

So, again, by Caratheodory Theorem we can say that the function $g \circ f$ is differentiable at p , and $(g \circ f)'(p) = g'(f(p))f'(p)$.

Thus if f is differentiable on J , and g is differentiable on I , then the composite function $g \circ f$ is differentiable on J , and we write

$$(g \circ f)' = (g' \circ f)f'.$$ ■

Here are some examples illustrating the use of this Chain Rule.

Example 6: Differentiate the following functions with respect to x:

i) $\cos(3x^2+1)$ ii) $(2x-9)^7$ iii) $\tan^2 5x$

Solution: i) Let $h(x) = \cos(3x^2+1)$, $f(x) = 3x^2+1$, and $g(x) = \cos x$. Then $h = g \circ f$. Therefore, $h'(x) = (g \circ f)'(x)' = g'(f(x))f'(x)$.

Now, $f'(x) = 6x$, and $g'(f(x)) = -\sin(f(x)) = -\sin(3x^2+1)$.

Thus, $h'(x) = -6x\sin(3x^2+1)$.

ii) Here let $h(x) = (2x-9)^7$, $f(x) = 2x-9$, and $g(x) = x^7$. Then $f'(x) = 2$, and $g'(f(x)) = 7(f(x))^6 = 7(2x-9)^6$.
Thus, $h'(x) = 14(2x-9)^6$.

iii) Here the situation is different. Let us see how.
Let $h(x) = \tan^2(5x)$, $r(x) = 5x$, $f(x) = \tan x$, and $g(x) = x^2$. Then $h(x) = (g \circ f \circ r)(x)$. Therefore, $h'(x) = g'((f \circ r)(x))\,(f \circ r)'(x)$.
$= g'((f \circ r)(x))f'(r(x))r'(x)$.

Note that we have twice applied the Chain Rule here.

Now we have $g'((f \circ r)(x)) = 2\tan(5x)$, $f'(r(x)) = \sec^2(5x)$, and $r'(x) = 5$.

Thus, $h'(x) = 10\tan(5x)\sec^2(5x)$.

***

It is time to try some exercises based on the theorems studied so far.

E2) Differentiate:

a) $f(x) = \frac{x}{1+2x^2}$ b) $f(x) = (\sin x)^5 + (\cos x)^3$

E3) Suppose $f(x) = \begin{cases} x^2, x > 0 \\ 0, x \le 0 \end{cases}$. Is f differentiable at $x = 0$? Write the derived function of f. Is this derived function continuous at $x = 0$?

E4) Differentiate the following functions with respect to x.

a) $\sin^5(x^7)$ b) $\sqrt{1+3x-4x^3}$ c) $\cos\sqrt{8x-1}$

In the next section we shall discuss a fundamental theorem in Analysis.

11.4 THE INVERSE FUNCTION THEOREM

We discuss Inverse Function Theorem in this section. It States that under certain conditions, the derivative of the inverse of a function can be obtained from the derivative of the original function. Graphically speaking, it makes sense that this is possible. We look at the graph of the inverse of a function f, it is the reflection of the graph of f about the origin line $y = f(x)$. Thus each tangent line of f is reflected through the line $y = f(x)$ into the tangent line of f^{-1}.

We next discuss some theorems about continuous functions, which will lead us to the proof of the Inverse Function Theorem. The theorems on continuous function are also important to establish since it leads to many other results which you will come across in later units.

We shall now prove two theorems on continuous functions.

Theorem 6: Let $f : [a,b] \to \mathbb{R}$ be continuous on its domain. Suppose $f(p) > 0$ for some $p \in (a,b)$. Then there exists a neighbourhood N of p, such that, $f(x) > 0$ for all $x \in N$.

Proof: Let $\varepsilon = \frac{f(p)}{2}$. Since f is continuous at p, for this ε, there exists a $\delta > 0$, such that $(p-\delta, p+\delta) \subset [a,b]$, and $|x-p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$.

That is,

$$|x-p| < \delta \Rightarrow f(p) - \varepsilon < f(x) < f(p) + \varepsilon$$
$$\Rightarrow \frac{f(p)}{2} < f(x) < \frac{3f(p)}{2}, \text{ since } \varepsilon = \frac{f(p)}{2}.$$

This means, $f(x) > \frac{f(p)}{2} > 0$, for all $x \in (p-\delta, p+\delta)$.

Thus, $N = (p-\delta, p+\delta)$ is the required neighbourhood. ■

Remark 2: You may note that in the earlier theorem we made the assumption that $f(p) > 0$ for some $p \in (a,b)$. Infact if $p = a$ i.e. $f(a) > 0$, then we can similarly prove that there exists $\delta > 0$, such that $[a, a+\delta) \subset [a,b]$, and $f(x) > 0$, for all $x \in [a, a+\delta)$. In the same way if $p = b$ i.e., if $f(b) > 0$, then we can similarly prove that there exists a $\delta > 0$, such that $(b-\delta, b] \subset [a,b]$, and $f(x) > 0$, for all $x \in (b-\delta, b]$.

We shall prove the second theorem now.

Theorem 7 (Bolzano Theorem): Let $f : [a,b] \to \mathbb{R}$ be continuous on its domain. Suppose $f(a)f(b) < 0$, that is, $f(a)$ and $f(b)$ have opposite signs. Then there exists a point $p \in (a,b)$, such that $f(p) = 0$.

Proof: Suppose $f(a) > 0$, and $f(b) < 0$. Let $S = \{x \in [a,b] | f(x) \geq 0\}$. Then S is a non-empty subset of $[a,b]$, since $a \in S$. Let $p = \sup S$.

Then we claim that $p \neq a$. By Theorem 5, there exists $\delta > 0$, such that $f(x) > 0$ for all $x \in [a, a+\delta)$. This means, $a + \frac{\delta}{2} \in S$. So, $a \neq \sup S$. Similarly, $b \neq \sup S$. we leave it for you to prove. See E6).

Therefore, $a < p < b$. Since f is continuous, and $p = \sup S$, $f(p) \geq 0$. Now we will show that $f(p) = 0$.

If $f(p) \neq 0$, then $f(p) > 0$, and by Theorem 5 there exists a neighbourhood of p, say, $(p-\delta, p+\delta) \subset (a,b)$, such that $f(x) > 0$ for all $x \in (p-\delta, p+\delta)$. But this means the point $p + \frac{\delta}{2} \in S$, since $f\left(p + \frac{\delta}{2}\right) > 0$. This contradicts the fact that $p = \sup S$.

Therefore, $f(p) = 0$. ■

Next we shall prove another theorem on similar lines as Theorem 7 which is a generalized form of the Bolzano theorem. This theorem is known as "Intermediate Value Theorem (IVT in brief).

Theorem 8 (Intermediate Value Theorem (IVT)): Suppose that f is continuous on $[a,b], f(a) \neq f(b),$ and k is any number between $f(a)$ and $f(b)$. Then there exists at least one point $c \in (a,b)$ such that $f(c) = k.$

Proof: Case 1: $f(a) < k < f(b)$

Consider $g(x) = f(x) - k.$ Then $g(a) = f(a) - k < 0$ and $g(b) = f(b) - k > 0.$ Therefore $g(a)$ and $g(b)$ are of different sign. By Theorem 7, there exist $p \in (a,b)$ such that $g(p) = 0$ which implies that $f(p) = k.$

On the other hand, if $f(b) < k < f(a)$, then $g(a)g(b) < 0.$ Appearing to Theorem 7, once again we obtain the result. ■

Note: Now we shall give the geometrical interpretations of Theorems 6 and 7.

Geometrically theorem 6 says that if for a continuous function f defined on the closed interval $[a,b]$, the values $f(a)$ and $f(b)$ are of opposite signs, then the graph of f cuts the real line x-axis at some point k such that $f(k)=0$. This is illustrated in the Figure given below.

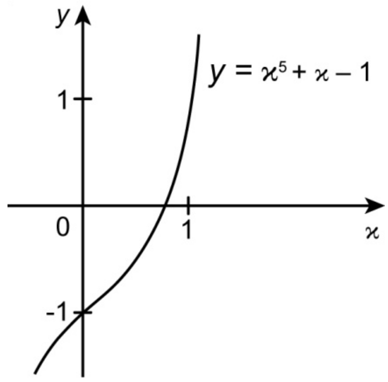
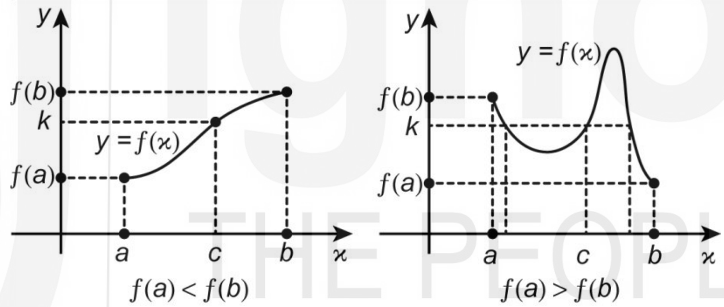


Fig. 3: The graph of f cuts the real line at $(p,0)$.

Whereas Theorem 7 geometrically shows that if for a continuous function f defined on a close bounded interval $[a,b]$ and if $f(a)\neq f(b)$, then given any k between $f(a)$ and $f(b)$ there exists c between a and b such that there exists the line $y=k$ exists the graph of f at (c,k).



(a) **(b)**

Fig. 4

The graph (b) above, shows that there may be more than one value of c. Does the converse of IVT hold? That means if a function satisfies the conclusions given by Theorem 8, is it necessary that the function is continuous? The answer is no. The following function along with the graph shown in Fig. 5 explains this.

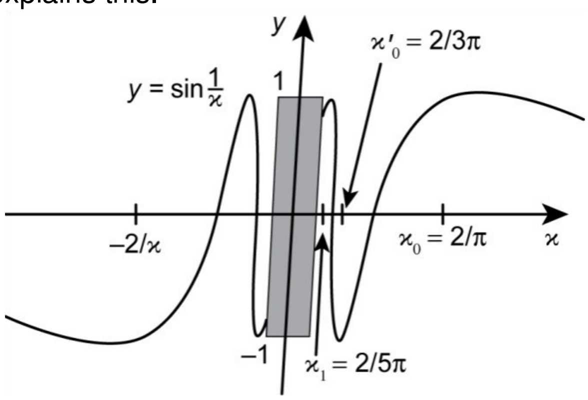


Fig. 5: The graph of the function $f(x)=\sin\left(\frac{1}{x}\right)$

The function given in graph, above, shows that there are non-continuous functions which satisfies the conclusions of Theorem 7. In view of the importance of this property, we consider the following class of functions.

Definition 4: A function f has the intermediate value property on an interval $[a,b]$, if for all $x < y$ in $[a,b]$ and L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x, y)$ where $f(c) = L.$
Thus every continuous function has the intermediate value property.

The IVT has lot of applications. One such application is to identify the points on the earth's surface which are 'exactly opposite of each other such as North Pole and South Pole. These points are called antipodal points. According to IVT theorem there is always a pair of antipodal points on the Equator of the Earth at which the temperature is the same. It is assumed that the temperature is a continuous function of position on Earth's surface. For more details you can refer to any website given in reference. The figure in the margin illustrates this.

Another application of IVT is that it sometimes helps us to locate some of the roots of the polynomials. We illustrate this in the following example.

Example 7: Show that the equation $x^4 + 2x - 11 = 0$ has a root lying between 1 and 2.

Solution: The function $f(x) = x^4 + 2x - 11$ is a continuous function on the closed interval $[1,2], f(1) = -8$ and $f(2) = 9.$ Hence, by Theorem 7, there exists an $x_0 \in]1,2[$ such that $f(x_0) = 0,$ i.e., x_0 is a root of the equation $x^4 + 2x - 11 = 0$ lying in the interval $]1,2[.$

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Remark 3: You should note that Theorem 8 only ensures the existence of a root of a polynomial. It does not specify the root of the polynomial.

Next we shall see another example.

Example 8: Any polynomial of odd degree must have at least one root.

Solution: To see this let $p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ where n is odd, and the coefficients a_i are constants with $a_n \neq 0.$ Assume for the moment that $a_n > 0.$ Then

$$p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = x^n \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + ... + a_n \right),$$

and as $x \to -\infty, x^n \to -\infty$ for all n and $\left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + ... + a_n \right) \to a_n > 0,$

and so $p(x) \to -\infty.$ Similarly, as $x \to +\infty, x^n \to +\infty$ and $p(x) \to +\infty.$ Thus, for any $M > 0,$ there exist points x_1 and x_2 such that $p(x_1) < -M < 0 < M < p(x_2),$ and from the IVT, there is a point c between x_1 and x_2 with $p(c) = 0.$

If $a_n < 0,$ then $p(x) \to +\infty$ as $x \to -\infty$ and $p(x) \to -\infty$ as $x \to +\infty,$ and the IVT can be applied as above.

Notice that this argument breaks down if n is even, since $x^n \to +\infty$ as $x \to \pm\infty.$

***

As a consequence of Theorem 8, we prove another theorem.

Theorem 9: Let $f : I \to \mathbb{R}$ be continuous on the closed and bounded interval, I. Let $f(I) = \{y | y = f(x), x \in I\} = J$. Then J is an interval in $\mathbb{R}$.

Proof: Now, I is not empty, and therefore, J is not empty. If J is a singleton set, then it is called a degenerate interval.

Suppose $c, d \in J, c < d$. Let $k \in \mathbb{R}, c < k < d$. Further, suppose $c = f(a)$, and $d = f(b)$, for some $a, b \in I$.

Now consider a new function, $g : I_1 \to \mathbb{R}$, defined by $g(x) = f(x) - k$. Here I_1 is the is the interval $[a, b]$, or $[b, a]$. Then $g(a) < 0$, and $g(b) > 0$. Therefore, applying Theorem 8 we can say that $g(x) = 0$, for some $x \in I_1$. That is, $f(x) = k$ for some $x \in I_1$. Thus, $k \in J$. This is true for all $k \in J$. Therefore this shows that J is an interval. ■

Remark 4: The theorem above, says that the image of a closed interval under a continuous function is a closed interval. The theorem is known as "The interval Image Theorem".

We are now ready to discuss the inverse function theorem for continuous function and then to differentiable function.
We shall prove two theorems.

Theorem 10 (Inverse Function Theorem for Continuous Functions): If the function, $f : I \to \mathbb{R}$ is an injective, continuous function defined on the closed and bounded interval I, and $f(I) = J$, then $f^{-1} : J \to I$ is also continuous.

Proof: We are going to use the sequential criterion of continuity to prove this theorem.

Let $y \in J$, and let (y_n) be a sequence in J, converging to y. Then $(x_n) = (f^{-1}(y_n))$ is a sequence in I. Since I is bounded, (x_n) is also bounded. Therefore, it has a convergent subsequence, (x_{n_k}), converging to, say, x. Since f is continuous, $(f(x_{n_k})) \to f(x)$. Now, $(f(x_{n_k}))$ is a subsequence of (y_n), and hence must converge to y. Therefore, $y = f(x)$. If (x_{m_q}) is any other subsequence of (x_n), converging to x', then $(f(x_{m_q})) \to f(x')$, and, again, $y = f(x')$. Since f is injective, we have $x = x'$.

This means that any subsequence of (x_n) converges to x. Therefore, $(x_n) \to x$.

That is, $(f^{-1}(y_n)) \to f^{-1}(y)$. Hence, f^{-1} is continuous at y. Since y was an arbitrary point of J, we get that f^{-1} is continuous on J. ■

Now we use the theorems on continuity and the Caratheodory Theorem, to prove the Inverse Function Theorem for differentiable function.
Roughly speaking it state that under certain conditions the derivative of the inverse of a function can be recovered from the derivative of the original function. Graphically it make sense that this information can be obtained since

the graph of the inverse of a function f is the reflection of f along the line $y = x$. Therefore each tangent line of f should be reflected into the tangent lines of f^{-1}. The following figures illustrates this.

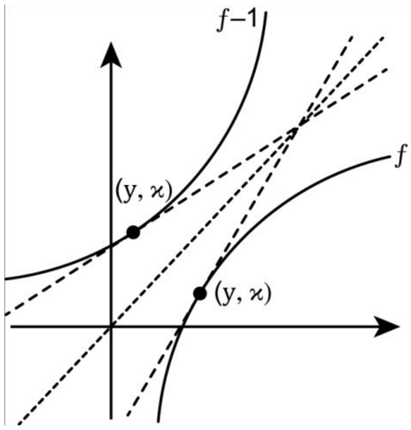

Fig. 6

Here is the statement and proof of the theorem.

Theorem 11 (Inverse Function Theorem): Let the function, $f : I \to \mathbb{R}$ be an injective, continuous function defined on the closed and bounded interval I, and let $f(I) = J$. If f is differentiable at $p \in I$, and $f'(p) \neq 0$, then $f^{-1} : J \to I$ is differentiable at $q = f(p)$, and $\left(f^{-1}\right)'(q) = \frac{1}{f'(p)}$.

Proof: By Caratheodory Theorem, there exists a function $\varphi : I \to \mathbb{R}$, continuous at p, such that

$$f(x) - f(p) = \varphi(x)(x - p) \qquad \text{... (5)}$$

for $x \in I$, and $\varphi(p) = f'(p)$. Since $\varphi(p) \neq 0$, by Theorem 6, there exists a $\delta > 0$, such that $\varphi(x) \neq 0$ for all $x \in (p - \delta, p + \delta) \cap I$.

Let $f((p - \delta, p + \delta) \cap I) = U$. Then for $y \in U$ we can write,
$y - q = f\left(f^{-1}(y)\right) - f(p) = \varphi\left(f^{-1}(y)\right).\left(f^{-1}(y) - f^{-1}(q)\right)$ by using Theorem 2.

Now, $\varphi\left(f^{-1}(y)\right) \neq 0$ for $y \in U$. Therefore, we can divide by $\varphi\left(f^{-1}(y)\right)$, and get

$$f^{-1}(y) - f^{-1}(q) = \frac{1}{\varphi\left(f^{-1}(y)\right)}(y - q).$$

Since the function, f^{-1} is continuous at q, and the function φ is continuous at $f^{-1}(q) = p$, the composite function, $\varphi \circ f^{-1}$ is continuous at q. This means the function, $\frac{1}{\varphi \circ f^{-1}}$ is also continuous at q.

Again, we apply Caratheodory Theorem, and conclude that $\left(f^{-1}\right)'(q)$ exists, and $\left(f^{-1}\right)'(q) = \frac{1}{\varphi\left(f^{-1}(q)\right)} = \frac{1}{\varphi(p)} = \frac{1}{f'(p)}$. ■

Remark 5: You may note in the theorem stated above we have the condition that f is differentiable at p. If we assume that the function is differentiable on the interval I, and if $f'(x) \neq 0$ for all $x \in I$, then we get that f^{-1} is differentiable on J, and also,

$$f^{-1} = \frac{1}{f' \circ f^{-1}}.$$

If $f'(x) = 0$ for some x_0 in the domain of an injective function, f, then f^{-1} is cannot be differentiable at x_0.

Now you learn the following examples very carefully so that you understand how to apply this theorem.

Example 9: Find the derivative of the inverse of $f(x) = x^n, n \in \mathbb{N}$, wherever it exists.

Solution: We will have to consider two cases: n even or odd.

Case 1: If n is even, we need to take the domain of f as $[0, \infty)$, so as to ensure the existence of its inverse. We know that $f'(x) = nx^{n-1}$. Hence, $f'(x) \neq 0$ for all $x > 0$.

The inverse of f is given by $f^{-1}(x) = x^{1/n}$. By Inverse Function Theorem, $\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{nx^{n-1}} = \frac{1}{ny^{(n-1)/n}} = \frac{1}{n} y^{(1/n)-1}$, since $y = x^n$.

Note that f^{-1} is not differentiable at $x = 0$, since $f'(0) = 0$.
If n is odd, then the inverse of f is given by $f^{-1}(x) = x^{1/n}$, and $f'(x) = nx^{n-1}$.

Then, for all $x \neq 0$, using Inverse Function Theorem again, we get

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{nx^{n-1}}$$
$$= \frac{1}{ny^{(n-1)/n}}$$
$$= \frac{1}{n} y^{(1/n)-1}$$

In this case also, f^{-1} is not differentiable at $x = 0$, since $f'(0) = 0$.

Hence the result for n is odd.

We have seen thus far, that if, $f(x) = x^r$, where either $r \in \mathbb{N}$, or $r = \frac{1}{n}, n \in \mathbb{N}$, then $f'(x) = rx^{r-1}$, for all $x > 0$. (Of course, $f'(0) = rx^{r-1}$ exists and is equal to 0, if $r \in N$.)

We can now use the chain rule to get the derivative of $x^{m/n}$, for $x > 0$.

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Example 10: Find the derivative of $f(x) = x^{m/n}, m, n \in \mathbb{N}, m, n > 0, x > 0$.

Proof: The function f is a composite of two functions. That is, $f = g \circ h$, where $g(x) = x^m$, and $h(x) = x^{1/n}, x > 0$. Then, using Chain Rule, we can write

$$f'(x) = g'(h(x))h'(x) = m\left(x^{1/n}\right)^{m-1}.\frac{1}{n}x^{(1/n)-1} = \frac{m}{n}x^{(m/n)-1}.$$

***

Example 11: Find the derivative of the inverse of $f(x) = \tan x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution: You can see in Fig. 8, that $\tan x$ is strictly increasing in its domain and its range is $(-\infty, \infty)$. Therefore it has an inverse,

$$\tan^{-1} : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

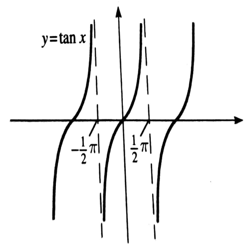

Fig. 8: Graph of $\tan x$

Function, f is differentiable on its domain, and

$$f'(x) = \sec^2 x \neq 0 \text{ for all } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Thus, by Inverse Function Theorem,

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + y^2}, \text{ for all } y \in (-\infty, \infty).$$

***

Working along these lines you can find the derivatives of all the inverse trigonometric functions.

