Indira Gandhi National Open University
School of Sciences

BMTC-133
REAL ANALYSIS

Block

6

SEQUENCES AND SERIES OF FUNCTIONS

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BLOCK INTRODUCTION

In Block 2, you were introduced to the notion of sequences of real numbers and their convergence. In Block 3, infinite series of real numbers and their convergence was considered. In this Block, we want to discuss sequences and series whose terms are functions defined on a subset of Real number. Such sequences and series are known as sequences or series of real functions. This Block is divided into two units.

Unit 17 covers sequences of functions. A sequence of functions is almost a straightforward generalization of a sequence of real numbers. We consider here the most basic form of convergence of a sequence of functions, called pointwise convergence. Then we go on to define a stronger form of convergence called uniform convergence. Whenever a sequence of functions is convergent, its limit is a function called limit function. The question arises whether the properties of continuity, differentiability, integrability of the individual functions in a sequence or series of functions are preserved by the limit function. We shall show that these properties are preserved by the uniform convergence and not by the pointwise convergence.

In Unit 18 we discuss the convergence of series of functions in terms of convergence of its sequence of partial sums. Since the sequence of functions can either converge pointwise or uniformly (or not at all), we can define pointwise convergence and uniform convergence of a series of functions. Next, we discuss a useful result known as the Wierstrass M-Test, which gives straightforward conditions that can determine if a series of functions is uniformly convergent. We close this unit with a brief introduction to power series.

NOTATIONS AND SYMBOLS (used in Block 6)

(Also see the notations used in Calculus and Differential Equations)

$(f_n)_{n\in\mathbb{N}}$	a sequence of functions
$f_n \to f$	f is the (pointwise) limit of $(f_n)_{n\in\mathbb{N}}$
$\sum f_n$	a series of functions
$\sum a_n (x-c)^n$	a power series about $x = c$
$\sum f_n = f$	f is the sum of the series $\sum f_n$

UNIT 17

SEQUENCES OF FUNCTIONS

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17.1 INTRODUCTION

You are familiar with sequences of real numbers in Block 2. You have studied their convergence characteristics. In this unit we are going to introduce you to sequences of functions which is a straight forward generalisation of sequences of real numbers. Once you are familiar with the idea, we shall talk about the convergence of these sequences also. We shall be using the concepts discussed in Block 2 to study the sequences of functions. So, it will be a good idea to go back and revise the definitions and major theorems of units of Block 2.

In Sec. 17.2 we start our discussion on sequences of functions as a generalisation to the concept of sequence of real numbers. We shall familiarise you with examples of sequences of functions and explain how it is related to sequences of real numbers. We shall then explain the convergence of these sequences by introducing the term point wise convergence.

In Sec. 17.3 we shall define another type of convergence of sequence of functions known as uniform convergence. We shall explain how it is different from point-wise convergence.

In Sec. 17.4 we discuss what properties such as continuity are preserved by uniform convergence. We shall present two theorems.

Objectives

After studying this unit, you should be able to

- define the pointwise limit of a sequence of functions;
- decide whether a given sequence of functions is pointwise convergent or not;

- define the uniform limit of a sequence of functions;
- show that the uniform limit of a sequence of contiuous/ differentiable/ integrable functions is contiuous/differentiable/integrable.

17.2 POINTWISE CONVERGENCE

In this section we consider a generalisation of sequences of real numbers, namely, sequences of real-valued functions, i.e. it is a sequence each of whose terms is a real valued function. We discuss the basic form of the convergence of these sequences known as pointwise convergence.

To understand the sequences of functions, we consider a geometric sequence with which you are already familiar.

Let $x \in \mathbb{R}$. For each $n \in \mathbb{N}$, let us consider a function f_n defined on $\mathbb{R}$ by $f_n(x) = x^n$.

Then for each n, f_n is a function and these functions share a common domain, in this case it is $\mathbb{R}$. Note that $f_1(x) = x, f_2(x) = x^2, \; f_3(x) = x^3, \ldots, f_n(x) = x^n, \ldots$ Thus we have a sequence of functions given by $f_1, f_2, \ldots, f_n, \ldots$ which we denote by $(f_n)_{n\in\mathbb{N}}$.

Note that here for each $n \in \mathbb{N}$, there is a function f_n defined on $\mathbb{R}$.

Suppose we fix $x \in \mathbb{R}$, say $x = 0$. Then we have $f_1(0) = 0, f_2(0) = 0, f_3(0) = 0, \ldots$ Thus we get the sequence $(0,0,0\ldots,0,\ldots)$ of real numbers. Similarly when we put $x = \frac{1}{2}$, then we get the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$ and for $x = 1$, we get the constant sequence, $(1,1,1,\ldots..)$.

Hence by evaluating the functions in the given sequence at each point of the domain, we get sequences of real numbers. We can also talk about the convergence of the given sequence of functions. The definition of convergence is closely linked to the convergence of the sequences of real numbers generated. Before we write the definition, we give some examples to make you familiar with the idea of sequences of functions.

Example 1: For the given sequence of functions on $\mathbb{R}$, find the sequence $(f_n(x))_{n\in\mathbb{N}}$ of real numbers corresponding to the values given against it.

i) $f_n(x) = \dfrac{x}{n}, x = 1, 0, \dfrac{1}{3}$

ii) $f_n(x) = \dfrac{x^n}{n!}, x = 1, -2, \dfrac{1}{2}$.

Solution: Let us try one by one.

i) For $x = 1, f_1(x) = 1, f_2(x) = \dfrac{1}{2}, f_3(x) = \dfrac{1}{3} \ldots f_n(x) = \dfrac{1}{n}, \ldots.$

Hence we get the sequence $\left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$

For $x = 0,$ the sequence is $(0)_{n\in\mathbb{N}}$.

For $x = \frac{1}{3}, f_1(x) = \frac{1}{3}, f_2(x) = \frac{1}{6}, f_3(x) = \frac{1}{9}, \ldots, f_n(x) = \frac{1}{3n}, \ldots$

Therefore the sequence is $\left(\frac{1}{3n}\right)_{n\in\mathbb{N}}$.

ii) For $x = 1, f_1(x) = 1, f_2(x) = \frac{1}{2!}, f_3(x) = \frac{1}{3!} \ldots.. f_n(x) = \frac{1}{n!} \ldots$

Therefore the sequence is $\left(\frac{1}{n!}\right)_{n\in\mathbb{N}}$

For $x = -2, f_1(x) = -2, f_2(x) = \frac{(-2)^2}{2!} = \frac{4}{2} = 2,$

$f_3(x) = \frac{(-2)^3}{3!} = \frac{(-1)^3 \times 8}{2 \times 3} = \frac{-4}{3}, \ldots f_n(x) = \frac{(-2)^n}{n!}$

Hence the sequence is $\left(\frac{(-2)^n}{n!}\right)_{n\in\mathbb{N}}$

For $x = \frac{1}{2}, f_1(x) = \frac{1}{2n!}, f_2(x) = \frac{1}{4 \times 2} = \frac{1}{8},$

$f_3(x) = \frac{1}{2^3 \times 3!} = \frac{1}{8 \times 6} = \frac{1}{48} \ldots f_n(x) = \frac{1}{2^n n!} \ldots$

Therefore sequence is $\left(\frac{1}{2^n n!}\right)_{n\in\mathbb{N}}$.

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You can solve this exercise now.

E1) Write the functions, f_1, f_2, f_3 in each of the sequences given below. Also find the sequences of real numbers obtained by evaluating the functions in the sequence at any two points of your choice from the given domain.

i) $(f_n)_{n\in\mathbb{N}}$, where $f_n(x) = 1 + nx, A = [0,2]$

ii) $(f_n)_{n\in\mathbb{N}}$, where $f_n(x) = 2 + \frac{1}{nx^2}, A =]0, 2[$.

You have realized that when a sequence of functions is given, by evaluating each of the functions at a given point of the domain, we get a sequence of real numbers. We can then decide whether this sequence is convergent or divergent.

Let us go back to the example of the sequence of functions $f_n(x) = x^n$, discussed at the beginning.

Here at $x = 0,\ f_n(0) = 0, \forall n$. Therefore, $\lim_{n\to\infty}(f_n(0)) = 0.$

Similarly at $x = 1,\ f_n(1) = 1, \forall n$. Therefore, $\lim_{n\to\infty}(f_n(1)) = 1.$

If $0 < x < 1$, then $f_n(x) = x^n \to 0$, as $n \to \infty$. (You can write the sequence for $x = 1/2$, as an example, and see that it tends to zero.)

So, if we define a function, $f : [0,1] \to R, f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$, then we can say that (f_n) converges pointwise to f on $[0,1]$ for each point of $[0,1]$.

In this case we say that the sequence of functions is pointwise convergent in the domain. Here is the precise definition.

Definition: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real valued functions defined on a subset A of $\mathbb{R}$. We say that the sequence, $(f_n)_{n\in\mathbb{N}}$ **converges pointwise to a function**, f on A, if, for each $x \in A$, the sequence, $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$.

In this case, the function f is called the **pointwise limit** of $(f_n)_{n\in\mathbb{N}}$ on A, and we write $f = \lim_{n\to\infty} f_n$ on A, or, $f_n \to f$ on A.

Let us try to understand this definition through some more examples.

Example 2: Find the pointwise limit of the following sequences of functions, if they exist.

i) $(f_n)_{n\in\mathbb{N}}$, where $f_n :]0,\infty[\to \mathbb{R}$, is given by $f_n(x) = \dfrac{x}{n}$.

ii) $(f_n)_{n\in\mathbb{N}}$, where $f_n : [0,\infty[\to \mathbb{R}$, is given by $f_n(x) = \dfrac{1}{1+x^n}$.

Solution: i) In this case it is easy to see that for any fixed $x \in]0,\infty[, f_n(x) = \dfrac{x}{n} \to 0$. Therefore, the pointwise limit of this sequence of functions is the zero function, $f :]0,\infty[\to \mathbb{R}$, given by $f(x) = 0 \ \forall \ x \in]0,\infty[$.

ii) Here, $f_n(0) = \dfrac{1}{1+0} = 1, \forall n$. Therefore, $f_n(0) \to 1$.

If $0 < x < 1$, then $f_n(x) \to 1$, since, in this case, $x^n \to 0$.

If $x = 1$, then $f_n(1) = \dfrac{1}{1+1} = \dfrac{1}{2} \forall n$, and therefore, $f_n(1) \to \dfrac{1}{2}$ as $n \to \infty$.

If $x > 1$, then $x^n \to \infty$, and therefore, $f_n(x) = \dfrac{1}{1+x^n} \to 0$ as $n \to \infty$.

Thus, if we define the function $f : [0,\infty[\to \mathbb{R}$, by $f(x) = \begin{cases} 1, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases}$,

then $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f on $[0,\infty[$.

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Now, to understand the convergence more, we look at the graphs of the functions given in the sequence.

For instance, let us look at the graphs of the functions $f_n : [0,1] \to \mathbb{R}, f_n(x) = x^n$ for some values of n. (See Fig. 1).