Example 12: The function, $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 4x + 3$ has an inverse. Find the values of $\left(f^{-1}\right)'(y)$, for the values of y corresponding to $x = 0, -2, 3$.

Solution: $f'(x) = 3x^2 + 4 \neq 0$ for all $x \in (-\infty, \infty)$.

Therefore, f^{-1} is differentiable at all $y \in (-\infty, \infty)$, and

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{3x^2 + 4}.$$

Then the values of $\left(f^{-1}\right)'(y)$, for $x = 0, -2, 3$ are, $1/4, 1/16, 1/31$, respectively.

***

Now try these exercises on your own.

E5) Find the derivatives at a point y_0 of the domain of the inverse function theorem f, where $x = \sin x, x \in \left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$.

E6) Given that $f : \mathbb{R} \to \mathbb{R}, f(x) = x^5 + 2x + 1$ has an inverse, find the values of $\left(f^{-1}\right)'(y)$, for the values of y corresponding to $x = 0, -1, 1$.

E7) Given that $f:\mathbb{R}\to\mathbb{R}, f(x)=x^3+8x-2$ has an inverse, find the values of $\left(f^{-1}\right)'(y)$, for the values of y corresponding to $x=1,2,3$.

That brings us to the end of this Unit. So let us summarize the points covered in it.

11.5 SUMMARY

In this unit we have done the following:

- introduced the concept of derivatives of functions;
- proved that a differentiable function is continuous, but the converse is not true;
- explained how to get the derivatives of some functions, if they exist, using the definition;
- given formulas for the derivatives of the sum, difference, scalar multiple, product, and quotient of differentiable functions, and used those to get the derivatives of many more functions;
- explained the chain rule, which is useful in finding the derivatives of composite functions;
- proved cartheodary Theorem for differentiable function;
- proved the following theorems on continuity;
  i) Bolzano Theorem
  ii) Intermediate Value Theorem (IVT)
  iii) Inverse Function Theorem
  iv) Interval Image Theorem
- Proved the Inverse Function Theorem for differentiability and discussed how to apply this theorem for finding the derivative of inverse functions.

11.6 SOLUTIONS AND ANSWERS

E1) $f(x)=x^{2/3}$

Now, $\dfrac{f(0+h)-f(0)}{h}=\dfrac{h^{2/3}-0}{h}=\dfrac{1}{h^{1/3}}$

Therefore, $\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$ does not exist.

So, $f(x)$ is not differentiable at $x=0$.

Thus, the tangent of $f(x)$ at $x=0$ does not exist.

E2) a) Let $h(x)=x$ and $g(x)=1+2x^2$

Therefore, $f(x)=\dfrac{h(x)}{g(x)}$

As, we can see $g(x) \neq 0$ for all $x \in \mathbb{R}$.

Therefore, using the result (iv) of Theorem 2, we can write

$$f'(x) = \left(\frac{h}{g}\right)'(x) = \frac{h'(x)g(x) - h(x)g'(x)}{(g(x))^2}$$

Now, $h'(x) = 1$ & $g'(x) = 0 + 2(2x) = 4x$

Therefore,

$$f'(x) = \frac{1(1+2x^2) - x(4x)}{(1+2x^2)^2} = \frac{1+2x^2-4x^2}{(1+2x^2)^2}$$

$$= \frac{1-2x^2}{(1+2x^2)^2}$$

b) $f(x) = (\sin x)^5 + (\cos x)^3$

Let $f_1(x) = (\sin x)^5$ & $f_2(x) = (\cos x)^3$

Therefore, $f(x) = f_1(x) + f_2(x)$

So, using Remark 1, we have

$$f'(x) = f_1'(x) + f_2'(x)$$

Now, let, $h_1(x) = x^5$ & $h_2(x) = x^3$
and $f_1(x) = \sin x$ & $g_2(x) = \cos x$

Therefore, using the definition of composite function, we have

$$f_1(x) = (h_1 o g_1)(x) \quad \& \quad f_2(x) = (h_2 o\, g_2)(x).$$

Definition: (Composite function): Suppose $f :] \to \mathbb{R}$ and $g : I \to \mathbb{R}$, where I and $]$ are subsets of $\mathbb{R}$ and $f(]) \subseteq I$.

The composite function $gof :] \to \mathbb{R}$ is defined by $(gof)(x) = g(f(x))$.

Therefore, using Theorem 5, we have

$$f_1'(x) = (h_1 o g_1)'(x) \text{ and } f_2'(x) = (h_2 o g_2)'(x)$$
$$= h_1'(g_1(x))g_1'(x) \qquad = h_2'(g_2(x))g_2'(x).$$

Now, $h_1'(x) = 5x^9, h_2'(x) = 3x^2, g_1'(x) = \cos x$ and $g_2'(x) = -\sin x$.

So, $f_1'(x) = 5(\sin x)^9(\cos x)$

$$= 5\sin^9 x\cos x$$

and $f_2'(x) = 3(\cos x)^2(-\sin x)$

$$= -3\sin x\cos^2 x$$

$$\therefore f'(x) = 5\sin^9 x\cos x - 3\sin x\cos^2 x.$$

E3) $f(x) = \begin{cases} x^2, & x > 0 \\ 0, & x \le 0 \end{cases}$

Now, $\dfrac{f(0+h) - f(0)}{h} = \begin{cases} \dfrac{h^2 - 0}{h}, & h > 0 \\ \dfrac{0-0}{h}, & h < 0 \end{cases}$

Therefore, $\lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^+} h = 0$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} 0 = 0$$

So, $\lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ exists.

Therefore, $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

Derived function of f,

$f'(x) = \begin{cases} 2x, & x > 0 \\ 0, & x \le 0 \end{cases}$

Now, $\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^+} 2x = 0$

and $\lim\limits_{x \to 0^-} f'(x) = \lim\limits_{x \to 0^-} 0 = 0$

So, $\lim\limits_{x \to 0} f'(x)$ exists and $\lim\limits_{x \to 0} f'(x) = 0 = f'(0)$

Therefore, the derived function $f'(x)$ is continuous at $x = 0$.

E4) a) Let $f(x) = \sin^5(x^7), g(x) = \sin^5(x)$ and $h(x) = x^7$

Therefore, $f(x) = (goh)(x)$

Now, using Theorem 5, we have $f'(x) = g'(h(x))h'(x)$.

Now, $g'(x) = 5\sin^9 x\cos x$ [See Problem (c) of (E3)]
and $h'(x) = 7x^6$

Therefore, $f'(x) = \{5\sin^9(x^7)\cos(x^7)\}\{7x^6)$

$$= 35x^6 \sin^9(x^7)\cos(x^7)$$

b) Let $f(x) = \sqrt{1 + 3x - 9x^3}, g(x) = \sqrt{x},$

and $h(x) = 1 + 3x - 4x^3$

Therefore, $f(x) = (g\, o\, h)(x)$

Now, using Theorem 5, we have

$f'(x) = g'(h(x))h'(x)$

Now, $g'(x) = \dfrac{1}{2\sqrt{x}}, x > 0$ [See E1) iv)]

So, $g'(h(x)) = \dfrac{1}{2\sqrt{1+3x-9x^3}}, h(x) > 0$

and $h'(x) = 0 + 3 - 12x^2$

$= 3 - 12x^2$

Therefore, $f'(x) = \dfrac{1}{2\sqrt{1+3x-4x^3}}(3-12x^2), (1+3x-4x^3) > 0$

$= \dfrac{3-12x^3}{2\sqrt{1+3x-4x^3}}, (1+3x-4x^3) > 0.$

E5) $f(x) = \sin x,\ x \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$

Let, $I = \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$

So, $f(I) =]-1,1[$

Now, $f'(x) = \cos x$

Let $y_0 = f(x_0)$

So, $f'(x_0) = \cos x_0 \neq 0 \ \forall\ x_0 \in \left(-\pi/2, \pi/2\right)$

Therefore, by Inverse Function Theorem, we can say,

$f^{-1} :]-1,1[\to \left]-\pi/2, \pi/2\right[$ is differentiable for a point $y_0 \in]-1,1[$ and

$(f^{-1})^1(y_0) = \dfrac{1}{f'(x_0)} = \dfrac{1}{\cos x_0}.$

E6) $f(x) = x^5 + 2x + 1$

Now, $f'(x) = 5x^4 + 2 \neq 0\ \forall\, x \in \mathbb{R}.$

Thus by Inverse Function Theorem, $f^{-1} : \mathbb{R} \to \mathbb{R}$ is differentiable and

$(f^{-1})^1(y) = \dfrac{1}{f'(x)}$, where $y = f(x) \in \mathbb{R}$

$$= \frac{1}{5x^4 + 2}$$

Then the values of $(f^{-1})^1(y)$ for $x = 0, -1, 1$ are $\frac{1}{2}, \frac{1}{7}, \frac{1}{7}$ respectively.

E7) $f(x) = x^3 + 8x - 2$

Now, $f'(x) = 3x^2 + 8 \neq 0$ for all $x \in \mathbb{R}$.

Therefore, by Inverse Function Theorem, $f^{-1} : \mathbb{R} \to \mathbb{R}$ is differentiable at all $y \in \mathbb{R}$ and $(f^{-1})(y) = \frac{1}{f'(x)} = \frac{1}{3x^2 + 8}$.

Then the values of $(f^{-1})^1(y)$ for $x = 1, 2, 3$ are $\frac{1}{11}, \frac{1}{20}$ and $\frac{1}{35}$ respectively.

UNIT 12

APPLICATIONS

Structure **Page No**

12.1 INTRODUCTION

Recall that in Unit 11, we have discussed the concept of differentiability. In that unit, we also established the rules for finding derivatives of combinations of differentiable functions. In this unit we shall concentrate on certain applications of differentiability such as whether the derivative possesses the Intermediate Value Property, or what the shape of a function is around a point at which its derivative vanishes. You must have seen some of these applications in your course Calculus. We shall also look at the geometrical significance of the properties possessed by derivatives, and apply them, for instance, in showing the existence of roots of equations.

We begin the unit with Darboux (read as "Daar-boo") Theorem in Section 12.2. We shall also discuss, in the section, the Interior Extremum Theorem which tells us how the derivative of a function behaves at an interior point of extrema. In Section 12.3, we shall discuss Rolle's Theroem, and its geometrical and algebraic interpretation. Section 12.4 introduces three mean value theorems namely, Lagrange's Mean value Theorem, Cauchy's Mean Value Theorem, and Generalised Mean Valued Theorem. Finally in Section 12.5 you will see how to prove whether a differentiable function is monotone or not in a given interval.

Objectives

After studying this unit you should be able to:

- describe and prove the Interior Extremum Theorem;
- state and prove Darboux's Theorem;

- state and prove Rolle's Theorem, and give its geometrical and algebraic interpretations;
- state, prove and apply Lagrange's Mean Value Theorem, Cauchy's Mean Value Theorem, and the Generalised Mean Value Theorem to obtain some known results;
- show when a function increases/decreases in its domain.

12.2 DARBOUX'S THEOREM

Fig. 1:Jean Gaston Darboux

In this section we shall discuss Darboux's Theorem, given by the French Mathematician Jean Gaston Darboux (1842-1917). But before that we shall discuss an important result known as Interior Extremum Theorem which is used in the proof of Darboux's Theorem.

We begin with an example. Consider the function $f:[-1,5]\to\mathbb{R}$ defined by $f(x)=\dfrac{(x-1)^2}{4}+2$ (see Fig. 2).

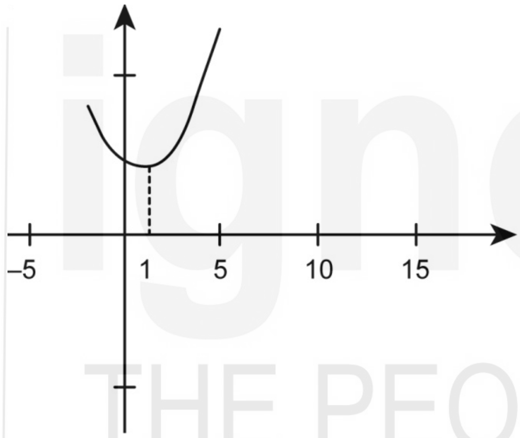

Fig. 2: Graph of $f(x)=\dfrac{(x-1)^2}{4}+2$

You know that since f is a polynomial function, it is differentiable on $]-1,5[$. Also you can see that f attains a minimum value at 1, i.e., 1 is a point of minimum of f. Now look at the derivative of f at 1. You should check that $f'(1)=0$. This means the tangent on the curve of f at 1 is parallel to the x-axis. This is not a coincidence. In fact, if you take any differentiable function which attains an extreme (minimum or maximum) value at an interior point of its domain, then its derivative is zero at that point. This is what the following theorem tells us.

Theorem 1 (Interior Extremum Theorem): Let $f:[a,b]\to\mathbb{R}$ be a function, differentiable on $\left]a,b\right[$. If f attains an extreme value at some interior point c of $[a,b]$, , that is, $c\in]a,b[$, then $f'(c)=0$.

Proof: We shall prove this by contradiction. So, without loss of generality, let us assume that f attains a maximum value at some $c\in]a,b[$ such that $f'(c)\neq 0$. Then either $f'(c)<0$ or $f'(c)>0$.

First let us consider the case $f'(c)<0$. By the definition of derivative, we have

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}.$$

Take $\varepsilon = \frac{-f'(c)}{2}$. Note that $\varepsilon > 0$. Then there exists some δ-neighbourhood of $c, N_\delta(c)$ in $]a,b[$ such that for all $x \in N_\delta(c)$

$$\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \varepsilon.$$

Now we have $-\varepsilon < \frac{f(x)-f(c)}{x-c} - f'(c) < \varepsilon$. This gives us

$$f'(c) - \varepsilon < \frac{f(x)-f(c)}{x-c} < f'(c) + \varepsilon \Leftrightarrow \frac{3}{2} f'(c) < \frac{f(x)-f(c)}{x-c} < \frac{f'(c)}{2}.$$

Since $f'(c) < 0$, we get $\frac{f(x)-f(c)}{x-c} < 0$. Now if $x \in N_\delta(c)$ is such that $x < c$, then we have $f(x) > f(c)$. This contradicts the fact that c is a point of maximum of f. Hence our assumption that $f'(c) < 0$ is false. So, $f'(c) \geq 0$. You can produce a similar argument to show that $f'(c) \leq 0$. Thus we have proved that $f'(c) = 0$. ■

Remark 1: Interior Extremum Theorem is a very useful result about differentiable functions, and you will see its application in proving many theorems that follow.

Now, let us recall the statement of the Intermediate Value Theorem from Unit 11. It essentially says that every continuous function attains all the values between any two of its values. Therefore, if the derivative of a function is continuous, then we can say that the derivative function too has the same property. However, as you know, the derivative of a function need not be continuous. Darboux proved that whether the derivative is continuous or not, it always possesses the Intermediate Value Property. This is contained in the following theorem.

Theorem 2 (Darboux's Theorem): Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on the closed interval $[a,b]$ and r be any number between $f'(a)$ and $f'(b)$. Then there exists a number c in the open interval $]a,b[$ such that $f'(c) = r$.

Proof: Suppose that $f'(a) < r < f'(b)$. Define a real valued function $g : [a,b] \to \mathbb{R}$ such that $g(x) = f(x) - rx$. Then g is differentiable on $[a,b]$, and $g'(x) = f'(x) - r$. This means, g is continuous on $[a,b]$. From Unit 10 you know that every continuous function attains its minimum on a closed bounded interval. Hence there exists a point $c \in [a,b]$ such that $g(c) = \min\{g(x) \mid a \leq x \leq b\}$.

First we shall show that $c \neq a$ and $c \neq b$. Note that $f'(a) < r < f'(b)$ implies that $g'(a) < 0 < g'(b)$. By the definition of derivative, you know that

$$g'(a) = \lim_{x \to a^+} \frac{g(x)-g(a)}{x-a}.$$

Let $\varepsilon = -\frac{g'(a)}{2} > 0$. Then there exists some $0 < \delta < b-a$ such that for all $x \in]a, a+\delta[$

$$\left|\frac{g(x)-g(a)}{x-a}-g'(a)\right|<\varepsilon \Rightarrow \frac{g(x)-g(a)}{x-a}<0$$

(See the argument given in the proof of Theorem 1.) Using $x>a$ we get $g(x)-g(a)<0$, which implies $g(x)<g(a)$. This means $g(a)$ is not the minimum value of g, and hence $c\neq a$. You should use similar arguments to show that $c\neq b$ (see E1). Thus, $c\in]a,b[$. Now the Interior Extremum Theorem (Theorem 1) implies that $g'(c)=0$. This means $f'(c)=r$. ■

The following corollary is an immediate consequence of the theorem above.

Corollary 1: If a function $f:[a,b]\to\mathbb{R}$ is differentiable on the closed interval $[a,b]$ and $f'(a)$ and $f'(b)$ are of opposite signs, then there exists a number c in the open interval $]a,b[$ such that $f'(c)=0$.

Remark 2: We can restate the Darboux's Theorem as follows.

If f is a differentiable function on a closed and bounded interval, then f' possesses the Intermediate Value Property.

Let us now consider some examples of application of Theorem 2.

Example 1: Consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0.\end{cases}$$

Show that f is differentiable on every closed bounded interval of $\mathbb{R}$. Hence conclude that f' satisfies the intermediate value property.

Solution: You can see that f is differentiable on $\mathbb{R}$, and $f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$, for $x\neq 0$. Also, $f'(0)=0$.

Hence, f' satisfies the Intermediate Value Property by Darboux's Theorem.

***

Example 2: Show that for the function $f(x)=x^3-8x^2-10$ there is some $c\in]1,2[$ such that $f'(c)=-15$. Find a value for such a point c.

Solution: Since $f'(x)=3x^2-16x$, we have $f'(1)=-13, f'(2)=-20$. Since $-20<-15<-13$, there is some $c\in]1,2[$ such that $f'(c)=-15$. This gives us $3c^2-16c+15=0$. Solving this equation, we get $c=\frac{8\pm\sqrt{19}}{3}$. Since $c\in]1,2[$, we choose $c=\frac{8-\sqrt{19}}{3}$.

***

Let us now look at another consequence of Darboux's Theorem.

Corollary 2: Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$, and suppose

$f'(x) \neq 0 \, \forall \, x \in [a,b]$. Then, either $f'(x) < 0 \, \forall \, x \in [a,b]$ or $f'(x) > 0 \, \forall \in [a,b]$.

Proof: Suppose there exist two points x_1 and x_2 in $[a,b]$ such that $x_1 < x_2$ and $f'(x_1) < 0 < f'(x_2)$. Then f is differentiable on $[x_1, x_2]$. Applying Darboux's Theorem we get a point $c \in \,]x_1, x_2[\, \subseteq [a,b]$ such that $f'(c) = 0$. This contradicts our assumption that $f'(x) \neq 0 \, \forall \, x \in [a,b]$. Hence, either $f'(x) < 0$ for all $x \in [a,b]$ or $f'(x) > 0$ for all $x \in [a,b]$. ■

Example 3: Let $f : [2,5] \to \mathbb{R}$ be a function such that

$$f(x) = \begin{cases} 3, & 2 \leq x < 3 \\ 0, & 3 \leq x \leq 4 \\ 7, & 4 < x \leq 5 \end{cases}.$$

Show that there is no a function $g : [2,5] \to \mathbb{R}$, differentiable on $[2,5]$ such that $g'(x) = f(x)$ for all $x \in [2,5]$.

Solution: On the contrary, assume that there exists a function $g : [2,5] \to \mathbb{R}$ such that $g'(x) = f(x)$ for all $x \in [2,5]$. We shall arrive at a contradiction. Note that g is differentiable on $[2,5]$. Also, note that $g'(2) = 3$ and $g'(5) = 7$. So 4 lies between $g'(2)$ and $g'(5)$. Hence by Darboux's Theorem there exists some $c \in \,]2,5[$ such that $g'(c) = f(c) = 4$. But f never achieves 4, which is a contradiction. Hence there exists no function $g : [2,5] \to \mathbb{R}$ which is differentiable on $2[,5]$ such that $g'(x) = f(x)$ for all $x \in [2,5]$.

***

Now, try the following exercises.

E1) Let $g : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. Show that if $g'(b) > 0$, then $g(b) \neq \min \{g(x) \mid a \leq x \leq b\}$.

E2) Prove that if a function $f : [-1,1] \to \mathbb{R}$ is differentiable on $[-1,1]$ and $f'(-1) > 0 > f'(1)$, then there exists a number $c \in \,]-1,1[$ such that $f'(c) = 0$.

E3) Let $f : [0,2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}.$$

Show that there is no real valued function defined on $[0,2]$ whose derivative is f.

Thus far we have only seen two fundamental properties of derivatives. In the next section we shall use them to explore more properties of derivatives.

12.3 ROLLE'S THEOREM

In this section, we will study Rolle's Theorem given by Michel Rolle (1652-1719), a French mathematician. This theorem can be treated as the foundation for the mean value theorems which you will study in the next section.