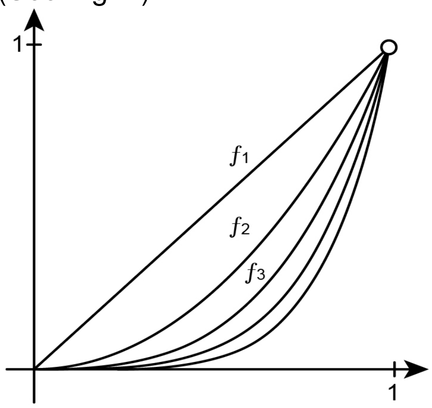

Fig. 1

All these graphs pass through the origin and also through the point $(1,1)$. As n increases, the graph seems to sag more and more between these two points. The limit is shown in red, where the graph coincides with the x-axis for $0 \le x < 1$, and also contains the point $(1,1)$.

Next we shall look the graphs of the functions $f_n(x) = \frac{x}{n}$ for some values of n .(See Fig. 2).

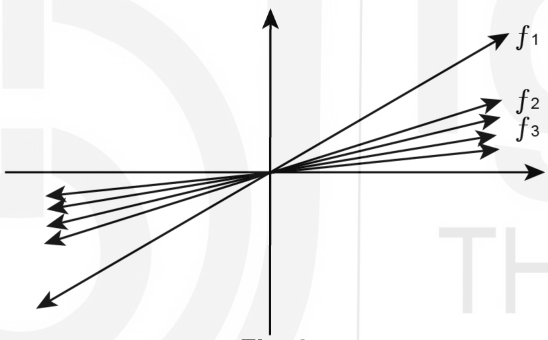

Fig. 2

These are all straight lines originating at the point $(0,0)$. You can also see that as n increases, the slope of the line decreases. Finally, as $n \to \infty$, we get a line with slope 0. This is the limit of the sequence of functions. In the graph you can see it in red.

All the functions in all the sequences given in examples above are continuous on their domains. But what about the limits of these sequences? The limit function for $f_n(x) = x^n$ is discontinuous at 1. Whereas the limit of function for $f_n(x) = \frac{x}{n}$ is continuous on $[0,1]$. Also note that the limit function for the sequence of functions in part (ii) in Example 2 is discontinuous. Indeed the limit function f is

$$f(x) = \begin{cases} 1, & if \quad 0 \le x < 1 \\ \frac{1}{2}, & if \quad x = 1 \\ 0, & if \quad x > 1 \end{cases}$$

So, the **pointwise limit of a sequence of continuous functions need not be continuous**.

So far we have considered the pointwise convergence of the sequence of functions at each point of the domain.

We rewrite the pointwise convergence in the following way also.

Definition: A sequence of functions $(f_n)_{n\in\mathbb{N}}$ **converges pointwise to** f on its domain, if for a given $\varepsilon > 0$, for every x in the domain, there exists a natural number, $N(\varepsilon, x)$, such that

$$n \geq N(\varepsilon, x) \Rightarrow |f_n(x) - f(x)| < \varepsilon .$$

The number, $N(\varepsilon, x)$ depends on ε, and also on x. That is, for a given $\varepsilon > 0$, the same natural number may not work for all x in the domain. We shall illustrate this with an example.

Let us consider the function $f_n(x) = x^n \; x \in [0,1], n = 1, 2....$ We know that $(f_n)_{n\in\mathbb{N}}$ converges to the function f given by

$$f(x) = \begin{cases} 0, & x \in [0,1[\\ 1, & x = 1 \end{cases}$$

Let us select some $x \in [0,1]$, say $x = \frac{1}{2}$. Since $f_n(x) \to f(x)$ for every x, we get that $f_n\left(\frac{1}{2}\right) \to f\left(\frac{1}{2}\right) = 0.$

Now we apply the definition with $\varepsilon = .001$. Then for this $\varepsilon > 0$ there exists an $N\left(\varepsilon, \frac{1}{2}\right)$ such that $\left|\left(\frac{1}{2}\right)^n\right| < .001 \; \forall \; n \geq N\left(\varepsilon, \frac{1}{2}\right)$. Then N should be at least 10.

Suppose we select some other $x \in [0,1]$, say $x = \frac{9}{10}$. Then we have $f_n\left(\frac{9}{10}\right) \to f\left(\frac{9}{10}\right) = 0$. Now, for $\varepsilon = .001$ there exists $N\left(\varepsilon, \frac{9}{10}\right)$ such that $\left|\left(\frac{9}{10}\right)^n\right| < .001$ for all $n \geq N\left(\varepsilon, \frac{9}{10}\right)$ Then N should be at least 66. If we select any other x, then N could be some other number. This shows that N varies according to the choice of x.

Now the question arises, are there cases where we can find a common value of N, for all x? The answer is yes. It leads us to a stronger convergence criteria, called uniform convergence. We shall present this in the next section.

Before that, it is very important to try some exercises on your own. Once you have completely understood pointwise convergence, you will be ready for the uniform convergence.

Here are some exercises.

E2) Find the pointwise limits of the sequences of functions, $(f_n)_{n\in\mathbb{N}}$ on the given domains, where f_n is given as follows:

i) $f_n(x) = \dfrac{x}{n}, x \in \mathbb{R}$ ii) $f_n(x) = \dfrac{\sin nx}{n}, x \in [0,1]$

iii) $f_n(x) = \dfrac{nx}{1+n^2x^2}, x \in \mathbb{R}$ iv) $f_n(x) = x + \dfrac{x}{n}\sin(nx), x \in [-1,1]$

E3) If $f_n(x) = (\cos \pi x)^n$, show that $(f_n)_{n\in\mathbb{N}}$ converges pointwise on $[0,1[$, but not on $[0,1]$.

E4) If $f_n(x) = \begin{cases} 1, & \text{if } x \in [-n,n] \\ 0, & \text{otherwise,} \end{cases}$

then show that $(f_n)_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{R}$.

(Draw the graphs of f_1, f_2, f_3 to understand the sequence.)

Now we are ready to discuss uniform convergence.

17.3 UNIFORM CONVERGENCE

In this section we shall introduce you to a stronger form of convergence for sequence of functions.

Let us start with a sequence of functions $(f_n)_{n\in\mathbb{N}}$, each with domain D. In the last section we noted that for a given sequence of functions, we get a number of sequences of real numbers, one for each point in the domain. If all these sequences of real numbers are convergent, then their limits define the pointwise limit of the given sequence of functions. We have expressed this using symbolic language towards the end of last section. A sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f on its domain, if for a given $\varepsilon > 0$, for every x in the domain, there exists a natural number, $N(\varepsilon, x)$, such that

$$n \geq N(\varepsilon, x) \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

We had also observed that $N(\varepsilon, x)$ depends not only on ε, but also on x, for instance, in the case of $f_n(x) = x^n$. This implies that even though $(f_n(x)) \to f(x)$ for every x, the 'speed' of convergence of each sequence may not be the same, For instance, $\dfrac{1}{3^n} \to 0$ faster than $\dfrac{1}{2^n} \to 0$.

But there are cases, where this is the same. That is, $N(\varepsilon, x)$ depends only on ε, and not on x. For example, if you take the sequence in E2) ii), we see that each of the sequences, $\left(\dfrac{\sin nx}{n}\right)_{n\in\mathbb{N}}$ is dominated by the sequence $\left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$.

For a given $\varepsilon > 0$ then we choose $n_0 > \dfrac{1}{\varepsilon}$, then $n \geq n_0 \Rightarrow \left|\dfrac{1}{n} - 0\right| = \dfrac{1}{n} < \dfrac{1}{n_0} < \varepsilon$.

This same n_0 works for the sequences for all x.

This type of convergence is called uniform convergence.

Here is the definition:

Definition: A sequence $(f_n)_{n\in\mathbb{N}}$ of real valued functions defined on a set S is said to **converge uniformly** to a function f on S, if for every $\varepsilon > 0$, there exists a natural number n_0 such that $n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon$ for every $x \in S$.

The number n_0 here depends only on ε, and not on x. It is the same for all $x \in S$.

In this case we say that $(f_n)_{n\in\mathbb{N}}$ is **uniformly convergent** to f on S.

(Do you see a similarity between this and pointwise continuity which we have discussed in Unit 10, Block-3).

From the definition it is obvious that **if a sequence of functions is uniformly convergent, then it is also pointwise convergent**. And we have also observed that every pointwise convergent sequence need not be uniformly convergent as discussed in the earlier section. We now present a few examples to bring out this point. In each of these cases we shall first get the pointwise limit of the given sequence of functions, and then find out if the sequence uniformly converges to that limit or not.

Before we come to the examples, let us see what it means to say that a given sequence is not uniformly convergent.

Remark 1: The negation of the statement given in the definition of uniform convergence is that there exists an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, there exists an $n_k \geq k$ and an $x_k \in S$ such that

$$\left|f_{n_k}(x_k) - f(x_k)\right| \geq \varepsilon \qquad \dots (1)$$

This shows that there exists an $\varepsilon > 0$ such that for each n_k we get f_{n_k} and $x_k \in S$ satisfying (1). Here $(f_{n_k})_{k\in\mathbb{N}}$ is a subsequence of the given sequence $(f_n)_{k\in\mathbb{N}}$ and $(x_{n_k})_{k\in\mathbb{N}}$ is sequence in S.

More precisely when a sequence is not uniformly convergent, we get a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of the given sequence, $(f_n)_{n\in\mathbb{N}}$, and a sequence $(x_k)_{k\in\mathbb{N}}$ in the domain, such that , $\left|f_{n_k}(x_k) - f(x_k)\right| \geq \varepsilon$ for some $\varepsilon > 0$.

Example 3: Examine the sequences given in Example 2 for uniform convergence.

Solution: i) $f_n : [0,1] \to \mathbb{R}, f_n(x) = x^n$. We have seen that the pointwise limit of this sequence is the function, $f : [0,1] \to \mathbb{R}, f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$.

If we take $\varepsilon = \dfrac{1}{4}$, then taking $n_k = k$, and $x_k = \left(\dfrac{1}{2}\right)^{1/k}$, we get a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of the given sequence and a sequence $(x_k)_{k\in\mathbb{N}}$ in the domain, such that,

$$\left|f_{n_k}(x_k)-f(x_k)\right|=\left|f_k\left(\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)-f\left(\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)\right|$$

$$=\left|\frac{1}{2}-0\right|=\frac{1}{2}>\varepsilon.$$

Therefore, by Remark 1, the sequence of functions is not uniformly convergent.

ii) $f_n:]0,\infty[\to\mathbb{R}, f_n(x)=\frac{x}{n}$. We have seen that the pointwise limit of this sequence is the function, $f:]0,\infty[\to\mathbb{R}$ such that $f(x)=0\ \forall\ x\in]0,\infty[$.