Fig. 3: Michel Rolle

Before we state the theorem, let us look at an example. Consider the graph of the function $f(x) = \sin(\pi x),\ x \in [0,1]$ shown in Fig. 4 below.

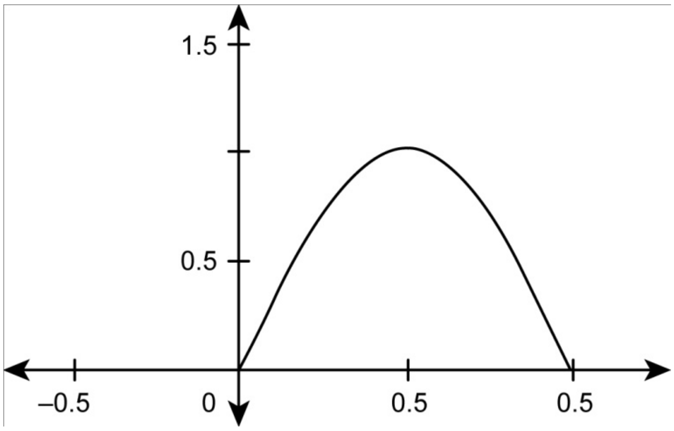

Fig. 4: Graph of $f(x) = \sin(\pi x),\ x \in [0,1]$

We say that a real number p is a root of an equation $f(x) = 0$ if $f(p) = 0$.

You know that f is continuous and differentiable in the interval $[0,1]$. Also you know that $f(0) = 0 = f(1)$. Now observe that there is a point c in $]0,1[$ at which f attains a maximum value. This point is $c = \frac{1}{2}$. From the Interior Extremum Theorem, you know that $f'(c) = 0$.

This raises a general question: given a continuous and differentiable function f defined on an interval $[a,b]$, does there always exist an interior point c of $[a,b]$ such that $f'(c) = 0$?

Michel Rolle was the first mathematician who found the answer of this question. It is stated in the following theorem, due to him.

Theorem 3 (Rolle's Theorem): Let $f : [a,b] \to \mathbb{R}$ be a function such that

i) f is continuous on $[a,b]$,
ii) f is differentiable on $]a,b[$, and
iii) $f(a) = f(b)$.

Then there exists a real number $c \in]a,b[$ such that $f'(c) = 0$.

Proof: Since f is continuous on the closed bounded interval $[a,b]$, f attains its extreme values. Let $\sup f = M$ and $\inf f = m$. Then there are points $c, d \in [a,b]$ such that $f(c) = M$ and $f(d) = m$. Only two possibilities arise now: either $M = m$ or $M \neq m$.

Case 1: When $M = m$. Then for some fixed real number k, $f(x) = k\ \forall\ x \in [a,b]$, which implies $f'(x) = 0 \,\forall\, x \in [a,b]$.

Case 2: When $M \neq m$. Then we proceed as follows:

Since $f(a) = f(b)$, at least one of the numbers M or m, is different from $f(a)$, and hence different from $f(b)$. Suppose $M \neq f(a)$. Then it follows that

$f(c) \neq f(a)$ which implies that $c \neq a$ as f is a function. Similarly $M \neq f(b)$ means $f(c) \neq f(b)$ which implies $c \neq b$.

Thus $c \in]a,b[$. Now the Interior Extremum Theorem implies that $f'(c) = 0$. ■

Let us consider an example.

Example 4: Check whether or not the following functions satisfy the conditions of Rolle's Theorem.

i) $f(x) = x^3 - 6x^2 + 11x - 6 \ \forall \ x \in [1,3]$.

ii) $f(x) = (x-a)^m (x-b)^n, \forall x \in [a,b]$, where m and n are positive intergers.

iii) $f(x) = x^{2/3}, \forall \ x \in [0,1]$.

If they do, verify the conclusion by finding a point c in their domain where the derivative vanishes.

Solution: i) We first note that the function $f(x) = x^3 - 6x^2 + 11x - 6$ is a polynomial function defined on $[1,3]$. Therefore, f is continuous on $[1,3]$ and derivable in $]1,3[$. Also $f(1) = f(3) = 0$. Thus, Conditions (i), (ii) and (iii) of Rolle's Theorem are satisfied. Therefore, by Rolle's Theorem, there is a real number $c \in]1,3[$ such that $f'(c) = 0$. To find the possible values of c, let us compute $f'(c)$. We have $f'(c) = 3c^2 - 12c + 11$. Thus

$$f'(c) = 0 \Rightarrow c = 2 + \frac{1}{\sqrt{3}}, 2 - \frac{1}{\sqrt{3}}.$$

Note that both the values of c lie in $]1,3[$. Hence the conclusion of the Rolle's Theorem is verified.

ii) Note that f is continuous in $[a,b]$ and derivable in $]a,b[$ as f is a polynomial. Also $f(a) = f(b) = 0$. So the hypothesis of Rolle's Theorem is satisfied. Now

$$f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1}$$
$$\therefore f'(x) = 0 \Rightarrow (x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)] = 0$$
$$\Rightarrow m(x-b) + n(x-a) = 0$$
$$\Rightarrow (m+n)x - (na + mb) = 0.$$

Let $c = \dfrac{na + mb}{m+n}$. Since m and n are positive integers, therefore, this point c lies in $]a,b[$. Indeed, c is a point which divides the closed interval $[a,b]$ in the ratio $m:n$.

iii) Since f is not differentiable at $x = 0, f$ does not satisfy the conditions of the Rolle's Theorem.

***

Let us now look at how the Rolle's Theorem can be understood geometrically.

Geometrical Interpretation of Rolle's Theorem

Look at the following graph of a function $f : [a,b] \to \mathbb{R}$ satisfying the conditions of Rolle's Theorem.

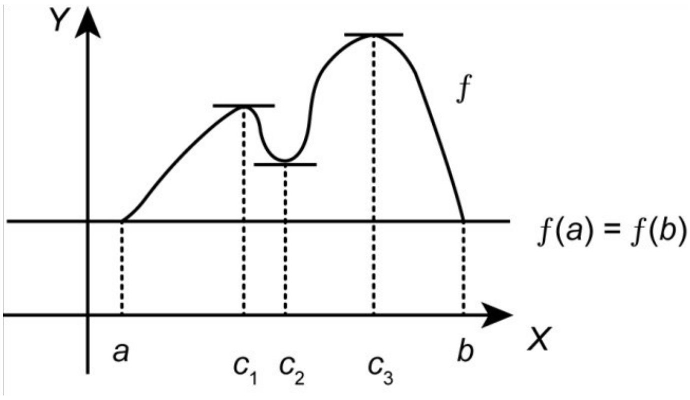

Fig. 5

Between a and b, there are some points, namely c_1, c_2 and c_3, at which f attains extreme values. These are the points where f' vanishes. Geometrically, this means that the slope of the tangents at these points are zero, or equivalently, the tangents at these points are parallel to the x-axis.

Intersetingly, Rolle's Theorem can also be interpreted algebraically.

Algebraic Interpretation of Rolle'sTheorem

Let a function f be continous on the closed interval $[a, b,]$ and differentiable on the open interval $]a, b[$. Then between any two roots a and b of $f(x) = 0,$ there exists at least one root c of $f'(x) = 0.$

We can prove this result by observing that when a and b are roots of $f(x) = 0,$ then $f(a) = 0$ and $f(b) = 0$ and therefore, $f(a) = f(b).$ Then by Rolle's Theorem there is a point c of $]a, b[$ such that $f'(c) = 0$, which means that c is a root of $f'(x) = 0.$

Let us look at some applications of Rolle's Theorem.

Example 5: Show that there is no real number λ for which the equation $x^3 - 27x + \lambda = 0$ has two distinct roots in $[0, 2]$.

Solution: Let $f(x) = x^3 - 27x + \lambda$. Suppose for some value of $\lambda, f(x) = 0$ has two distinct roots α and $\beta,$ in $[0, 2].$ Without any loss of generality, suppose $\alpha < \beta$. Then, $[\alpha, \beta] \subseteq [0, 2].$ Now f is continuous on $[\alpha, \beta],$ derivable in $]\alpha, \beta[$ and $f(\alpha) = f(\beta) = 0.$ Therefore, by Rolle's Theorem, there exists $c \in]\alpha, \beta[$ such that $f'(c) = 0$. That is, $3c^2 - 27 = 0.$ This implies $c = \pm 3.$ But neither 3 nor -3 lies in $]0, 2[,$ that is, $-3, 3 \notin]\alpha, \beta[.$ Thus we arrive at a contradiction. Hence the result follows.

***

Example 6: Show that if all the roots of a polynomial function P of degree $n (\geq 2)$ are real, then all the roots of P' are also real.

Solution: Let P be a polynomial of degree $n (\geq 2)$ with all roots real. You know that a polynomial of degree n has n roots. Thus P has n real roots, say $\alpha_1, \alpha_2, \ldots, \alpha_n$ sorted in ascending order. The algebraic interpretation of

Rolle's Theorem tells us that between any two roots of P, there is a root of P'. Therefore, for each $i \in \{2,3,\ldots,n\}$ there is a root β_{i-1} of P' such that $\beta_{i-1} \in [\alpha_{i-1}, \alpha_i]$. Thus the roots $\beta_1, \beta_2, \ldots, \beta_{n-1}$ of P' are real. Since P' has degree $n-1$, P' has no other roots. Consequently, all the roots of P' are real.

***

Now you try the following exercises.

E4) Verify Rolle's Theorem for the function f where

$$f(x) = \sin x, x \in [-2\pi, 2\pi].$$

E5) Examine the validity of the hypothesis and the conclusion of Rolle's Theorem for the functions f defined by

i) $f(x) = \cos x,\ x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

ii) $f(x) = 1 + (x+1)^{\frac{2}{3}}, x \in [0,2].$

E6) Show that the equation $x^3 - 25x + 9 = 0$ does not have two distinct roots in the interval $[-2,2]$.

E7) Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$.

E8) Prove that if $a_0, a_1, ..., a_n \in \mathbb{R}$ be such that $\frac{a_0}{n+1} + \frac{a_1}{n} + ... + \frac{a_{n-1}}{2} + a_n = 0,$ then there exists at least one real number x between 0 and 1 such that $a_0 x^n + a_1 x^{n-1} + ... + a_n = 0.$

E9) If a function f is such that its derivative, f', is continuous on $[a,b]$ and derivable on $]a,b[$, then show that there exists a number $c \in]a,b[$ such that $f(b) = f(a) + (b-a) f'(a) + \frac{1}{2}(b-a)^2 f''(c).$

If you have gone through the exercises above you must have understood how important Rolle's Theorem is in applied problems. Next we shall see its generalisations.

12.4 MEAN VALUE THEOREMS

In this section, we shall discuss mean-value theorems given by the two famous French mathematicians A-L. Cauchy (1789-1857) with whom you are already familiar from the previous block, and Joseph-Louis Lagrange (1736-1813).

Fig. 6: Augustin-Louis Cauchy

Let us consider a continuous function $f : [a,b] \to \mathbb{R}$ which has a slope at every interior point, except possibly at a and b. Look at the graph of the function, that is, the curve representing this function in Fig. 8 given below.

Fig. 7: Joseph-Louis Lagrange

We shall write MVT in short for Mean Value Theorem. The terminology Mean Value Theorem stems from the fact that all these theorems relate the mean value of the derivative over an interval $[a,b]$ to the actual derivative at an interior point of $[a,b]$.

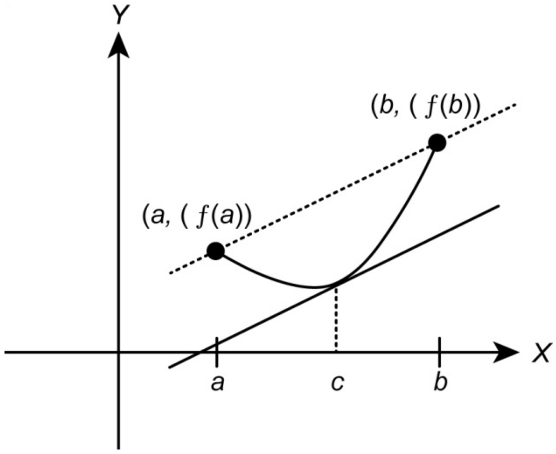

Fig. 8: The tangent at c has the slope $\dfrac{f(b)-f(a)}{b-a}$.

Then $(a, f(a))$ and $(b, f(b))$ are two points on the curve. The line passing through them is a secant line of the curve, which has slope $\dfrac{f(b)-f(a)}{b-a}$. The question that concerned Lagrange and his contemporaries is: Does there exists a tangent at some point c to this curve that is parallel to this secant line? Lagrange's Mean Value Theorem provides the answer.

Theorem 4 (Lagrange's Mean Value Theorem): If a function $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and derivable on $]a,b[$, then there exists a point $c\in\,]a,b[$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Proof: Consider a function $f:[a,b]\to\mathbb{R}$, which is continuous on $[a,b]$ and derivable on $]a,b[$. Define a function $\phi:[a,b]\to\mathbb{R}$ such that $\phi(x)=f(x)+Ax$ for all $x\in[a,b]$, where A is a constant to be chosen such that $\phi(a)=\phi(b)$. Now

$$\phi(a)=\phi(b)\Rightarrow f(a)+Aa=f(b)+Ab\Rightarrow A=-\frac{f(b)-f(a)}{b-a}.$$

Then, the function ϕ, being the sum of two continuous and derivable functions satisfies the conditions of the Rolle's Theorem. That is, ϕ is continuous on $[a,b]$, derivable on $\left]a,b\right[$, and, of course, $\phi(a)=\phi(b)$.

Therefore by Rolle's Theorem there is a real number $c\in\,]a,b[$ such that $\phi'(c)=0$. But $\phi'(x)=f'(x)+A$. So, we have $f'(c)=-A=\dfrac{f(b)-f(a)}{b-a}$. ■

Langrange's Mean Value Theorem is often stated in the following form:

Let $f:[a,a+h]\to\mathbb{R}$ be a function which is continuous on $[a,a+h]$ and derivable on $]a,a+h[$, where $h>0$. Then there exists a real number θ in $\left]0,1\right[$ such that $f(a+h)=f(a)+hf'(a+\theta h)$.

Lagrange's Mean-Value Theorem has many important implications. Let us first look at the following corollary.

Corollary 3: If a function f is continuous on $[a,b]$, derivable on $]a,b[$ and $f'(x)=0$ for all $x\in\,]a,b[$, then $f(x)=k\ \forall\ x\in[a,b]$, where k is some fixed real number, i.e., f is a constant function.

Proof: Let λ be any point of $[a,b]$. Then $[a,\lambda]\subseteq[a,b]$. Now f is continuous on $[a,\lambda]$ and is derivable on $]a,\lambda[$.

Therefore, by Lagrange's Mean Value Theorem, there exists some $c\in]a,\lambda[$ such that $f'(c)=\dfrac{f(\lambda)-f(a)}{\lambda-a}$. Now

$$f'(x)=0\,\forall\, x\in\,]a,b[\;\Rightarrow f'(c)=0\Rightarrow f(\lambda)=f(a)$$

But λ is an arbitrary point of $[a,b]$. Therefore $f(x)=f(a)=k$ (say) $\forall\, x\in[a,b]$. Thus, f is a constant function. ■

Yet another application is given below.

Assume that an object moves through some geographical area from time t_1 to time t_2, and $f(t)$ is its position at time t. Then at some point of time between t_1 and t_2, the object must have moved at its average speed $\dfrac{f(t_2)-f(t_1)}{t_2-t_1}$.

The following examples show that the conditions of Lagrange's Mean Value Theorem cannot be weakened. That is, if we restrict the continiuity to a proper subset of $[a,b]$, or differentiablitiy to a proper subset of $]a,b[$, the conclusion may no longer hold.

Example 7: Let a function f be defined on $[1,2]$ as follows:

$$f(x)=\begin{cases}1, & \text{if } x=1\\ x^2, & \text{if } 1<x<2\\ 2, & \text{if } x=2\end{cases}$$

Show that f does not satisfy the conditions of Lagrange's MVT. Does the conclusion of the theorem hold in this case?

Solution: Note that f is continuous on the semi-open interval $[1,2[$ and derivable on the open interval $]1,2[$. However, f is discontinuous at $x=2$, because $\lim\limits_{x\to2} f(x)=4\neq f(2)$. So the first condition of Lagrange's Mean Value Theorem is violated. Note that the conclusion is also not true, as $\dfrac{f(2)-f(1)}{2-1}\neq f'(c)$ for any $c\in\left]1,2\right[$. (Indeed, $f'(x)=2x$ for all $x\in\left]1,2\right[$. So, $f'(c)=2c$, whereas $\dfrac{f(2)-f(1)}{2-1}=1$.)

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Example 8: Let $f(x)=|x|\,\forall\, x\in[-1,2]$. Does f satisfy all the conditions of Lagranges's MVT? Does the conclusion of the theorem hold? Justify your answer.

Solution: Here f is continuous on $[-1,2]$ and derivable at all points of $[-1,2]$ except at $x=0$. So, the second condition of Lagrange's Mean Value Theorem is violated.

We can write $f(x) = \begin{cases} x & \text{if } 0 \le x \le 2 \\ -x & \text{if } -1 \le x < 0 \end{cases}$

So, $f'(x) = \begin{cases} 1 \text{ if } 0 < x < 2 \\ -1 \text{ if } -1 < x < 0 \end{cases}$

Also $\frac{f(2) - f(-1)}{2-(-1)} = \frac{2-1}{2+1} = \frac{1}{3}.$

Thus $\frac{f(2) - f(-1)}{2-(-1)} \ne f'(x)$ for any x in $]-1, 2[$. Hence, the conclusion of the theorem does not hold.

***

Remark 3: Note that the conditions of Lagrange's Mean Value Theorem are only sufficient. They are not necessary. This is evident from the following function.

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{4} \\ x & \text{if } \frac{1}{4} \le x < \frac{1}{2} \\ \frac{x}{2} + 1 & \text{if } \frac{1}{2} \le x \le 2 \end{cases}$$

You can see that $\frac{3}{8}$ lies in $]0, 2[$ and $f'\left(\frac{3}{8}\right) = 1 = \frac{f(2) - f(0)}{2-0}$.

However, f is neither continuous on $[0, 2]$, nor differentiable on $]0, 2[$.

Example 9: Verify the hypothesis of Lagrange's Mean Value Theorem for the following functions. Hence, for each of the functions find a point c that satisfies the conclusion of the theorem.

i) $f(x) = \frac{1}{x},\ x \in [1, 4]$

ii) $f(x) = \ln x,\ x \in \left[1, 1 + \frac{1}{e}\right]$

Solution: i) Here $f(x) = \frac{1}{x}, x \in [1, 4]$. You know that f is continuous in $[1, 4]$ and derivable in $]1, 4[$. So, f satisfies the hypothesis of Lagrange's M.V.T, and, hence there exists a point $c \in]1, 4[$ satisfying $f'(c) = \frac{f(4) - f(1)}{4-1}$. Putting the values of f and f', you get

$$-\frac{1}{c^2} = \frac{\left(\frac{1}{4}\right) - 1}{3} \Rightarrow c = \pm 2.$$

We can choose $c = 2$ as it belongs to $]1, 4[$.

ii) Here $f(x) = \ln x,$ where $x \in [1, 1+e^{-1}]$. You know that f is continuous in $[1, 1+e^{-1}]$ and derivable in $]1, 1+e^{-1}[$. Therefore, the hypotheses of Lagrange's Mean Value Theorem are satisfied by f. So, there exists a point $c \in]1, 1+e^{-1}[$ such that

$$f'(c) = \frac{f(1+e^{-1}) - f(1)}{(1+e^{-1}) - 1}.$$

Putting the values of f and f', you get

$$\frac{1}{c} = \frac{\ln(1+e^{-1}) - \ln 1}{e^{-1}} \Rightarrow c = \frac{1}{e \ln(1+e^{-1})}$$

Now use the inequality

$$\frac{x}{1+x} < \ln(1+x) < x \text{ for all } x > 0 \quad \text{(See Example 17.)}$$

to show that $c \in]1, 1+e^{-1}[$. (Indeed, put $x = e$ in the inequality above, and simplify.)

***

Example 10: Given any real numbers $a < b,$ show that there exists a real number c between a and b such that

$$c^2 = \frac{1}{3}(a^2 + ab + b^2).$$

Solution: Consider the function f, defined by $f(x) = x^3$ for all $x \in [a, b]$.

Note that f satisfies the hypothesis of Lagrange's Mean Value Theorem. Hence, there exists $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 3c^2 = \frac{b^3 - a^3}{b - a} \Rightarrow c^2 = \frac{1}{3}(a^2 + ab + b^2)$$

***

Now it is time for you to apply the results learnt to some exercises.

E10) Check whether or not the Lagrange's Mean Value Theorem is applicable for the following functions. If yes, find a suitable point 'c' in the interior of their domain.

i) $f(x) = \cos x$ for all $x \in \left[0, \frac{\pi}{2}\right]$.

ii) $f(x) = x + |x+1|$ for all $x \in [-2, 2]$.

E11) Consider a function f defined as $f(x) = x(x-1)(x-2)(x-3)$ for all $x \in [0, 3]$. How many points are there in $]0, 3[$ at which the slope of f is equal to the slope of the line passing through $(0, f(0))$ and $(3, f(3))$?

E12) If the functions f and g are continuous in $[a,b]$, differentiable in $]a,b[$ and $f'(x) = g'(x)$ for all $x \in]a,b[$, then show that $f - g$ is constant.

E13) Show that on the curve, $y = ax^2 + bx + c, (a,b,c \in \mathbb{R}, a \neq 0)$, the secant line passing through the points whose abscissa are $x = m$ and $x = n$, is parallel to the tangent at the point whose abscissa is given by $x = (m+n)/2$.

E14) Let a function f be defined and continuous on $[a-h, a+h]$, and derivable on $]a-h, a+h[$, where $a \in \mathbb{R}$ and $h > 0$. Prove that there exists a real number $\theta (0 < \theta < 1)$ for which $f(a+h) + f(a-h) - 2f(a) = h[f'(a+\theta h) - f'(a-\theta h)]$.

Now let us discuss Cauchy's Mean Value Theorem which is a generalised form of Lagrange's Mean Value Theorem by using two functions.

Theorem 5 (Cauchy's Mean Value Theorem): Let f and g be two functions defined on $[a,b]$ such that

i) f and g are continuous on $[a,b]$,
ii) f and g are derivable on $]a,b[$, and
iii) $g'(x) \neq 0 \; \forall \; x \in]a,b[$.

Then there exists a number $c \in]a,b[$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

Proof: Let us define a function ϕ by

$$\phi(x) = f(x) + A\, g(x) \text{ for all } x \in [a,b],$$

where A is a constant to be chosen such that $\phi(a) = \phi(b)$. If $\phi(a) = \phi(b)$, then $f(a) + A\, g(a) = f(b) + A\, g(b)$ which gives $A = -\frac{f(b) - f(a)}{g(b) - g(a)}$.

Note that $g(b) - g(a) \neq 0$. (Because, if $g(b) = g(a)$ then g satisfies the conditions of Rolle's Theorem. Consequently, $g'(c) = 0$ for some $c \in]a,b[$. This contradicts the hypothesis (iii).) Observe that

i) ϕ is continuous on $[a,b]$,
ii) ϕ is derivable on $]a,b[$,
iii) $\phi(a) = \phi(b)$.

This means, ϕ satisfies the conditions of Rolle's Theorem, and hence there is a point $c \in]a,b[$ such that $\phi'(c) = 0$. This implies $f'(c) + A\, g'(c) = 0$

i.e. $\frac{f'(c)}{g'(c)} = -A = \frac{f(b) - f(a)}{g(b) - g(a)}$. ■

If in the statement of the theorem above, b is replaced by $a + h$, then the number $c \in]a,b[$ can be written as $a + \theta h$, where $0 < \theta < 1$. So, Cauchy's MVT can be restated as follows:

Alternative statement of Cauchy's Mean Value Theorem

Let f and g be defined and continuous on $[a, a+h]$, derivable on $]a, a+h[$ and $g'(x) \neq 0 \forall x \in]a, a+h[$. Then there exists a real number $\theta \in]0,1[$ such that

$$\frac{f'(a+\theta h)}{g'(a+\theta h)} = \frac{f(a+h)-f(a)}{g(a+h)-g(a)}.$$

Lagrange's Mean Value Theorem can be deduced from Cauchy's Mean Value Theorem by taking the function g as $g(x) = x$.

Now let us look at some applications of Cauchy's Mean Value Theorem.

Example 11: Verify Cauchy's Mean Value Theorem for the functions f and g defined as $f(x) = x^2, g(x) = x^4 \ \forall\ x \in [2,4]$.

Solution: The functions f and g, being polynomial functions, are continuous in $[2,4]$ and derivable in $]2,4[$. Also $g'(x) = 4x^3 \neq 0 \forall x \in]2,4[$. So, all the conditions of Cauchy's Mean Value Theorem are satisfied. Therefore, there exists a point $c \in]2,4[$ such that

$$\frac{f(4)-f(2)}{g(4)-g(2)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{12}{240} = \frac{2c}{4c^3} \Rightarrow c = \pm\sqrt{10}$$

We see that $c = \sqrt{10}$ lies in $]2,4[$. So Cauchy's Mean Value Theorem is verified.

***

Example 12: Let $\alpha, \beta \in \left]0, \frac{\pi}{2}\right[$. Show that $\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$, for some θ such that $\alpha < \theta < \beta$.

Solution: Let $f(x) = \sin x$ and $g(x) = \cos x$, where $x \in [\alpha, \beta] \subset \left]0, \frac{\pi}{2}\right[$.

Now $f'(x) = \cos x$ and $g'(x) = -\sin x$. Functions f and g are both continuous on $[\alpha, \beta]$, derivable on $]\alpha, \beta[$, and $g'(x) \neq 0 \forall x \in]\alpha, \beta[$. Therefore, by Cauchy's Mean Value Theorem, there exists some $\theta \in]\alpha, \beta[$ such that

$$\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta} \Rightarrow \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta.$$

***

The next theorem generalises both Lagrange's and Cauchy's Mean Value Theorems. In this theorem, three functions f, g, h are involved.

Theorem 6 (Generalised Mean Value Theorem): If the functions f, g and h are continuous in $[a,b]$, and derivable in $]a,b[$, then there exists a real number $c \in]a,b[$ such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Proof: Define the function $\phi : [a,b] \to \mathbb{R}$ by

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}.$$

Observe that ϕ is a linear combination of the fuctions f, g and h with constanct coefficients. Since each of the functions f, g and h is continuous on $[a,b]$ and derivable on $]a,b[$, therefore ϕ is also continuous on $[a,b]$ and derivable on $]a,b[$. Now

$$\phi(a) = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0 = \phi(b).$$

Therefore, ϕ satisfies all the conditions of Rolle's Theorem. So there exists $c \in]a,b[$ such that $\phi'(c) = 0,$ i.e.,

$$\phi'(c) = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

■

Now you may try the following exercises.

E15) Verify the Cauchy's Mean Value Theorem for the functions, $f(x) = \sin x, g(x) = \cos x$ in the interval $\left[-\frac{\pi}{2}, 0\right]$.

E16) Check whether or not the functions f and g be defined on $[a,b]$ by $f(x) = e^x$ and $g(x) = e^{-x}$ satisfy the conditions of Cauchy's Mean Value Theorem.

E17) Let $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}} \forall x \in [a,b]$ given that $a > 0.$ Verify]

Cauchy's Mean Value Theorem and show that the point c obtained thus, is the geometric mean of a and b.

E18) Let a function f be continuous on $[a,b]$, and differentiable on $]a,b[$, where $a > 0.$ Show that there exists some $c \in]a,b[$ such that $\frac{b\,f(a) - a\,f(b)}{b-a} = f(c) - c\,f'(c).$

E19) Derive Cauchy's M.V.T. and Lagrange's M.V.T. from the Generalised M.V.T.

We hope by now you must have learnt how to apply mean value theorems in specific problems. Next we shall see some applications of mean value theorems in establishing monotonicily of functions, and of inequalities involving monotone functions.

12.5 INCREASING AND DECREASING FUNCTIONS

In the course Calculus (BMTC-131) you have studied monotone functions, i.e, increasing or decreasing functions, and certain criteria for showing whether a given function is increasing or decreasing. We shall study and discuss these concepts again as they are central to real analysis, and use them to prove certain inequalities.

A function that is either increasing or decreasing on its domain is called a **monotone function**.

A function that is strictly increasing or strictly decreasing on its domain is called a **strictly monotone function.**

Let us recall the following definitions.

Definition: A function $f : \mathbb{R} \to \mathbb{R}$ said to be **increasing** if for all $x, y \in \mathbb{R}, x < y$ implies $f(x) \le f(y)$. In case $x < y$ implies the strict inequality $f(x) < f(y)$ we call f **strictly increasing**.

Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **decreasing** if for all $x, y \in \mathbb{R}, x < y$ implies $f(x) \ge f(y)$. In case, $x < y$ implies the strict inequality $f(x) > f(y)$ we call f **strictly decreasing**.

It is easy to see that if f is an increasing function that is differentiable at every point of an interval, say $]a,b[$, then $f'(x) \ge 0 \ \forall \, x \in \,]a,b[$. And, if f is strictly increasing on $[a,b]$, besides being differentiable on $]a,b[$, then $f'(x) > 0 \ \forall \, x \in \,]a,b[$. Thus the differentiable functions that are increasing (or strictly increasing) have the property that their graphs have always nonnegative (or positive) slopes. Below we have plotted one such function.

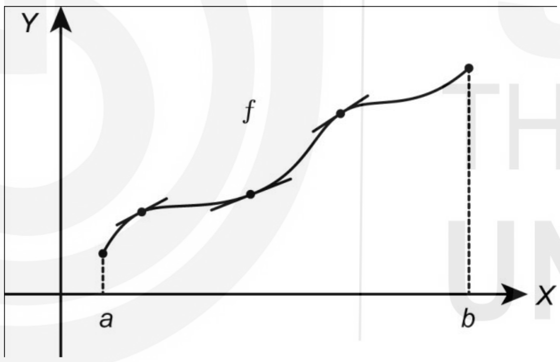

Fig. 9: A graph of an increasing function.

Now consider the following result.

Theorem 7: If a function $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, derivable in $]a,b[$ and $f'(x) > 0$ for all $x \in]a,b[$, then f is strictly increasing on $[a,b]$.

Observe that in Theorem 6 if we replace $f'(x) > 0$ by $f'(x) \ge 0$, then from the conclusion the word 'strictly' can be

Proof: Let x_1 and x_2 be any two points of $[a,b]$ such that $x_1 < x_2$. Then f is continuous in $[x_1, x_2]$ and derivable in $]x_1, x_2[$. So by Lagrange's Mean Value Theorem,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0,$$

for some point c such that $x_1 < c < x_2$.

This implies that $f(x_2) - f(x_1) > 0$, i.e., $f(x_2) > f(x_1)$. Therefore, f is strictly increasing on $[a,b]$. ■

Now consider the following examples.

Example 13: Show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^3-3x^2+3x-5$ is increasing on $\mathbb{R}$.

Solution: We have $f(x)=x^3-3x^2+3x-5$. Note that f is continuous, and differentiable on $\mathbb{R}$. Now,

$$f'(x)=3x^2-6x+3=3(x-1)^2\geq 0 \text{ for all } x\in\mathbb{R}.$$

Therefore, f is increasing on $\mathbb{R}$.

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Example 14: Find the intervals in which the function f defined on $\mathbb{R}$ by $f(x)=2x^3-15x^2+36x+5$ is increasing or decreasing.

Solution: Here $f(x)=2x^3-15x^2+36x+5$. Since f is a polynomial, f is differentiable, and

$$f'(x)=6x^2-30x+36=6(x-2)(x-3).$$

So $f'(x)>0$, whenever $x>3$ or $x<2$. Thus f is increasing in the intervals $]-\infty,2[$ and $]3,\infty[$.

On the other hand, $f'(x)<0$, for $2\leq x\leq 3$. Therefore f is decreasing in $[2,3]$.

***

Now with the help of increasing and decreasing functions we shall prove some inequalities involving real valued functions.

Example 15: Prove that $\sin x<x$ for $0<x\leq\frac{\pi}{2}$.

Solution: Let $f(x)=x-\sin x$, where $0\leq x\leq\frac{\pi}{2}$. Note that f is continuous in $\left[0,\frac{\pi}{2}\right]$ and derivable in $\left]0,\frac{\pi}{2}\right[$. Also $f'(x)=1-\cos x>0$ for $0<x<\frac{\pi}{2}$.

Therefore, by Theorem 6, f is strictly increasing in $\left[0,\frac{\pi}{2}\right]$. So, for all

$0<x\leq\frac{\pi}{2}$, we get

$f(0)<f(x)\Rightarrow 0<x-\sin x \Rightarrow \sin x<x.$

***

Example 16: Prove that $\tan x>x$, whenever $0<x<\frac{\pi}{2}$.

Solution: Let c be any real number such that $0<c<\frac{\pi}{2}$. Consider the function, f defined by $f(x)=\tan x-x\ \forall\, x\in[0,c]$.

You know that f is continuous as well as derivable on $[0,c]$.

Also, $f'(x)=\sec^2 x-1=\tan^2 x>0\ \forall\, x\in\ ]0,c[$. Therefore, f is strictly increasing in $[0,c]$.

Consequently, $f(0) < f(c) \Rightarrow 0 < f(c) \Rightarrow \tan c > c$

Since c arbitrary, the inequality follows.

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Example 17: Show that $x > \ln(1+x) > \dfrac{x}{1+x}$ for all $x > 0$.

Solution: Let us prove the first inequality first. So, let $f(x) = x - \ln(1+x)$, for $x \geq 0$.

Therefore $f'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x}$.

Thus we have $f'(x) > 0$ for all $x > 0$. Therefore f is strictly increasing in $[0, \infty[$.

Now $x > 0 \Rightarrow f(x) > f(0) \Rightarrow x > \ln(1+x)$.

To prove the second inquality, let

$$g(x) = \ln(1+x) - \frac{x}{1+x}, \text{ for } x \geq 0.$$

Then g is differentiabe, and

$$g'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{1}{(1+x)^2}.$$

So, $g'(x) > 0 \,\forall\, x > 0$, and hence g is strictly increasing on $[0, \infty[$.

Now $x > 0 \Rightarrow g(x) > g(0) \Rightarrow \ln(1+x) > \dfrac{x}{1+x}$.

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Now try the following exercises.

E20) Find the intervals in which the function, f, defined on $\mathbb{R}$ by $f(x) = x^3 - 6x^2 + 9x + 4, \forall\, x \in \mathbb{R}$, is increasing or decreasing.

E21) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, derivable on $]a,b[$, and $f'(x) < 0$ for all $x \in]a,b[$. Show that f is strictly decreasing on $[a,b]$.

E22) Show that the function f, defined on $\mathbb{R}$ by $f(x) = 9 - 12x + 6x^2 - x^3, \forall\, x \in \mathbb{R}$, is decreasing in every interval.

E23) Let $f : [a,b] \to \mathbb{R}$ be derivable in the interior of $[a,b]$ such that $f'(x) > 0$ for all interior points x of $[a,b]$. What can you conclude about the monotonicity of f on $[a,b]$?, on $]a,b[$? Justify.

E24) Prove that

i) $x - x^3 < \tan^{-1} x$ if $x > 0$

ii) $e^{-x} > 1 - x$ if $x > 0$.

We end our discussion on increasing and decreasing functions here. You can go through the Block 4 of our course BMTC-131 to learn more about them. Let us now summarise what we have covered in this unit.

12.6 SUMMARY

In this unit, we have covered the following points:

1) If a differentiable function attains an extreme value at an interior point, then its derivative vanishes at the point.
2) Derivatives possess Intermediate Value Property.
3) We have discussed Rolle's Theorem, and its geometrical and algebraic interpretations, and some applications.
4) We have discussed Langrange's M.V.T., Cauchy's M.V.T., and the Generalised M.V.T.; and their applications.
5) We have seen how to use the sign of derivative in deducing the monotonicity of functions.
6) Finally, we have discussed how to prove some inequalities involving real-valued functions.

12.7 SOLUTIONS/ANSWERS

E1) We are given that g is a differentiable function on $[a,b]$. Since $g'(b) > 0$, let us take $\varepsilon = \frac{g'(b)}{2}$. Now, by the definition of derivative, there exists some $0 < \delta < b - a$ such that for all $x \in]b-\delta, b[$

$$\left| \frac{g(x)-g(b)}{x-b} - g'(b) \right| < \varepsilon \Rightarrow \frac{g(x)-g(b)}{x-b} > 0$$
$$\Rightarrow g(x) - g(b) < 0$$
$$\Rightarrow g(x) < g(b)$$

This proves that $g(b) \neq \min\{g(x) | \ a \leq x \leq b\}$.

E2) It follows directly from Darboux's Theorem.

E3) On the contrany assume that $g : [0,2] \to \mathbb{R}$ is a function such that $g'(x) = f(x) \ \forall \, x \in [0,2]$. Now consider the number $\frac{1}{2}$ which lies between $g'(0)$ and $g'(2)$. Since g is differentiable in $[0,2]$, by Darboux's Theorem there exists some $c \in]0,2[$ such that $g'(c) = \frac{1}{2} = f(c)$. But, by the definition of f, there is no such c. Hence there exists no real valued function on $[0,2]$ whose derivative is f.

E4) We note that the function sine is continuous and differentiable on $\mathbb{R}$. Therefore, f is continuous as well as differentiable on $]-2\pi, 2\pi[$. Also $f(-2\pi) = 0 = f(2\pi)$. Thus f satisfies the conditions of Rolle's Theorem. Hence, there exists a point $c \in]-2\pi, 2\pi[$ such that $f'(c) = 0$.

To find $c,$ we solve the equation $f'(c)=0.$ So

$$f'(c)=0 \Rightarrow \cos c = 0$$
$$\Rightarrow c = \left(\frac{2n+1}{2}\right)\pi,\ n\in \mathbb{Z}$$
$$\Rightarrow c = \frac{-3\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \qquad (\because c \in \left]-2\pi, 2\pi\right[)$$

E5) i) We have $f(x)=\cos x, x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. So, f is continuous in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and differentiable in $-\frac{\pi}{2},\frac{\pi}{2}$. Further, $f\left(-\frac{\pi}{2}\right)=0=f\left(\frac{\pi}{2}\right)$. Thus, the conditions of Rolle's Theorem are satisfied. Therefore, these exists a number $c\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ such that $f'(c)=0,$ i.e., $-\sin c=0.$ This implies $c=0.$

ii) Here $f(x)=1+(x+1)^{\frac{2}{3}}, x\in[0,2].$ Clearly f is continuous on $[0,2]$ and differentiable on $\left]0,2\right[$. But we can see that $f(0)=2,$ and $f(2)=1+3^{\frac{2}{3}}\neq 2.$ Hence f does not satisfy the hypotheses of Rolle's Theorem. Now let us examine whether f satisfies the conclusion of Rolle's Theorem or not. We can compute $f'(x)=\frac{2}{3(x+1)^{\frac{1}{3}}}$. Since $x\geq 0, f'(x)>0.$ Thus, there is no point $c\in\left]0,2\right[$ such that $f'(c)=0.$ That is, the conclusion of Rolle's Theorem does not hold.

E6) Let us define $f(x)=x^3-25x+9$ for all $x\in[-2,2].$ If possible, let $\alpha<\beta$ be two roots of the equation $x^3-25x+9=0$ in $[-2,2].$ Then $f(\alpha)=f(\beta)=0.$ Since f is continuous and differentiable on $[\alpha,\beta],$ f satisfies the hypotheses of Rolle's Theorem. Hence there exist some $c\in\left]\alpha,\beta\right[$ such that

$$f'(c)=0\Rightarrow 3c^2-25=0\Rightarrow c=\pm\frac{5}{\sqrt{3}}.$$

But none of these values of c lies in $\left]-2,2\right[$.

Since $\left]\alpha,\beta\right[\subseteq\left]-2,2\right[$, none of the values of c lies in $\left]\alpha,\beta\right[$. This is a contradiction. Hence the given equation does not have two distinict roots in $[-2,2].$

E7) Let a and b be any two roots of the equation $e^x\sin x=1.$ We know that $e^x\sin x=1\Leftrightarrow \sin x=e^{-x}\Leftrightarrow \sin x-e^{-x}=0.$

So, let $f(x)=\sin x-e^{-x}, \forall x\in\mathbb{R}.$Then $f(a)=f(b)=0.$ Since f is differentiable, f is continuous as well. So f satisfies the conditions of Rolle's Theorem. Therefore, there exists some $c\in\left]a,b\right[$ such that

$$f'(c) = 0 \Rightarrow \cos c + e^{-c} = 0$$
$$\Rightarrow e^c \cos c + 1 = 0$$

Thus, c is a root of the equation $e^x \cos x + 1 = 0.$ This is what we wanted to prove.

E8) The idea is to define a function f such that its derivative is $f'(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n.$

So, let us define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \frac{a_0 x^{n+1}}{n+1} + \frac{a_1 x^n}{n} + \cdots + \frac{a_n x}{1}.$$

Then $f(0) = 0,$ and

$$f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a_n = 0.$$

Since f is a polynomial, f is continuous and differentiable on $[0,1].$ Hence f satisfies the hypothetis of Rolle's Theorem. Therefore, there exists some $x \in \left]0,1\right[$ such that

$$f'(x) = 0 \Rightarrow a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0.$$

E9) Let us define a function $\phi : [a,b] \to \mathbb{R}$ by

$$\phi(x) = f(b) - f(x) - (b-x) f'(x) - (b-x)^2 A, \forall x \in [a,b],$$

where A is chosen in such a way that $\phi(a) = \phi(b).$

Since f is differentiable, and f' is continuous on $[a,b],$ the function ϕ is continuous on $[a,b]$ and differentiable on $\left]a,b\right[.$ Also $\phi(a) = \phi(b) = 0$ by the choice of $A.$ Thus ϕ satisfies the hypothesis of Rolle's Theorem.

Hence, there exists a point $c \in \left]a,b\right[$ such that

$$\phi'(c) = 0 \Rightarrow -f'(c) + f'(c) - (b-c) f''(c) + 2(b-c)A = 0$$
$$\Rightarrow A = \frac{1}{2} f''(c).$$

Substituting $x = a$ and $A = \frac{1}{2} f''(c)$ in the definition of $\phi(x),$ we get the desired result.