Here, if we take $\varepsilon=\frac{1}{2}$, then taking $n_k=k$, and $x_k=k$, we get a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of the given sequence and a sequence $(x_k)_{k\in\mathbb{N}}$ in the domain, such that , $\left|f_{n_k}(x_k)-f(x_k)\right|=1>\varepsilon$. Therefore, by Remark 1, the sequence of functions is not uniformly convergent.

iii) $f_n:[0,\infty[\to\mathbb{R}, f_n(x)=\frac{1}{1+x^n}$. We have seen that the pointwise limit of this sequence is the function, $f:[0,\infty[\to\mathbb{R}, f(x)=\begin{cases}1, & 0\le x<1\\ \frac{1}{2}, & x=1\\ 0, & x>1.\end{cases}$

If we take $\varepsilon=\frac{1}{4}$, then taking $n_k=k$, and $x_k=2^{1/k}$, we get a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of the given sequence and a sequence $(x_k)_{k\in\mathbb{N}}$ in the domain, such that , $\left|f_{n_k}(x_k)-f(x_k)\right|=\left|\frac{1}{1+\left(2^{1/k}\right)^k}-0\right|=\frac{1}{3}>\varepsilon$. Therefore, by Remark 1, the sequence of functions is not uniformly convergent.

iv) $f_n:[0,1]\to\mathbb{R}, f_n(x)=\frac{x^n}{1+x^n}$. We have seen that the pointwise limit of this sequence is the function, $f:[0,1]\to\mathbb{R}, f(x)=\begin{cases}0, & 0\le x<1\\ \frac{1}{2}, & x=1.\end{cases}$

Again, if we take $\varepsilon=\frac{1}{4}$, then taking $n_k=k$, and $x_k=\frac{1}{2^{1/k}}<1$, we get a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of the given sequence and a sequence $(x_k)_{k\in\mathbb{N}}$ in the domain, such that, $\left|f_{n_k}(x_k)-f(x_k)\right|=\left|\frac{\left(\frac{1}{2^{1/k}}\right)^k}{1+\left(\frac{1}{2^{1/k}}\right)^k}-0\right|=\frac{1/2}{1+\frac{1}{2}}=\frac{1}{3}>\varepsilon$.

Therefore, by Remark 1, the sequence of functions is not uniformly convergent.

Thus, all the sequences in this example are pointwise convergent, but not uniformly convergent.

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In the next example we give some sequences which are uniformly convergent.

Example 4: Show that the following sequences $(f_n)_{n\in\mathbb{N}}$ are uniformly convergent on their domains.

i) $f_n : [0,\infty[\to \mathbb{R}, f_n(x) = \dfrac{x}{1+nx}$

ii) $f_n : [0,1] \to \mathbb{R}, f_n(x) = \dfrac{\sin nx}{n^2}$

Solution: i) Here you can easily check that $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent to the function $f : [0,\infty[\to \mathbb{R}, f(x) = 0$. Now, for any $x \in [0,\infty[,\ |f_n(x)| = \dfrac{x}{1+nx} \le \dfrac{1}{n}$, and the sequence $\left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$ tends to zero.

Therefore, for a given $\varepsilon > 0$, we can choose n_0, such that $n \ge n_0 \Rightarrow \left|\dfrac{1}{n}\right| < \varepsilon.$

Then, for every $x \in [0,\infty[, |f_n(x)| = \dfrac{x}{1+nx} \le \dfrac{1}{n} < \varepsilon,$ whenever $n \ge n_0$.

This shows that $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to the zero function.

ii) We are going to apply a similar argument here. You can check that this sequence of functions converges pointwise to the function,

$f : [0,1] \to \mathbb{R}, f(x) = 0$.

Now, for every $x \in [0,1],\ |f_n(x)| = \left|\dfrac{\sin nx}{n^2}\right| \le \dfrac{1}{n^2}$, and the sequence $\left(\dfrac{1}{n^2}\right)_{n\in\mathbb{N}}$ tends to zero.

Therefore, for a given $\varepsilon > 0$, we can choose n_0, such that $n \ge n_0 \Rightarrow \left|\dfrac{1}{n^2}\right| < \varepsilon$.

Then, for every $x \in [0,1], |f_n(x)| = \left|\dfrac{\sin nx}{n^2}\right| \le \dfrac{1}{n^2} < \varepsilon,$ whenever $n \ge n_0$.

So $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to the function, $f : [0,1] \to \mathbb{R}, f(x) = 0$.

***

In both the cases in the Example 4 each of the functions, f_n, was a bounded function, and their bounds formed a convergent sequence.

We now introduce the concept of uniform norm for bounded functions.

Definition: A function, $g : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}$, is said to be **bounded**, if the set $g(S)=\{g(x) \mid x\in S\}$ is a bounded subset of $\mathbb{R}$. If g is bounded, then $\|g\|=\sup\{|g(x)| \, |x\in S\}$ is called the **norm** of g.

You would agree that $\|g\|\le \varepsilon \Leftrightarrow |g(x)|\le \varepsilon$ for all $x\in S$.

The following theorem gives us a criterion for uniform convergence.

Theorem 1: A sequence of bounded functions, $(f_n)_{n\in\mathbb{N}}$, defined on $S\subseteq\mathbb{R}$ converges uniformly on S to a function $f: S\to\mathbb{R}$, if and only if, $\|f_n - f\|\to 0$.

Proof: We start with the 'if part'. First let us suppose that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. This means that if for every $\varepsilon>0$, there exists a natural number, n_0, such that

$n\ge n_0 \Rightarrow |f_n(x)-f(x)|<\frac{\varepsilon}{2}$ for every $x\in S$.

This implies that $\|f_n - f\|=\sup\{|f_n(x)-f(x)| \, |x\in S\}\le\frac{\varepsilon}{2}<\varepsilon$ for all $n\ge n_0$.

That is, $\|f_n - f\|\to 0$.

Now we prove the 'only if' part.