E10) i) We have $f(x) = \cos x, \forall x \in \left[0, \frac{\pi}{2}\right].$ We can see that f is continuous on $\left[0, \frac{\pi}{2}\right]$ and differentiable on $\left]0, \frac{\pi}{2}\right[.$ Thus f satisfies the hypotheses of Lagrange's M.V.T. Therefore, there exists a point $c \in \left]0, \frac{\pi}{2}\right[$ such that

$$f'(c) = \frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0} \Rightarrow -\sin c = \frac{0-1}{\frac{\pi}{2}}$$

$$\Rightarrow c = \sin^{-1}\left(\frac{2}{\pi}\right)$$

Since we know that $0 < \frac{2}{\pi} < 1$, we get $\sin^{-1}\left(\frac{2}{\pi}\right) \in \left]0, \frac{\pi}{2}\right[$.

ii) Here $f(x) = x + |x+1|, \forall x \in [-2, 2]$.

We know that f is continuous as it is the sum of two continuous functions. However, f is not differentiable at $x = -1$. This is evident from the following arguments

$$\lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \frac{x + |x+1| + 1}{x+1}$$

$$= \lim_{x \to -1} \left(1 + \frac{|x+1|}{x+1}\right)$$

$$= 1 + \lim_{x \to -1} \frac{|x+1|}{x+1}$$

$$= 1 + \lim_{y \to 0} \frac{|y|}{y} \qquad \text{[Taking } x+1 = y \text{]}$$

Since $\lim_{y \to 0} \frac{|y|}{y}$ does not exists, f is not differentiable at $x = -1$.

Thus f does not satisfy hypotheses of Lagrange's M.V.T., and hence Lagrange's M.V.T. is not applicable.

E11) We are given the function

$$f(x) = (x-1)(x-2)(x-3), \forall x \in [0, 3].$$

Since f is a polynomial, f is continuous on $[0, 3]$, and differentiable on $]0, 3[$. Thus, f satisfies the hypothesis of Lagrange's M.V.T. Therefore, there exists some $c \in]0, 3[$ such that

$$f'(c) = \frac{f(3) - f(0)}{3 - 0} \Rightarrow 3c^2 - 12c + 11 = 2$$

$$\Rightarrow (c-1)(c-3) = 0$$

$$\Rightarrow c = 1, 3$$

Since $3 \notin]0, 3[$, there is only one point, namely, $c = 1 \in]0, 3[$ at which the slope of the tangent on the graph of f is equal to the slope of the line passing through $(0, f(0))$ and $(3, f(3))$.

E12) Let $\lambda \in [a, b]$ be arbitrary. Define $h : [a, \lambda] \to \mathbb{R}$ by

$$h(x) = f(x) - g(x), \forall x \in [a, \lambda].$$

Since both f and g are continuous on $[a, \lambda]$, h is also continuous on

$[a,\lambda]$. Since both f and g are differentiable on $]a,\lambda[$, h is also differentiable on $]a,\lambda[$. Thus h satisfies the hypothesis of Lagrange's Mean Value Theorem. Therefore, there exists some $c\in\ ]a,\lambda[$ such that

$$h'(c)=\frac{h(\lambda)-h(a)}{\lambda-a}.$$

But $h'(c)=f'(c)-g'(c)=0$. This implies $h(\lambda)=h(a)$. Since λ is arbitrary, $h(x)=h(a)$ for all $x\in[a,b]$. Consequently h is a constant, i.e. $f-g$ is a constant.

E13) Assume, without loss of generality, that $m<n$. Note that the given curve represents the function $f(x)=ax^2+bx+c$ defined on $[m,n]$. Since f is a polynomial, f is continuous on $[m,n]$, and derivable on $]m,n[$. Thus f satisfies the hypothesis of Lagrange's Mean Value Theorem. Therefore, there exists some $x\in\ ]m,n[$ such that

$$f'(x)=\frac{f(n)-f(m)}{n-m}\Rightarrow 2ax+b=\frac{(an^2+bn+c)-(am^2+bm+c)}{n-m}$$
$$\Rightarrow 2ax+b=a(m+n)+b$$
$$\Rightarrow x=\frac{m+n}{2}$$

Thus $x=\frac{m+n}{2}$ is the abscissa of the point where the tangent to the curve has a slope equal to the slope of the secant line passing through the point with abscissa are $x=m$ and $x=n$.

E14) Define $g:[0,1]\to\mathbb{R}$ by

$$g(x)=f(a+xh)+f(a-xh),\forall\, x\in[0,1].$$

Since $f(a+xh)$ and $f(a-xh)$ are continuous on $[0,1]$, and differentiable on $]0,1[$, the function g also is continuous on $[0,1]$ and differentiable on $]0,1[$. Therefore, there is some $\theta\in\ ]0,1[$ such that

$$g'(\theta)=\frac{g(1)-g(0)}{1-0}.$$

But $g'(\theta)=f'(a+\theta h)h-f'(a-\theta h)h,\ g(0)=2f(a)$, and

$$g(1)=f(a+h)+f(a-h).$$

Thus

$$h[f'(a+\theta h)-f'(a-\theta h)]=f(a+h)+f(a-h)-2f(a).$$

E15) We know that both f and g are continuous in $\left[-\frac{\pi}{2},0\right]$ and differentiable in $\left]-\frac{\pi}{2},0\right[$. Also, $g'(x)-\sin x\neq 0$ for any $x\in\left]-\frac{\pi}{2},0\right[$.

Thus f and g satisfy the condition of Cauchy's Mean Value Theorem.

Therefore, these exists some $c \in \left]-\frac{\pi}{2}, 0\right[$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(0) - f\left(-\frac{\pi}{2}\right)}{g(0) - g\left(-\frac{\pi}{2}\right)},$$

i.e.,

$$\frac{\cos c}{-\sin c} = \frac{0+1}{1-0} \Rightarrow \tan c = -1 \Rightarrow c = -\frac{\pi}{4}.$$

Since $c \in \left]-\frac{\pi}{2}, 0\right[$, Cauchy's Mean Value Theorem is verified.

E16) We know that f and g are continuous on $\mathbb{R}$, and hence on $[a,b]$. Likewise, f and g are differentiable on $\mathbb{R}$, and hence on $]a,b[$. Further, $g'(x) = -e^{-x}$. So, $g'(x) \neq 0$ for any $x \in]a,b[$. Thus f and g satisfiy the conditions of Cauchy's Mean Value Theorem. Therefore, there exists some $c \in]a,b[$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

This implies

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}} \Rightarrow -e^{2c} = \frac{e^b - e^a}{\left(\frac{1}{e^b} - \frac{1}{e^a}\right)}$$

$$\Rightarrow -e^{2c} = \frac{e^a \cdot e^b\,(e^b - e^a)}{e^a - e^b}$$

$$\Rightarrow e^{2c} = e^{a+b} \qquad (\because a \neq b)$$

$$\Rightarrow c = \frac{a+b}{2}$$

Since $c \in]a,b[$, Cauchy's Mean Value Theroem is verified.

E17) We know that f and g are continuous on $\mathbb{R}^+$. Since $a > 0$, $[a,b] \subseteq \mathbb{R}^+$. So, f and g are continuous on $[a,b]$. Also, f and g are differentiable on $\mathbb{R}^+$, and therefore on $]a,b[$. Further, $g'(x) = -\frac{1}{2x^{3/2}}$. Thus $g'(x) \neq 0$ for any $x \in]a,b[$. Thus f and g satisfy the conditions of Cauchy's Mean Value Theorem. Therefore, there exists some $c \in]a,b[$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

This implies

$$\frac{\frac{1}{2}c^{-\frac{1}{2}}}{-\frac{1}{2}c^{-\frac{3}{2}}} = \frac{\sqrt{b}-\sqrt{a}}{\left(\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}\right)}.$$

Simplifying this equation, we get $c = \sqrt{ab}$ as desired.

E18) Observe that

$$\frac{bf(a)-af(b)}{b-a} = \frac{\frac{f(a)}{a}-\frac{f(b)}{b}}{\frac{1}{a}-\frac{1}{b}}.$$

Now, let $F(x) = \frac{f(x)}{x}$, and $G(x) = \frac{1}{x}$, for all $x \in [a,b]$. Since f is continuous on $[a,b]$, and $a > 0$ the function F is also continuous on $[a,b]$. Since f is differentiable on $]a,b[$, F is differentiable on $]a,b[$. Similarly, the function G is also continuous on $[a,b]$ and differentiable on $]a,b[$. Now $G'(x) = -\frac{1}{x^2}$. So, $G'(x) \neq 0$ for any $x \in]a,b[$.

Thus F and G satisfy the conditions of Cauchy's Mean Value Theorem. Therefore, there exists some $c \in]a,b[$ such that

$$\frac{F'(c)}{G'(c)} = \frac{F(b)-F(a)}{G(b)-G(a)} = \frac{bf(a)-af(b)}{b-a}.$$

We know that $F'(c) = \frac{f'(c)}{c} - \frac{f(c)}{c^2}$, and $G'(c) = -\frac{1}{c^2}$. Substituting these valued in the equation above, and simplifying we get

$$f(c) - cf'(c) = \frac{bf(a)-af(b)}{b-a}.$$

E19) Let us take $h(x) = 1$ in the statement of Generalised Mean Value Theorem. Then we have a point $c \in]a,b[$ such that

$$\begin{vmatrix} f'(c) & g'(c) & 0 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix} = 0.$$

This implies,

$f'(c)[g(a)-g(b)] - g'(c)[f(a)-f(b)] = 0$, i.e.,

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)},$$

which is the conclusion of Cuchy's Mean Value Theorem.

Similarly, if you take $h(x) = 1$, and $g(x) = x$ in the Generalised Mean Value Theorem, you will get Lgrange's Mean Value Theorem.

E20) We have $f(x) = x^3 - 6x^2 + 9x + 4,$ for $x \in \mathbb{R}$. Since f is a polynomial function, f is differentiabe. Now $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3).$ For $x < 1$ or $x > 3,$ we find that $f'(x) > 0.$ This means f is increasing in $]-\infty, 1[$ and $]3, \infty[$. On the other hand, for $1 \le x \le 3$ we get $f'(x) \le 0.$ This means f is decreasing on $[1, 3]$.

E21) Let $x_1, x_2 \in [a, b]$ be such that $x_1 < x_2$. Then f is continuous on $[x_1, x_2],$ and differentiable on $]x_1, x_2[$. Hence by Lagrange's Mean Value Theorem, we get some point $c \in \,]x_1, x_2[$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \Rightarrow f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

We are given that $f'(x) < 0$ for all $x \in \,]a, b[$. This implies $f'(c) < 0.$ Now $(x_2 - x_1) > 0$ implies that

$$f(x_2) - f(x_1) < 0, \text{ i.e., } f(x_2) < f(x_1).$$

Since x_1, x_2 are arbitrary, we have proved that f is strictly decreasing.

E22) We are given that $f(x) = 9 - 12x + 6x^2 - x^3,$ for $x \in \mathbb{R}$. Clearly f is differentiable and $f'(x) = -12 + 12x - 3x^2 = -3(x-2)^2$. We can see that $f'(x) \le 0$ for all $x \in \mathbb{R}$. Thus f is decreasing in every interval of $\mathbb{R}$.

E23) The function f need not be monotone on $[a, b]$. For instance, let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } \quad x = 0 \\ x^2, & \text{if } \quad 0 < x < 1 \\ 2, & \text{if } \quad x = 1. \end{cases}$$

Then f is derivable on $]0,1[$, and $f'(x) = 2x.$ So $f'(x) > 0$ for all $x \in \,]0,1[$. However, we can see that f is neither increasing nor decreasing on $[0,1]$. Of course, f is increasing on $]0,1[$.

For the second part, the answer is f is strictly increasing. The proof is same as the proof of Theorem 6.

E24) i) Let us define $f(x) = x - x^3 - \tan^{-1} x,$ for all $x \ge 0.$ We know that f is continuous and differentiable on $[0, \infty[$. Also

$$f'(x) = 1 - 3x^2 - \frac{1}{1+x^2} = \frac{-3x^4 - 2x^2}{1+x^2}.$$

Thus $f'(x) < 0$ for all $x > 0$ and $f'(0) = 0.$ Therefore, f is strictly decreasing on $]0, \infty[$.

So, $x > 0$ implies $f(x) < f(0),$ i.e., $x - x^3 < \tan^{-1} x.$

ii) Let $f(x) = e^{-x} - 1 + x,$ for all $x \geq 0$. We know that f is continuous and differentiable on $[0, \infty[$. Also, $f'(x) = -e^{-x} + 1$. So, for $x > 0$ we have $f'(x) < 0$ and $f'(0) = 0$. Thus f is strictly decreasing on $]0, \infty[$. Therefore,

$$\begin{aligned} x > 0 &\Rightarrow f(x) < f(0) \\ &\Rightarrow e^{-x} - 1 + x < 0 \\ &\Rightarrow e^{-x} < 1 - x. \end{aligned}$$

UNIT 13

HIGHER ORDER DERIVATIVES

Structure **Page No**

13.1 INTRODUCTION

In your Calculus course, you have come across various functions. In this course also you have examined many different types functions for continuity and differentiability such as, polynomial functions, trigonometric functions, step functions, exponential functions, and so on. You may have realised that polynomial functions are the easiest to handle. We can easily find the value of such a function at a point of its domain. Further, these functions are continouous and differentiable at all points of the domain. So, if we can find a polynomial function, which is approximately equal to a given function, f, then we are in a better position to understand this f. Such an approximation by a polynomial function may also help us in finding approximate values of f at different points of its domain.

In this unit we are going to see how a given functions can be approximated by a polynomial function. Taylor's theorem will help us in this quest. But, before that we shall introduce the concept of higher order derivatives in Sec 13.2. We shall then discuss Taylor's theorem and illustrate how we use it to obtain useful approximations and their values.

Afterwards in Sec. 13.3 we state and prove some necessary and sufficient conditions for a function to have a local extremum at a point of its domain.

Objectives

After studying this unit you should be able to:

- find the nth derivative $(n \geq 1)$ of a function, whenever it exists;
- state and prove Taylor's theorem;

- Use Taylor's theorem to find the approximate values of some functions;
- Use Taylor's theorem to find the series expansions of some functions;
- find and classify the extreme points of a function, if they exists.

13.2 DERIVATIVES OF HIGHER ORDERS

In this section we shall discuss Higher order derivatives and Taylor's theorem.

Let us start with an example.

We have introduced "derived function" in Unit 11.

Consider the function f, defined on the interval $[1,8]$ by $f(x) = x^3 + 5x - 3.$ This is a polynomial function, and you will agree that it is differentiable at all points in $[1,8]$. Recall, that at $1 \in [1,8]$, we will consider the right-hand limit of the difference quotient $\frac{f(x) - f(1)}{x-1}$ while finding the derivative. Similarly, at $8 \in [1,8]$, we shall consider the left-hand limit of the difference quotient $\frac{f(x) - f(8)}{x-8}$. At any point $x \in [1,8]$, the derivative of this function is $3x^2 + 5$, we get the derived function of f as $f' : [1,8] \to \mathbb{R}$, $f'(x) = 3x^2 + 5$. Now, this is again a polynomial function, and so it is differentiable. The derivative, f'', of f' at a point $x \in [1,8]$ is given by $6x$ we have a new function, $f'' : [1,8] \to \mathbb{R}$, $f''(x) = 6x.$ This is the derived function of f'. This $f''(x) = 6x$ is called the second order derivative of f at x. Our new function f'' is again differentiable on $[1,8]$ and its derived function, f''' is given by $f''' : [1,8] \to \mathbb{R}, f'''(x) = 6.$ $f'''(x)$ is the third order derivative of f at x. This f''' is a constant function, and its derivative, $f^{(4)}(x) = 0, \forall\, x \in [1,8]$. This is the fourth order derivative of f at x.

Now, if we keep on differentiating further, we will get all successive derivatives, $f^{(5)}(x), f^{(6)}(x), f^{(7)}(x), \ldots$ of f at x equals zero. The derivative f'', f''', f^4 and so on are called **higher order derivatives**.

The formal definition of the higher order derivatives as follows.

Definition 1: If the derivatives f' of a function f exists on an interval I containing a point p, then this derivative of f' is called **second derivative** and is denoted by $f''(p)$ or $f^{(2)}(p)$.

In a similar way we define the third derivative, the fourth derivative and so on. The existence of the $(n-1)^{\text{th}}$ derivative is necessary to define n^{th} derivative. Then we have

$$f^{(n)}(p) = \lim_{x \to p} \frac{f^{(n-1)}(x) - f^{(n-1)}(p)}{x - p} \qquad \ldots (1)$$

Let us find the higher order derivatives of some functions.

Example 1: Find i) fourth order derivative of $f(x) = e^{3x}$ at $x = 2.$
ii) fifth order derivative of $f(x) = \sin(4x+1)$ at $x = 1.$

Solution: i) f is an exponential function and is differentiable on $\mathbb{R}$.

$f'(x) = 3e^{3x}$

$f''(x) = f^{(2)}(x) = 9e^{3x}$

$f'''(x) = f^{(3)}(x) = 27e^{3x}$

$f^{(4)}(x) = 81e^{3x}$, and $f^{(4)}(2) = 81e^{6}$.

ii) The sine function is also differentiable on $\mathbb{R}$.

$f'(x) = 4\cos(4x+1)$

$f^{(2)}(x) = 4^2\cos(4x+1)$

$f^{(3)}(x) = 4^3\cos(4x+1)$

$f^{(4)}(x) = 4^4\cos(4x+1)$

$f^{(5)}(x) = 4^5\cos(4x+1)$,

and $f^{(5)}(1) = 4^5\sin 5$.

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The next example will test whether you have understood differentiability of functions. Study it carefully.

Example 2: Find $f^{(3)}(x)$, if possible, when $f:\mathbb{R}\to\mathbb{R}, f(x) = |x^3|$.

Solution: Now $f(x) = |x^3|$.

So $f(x) = \begin{cases} -(x^3), & x < 0 \\ x^3, & x \ge 0. \end{cases}$

Let us first consider $x \in (0,\infty)$. For all $x \in (0,\infty), f(x) = x^3$. Therefore, $f'(x) = 3x^2, f''(x) = 6x$ and $f^{(3)}(x) = 6$.

Now for all $x \in (-\infty,0), f(x) = -x^3$. Therefore, $f'(x) = -3x^2, f''(x) = -6x$, and $f^{(3)}(x) = -6$.

If $x = 0$, then $f'(0)$ will exist, if $\lim_{h\to 0}\frac{f(h)-f(0)}{h}$ exists.

Now $\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h^3-0}{h}$

$= \lim_{h\to 0^+} h^2$

$= 0.$

And, $\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{-h^3-0}{h}$

$= \lim_{h\to 0^-}(-h^2)$

$= 0.$

This means, $f'(0)$ exists, and is equal to zero.

So, $f'(x) = \begin{cases} -3x^2, & x < 0 \\ 0, & x = 0 \\ 3x^2, & x > 0 \end{cases}$

Let us now find the second order derivative.

Again, for $x<0, f''(x)=-6x.$
for $x>0, f''(x)=6x$

For $x=0,$ we need to find if $\lim\limits_{h\to 0}\dfrac{f'(h)-f'(0)}{h}$ exists.

Proceeding as before, check that $\lim\limits_{h\to 0^+}\dfrac{f'(h)-f'(0)}{h}=\lim\limits_{h\to 0^+}3h=0,$ and

$\lim\limits_{h\to 0^-}\dfrac{f'(h)-f'(0)}{h}=\lim\limits_{h\to 0^-}(-3h)=0.$

Therefore we conclude that $f''(0)=0.$

So $f''(x)=\begin{cases}-6x, & x<0\\ 0, & x=0\\ 6x, & x>0\end{cases}$

Moreover we can write $f''(x)=6|x|, x\in\mathbb{R}.$

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Recall that in Example 1 of Unit 11 we have seen that the absolute value function is differentiable at all $x\in\mathbb{R}, x\neq 0.$ It is not differentiable at $x=0.$

Using that result we can say, that $f^{(3)}=\begin{cases}-6, & x<0\\ 6, & x>0\end{cases},$ and $f^{(3)}(0)$ does not exist.

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In the next example we consider some functions which you will see often in your study of Analysis.

Example 3: Find the following derivatives:

i) $f^{(5)}(x),$ when $f(x)=x^5.$

ii) $g^{(n)}(x),$ when $g(x)=(ax+b)^m,\ m,n\in\mathbb{N}.$

iii) $h^{(n)}(x),$ when $h(x)=\sin(ax+b),\ n\in\mathbb{N}.$

iv) $p^n(x),$ when $p(x)=\sin 3x\cos x.$

Solution: i) $f(x)=x^5.$ Therefore, we get $f'(x)=5x^4, f''(x)=20x^3,$
$f'''(x)=60x^2, f^{(4)}(x)=120x, f^{(5)}(x)=120.$ You can see that
$f^{(n)}(x)=0\ \forall n>5.$

ii) $g(x)=(ax+b)^m$. So,
$g'(x)=ma(ax+b)^{m-1}$
$g''(x)=m(m-1)a^2(ax+b)^{m-2}$
$g'''(x)=m(m-1)(m-2)a^3(ax+b)^{m-3}$

Now, proceeding in this manner, we can guess that

$g^{(n)}(x)=m(m-1)(m-2).....(m-(n-1))a^n(ax+b)^{m-n},$ when $m\geq n.$

In particular, when $m=n, g^{(m)}(x)=m(m-1)(m-2)...1.a^m,$ as constant and then $g^{(n)}(x) = 0$ for $n > m$.

Here, for $m \ge n$, we have guessed the form of $g^{(n)}(x)$, based on our knowledge of the first three derivatives of g. But it has to be proved. This can be easily done with the help of induction.

Suppose $P(n): g^{(n)}(x) = m(m-1)(m-2)...(m-(n-1))a^n (ax+b)^{m-n}$.

Then $P(1) = g'(x) = m\,a(ax+b)^{m-1}$ is true.

Let us assume that $p(k)$ is true, $k < m$. Therefore,

$$g^{(k)}(x) = m(m-1)(m-2)...(m-(k-1))a^k (ax+b)^{m-k} \qquad \text{... (2)}$$

If we differentiate both sides of Eqn. (2), we get

$$g^{(k+1)}(x) = m(m-1)(m-2)...(m-(k-1))a^k (m-k).a(ax+b)^{m-k-1}$$

This tells us that $p(k+1)$ is true.

Hence, by induction, $p(n)$ is true for all n.

iii) $h(x) = \sin(ax+b)$. Therefore,

$$h'(x) = a\cos(ax+b)$$
$$h''(x) = -a^2 \sin(ax+b)$$
$$h^{(3)}(x) = -a^3 \cos(ax+b)$$
$$h^{(4)}(x) = a^4 \sin(ax+b)$$

We observe the pattern of these derivatives, and write,

$$h'(x) = a\sin(ax+b+\pi/2)$$
$$h''(x) = a^2 \sin(ax+b+2.\pi/2)$$
$$h^{(3)}(x) = a^3 \sin(ax+b+3.\pi/2)$$
$h^{(4)}(x) = a^4 \sin(ax+b+4.\pi/2)$, and so on.

Based on this, we can write $h^{(n)}(x) = a^n \sin(ax+b+n\pi/2)$.

This statement needs to be proved by induction. We leave it as an exercise for you to try.

iv) $p(x) = \sin 3x \cos x = \frac{1}{2}(2\sin 3x \cos x)$

$$= \frac{1}{2}[\sin 4x + \sin 2x]$$

$$\therefore p^{(n)}(x) = \frac{1}{2}[4^n \sin(4x+n\pi/2) + 2^n \sin(2x+n\pi/2)]$$

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You should now try to find the higher order derivatives of some functions on your own.

Try these exercises now.

E1) Let $h(x) = \sin(ax+b)$. Use induction to prove $h^{(n)}(x) = a^n \sin(ax+b+n\pi/2)$.

E2) If $f(x) = \cos ax, x \in \mathbb{R}, a \neq 0,$ find $f^{(3)}(x),$ and $f^{(n)}(x),\ n \in \mathbb{N}.$

E3) If $f(x) = \ln(ax+b), a \neq 0, x \in \mathbb{R},$ find $f'(x), f''(x), f^{(3)}(x),$ and $f^{(n)}(x), n \in \mathbb{N}.$

R.H.S is the short form for Right Hand side of an equality sign in an equation and L.H.S for the Left Hand Side.

You recall that in the introduction to this unit we talked about finding a polynomial which is approximately equal to a given function in a neighbourhood of a given point. We now come to this problem.

Suppose a function f is defined on an interval I containing a point $a.$ If we take the function $P_0(x) = f(a) \forall a \in I,$ then P_0 is a constant function, and agrees with the function f at $a.$ Now we suppose that f is differentiable at $'a'$ and consider the function P_1 on I defined by $P_1(x) = f(a) + (x-a) f'(a), x \in I.$

The expression on the R.H.S of equation above is a polynomial function (of degree 1, and we have $P_1(a) = f(a),$ and $P_1'(a) = f'(a).$

If the second order derivative of f also exists at $a,$ then we define a function P_2 on I given by $P_2(x) = f(a) + (x-a) f'(a) + \dfrac{(x-a)^2}{2!} f''(a).$

This is also a polynomial function, and we have

$$P_2(a) = f(a), P_2'(a) = f'(a) \text{ and } P_2''(a) = f''(a).$$

Do you observe a pattern in the process of defining these polynomial functions P_0, P_1 and P_2? Infact by this process we are finding polynomial functions defined on $I,$ which seem to agree more and more with the function f at the point $'a'.$ This leads us to believe that we are getting better and better polynomials, as an approximate of f at the point $'a'$.

You have studied curve tracing in your Calculus course. In the light of that, let us see the current discussion.

We have $f(a) = P_0(a).$ So this means that the curves for f and P_0 cross at the point $(a, f(a)).$ For $P_1,$ in addition we have $f'(a) = P_1'(a).$ So we know that the curves for f and P_1 have the same slope at $'a'.$
Extending this process further, if the given function f has derivatives of order n at the point $a,$ then we have

$$P_n(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f''(a) \qquad \ldots (3)$$

Thus, $P_n(a) = f(a), P_n'(a) = f'(a), P_n''(a) = f''(a),$ and, in general, $P_n^{(m)}(a) = f^m(a) \forall\, m \leq n.$

We hope that the polynomial function on the R.H.S of Eqn (3) gives us a "good" approximation of f at $a,$ and that the "goodness" will get better as we increase $n.$

A word of caution though!

In the previous two paragraphs we have been talking about our "beliefs" and "hopes". Are we justified in believing and hoping? The answer to this question is given to us by Taylor's Theorem, which we state now. In fact, the polynomials $P_0, P_1, ..., P_n, n \in \mathbb{N}$ are called **Taylor Polynomials** of f at a.

Theorem 1(Taylor's Theorem): Let $f : I \to \mathbb{R}$, where $I = [p, q]$ is closed a interval in $\mathbb{R}$. Let $n \in \mathbb{N}$. Suppose that $f', f'', ..., f^{(n)}$ exist and continuous on I, and $f^{(n+1)}$ exists on the open interval (p, q). If $a \in I$, then for any $x > a$, $x \in I$, there exists a point c between a and x, such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad ... (4)$$

Proof: Let J be the interval with end points x and a. If $x = a$, then Eqn. (4) is true. Suppose $x \neq a$, and $x > a$. Let $J = [a, x]$.

We define a new function $F : J \to \mathbb{R}$, as

$$F(t) = f(x) - f(t) - (x-t)f'(t) - \frac{(x-t)^2}{2!}f''(t) - ... - \frac{(x-t)^n}{n!}f^{(n)}(t) - \frac{A(x-t)^{n+1}}{(x-a)^{n+1}}, ... \quad ... (5)$$

where A is a constant, such that $F(x) = 0$.

If $F(a) = 0$, then from (5) we get

$$F(a) = f(x) - f(a) - (x-a)f'(a) - \frac{(x-a)^2}{2!}f''(a) - .... - \frac{(x-a)^n}{n!}f^{(n)}(a) - A = 0.$$

Therefore, we get

$$A = f(x) - f(a) - (x-a)f'(a) - \frac{(x-a)^2}{2!}f''(a) - ... - \frac{(x-a)^n}{n!}f^{(n)}(a), \quad ... (6)$$

a constant.

(Recall that a and x are two fixed points in $[p, q]$).

For that, we proceed as follows: we know that $f, f', f'', ..., f^{(n)}$ and $(a-x)^m, m \in \mathbb{N}$ are all continuous on $[a, x]$ and differentiable on (a, x). So F is continuous on $[a, x]$, and differentiable on (a, x). Therefore, F satisfies all the conditions of Rolle's Theorem. Therefore by Rolle's theorem there exists $c \in]a, x[$, such that $F'(c) = 0$.

Now we find $F'(t)$. For that we differentiate both sides of Eqn. (5) with respect to t we get

$$F'(t) = -f'(t) + f'(t) - (x-t)f''(t) + (x-t)f''(t) - \frac{(x-t)^2}{2!}f'''(t) - \cdots - \frac{(x-t)^n}{n!} - f^{(n+1)}(t) + \frac{A(n+1)(x-t)^n}{(x-a)^{n+1}}$$

$$= \frac{-(x-t)^n}{n!} f^{(n+1)}(t) + \frac{A(n+1)(x-t)^n}{(x-a)^{n+1}}$$ [All other terms get cancelled except these two terms]. ... (7)

Now we put c on both sides of Eqn. (7) and get

$$F'(c) = \frac{-(x-c)^n}{n!} f^{(n+1)}(c) + \frac{A(n+1)(x-c)^n}{(x-a)^{n+1}} = 0$$ (by Rolle's theorem ... (8)

From Eqn. (8), we get

$$A = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c) \qquad \text{... (9)}$$

Substituting this value of A in Eqn. (6) we get

$$f(x) - f(a) - (x-a)f'(a) - \frac{(x-a)^2}{2!} f''(a) - \ldots - \frac{(x-a)^n}{n!} f^{(n)}(a)$$

$$= \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

And hence,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a)$$

$$+ \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$ ■

Alternate Proof: Let $J = \{t : a \le t \le x\}$. For $t \in J$, put

$$F(t) = f(x) - f(t) - (x-t)f'(t) - \frac{(x-t)^2}{2!} f''(t) - \ldots - \frac{(x-t)^n}{n!} f^n(t) - A\left(\frac{(x-t)}{(x-a)}\right)^{n+1}$$

where A is a constant to be so chosen such that $F(x) = F(a)$. If $F(a) = 0$ then

$$f(x) = f(a) + (x-a)f'(a) + \ldots + \frac{(x-a)^n}{n!} f^n(a) + A \qquad \text{... (10)}$$

The function F satisfies the conditions of Rolle's Theorem.

i) $f', f'', \ldots, f^{(n)}$ are continuous on $[a, x]$

ii) Are differentiable on $]a, x[$

iii) $F(a) = F(x)$.

So $\exists\ c \in\ ]a, x[$ such that $F'(c) = 0$. Observe that

$$0 = F'(c) = \frac{-(x-c)^n}{n!} \overset{(n+1)}{f}(c) + \frac{A(n+1)(x-c)^n}{(x-a)^{n+1}}$$

Which implies

$$A = \frac{\overset{(n+1)}{f}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{... (11)}$$

Substituing for (11), we get in (10), we get

$$f(x) = f(a) + (x-a) f'(a) + \ldots + \frac{(x-a)^n}{n!} f^n(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(c) \qquad \blacksquare$$

Remark 1: The Theorem 1 also holds for $x < a$, in that case the interval will be $[x, a]$.

Remark 2: Using the form of expression for P_n given in Eqn (3), we can write

$$f(x) = P_n(x) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c),$$

where $P_n(x)$ is the **nth Taylor polynomial of** f. We also write

$f(x) = P_n(x) + R_n(x)$, where $R_n(x) = \dfrac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$, where c is some point between a and x (obtained from Rolle's Theorem). This formula of R_n is called the **Lagrange Form of Remainder**.

So, when we try to approximate a given function by a Taylor polynomial, we know that there is an "error" of $R_n(x) = f(x) - P_n(x)$.

Here are some examples to clarify these concepts.

Example 4: If $f(x) = \sqrt{1+x}, x \geq 0$, find $P_1(x), R_1(x), P_2(x)$ and $R_2(x)$, for $x > 0$. Show that, for $x > 0$, the inequality $1 + \dfrac{x}{2} - \dfrac{x^2}{8} < \sqrt{1+x} < 1 + \dfrac{x}{2}$ holds.

Solution: $f(x) = \sqrt{1+x} = (1+x)^{1/2}$. Therefore,

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$$

$$f'''(x) = \frac{3}{8}(1+x)^{-5/2}.$$

So, $f(0) = 1, f'(0) = \dfrac{1}{2}, f''(0) = -\dfrac{1}{4}$ and $f'''(0) = \dfrac{3}{8}$.

$$\boldsymbol{P_1(x)} = f(0) + x . f'(0) = 1 + \frac{x}{2}$$

$$\boldsymbol{R_1(x)} = \frac{x^2}{2!} f''(c) = \frac{x^2}{2}\left(-\frac{1}{4}\right)(1+c)^{-3/2}, 0 < c < x$$

$$\boldsymbol{P_2(x)} = f(0) + xf'(0) + \frac{x^2}{2!} f''(0)$$

$$= 1 + \frac{x}{2} - \frac{1}{8}x^2$$

$$\boldsymbol{R_2(x)} = \frac{x^3}{3!} f'''(c) = \frac{x^3}{6}\left(\frac{3}{8}\right)(1+c)^{-5/2}, \ 0 < c < x.$$

Now, $\sqrt{1+x} = P_1(x) + R_1(x)$

$$= 1 + \frac{x}{2} - \frac{x^2}{8}(1+c)^{-3/2}, \ 0 < c < x.$$

Since $\frac{x^2}{8}(1+c)^{-3/2} > 0,$

$$\sqrt{1+x} < 1+\frac{x}{2} \qquad \text{... (12)}$$

Again, $\sqrt{1+x} = P_2(x) + R_2(x)$

$$= 1+\frac{x}{2}-\frac{1}{8}x^2+\frac{x^3}{16}(1+c)^{-5/2},\ 0<c<x$$

Here, $\frac{x^3}{16}(1+c)^{-5/2} > 0.$ Therefore,

$1+\frac{x}{2}-\frac{1}{8}x^2 < \sqrt{1+x}.$ This together with Eqn. (12) gives

$1+\frac{x}{2}-\frac{1}{8}x^2 < \sqrt{1+x} < 1+\frac{x}{2},\ \forall\, x>0.$

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Note that we have used Taylor's Theorem to establish the inequality.

In the next example you will see how to find an approximate value and how to estimate the error in approximation using Taylor's theorem.

Example 5: Find the approximate value of $\sqrt[3]{1.2}$ using Taylor's Theorem with $n = 2.$ Also estimate the error.

Solution: We first note that $1.2 = 1+0.2$ i.e. $(1.2)^{1/3} = (1+0.2)^{1/3}.$ This is of the form $(1+x)^{1/3}$ where $x = 0.2.$ Therefore we consider the function $f(x) = (1+x)^{1/3},\ x \geq 0$ and apply Taylor's theorem for $n = 2$ at $a = 0.$ Then we have for $x > 0,$ Then, if $f(x) = P_2 + R_2,$ where

$P_2(x) = f(0) + x\ f'(0) + \frac{x^2}{2!}f''(0),$ and $R_2(x) = \frac{x^3}{3!}f^{(3)}(c),\ 0<c<x.$

Here $f'(x) = \frac{1}{3}(1+x)^{-2/3}, f''(x) = \frac{-2}{9}(1+x)^{-5/3}, f'''(x) = \frac{10}{27}(1+x)^{-8/3}.$

Therefore, $f(0) = 1, f'(0) = \frac{1}{3}, f''(0) = \frac{-2}{9}.$

$\therefore P_2(x) = 1 + x \times \frac{1}{3} + \frac{x^2}{2!}\left(\frac{-2}{9}\right)$

$1.2 = 1 + 0.2.$

Hence, $P_2(0.2) = 1 + 0.2\left(\frac{1}{3}\right) - \frac{(0.2)^2}{2}(+2/9)$

$$= 1 + \frac{0.2}{3} - \frac{0.04}{9} = \frac{9.56}{9} = 1.062$$

So, the approximate value of $\sqrt[3]{1.2}$ is 1.062.

The error, given by $R_2(x) = \frac{(0.2)^3}{6}\left(\frac{10}{27}\right)(1+c)^{-8/3} < \frac{0.08}{162} = 0.00049.$

Here, since $c > 0, 1 + c > 1,$ hence $(1+c)^{-8/.3} < 1.$

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Example 6: Find the approximate value of $e,$ with error less than $10^{-3}.$

Solution: We consider $f(x) = e^x,$ defined on $\mathbb{R},$ and take $a = 0, x = 1.$

Now $f'(x) = f''(x) = \ldots = f^{(x)}(x) = e^x, n \in \mathbb{N}.$

Therefore, $f(0) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = 1.$

Hence $P_n(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!},$ and $R_n(x) = \frac{e^c}{(n+1)!}, 0 < c < 1.$

We want $|R_n(x)| = \left|\frac{e^c}{(n+1)!}\right| < 10^{-3}.$

For $0 < c < 1, |e^c| < 3$ happens if we choose $n,$ so that

$$|R_n(x)| = \left|\frac{e^c}{(n+1)!}\right| < \frac{3}{(n+1)!} < 10^{-3},$$ or, whenever $(n+1)! > 3 \times 10^3 = 3000.$

Now, $7! = 5040 > 3000.$ So, it is enough to take $n = 6.$

$\therefore$ The approximate value of e with $n = 6,$ is

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2.718055\ldots,$$ and the error is less than $10^{-3}.$

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A special case of Taylor's Theorem, with $a = 0,$ is known as **Maclaurin's Theorem**.

Theorem 2 (Maclaurin's Theorem): Let $f : I \to \mathbb{R},$ where $I = [p, q].$ Let $n \in \mathbb{N}.$ Suppose $f', f'', \ldots, f^{(n)}$ are continuous on $I,$ and $f^{(n+1)}$ exists on (p, q).
If $0 \in I,$ then for any $x \in I,$ there exists a point $c, 0 < c < x,$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

Remark: If $x \in [p, q], x = a + h,$ then Taylor's theorem can be written as

Note that h could be negative also.

$$f(x) = f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a)$$

$$+ \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(c),\ a < c < a + h,$$ or

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(a + \theta h), 0 < \theta < 1.$$

The nth Taylor Polynomial of a function which is n times differentiable, is

$$P_n(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a)$$

The coefficients, $f'(a), \frac{f''(a)}{2!}, \ldots, \frac{f^{(n)}(a)}{n!}$ are called **Taylor coefficients of f**.

Now, if a function f is infinitely differentiable, that is, if the derivatives of all orders of f exist, then we can write it as a series,

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots \qquad \ldots (13)$$

This series is generally called a **power series**.

We do not know whether this power series converges or not. It has been proved that the series (13) will converge if and only if the sequence $\{R_n(x)\}$ of remainder converges to zero in a neighbourhood of $'a'$. In that case, the series (13) is called the **Taylor Series** or **Taylor Expansion** of f. Infact you might have observed that the Taylor polynomials are nothing but the partial sums of this Taylor series. Again, if $a = 0$, the series (8) becomes

$$f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots. \qquad \ldots (14)$$

This is called the **Maclaurin's Series or Maclaurin's Expansion** of f.

Let us see some examples.

Example 7: Find the Maclaurin's series for

i) $e^x, x \in \mathbb{R}$, ii) $\cos x \ \forall x \in \mathbb{R}$, iii) $\log(1+x), x \in [0,1]$

Solution: i) Let $f(x) = e^x \ \forall \ x \in \mathbb{R}$. Then $f^{(n)}(x) = e^x, \ \forall n \in \mathbb{N}, x \in \mathbb{R}$.

So, $f(0) = e^0 = 1$, and $f^{(n)}(0) = 1 \ \forall \ n \in \mathbb{N}$.

Therefore, the Macluarin's series of f is

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

Using Ratio test, you can easily show that the series

$$1 + |x| + \frac{|x|}{2!} + \ldots + \frac{|x|^4}{n!} + \ldots$$

is convergent, and hence the series

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

is also convergent. Therefore,

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \forall x \in \mathbb{R}.$$

ii) Let $f(x) = \cos x,\ x \in \mathbb{R}$.

Then $f(0) = 1,\ f'(0) = -\sin 0 = 0, f''(0) = -\cos 0 = -1,$ and so on. In general, $f^{(n)}(0) = \cos\left(\frac{n\pi}{2}\right), n \in \mathbb{N}.$ So we write the Maclaurin's series,

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \qquad \ldots (15)$$

We know that $\cos x = P_n(x) + R_n(x),$ where

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x),\ 0 < \theta < 1$$

$$= \frac{x^{n+1}}{(n+1)!} \cos\left(\theta x + \frac{(n+1)\pi}{2}\right)$$

Therefore, $|R_n(x)| \le \frac{|x^{n+1}|}{(n+1)!},\quad (\because \cos t \le 1\ \forall\ t \in \mathbb{R})$ and hence $\lim_{n\to\infty} R_n(x) \to 0$ as $n \to \infty\ \forall\ x \in \mathbb{R}.$

This means that (15) is a convergent series, and we can write

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \qquad \forall x \in \mathbb{R}.$$

iii) Let $f(x) = \log(1+x), x \in [0,1].$

Then $f'(x) = \frac{1}{1+x}, f''(x) = \frac{-1}{(1+x)^2}, f^{(3)}(x) = \frac{2}{(1+x)^3}$

$$f^{(4)}(x) = \frac{-2.3}{(1+x)^4}, \ldots, f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, n \in \mathbb{N}.$$

So, $f(0) = \log 1 = 0, f'(0) = 1, f''(0) = -1, f^{(3)}(0) = 2, f^{(4)}(0) = -3!, \ldots,$ $f^{(n)}(x) = (-1)^{n-1}(n-1)!$

Now we can write the Maclaurin's series,

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1}x^n}{n} + \ldots$$

In this case,

$$R_n(x) = \log(1+x) - \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^n x^n}{n}\right\}$$

$$= \frac{x^{n+1}}{(n+1)!} f^{(n)}(\theta x)$$

$$= \frac{x^{n+1}}{(n+1)!}\left[\frac{(-1)^n n!}{(1+\theta x)^{n+1}}\right]$$

$$= \frac{(-1)^n x^{n+1}}{(n+1)(1+\theta x)^{n+1}}$$

Therefore, $|R_n(x)| \le \frac{x^{n+1}}{n+1}$. Since $0 \le x \le 1, (1+\theta x) \ge 1,$ and $\frac{1}{1+\theta x} \le 1.$

So, we conclude that $R_n(x) \to 0$ as $n \to \infty$, and write

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, 0 \le x \le 1.$$

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Note: From these examples you must have observed that it is easy to find the Taylor polynomials which gives an approximation to the function. But the approximation includes some measurement of the error. Taylor's theorem can be used to estimate the error in approximation. If the remainder tends to $0,$ then we get the Taylor series expansion. On the other hand, if certain accuracy is specified like in Example 6, then the question involves finding a suitable $n.$

It is time to try solve some exercises now.