We suppose that $\|f_n - f\|\to 0$ as $n\to\infty$. Then for every $\varepsilon>0$, there exists a natural number, n_0, such that $n\ge n_0\Rightarrow \|f_n - f\|<\varepsilon$.

Now, $\|f_n - f\|=\sup\{|f_n(x)-f(x)| \, |x\in S\}<\varepsilon \Rightarrow |f_n(x)-f(x)|<\varepsilon$ for every $x\in S$, which means that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. ■

In Block 2 you have learned that there is a strong relationship between convergence of sequences and sequences being Cauchy. Here we shall discuss a concept analogous to Cauchy sequences for a sequence of functions.
We shall make a definition.

Definition: A sequence of functions, $(f_n)_{n\in\mathbb{N}}$ defined on a set S, is **uniformly Cauchy**, if $\forall\varepsilon>0, \exists n_0\in\mathbb{N}$, such that $n,m\ge n_0\Rightarrow |f_n(x)-f_m(x)|<\varepsilon$, for every $x\in S$.

Next we shall prove a theorem which gives a useful characterisation of uniform convergence in terms of Cauchy sequences.

Theorem 2 (Cauchy Criterion): If $(f_n)_{n\in\mathbb{N}}$ is sequence of real-valued functions defined on a set S, then $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on S, if and only if it is uniformly Cauchy.

Proof: We start with 'if' part.

Suppose $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to f. Then, given $\varepsilon>0, \exists n_0$, such that $n\ge n_0\Rightarrow |f_n(x)-f(x)|<\frac{\varepsilon}{2}$ for every $x\in S$. Then

$$n, m \geq n_0 \Rightarrow |f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy.

Only if part

Conversely, suppose $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy. Then for every $x \in S$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers, and therefore, is convergent. We define $f(x) = \lim_{n\to\infty} f_n(x),\ x \in S$.

We now show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on S.

Given $\varepsilon > 0, \exists n_0$, such that $n, m \geq n_0 \Rightarrow |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$, for every $x \in S$. We fix n, and take the limit as $m \to \infty$

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|.$$

Therefore, $n \geq n_0 \Rightarrow |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$, for every $x \in S$.

This means that $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent. ■

Sometimes we come across sequences of functions, which are not uniformly convergent on the given domain, but which may be uniformly convergent on a restricted domain. You can see one such sequence in the next example.

Example 5: Show that the sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n : [0,\infty[\to \mathbb{R}, f_n(x) = \frac{x}{x+n}$, is not uniformly convergent on $]0,\infty[$, but is uniformly convergent on $[0,5]$.

Solution: For a fixed x, $f_n(x) = \frac{x}{x+n} \to 0$ as $n \to \infty$. Therefore, the pointwise limit of the given sequence is the zero function, defined by $f : [0,\infty[\to \mathbb{R}, f(x) = 0$.

If we take $\varepsilon = \frac{1}{4}$, then taking $n_k = k$, and $x_k = k$, we get a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of the given sequence and a sequence $(x_k)_{k\in\mathbb{N}}$ in the domain, such that, $|f_{n_k}(x_k) - f(x)| = \frac{k}{k+k} = \frac{1}{2}$. Therefore, by Remark 1, $(f_n)_{n\in\mathbb{N}}$ is not uniformly convergent.

Now, if we restrict the domain to $[0,5]$, then

$$\|f_n - f\| = \|f_n\| = \sup\left\{\frac{x}{x+n} \mid x \in [0,5]\right\} = \frac{5}{5+n} \leq \frac{5}{n}.$$

And therefore, $\|f_n - f\| \to 0$ on $[0,5]$. Then, by Theorem 1, we conclude that $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on $[0,5]$.

In this case do you realise that the sequence will be uniformly convergent on $[0, a]$, where $a \in \mathbb{R}^+$?

***

We now give one more example of this type.

Example 6: We have shown in Example 3 iv), that the sequence of functions, $(f_n)_{n\in\mathbb{N}}$, where $f_n : [0,1] \to \mathbb{R}, f_n(x) = \frac{x^n}{1+x^n}$, is not uniformly convergent on $[0,1]$. Show that if $0 < b < 1$, then the sequence is uniformly convergent on $[0, b]$.

Solution: We have seen in Example 3 (iv), that the pointwise limit of this sequence is $f : [0,1] \to \mathbb{R}, f(x) = \begin{cases} 0, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1 \end{cases}$.

Now, if $x \in [0, b]$, then the pointwise limit on this restricted domain is $f : [0, b] \to \mathbb{R}, f(x) = 0.$

Then, $\|f_n - f\| = \|f_n\| = \sup\left\{\frac{x^n}{1+x^n} \mid x \in [0, b]\right\} \le b^n$

Since $0 < b < 1$, therefore, $b^n \to 0$ which implies $\|f_n - f\| \to 0$. Thus, we can conclude that is uniformly convergent on $[0, b]$.

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Now it's time you do some exercises.

E5) Show that the sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n : [0, \infty[\to \mathbb{R}, f_n(x) = \frac{nx}{1+nx}$, is not uniformly convergent on $[0, \infty[$, but is uniformly convergent on $[a, \infty[$, where $a > 0$.

E6) Show that the sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n : [0, \infty[\to \mathbb{R}, f_n(x) = \frac{\sin nx}{1+nx}$, is not uniformly convergent on $[0, \infty[$, but is uniformly convergent on $[a, \infty[$, where $a > 0$.

E7) Show that the sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n : \mathbb{R} \to \mathbb{R}, f_n(x) = x + \frac{1}{n}$, is uniformly convergent on $\mathbb{R}$.