E4) Find the fifth Taylor polynomial, $P_3(x)$ of $\sin x$ on $[-1,1], a = 0.$ Show that $|\sin x - P_5(x)| < \frac{1}{5040}$ for $|x| < 1.$

E5) Find the approximate value of $\sqrt{1.2}$ and $\sqrt{2},$ using Taylor's theorem with $n = 2.$

E6) Using Taylor's theorem, show that $x - \frac{x^3}{3!} \le \sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!}, x \ge 0,$ and $x - \frac{x^3}{3!} \ge \sin x \ge x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $x < 0.$

E7) Assuming the convergence of the power series,

i) show that $\tan^{-1} x = \tan^{-1} \pi/4 + \frac{(x - \pi/4)}{1 + \pi^2/16} - \frac{\pi(x - \pi/4)^2}{4(1+\pi^2/16)^2} + \ldots \forall\, x \in \mathbb{R}$

Hint: Apply Taylor's theorem to $f(x) = \tan^{-1} x$ with $a = \pi/4.$

ii) Expand $\cos x$ in powers of $(x - \pi/4).$

So far you have learnt a fundamental theorem known as Taylor's Theorem which gives a useful technique for approximating a function via Taylor polynomial. This was done by finding the higher order derivatives of the function. In view of this sometimes Taylor's Theorem is considered as an extension of mean value theorem which relates a function to its higher order derivatives.

In the next section we shall study another important concept called 'extrema', which also involves higher order derivatives.

13.3 EXTREMA OF A FUNCTION

In this section we deal with the concept of extrema which is a common term for maxima and minima of a function.

In Calculus you must have learnt that the study of extrema are useful in drawing the graph of a function. There you must have learnt that it is useful to study local extrema (local maxima and local minima) rather than absolute extrema (a term used to differentiate an extrema from local extrema).

We begin the study of local extrema by considering an example of a function.

The term 'relative' and 'local' are used commonly in the text book.

We mainly use the term local.

Take the function $f:[0,4\pi], f(x)=\sin x.$ We know that $-1 \le \sin x \le 1$ for all $x.$ The highest value of $\sin x,$ which is is attained at two points, $\pi/2$ and $5\pi/2$ in $[0,4\pi].$ Its lowest value, -1 is attained at $3\pi/2$ and $7\pi/2$ in $[0,4\pi].$ We say that $\sin x$ has a relative or local maximum at $x=\pi/2$ and at $x=5\pi/2.$ Likewise, we say that $\sin x$ has a relative or **local minimum** at $3\pi/2$ and $7\pi/2.$ We also say that $\sin x$ has local extrema at $\pi/2, 3\pi/2, 5\pi/2$ and $7\pi/2.$

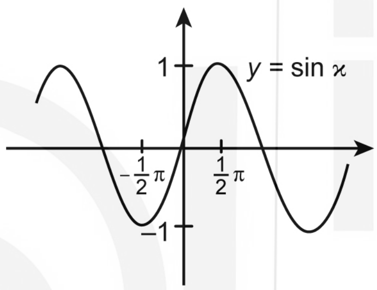

Fig. 1: Graph of $f(x)=\sin x.$

We formally give the definition now.

Definition 1: Let f be a function defined on an interval $I,$ and let c be an interior point of $I.$

i) f is said to have a local or relative maximum at $x=c,$ if $\exists a \ \delta>0,$ such that $f(x)<f(c) \ \forall \ x\in]c-\delta, c+\delta[, x\ne c.$

ii) f is said to have a local or relative minimum at $x=c,$ if $\exists a\, \delta>0,$ such that $f(x)>f(c) \forall x\in]c-\delta, c+\delta[, x\ne c.$

iii) f is said to have a local extremum, if it has a local maximum or a local minimum at $x=c.$

If you look at Fig. 1, you see that, the tangents at the points P, Q, R and S are all parallel to x-axis. That means the derivative $f'(x)$ of the function at each of these points is $0.$ We shall now prove this fact in the following theorem.

Theorem 3: Let f be a function defined on an interval $I.$ Suppose f has a local extremum at an interior point $c\in I.$ If the derivative of f at c exits, then $f'(c)=0.$

Proof: Suppose f has a local maximum at $c.$ Therefore, $\exists a \ \delta>0,$ such that $f(x)<f(c) \ \forall x\in]c-\delta, c+\delta[, x\ne c.$

That is, $f(x)-f(c)<0\ \forall\ x\in]c-\delta, c+\delta[, x\neq c.$ Now, if $x\in]c, c+\delta[,$ then $\frac{f(x)-f(c)}{x-c}<0.$ (Note that $f(x)-f(c)<0$ and $x-c>0$). Therefore,

$$\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}\leq 0 \qquad \text{... (16)}$$

On the other hand, if $x\in]c-\delta, c[,$ then $x-c<0$ and $f(x)-f(c)<0.$ Thus, $\frac{f(x)-f(c)}{x-c}>0.$

Therefore $\lim\limits_{x\to c^-}\frac{f(x)-f(c)}{x-c}\geq 0$ ... (17)

We know that $f'(c)=\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}$, exists. This will happen only if the left hand and right hand limits of $\frac{f(x)-f(c)}{x-c}$ as $x\to c$ given in (16) and (17) are equal. And these limits will be equal only if both are zero.

Therefore, $f'(c)=0.$

Similar argument shows that if f has a local minimum at c then $f'(c)=0.$ We leave this to you as an exercise. (See E8). ■

Theorem 3 gives us a **necessary condition** for f to have a local extremum at $c.$ Is this sufficient? That is, if $f'(c)=0,$ can we say that c is a local extremum? Let us see.

Here is a simple example to illustrate that the answer is no. Consdier the function $f:[-1,1]\to\mathbb{R}, f(x)=x^3.$ Then 0 is an internal point of $[-1,1].$ Further, $f'(x)=3x^2$, and $f'(0)=0.$ We shall check if f has a local minimum or maximum at 0? Note that $f(0)=0.$ Any neighbourhood of 0 will contain some positive and some negative numbers. The value of f at a positive number is positive and that at a negative number is negative. So every neighbourhood has function values which are less than $f(0)$ and also which are more than $f(0).$ Therefore by Definition 1, 0 is not a local extrema. This is also clear in the graph of this function. See Fig. 2.

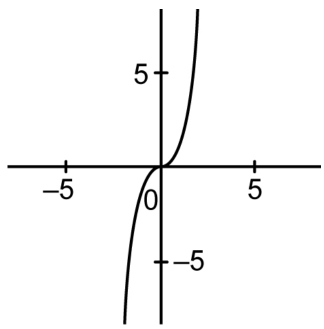

Fig. 2: Graph of $f(x)=x^3$ in $]-1,1[.$

This shows that the condition $f'(c)=0$ is not a sufficient condition for local extrema. You should also note that the condition $f'(c)=0$ can be checked only when f is differentiable at c. What about the points in the domain of f where f is not differentiable? Can a local extremum of f occur at such a point? The answer is yes. Take the example of the absolute value function $f : \mathbb{R} \to \mathbb{R}, f(x) = |x|$. See its graph in Fig. 3.

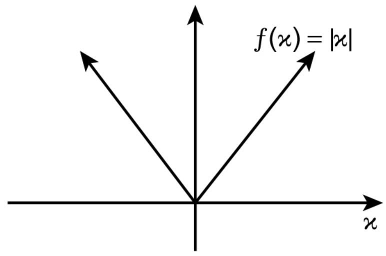

Fig. 3: Graph of $|x|$.

You know that this function is continuous on $\mathbb{R}$, and differentiable on $\mathbb{R} - \{0\}$. This function is not differentiable at $x = 0$. But isn't it clear from Fig. 3, that f has a minimum at $x = 0$?

Recall the following definition from the calculus course.

Definition 2: A point $x = c$ is called a **critical point** for the function f, if either f does not have a derivative at c or if the derivative of f exists, then $f'(c) = 0$. If $f'(c) = 0$, then, $f(c)$ is called a **stationary value**.

So what Theorem 3 does is this: From amongst the points at which the given function is differentiable, it helps us pick up some possible points at which a local extremum can occur.

Among the possible points indicated by this theorem, how do we find the actual local extrema? The theorems which follow will help us in this. Both these theorems (Theorem 4 and 5) give us sufficient conditions for the existence of local extreme points.

Theorem 4 (First Derivative Test): Let f be continuous on the interval $I = [a, b]$, and let c be an interior point of I. Assume that f is differentiable on $]a, c[$ and $]c, b[$. Then,

Then

i) if $\exists\ a\ \delta > 0$, such that $]c - \delta, c + \delta[\subseteq I$, and $f'(x) \geq 0$ for $c - \delta < x < c$, and $f'(x) \leq 0$ for $c < x < c + \delta$ then f has a **local maximum** at c.

ii) if $\exists\ a\ \delta > 0$, such that $]c - \delta, c + \delta[\subseteq I$, and $f'(x) \leq 0$ for $c - \delta < x < c$ and $f'(x) \geq 0$ for $c < x < c + \delta$, then f has a **local minimum** at c.

Proof: i) If $x \in]c - \delta, c[$, then we apply the Mean Value Theorem (MVT, Theorem) to the function f on $[x, c]$. We know that f is continuous on $[x, c]$, and differentiable on $]x, c[$ since $]x, c[\subseteq]a, c[$.

Therefore, by MVT, $\exists\ x_0 \in]x, c[$, such that $\dfrac{f(c) - f(x)}{c - x} = f'(x_0)$.

Since $x_0 \in]x, c[\subseteq]c - \delta, c[, f'(x_0) \geq 0$.

Hence, $\frac{f(c)-f(x)}{c-x} \geq 0,$

And $f(c)-f(x)=(c-x)f'(x_0) \geq 0,$ since $c-x \geq 0$ and $f'(x_0) \geq 0.$

Therefore, $f(x) \leq f(c)$ for $x \in]c-\delta, c[$ ... (18)

Now suppose $x \in]c, c+\delta[$. Then f is continuous on $[c, x]$ and differentiable on $]c, x[$. So, again, by MVT, $\exists\, x_1 \in]c, x[$, such that

$$\frac{f(x)-f(c)}{x-c} = f'(x_1)$$

Therefore, $f(x)-f(c) = f'(x_1)(x-c).$
Now $x-c>0$ and $f'(x_1) \leq 0.$ Thus, $f(x)-f(c) \leq 0,$ and we conclude that $f(x) \leq f(c),$ for
$x \in]c, c+\delta[$ ... (19)

Combining (18) and (19), we get that $f(x) \leq f(c)\ \forall\ x \in]c-\delta, c+\delta[$. This means that f has a local maximum at c.

ii) The proof of this part is similar to that of i). We leave it to you as an exercise. (See E9). ■

You may note that Theorem 4 gives us a **sufficient condition** for the existence of a local extremum.

Now, go through the following examples.

Example 8: Let us consider the following function

$$f(x) = \begin{cases} x^4\left(2+\sin\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Check whether f has a local minimum at 0. Also check if f satisfies the condition (ii) in Theorem 4.

Solution: We shall first check whether f has a local extrema. We first note that this function is differentiable on $\mathbb{R}$. Since $|\sin\theta| \leq 1$ for all θ we get that $\left|2+\sin\frac{1}{x}\right| \geq 1 \geq 0$ for all $x \neq 0$. Also, $x^4 \geq 0\ \forall\, x.$

Hence $f(x) \geq 0\ \forall\ x \in \mathbb{R}.$

So we can say that $x=0$ is a local (and also global) minimum of f. Infact it is a minimum point. Now we shall check whether the condition (ii) in Theorem 4 is satisfied.

$$f'(x) = 4x^3\left(2+\sin\frac{1}{x}\right) - x^2\cos\frac{1}{x}$$
$$= x^2\left[8x+4x\sin\frac{1}{x} - \cos\frac{1}{x}\right].$$

Consider $8x+4x\sin\frac{1}{x}-\cos\frac{1}{x}$

If $x=\frac{2}{(4n+1)\pi}, \sin\left(\frac{1}{x}\right)=\sin(4n+1)\frac{\pi}{2}=1$ and $\cos\left(\frac{1}{x}\right)=0.$

Therefore, $8x+4x\sin\frac{1}{x}-\cos\frac{1}{x}=\frac{16}{(4n+1)\pi}+\frac{8}{(4n+1)\pi}>0$ ... (20)

If $x=\frac{1}{2n\pi}, \sin\frac{1}{x}=\sin 2n\pi=0,$ and $\cos\frac{1}{x}=1.$

Thus, $8x+4x\sin\frac{1}{x}-\cos\frac{1}{x}=\frac{8}{2n\pi}-1<0 \;\forall\; n\geq 2$ ... (21)

From Eqn. (20) and Eqn. (21), we observe that $f'\left(\frac{2}{(4n+1)\pi}\right)>0$ and $f'\left(\frac{1}{2n\pi}\right)<0$ for all $n\geq 2.$

Now, given any δ-neighbourhood, $]-\delta,\delta[$ of $0,$ we can choose n large enough, such that $\frac{2}{(4n+1)\pi}<\delta$ and $\frac{1}{2n\pi}<\delta.$ So f' takes both positive and negative values on the right side of $0,$ no matter how small δ neighbourhood is chosen.

Thus the condition (ii) given in Theorem 4 for a local minimum is **not** satisfied. ■

This example tells us that the conditions given in Theorem 4 are only sufficient but not necessary for the existence of local extrema of a function.

So, Theorem 3 gives us a necessary condition, and Theorem 4 gives us a sufficient condition for the existence of the extrema of a function. We now apply both these conditions to arrive at the extrema in our next example.

Example 9: Examine the following functions for relative extrema:

i) $f(x)=x^3-3x-4,\; x\in\mathbb{R}$

ii) $f(x)=(x-2)^4(x+1)^5, x\in\mathbb{R}$

iii) $\sqrt{x}-2\sqrt{x+2}, x>0.$

Solution: i) $f(x)=x^3-3x-4\Rightarrow f'(x)=3x^2-3=3(x-1)(x+1).$

$f'(x)=0\Rightarrow x=\pm 1.$

So, $x=1$ and $x=-1$ are the two possible extremum points.

First consider $x=1.$ If $0<x<1, x+1>0$ and $x-1<0.$ Therefore, $f'(x)<0.$

If $x>1, x+1>0$ and $x-1>0.$ Therefore, $f'(x)>0.$ Applying Theorem 4, we conclude that f has a local minimum at $x=1.$

Now consider $x=-1.$ If $x<-1, x-1<0$ and $x+1<0.$ Therefore, $f'(x)>0.$

Theorem 4 tells us that f has a local maximum at $x = -1$.

ii) $f(x) = (x-2)^4(x+1)^5, \quad x \in \mathbb{R}.$

So, $f'(x) = 4(x-2)^3(x+1)^5 + 5(x-2)^4(x+1)^4$

$$= (x-2)^3(x+1)^4(9x-6). \quad \text{… (22)}$$

$f'(x) = 0$ at $x = 2, -1$ and $2/3$.

Let us take these stationary points one by one.

Case 1: $x = -1$

If $x < -1$, then $x - 2 < 0$, and $(x-2)^3 < 0$. Also $x + 1 < 0$, and therefore $(x+1)^4 > 0$ and $9x - 6 < 0$.

Thus from Eqn (16), we get, $f'(x) > 0$ for $x < -1$, whereas, if $-1 < x < 0$, then $x - 2 < 0$, and $(x-2)^3 < 0$. Since $x + 1 > 0$, $(x+1)^4 > 0$, and $9x - 6 < 0$.

This means, that f' does not change sign while passing through $x = -1$. Hence, f has neither a relative maximum, nor a minimum at $x = -1$.

Case 2: $x = 2/3$

For $0 < x < 2/3, (x-2)^3 < 0, (x+1)^4 > 0$ and $(9x-6) < 0$. This shows that $f'(x) > 0$ for $0 < x < 2/3$.

If $2/3 < x < 1$, then $(x-2)^3 < 0, (x+1)^4 > 0$, and $(9x-6) > 0$. This shows that $f'(x) < 0$ for $2/3 < x < 1$. Therefore, since f' changes sign from positive to negative while passing through $x = 2/3$, we can say that f has a relative maximum at $2/3$.

Case 3: $x = 2$

$2/3 < x < 2 \Rightarrow (x-2)^3 < 0, (x+4)^4 > 0$, and $(9x-6) > 0$. So $f'(x) < 0$ for $2/3 < x < 2$.

$x > 2 \Rightarrow f'(x) > 0$.

So we conclude that f has a local minimum at $x = 2$.

iii) $f(x) = \sqrt{x} - 2\sqrt{x+2}, x > 0.$

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{x+2}} = \frac{\sqrt{x+2} - 2\sqrt{x}}{2\sqrt{x}\sqrt{x+2}}.$$

$$f'(x) = 0 \Rightarrow \sqrt{x+2} = 2\sqrt{x} \Rightarrow x + 2 = 4x \Rightarrow x = 2/3.$$

Now the sign of $f'(x)$ will be determined by the sign of $\sqrt{x+2} - 2\sqrt{x}$.

$$\sqrt{x+2} - 2\sqrt{x} = \frac{x+2-4x}{\sqrt{x+2} + 2\sqrt{x}} = \frac{3(2/3 - x)}{\sqrt{x+2} + 2\sqrt{x}}.$$

So $f'(x) > 0$ for $0 < x < 2/3$, and $f'(x) < 0$ for $x > 2/3$. Therefore by the first derivative test we conclude that f has a local maximum at $2/3$.

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You have seen that the sign of the first derivatives helps us to decide whether a stationary point is a local maximum, minimum, or neither. Higher order derivatives of a function can also be used to decide the nature of stationary points. The next theorem shows us how the second derivative is used in this.

Theorem 5 (The Second Derivative Test): Let I be an interval, and c be an interior point of I. Suppose $f : I \to \mathbb{R}$ is differentiable on I, and $f'(c) = 0$.

i) If $f''(c) < 0$, then f has a local maximum at c.

ii) If $f''(c) > 0$, then f has a local minimum at c.

Proof: Since $f''(c)$ exists, it implies that f and f' exist and are continuous in a neighbourhood, $(c-\delta, c+\delta)$ of c, where $\delta > 0$, and $(c-\delta, c+\delta) \subseteq I$.

i) Since $f''(c) < 0$, f' is strictly decreasing at $x = c$. Thus, $\exists\ \delta_i > 0, \delta_1 < \delta$, such that $f'(x) < f'(c)\ \forall\ x \in (c, c+\delta_1)$, and $f'(x) > f'(c)\ \forall\ x \in]c-\delta_1, c[$. Since $f'(c) = 0$, this means $f'(x)$ is negative in $]c, c+\delta_1[$, and positive in $]c-\delta_1, c[$. In other words, f' changes sign from positive to negative in passing through c. Therefore, by the first derivative test, Theorem 4, we conclude that f has a local maximum at c.

ii) The proof of this is on exactly similar lines. We leave it to you as an exercise. See E10). ■

Note: The Theorems 3, 4 and 5 illustrates that the first and second derivatives, when they exists, help us identify the local minima and maxima of a function.

Here are a few examples to illustrate the use, and the limitations of Theorem 5.

Example 10: Examine the following functions for local extrema.

i) $f(x) = \sin x(1+\cos x), x \in [0, 2\pi]$.

ii) $f(x) = \dfrac{4}{x} - \dfrac{1}{x-1}, x \in \mathbb{R} \setminus \{0,1\}$.

iii) $f(x) = \left(\dfrac{1}{x}\right)^x, x > 0$.

Solution: i) $f'(x) = \cos x(1+\cos x) - \sin^2 x$
$$= \cos x + \cos 2x$$

$$\therefore f'(x) = 0 \Rightarrow \cos x + \cos 2x = 0 \Rightarrow \cos x = -\cos 2x.$$

$$\Rightarrow x = \frac{\pi}{3}.$$

Now $f''(x) = -\sin x - 2\sin 2x$, and

$$f''\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} - 2\sin 2\frac{\pi}{3}$$

$$= -\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2} = \frac{-3\sqrt{3}}{2} < 0.$$

So, f has a local maximum at $\frac{\pi}{3}$.

ii) $f'(x) = \frac{-4}{x^2} + \frac{1}{(x-1)^2} = \frac{-4(x-1)^2 + x^2}{x^2(x-1)^2} = 0$

$\Rightarrow x^2 - 4(x-1)^2 = 0 \Rightarrow 3x^2 - 8x + 4 = 0$

$\Rightarrow (3x-2)(x-2) = 0.$

$\Rightarrow x = \frac{2}{3}$ or $x = 2.$

$f''(x) = \frac{8}{x^3} - \frac{2}{(x-1)^3}$

$f''(2) = 1 - 2 = -1 < 0$ and

$f''(2/3) = 27 + 54 = 81 > 0.$

Hence f has a local maximum at 2, and a local minimum at $2/3$.

iii) Let $y = \left(\frac{1}{x}\right)^x$. Then $\log y = x\,\ell n\left(\frac{1}{x}\right) = -x\,\ell n.$

$\therefore \frac{1}{y}\cdot y' = -\log x - 1$

$\therefore y' = -y(1+\log x).$

Now $f'(x) = y' = 0$ if $-y(1+\log x) = 0$ if $\log x = -1$ that is, if $x = \frac{1}{e}.$

Since $y' = -y\,(1+\log x),$

$y'' = -y'(1+\log x) - y\cdot\frac{1}{x}.$

$$\therefore y''\left(\frac{1}{e}\right) = \frac{-f\left(\frac{1}{e}\right)}{\frac{1}{e}} = -e\cdot e^{1/e} < 0.$$

So, has a local maximum at $\frac{1}{e}$.

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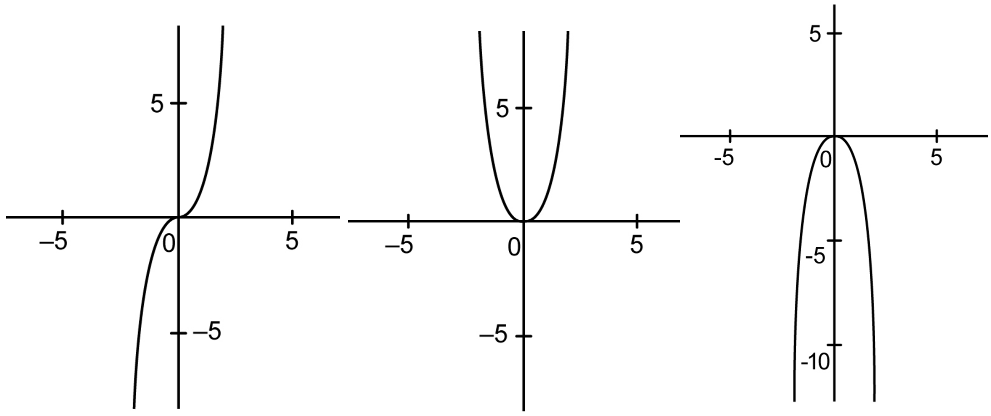
Example 11: Examine the nature of the stationary point $x = 0$ for function $f(x) = x^3, g(x) = x^4$ and $h(x) = -x^4,\ x \in \mathbb{R}.$

Solution: It is easy to check that $f'(0) = f''(0) = 0, g'(0) = g''(0) = 0$ and $h'(0) = h''(0) = 0.$

Theorem 5 cannot be used here. We now look at the graphs of these function given in Fig. 4 a), b) and c), below:



(a) **Graph of** f (b) **Graph of** g (c) **Graph of** h

Fig. 4

Isn't it clear that $x = 0$ is neither a local maximum, nor a minimum for f.

$x = 0$ is a local (and global) minimum for g, and it is local (and global) maximum for h.

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This example shows that Theorem 5 is of no help if the second derivative is zero. We now state without proof, a general theorem, that uses higher derivatives of a function to classify its stationary points.

Theorem 6: Let I be an interval. Let $c \in I$, and let $n \geq 2$. Suppose that the derivatives $f', f'', ..., f^{(n)}$ exist and are continuous in a neighbourhood of c, and that $f'(c) = f''(c) = ... = f^{(n-1)}(c) = 0$, but $f^{(n)}(c) \neq 0$.

i) If n is even and $f^{(n)}(c) > 0$, then f has a local minimum at c.

ii) If n is even and $f^{(n)}(c) < 0$, then f has a local maximum at c.

iii) If n is odd, then f has neither a local minimum nor a local maximum at c. ■

We are not including the proof of this theorem. But it can be easily proved by using Taylor's Theorem. You may note that the second derivative test, Theorem 5, is a special case of this theorem, when $n = 2$.

In the next example we use Theorem 6 to find the extreme points.

Example 12: Examine the function $f(x) = (x-3)^5(x+1)^4$ for extreme values.

Solution: $f'(x) = (x-3)^4(x+1)^3(9x-7)$

So the stationary points of f are $-1, 7/9$ and 3.

Now, $f''(x) = 8(x-3)^3(x+1)^2(9x^2 - 14x + 1)$, therefore,

$f''(-1) = f''(3) = 0$, and

$$f''(7/9) = 8\left(\frac{-20}{9}\right)^3\left(\frac{16}{9}\right)^2\left[\frac{49}{9} - 14 \times \frac{7}{9} + 1\right]$$

$$= -8\left(\frac{20}{9}\right)^3\left(\frac{16}{9}\right)^2\left(\frac{-40}{9}\right) > 0.$$

$\therefore x = \frac{7}{9}$ is a local minimum.

$$f'''(x) = 24(x-3)^2(x+1)(21x^3 - 49x^2 + 7x + 13)$$

Again, $f'''(-1) = f''(3) = 0$.

$$\begin{aligned} f^{(4)}(x) = 24[&2(x-3)(x+1)(21x^3 - 49x^2 + 7x + 13) \\ &+ (x-3)^2(x+1)(63x^2 - 98x + 7) \\ &+ (x-3)^2(21x^3 - 49x^2 + 7x + 13] \end{aligned}$$

Thus, $f^{(4)}(3) = 0$ and $f^{(4)}(-1) = 24.16(-64) < 0$.

Hence f has a local maximum at $x = -1$.

Now, $$\begin{aligned} f^{(5)}(x) = 24[&2(x+1)(21x^3 - 49x^2 + 7x + 13) \\ &+ 14(x-3)(3x-1)(9x^2 - 14x + 1) \\ &+ 14(x-3)^2(x+1)(9x-7)] \end{aligned}$$

So $f^{(5)}(3) \neq 0$.

Since the first non-zero derivative at $x = 3$ is of odd order 5, f has neither a maximum, nor a minimum at $x = 3$.

***

You must have learnt from the course Calculus that the absolute or global maximum (or minimum) value of a function, defined on the interval $[a,b]$, is the greatest (or smallest) value taken by the function in that interval. Thus a maximum (or minimum) is always a local maximum (or minimum) value. A function can have more than one local maximum (or local minimum) value. Among these values there can only be one value which is maximum and same is the cae with minimum. Therefore if the function is differerentiable at all interior points of the interval, we first find all the stationary points of the function, say, $c_1, c_2, ..., c_k$. Then we look at the values $f(a), f(c_1), f(c_2), ..., f(c_k), f(b)$. The greatest among these is the absolute maximum value of the function, and the least is the absolute minimum value of the function. The following example will make this clear.

Example 12: Find the absolute maximum and minimum values of the function $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$ on $[0, 2]$.

Solution: We first find the stationary points.

$$f'(x) = 12x^3 - 6x^2 - 12x + 6.$$

So $f'(x) = 0 \Rightarrow 2x^3 - x^2 - 2x + 1 = 0$ or $(x-1)(x+1)(2x-1) = 0$

That is, $x = -1, \frac{1}{2}$ or 1.

$-1 \notin [0,2]$. So the stationary points of f in $[0,2]$ are $x = \frac{1}{2}$ and $x = 1$.

Now $f(0) = 1, f\left(\frac{1}{2}\right) = \frac{39}{16}, f(1) = 2$ and $f(2) = 21$.

So, the absolute maximum value is 21 and absolute minimum value is 1.

***

You should now try your hand at the following exercises.

E8) If $f : I \to \mathbb{R}$, has a local minimum at d, show that $f'(d) = 0$, if it exists.

E9) Use the first derivative test to find the relative extremes of the function given by

i) $f(x) = 10x^6 - 24x^5 + 15x^4 - 40x^3 + 108, \ \forall x \in \mathbb{R}$,

ii) $f(x) = x^4 + 2x^2 - 4, x \in \mathbb{R}$.

E10) Prove Theorem 5 ii).

E11) Find the local maximum and minimum points of the function f defined by

i) $f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x, x \in [0, \pi]$

ii) $f(x) = 2x + \frac{1}{x^2}, x \neq 0$.

E12) Determine whether $x = 0$ is a relative extremum of the following functions:

i) $f(x) = x^3 + 2$

ii) $g(x) = \sin x - x$

iii) $h(x) = \sin x + \frac{x^3}{6}$

iv) $k(x) = \cos x - 1 + \frac{x^2}{2}$

E13) Find the absolute maximum and minimum values of $f(x) = x^4 - 4x^3 - 2x^2 + 12x + 1$ in the interval $[-2, 5]$.

That brings us to end of the unit. Let us briefly recall what we have covered in it.

13.4 SUMMARY

In this unit we have

i) introduced the notion of higher order derivatives of a function,

ii) proved Taylor's theorem,

iii) stated Maclaurin's theorem, which is a special case of Taylor's theorem,

iv) defined relative (or local) maximum and minimum and stationary points of a function,

v) proved that if the derivative of a function exists at its local extremum point, it has to be zero,

vi) observed that the condition in v) is necessary, and not sufficient.

vii) proved the first derivative test to find the local extrema of a function, and noted that the condition is a sufficient one.

viii) proved the second derivative test to find the local extremum of a function,

ix) stated a general test using the higher order derivatives to find the local extrema of a function.

13.5 SOLUTIONS AND ANSWERS

E1) $h(x) = \sin(ax+b)$.

Let $P_n : h^{(n)}(x) = a^n \sin(ax+b+n\pi/2)$.

Then $P_1 : h'(x) = h^{(1)}(x) = a\cos(ax+b)$
$= a\sin(ax+b+\pi/2)$.

So P_1 is true.

Suppose P_k is true.

Therefore, $h^{(k)}(x) = a^k \sin(ax+b+k\pi/2)$.

$$\therefore h^{(k+1)}(x) = \frac{d}{dx}h^{(k)}(x) = a^k.a\cos(ax+b+k\pi/2)$$
$$= a^{k+1}\sin(ax+b+k\pi/2+\pi/2)$$
$$= a^{k+1}\sin(ax+b+(k+1)\pi/2).$$

$\therefore P_{k+1}$ is true.

Hence by induction, P_n is true for all $n \in \mathbb{N}$.

E2) $f(1) = \cos ax$ $\quad f'(x) = -a\sin ax = a\cos(ax+\pi/2)$
$f''(x) = -a^2\cos ax = a^2\cos(ax+\pi)$
$f^{(3)}(x) = a^3\sin ax = a^3\cos(ax+3\pi/2)$

So, in general, $f^{(n)}(x) = a^n\cos(ax+n\pi/2)$, which can be proved by Induction as in E1).

E3) $f(x) = \log(ax+b) \cdot f'(x) = \dfrac{a}{ax+b}$

$$f''(x) = \frac{-a^2}{(ax+b)^2} \qquad f^{(3)}(x) = \frac{2a^3}{(ax+b)^3}$$

$\therefore f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$, to be proved by induction.

E4) $f(x) = \sin x,\ a = 0 \qquad f(0) = 0$
$f'(x) = \cos x, \qquad f'(0) = 1$
$f''(x) = -\sin x, \qquad f''(0) = 0$
$f^{(3)}(x) = -\cos x, \qquad f^{(3)}(0) = -1$
$f^{(4)}(0) = f^{(6)}(0) = 0 \quad f^{(5)}(0) = 1$

$\left|\sin x - P_6(x)\right| = \left|\dfrac{x^7}{7!}f^{(7)}(c)\right|$, where $0 < |c| < 1$.

$$= \left|\frac{x^7}{7!}(-\sin c)\right| \le \frac{1}{7!} = \frac{1}{5040}$$

E5) Let $f(x)=\sqrt{1+x},\quad x>-1,\ a=0.\ f(0)=1$

$$f'(x)=\frac{1}{2\sqrt{1+x}},\ f'(0)=\frac{1}{2}$$

$$f''(x)=\frac{-1}{4(1+x)^{3/2}},\ f''(0)=-1/4$$

$$f(x)\approx f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)$$

$$\sqrt{1.2}\approx 1+0.2\left(\frac{1}{2}\right)+\frac{(0.2)^2}{2!}\left(\frac{-1}{4}\right)$$

$$=1+0.1-\frac{0.01}{2}$$

$$=1.095$$

$$\sqrt{2}=\sqrt{1+1}\approx 1+1\left(\frac{1}{2}\right)+\frac{1}{2!}\left(\frac{-1}{4}\right)$$

$$=1.375.$$

E6) Let $f(x)=\sin x, a=0.\quad f(0)=0.$

$f'(x)=\cos x,\ f'(0)=1$

$f''(x)=-\sin x,\ f''(0)=0$

$f^{(3)}(x)=-\cos x, f^{(3)}(0)=-1$

$f^{(4)}(x)=\sin x \quad f^{(4)}(0)=0$

$f^{(5)}(x)=\cos x \quad f^{(5)}(0)=1$

Now $f(x)=f(0)+x\,f'(0)+\dfrac{x^2}{2!}f''(0)+\dfrac{x^3}{3!}f'''(c)$, where $0<c<x$.

$$\therefore \sin x = x-\frac{x^3}{3!}\cos(c)\ge x-\frac{x^3}{3!} \text{ if } x\ge 0,\ \cos c\le 1$$

$$\le x-\frac{x^3}{3!}+\frac{x^5}{5!}, \text{for } x\ge 0, \text{since } \cos c\le 1.$$

$$\therefore x-\frac{x^3}{3!}\le \sin x\le x-\frac{x^3}{3!}+\frac{x^5}{5!},\ x\ge 0.$$

Now suppose $x<0$. Then $y=-x>0$, and we have

$$y-\frac{y^3}{3!}\le \sin y\le y-\frac{y^3}{3!}+\frac{y^5}{5!}$$

$$\therefore -x+\frac{x^3}{3!}\le \sin(-x)\le -x-\frac{x^3}{3!}+\frac{x^5}{5!}$$

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}\le \sin x\le x-\frac{x^3}{3!}$$

E7) i) Let $f(x)=\tan^{-1}x \qquad a=\dfrac{\pi}{4}.\quad f(a)=\tan^{-1}\left(\dfrac{\pi}{4}\right)$

$$f'(x)=\frac{1}{1+x^2},\ f'\left(\frac{\pi}{4}\right)=\frac{1}{1+\dfrac{\pi^2}{16}}$$

$$f''(x) = \frac{-2x}{(1+x^2)^2}, f''\left(\frac{\pi}{4}\right) = \frac{-\pi}{2\left(1+\frac{\pi^2}{16}\right)^2}$$

So, by Taylor's theorem,

$$\tan^{-1} x = \tan^{-1}\left(\frac{\pi}{4}\right) + \frac{(x-\pi/4)}{1+\pi^2/16} - \frac{\pi(x-\pi/4)^2}{4(1+\pi^2/16)^2} + \dots$$

ii) Let $f(x) = \cos x,\ a = \pi/4.$ $\qquad f(a) = 1/\sqrt{2}$

$f'(x) = -\sin x,\ \ f'(a) = -1/\sqrt{2}$

$f''(x) = -\cos x,\ \ f''(a) = -1/\sqrt{2}$

$f^{(3)}(x) = \sin x,\ \ f^{(3)}(a) = 1/\sqrt{2}\dots.$

$$\therefore \cos x = \frac{1}{\sqrt{2}} - \frac{(x-\pi/4)}{\sqrt{2}} - \frac{(x-\pi/4)^2}{2\sqrt{2}} + \frac{(x-\pi/4)^3}{3!\sqrt{2}} \dots$$

$$= \frac{1}{\sqrt{2}}\left[1 - \left(x-\frac{\pi}{4}\right) - \frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3 + \dots\right]$$

E8) f has a local minimum at d. Therefore, $\exists\ \delta_1 > 0$ such that $f(x) > f(d)\ \forall\ x \in]d-\delta_1, d+\delta_1[, x \neq d.$ That is, $f(x) - f(d) > 0\ \forall\ x \in]d-\delta_1, d+\delta_1[, x \neq d.$

Now, if $x \in]d-\delta_1, d[$, then $\frac{f(x)-f(d)}{x-d} < 0,$ since $x - d < 0.$ Therefore, $\lim\limits_{x\to d^-} \frac{f(x)-f(d)}{x-d} \leq 0$.

If $x \in]d, d+\delta_1[$, then $\frac{f(x)-f(d)}{x-d} > 0.$ Therefore, $\lim\limits_{x\to d^+} \frac{f(x)-f(d)}{x-d} \geq 0.$

Since $f'(d) = \lim\limits_{x\to d} \frac{f(x)-f(d)}{x-d}$ exists,

$$\lim_{x\to d^-} \frac{f(x)-f(d)}{x-d} = \lim_{x\to d^+} \frac{f(x)-f(d)}{x-d} = 0. \quad f'(d) = 0.$$

E9) i) $f(x) = 10x^6 - 24x^5 + 15x^4 - 40x^3 + 108.$

$$\therefore f'(x) = 60x^5 - 120x^4 + 60x^3 - 120x^2$$
$$= 60x^2[x^3 - 2x^2 + x - 2]$$
$$= 60x^2(x^2+1)^2(x-2).$$

$\therefore f'(x) = 0$ for $x = 0$ and $2.$

If $x < 0, x - 2 < 0.$ Therefore $f'(x) < 0.$

$0 < x < 2 \Rightarrow x - 2 < 0 \Rightarrow f'(x) < 0.$

$\therefore f'(x)$ does not change sign at $x = 0.$

Hence f does not have a local extremum at $x = 0.$

$x < 2, f'(x) < 0$ and if $x > 2, f'(x) > 0.$

$\therefore f'$ changes sign from $-ve$ to $+ve$ while passing through $x = 2.$ Hence f has a local minimum at $x = 2.$

ii) $f(x) = x^4 + 2x^2 - 4 \Rightarrow f'(x) = 4x^3 + 4x = 4x(x^2+1)$
$f'(x) = 0$ if $x = 0,$ and changes sign from $-ve$ to $+ve$ while passing through $x = 0.$ Hence f has a local minimum at $x = 0.$

E10) To prove: If $f''(c) > 0,$ then f has a local minimum at $c,$ when $f'(c) = 0.$

Now, f and f' exist and are continuous in a neighbourhood $(c-\delta, c+\delta), \delta > 0, (c-\delta, c+\delta) \subseteq I.$ Since $f''(c) > 0, f'$ is strictly increasing at $c,$ and $\exists\ \delta_1 > 0, \delta_1 < \delta,$ such that $f'(x) < f'(c) \forall x \in (c-\delta_1, c)$ and $f'(x) > f'(c) \forall x \in (c, c+\delta_1).$ Since $f'(c) = 0,$ we have $f'(x)$ is negative $\forall\ x \in (c-\delta_1, c),$ and $f'(x)$ is positive $\forall x (c, c+\delta_1).$ So f' changes sign from negative to positive in passing through $c.$

Therefore, by the first derivative test, we conclude that f has a local minimum at $c.$

E11) i) $f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x, \qquad x \in [0, \pi].$

$$\therefore f'(x) = \cos x + \cos 2x + \cos 3x = 0$$
$$\Rightarrow \cos 2x + 2\cos x \cos 2x = 0 \qquad \begin{pmatrix}\cos 2x + \cos 3x \\ = 2\cos 2x \cos x\end{pmatrix}$$
$$\Rightarrow \cos 2x(1 + 2\cos x) = 0$$
$$\Rightarrow x = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}$$

$$f''(x) = -\sin x - 2\sin 2x - 3\sin 3x$$
$$f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}} < 0$$
$$f''\left(\frac{2\pi}{3}\right) = \frac{-\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} - 0 > 0$$
$$f''\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}} + 2 - \frac{3}{\sqrt{2}} < 0.$$

$\therefore f$ has local maximum at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, and local minimum at $\frac{2\pi}{4}$.

ii) $f(x) = 2x + \frac{1}{x^2}$

$$f'(x) = 2 - \frac{2}{x^3} = 2\left(1 - \frac{1}{x^3}\right) = 0 \Rightarrow x = 1$$

$$f''(x) = \frac{6}{x^4} \Rightarrow f''(1) = > 0$$

Hence f has a local minimum at $x = 1.$

E12) i) $f(x) = x^3 + 2 \Rightarrow f'(x) = 3x^2 \Rightarrow f'(0) = 0$

$f''(x) = 6x \Rightarrow f''(0) = 0.$
$f'''(x) = 6 \Rightarrow f'''(0) \neq 0.$
$\therefore x = 0$ is not a local extremum.

ii) $g(x) = \sin x - x \Rightarrow g'(x) = \cos x - 1 \Rightarrow g'(0) = 0$
$g''(x) = -\sin x \Rightarrow g''(0) = 0.$
$g'''(x) = \cos x \Rightarrow g'''(0) = 1 \neq 0.$

So g has no local extremum at $x = 0$.

iii) $h(x) = \sin x + \dfrac{x^3}{6}.$

$h'(x) = \cos x + \dfrac{x^2}{2} \Rightarrow h'(0) = 1 \neq 0.$
$\therefore x = 0$ is not a local extremum

iv) $k(x) = \cos x - 1 + \dfrac{x^2}{2}$
$k'(x) = -\sin x + x \Rightarrow k'(0) = 0$

$k''(x) = -\cos x + 1 \Rightarrow k''(0) = 0$

$k'''(x) = \sin x \Rightarrow k'''(0) = 0$

$k^{(4)}(x) = \cos x \Rightarrow k^{(4)}(0) = 1 \neq 0.$

Since 4 is even, k has a local minimum at $x = 0.$

E13) $f(x) = x^4 - 4x^3 - 2x^2 + 12x + 1, \quad x \in [-2, 5].$

$f'(x) = 4x^3 - 12x^2 - 4x + 12 = 0 \Rightarrow x^3 - 3x^2 - x + 3 = 0$

$\Rightarrow (x-1)(x-3)(x+1) = 0$
$\Rightarrow x = -1, 1, 3.$

We look for absolute max. and min. values among
$f(-2), f(-1), f(1), f(3), f(5).$

$f(-2) = 17, f(-1) = -8, f(1) = 8, f(3) = -8, f(5) = 136.$

$\therefore f$ has a global or absolute maximum at $x = 5,$ and a global minimum at -1 and $3.$