E8) Is the convergence of $(f_n)_{n\in\mathbb{N}}$, where $f_n : [0,1] \to \mathbb{R}, f_n(x) = \frac{1}{1+nx}$ uniform? Justify your answer.

We have observed earlier that a pointwise limit of a sequence of continuous functions need not be continuous. But many of the properties of functions are preserved under uniform convergence. We deal with this in the next section.

17.4 SOME THEOREMS ON CONSEQUENCES OF UNIFORM CONVERGENCE

In this section we shall see that some properties of functions in a sequence are carried over to the uniform limit of the sequence, of course, if the limit exists.

We shall start with the property of "boundedness".

A sequence $(f_n)_{n\in\mathbb{N}}$ is **uniformly bounded** on set S, if there is some $M > 0$ such that $\|f_n\| \le M \ \ \forall\, n$ and $\forall\, x \in S$.

Theorem 3: If $(f_n)_{n\in\mathbb{N}}$ is a sequence of uniformly bounded real valued functions converging uniformly to a function f on a set S, then f is also bounded on S.

Proof: Since $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, for $\varepsilon = 1$, there exists $n_0 \in \mathbb{N}$, such that $n \ge n_0 \Rightarrow |f_n(x) - f(x)| < 1$ for every $x \in S$. Then $|f(x)| \le |f(x) - f_{n_0}(x)| + |f_{n_0}(x)| \le 1 + M$, where M is an upper bound of f_{n_0} on S.

Therefore, we have proved that f is bounded on S. ■

Next we shall consider the property of "Continuity".

Theorem 4: If $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions converging uniformly to a function f on a set S, then f is also continuous on S.

Proof: Let $p \in S$. We shall first prove that f is continuous at p.
Since $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, for every $\varepsilon > 0, \exists n_0$, such that

$$n \ge n_0 \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3} \text{ for every } x \in S.$$

Since f_{n_0} is continuous at p, for the above said $\varepsilon > 0, \exists \delta > 0$, such that

$$\begin{aligned} x \in S, |x - p| < \delta \Rightarrow & |f(x) - f(p)| \\ & \le |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(p)| + |f_{n_0}(p) - f(p)| \\ & < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ & = \varepsilon \end{aligned}$$

This means that f is continuous at p. Since p was any arbitrary point of S, f is continuous on S. ■

We shall make a remark now.

Remark 2: Actually what we have proved in the theorem above is that the limit function f is continuous. That is

$$\lim_{x\to p} f(x) = f(p)$$

This together with the fact that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly implies that

$$\lim_{x\to p}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(p) = \lim_{n\to\infty}\lim_{x\to p} f_n(x).$$

This amounts to interchanging the order in which the limits are taken. Thus, **uniform convergence allows us to interchange the order in which the limits are taken**.

So, we have seen that if a sequence of continuous functions is uniformly convergent, then its limit is also continuous.

Using this theorem we can see that the convergence in Example i), iii) and iv) cannot be uniform. Because in each of these cases we have a sequence of continuous functions, whose limit is not continuous.

So, we have proved that continuity is preserved under uniform convergence. Now what about differentiability and integrability?

If differentiation was preserved by uniform convergence, then we should have

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty}\left(\frac{d}{dx} f_n(x)\right) = \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) = \frac{d}{dx} f(x) = f'(x).$$

But the following example shows that this is not true.

Example 7: Let $f_n(x) = \dfrac{x}{1+nx^2}$ on $\mathbb{R}$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ convergence to $f(x)=0$ uniformly on $\mathbb{R}$, but $(f_n')_{n\in\mathbb{N}}$ does not converge to f' uniformly.

Solution: It is easy to see that the maximum and minimum values of f_n are $\dfrac{1}{2\sqrt{n}}$ and $-\dfrac{1}{2\sqrt{n}}$, respectively. That is $-\dfrac{1}{2\sqrt{n}} \le f_n(x) \le \dfrac{1}{2\sqrt{n}}\ \forall\, x\in\mathbb{R}$. Thus for $\varepsilon>0$, we take $n_0 = \left[\dfrac{1}{4\varepsilon^2}\right]$ so that $n \ge n_0 \Rightarrow |f_n(x)| < \varepsilon, \forall\, x \in \mathbb{R}$. Therefore, $\left(f_n(x)_{n\in\mathbb{N}}\right)$ is uniformly convergent to the zero function.

We differentiate f_n and find

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} = \frac{1-nx^2}{1+2nx^2+n^2x^4}$$

If $x \ne 0$, then $f_n'(x) \to 0$ while if $x=0$, then $f_n'(0)=1\to 1$. Thus

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0 & \text{if} \quad x \ne 0 \\ 1 & \text{if} \quad x = 0. \end{cases}$$

The following figure gives an idea how $f_n'(0)\to 1$ as $n\to\infty$. But for $x\ne 0$, $f_n'(x)\to 0$ as $n\to\infty$. You may note that for $x\ne 0$, the slope (for example for $x=6$ in the figure) of the function (i.e. the inclination with the x-axis) is approaching to 0 where as for $x=0$, the slope approaches one i.e. the inclination is more towards y-axis.

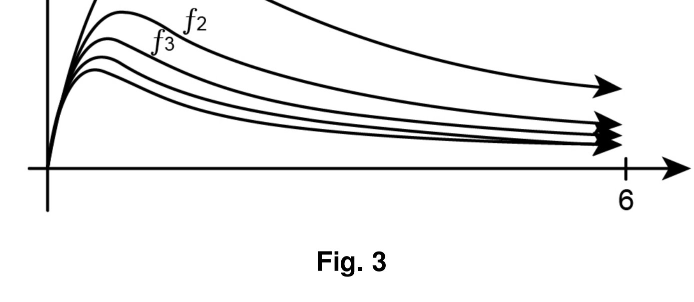

Fig. 3

This shows that $\lim_{n\to\infty} f_n'(x)$ is not equal to $f'(x)=0$ for all x. That means $f_n'(x)$ does not even converge to $f'(x)=0$ pointwise. Obviously, then the convergence is not uniform.

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Remark 3: The example above shows that the uniform convergence of $(f_n)_{n\in\mathbb{N}}$ to f and the differentiability of the members of the sequence is not enough to deduce that $(f_n')_{n\in\mathbb{N}}$ converge to f', even pointwise.

Actually, the uniform limit of a sequence of differentiable functions need not be differentiable. But the uniform limit of a sequence of integrable functions is integrable.

In the case of a sequence of differentiable functions, if we put some additional condition, then the uniform limit does become differentiable.

We present these facts in the following theorems. We are not going to prove these theorems here.

Theorem 5: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined and differentiable on a bounded interval, I, of $\mathbb{R}$. Suppose the sequence $(f_n(x_0))_{n\in\mathbb{N}}$ converges for some $x_0 \in I$. Suppose further, that the sequence $\left(f_n'\right)_{n\in\mathbb{N}}$ converges uniformly to a function, g on I. Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function f on I, such that f is differentiable on I, and $f' = g$.

The next theorem shows that the definition integral is preserved by uniform convergence.

Theorem 6: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined and integrable on an interval, $[a,b]$, of $\mathbb{R}$. Suppose $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function f on $[a,b]$. Then f is integrable on $[a,b]$, and $\int_a^b f = \lim_{n\to\infty}\int_a^b f_n$.

The Theorem above is very useful in showing that the convergence is not uniform as the next example shows:

Example 8: Consider the sequence $(f_n)_{n\in\mathbb{N}}$ where $f_n(x)$ is given by

$$f_n(x) = nxe^{-nx^2}, n\in \mathbb{N}, x\in [0,1]$$

Show that $(f_n)_{n\in\mathbb{N}}$ is not uniformly convergent on $[0,1]$.

Solution: Let $\varepsilon > 0$ be given. Then given $x \in [0,1]$,

$$\left|n\,xe^{-nx^2}\right| \le n\,e^{-n\,x^2} \qquad \ldots (2)$$

Since $ne^{-nx^2} \to 0$ as $n\to\infty$, there exists N_0 such that $\left|ne^{-nx^2}\right| < \varepsilon$ for $n \ge N_0$. This together with the inequality (2) gives us

$$\left|n\,x\,e^{-n\,x^2}\right| < \varepsilon \qquad \forall\ n \ge N_0.$$

This shows that the sequence $(f_n(x)) \to f(x) = 0$ pointwise. Also we have

$\int_0^1 f(x) = 0.$ Therefore if the convergence is uniform by Theorem 6

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

But $\int_0^1 f_n(x)dx = \frac{1}{2}\left[-e^{-nx^2}\right]_0^1 = \frac{1}{2}(1-e^{-n})$. Therefore

$\lim_{n\to\infty} f_n(x)dx = \frac{1}{2} \neq \int_0^1 f(x)dx = 0.$ This is not possible. Hence the convergence is not uniform.

***

You can try this exercise now.

E9) Show that the sequence $(f_n)_{n\in\mathbb{N}}$ where

$$f_n(x) = \begin{cases} n^2 x, & 0 \le x \le \frac{1}{n} \\ -n^2 x + 2n, & \frac{1}{n} \le x \le \frac{2}{n} \\ 0, & \frac{2}{n} \le x \le 1 \end{cases}$$

does not converge to $f(x) = 0$ uniformly in $[0,1]$.

That brings us to the end of this unit.

17.5 SUMMARY

1. In this unit we have introduced sequences of functions.
2. We have defined two types of convergence: pointwise and uniform.
3. We have noted that if a sequence of functions is uniformly convergent, then it is also pointwise convergent, but the converse is not true.
4. We have proved that the properties of boundedness, continuity, integrability are preserved under uniform convergence. Differentiabillity requires some more restrictions.

17.6 SOLUTIONS/ANSWERS

E1) i) $f_1(x) = 1 + xn,\ f_2(x) = 1 + 2n,\ f_3(x) = 1 + 3n$

The sequence corresponding to the function f_1 at $x = 1$ is $(2,3,4,....)$

The sequence corresponding to the function f_2 at $x = 1$ is $(3,5,7,....)$

The sequence corresponding to the function $f_3(x)$ at $x = 1$ is $(4,7,10,....)$

The sequence corresponding to the function f_1, f_2, and f_3 at $x = 0$ is $(1)_{n\in\mathbb{N}}$.

ii) $f_1(x) = 2 + \frac{1}{x^2}, f_2(x) = 2 + \frac{1}{2x^2}, f_3(x) = 2 + \frac{1}{3x^2}$

Take $x = 1$

Then, the sequence corresponding to f_1 is given by $f_1(1) = 3$,

$f_1(2) = 2 + \frac{1}{4} = \frac{9}{4},\ f_1(3) = \frac{19}{9}, ..., f_1(n) = 2 + \frac{1}{n^2} + ...$

That is $\left(3, \frac{9}{4}, \frac{19}{9}, ...\right)$

The sequence corresponding to $f_2(x)$ at $x = 1$ is given by

$f_2(1) = 2 + \frac{1}{2} = \frac{5}{2},\ f_2(2) = \frac{17}{8},$

$f_2(3) = 2 + \frac{1}{18} = \frac{37}{18}, ...$

$\therefore$ The sequence is $\left(\frac{5}{2}, \frac{17}{8}, \frac{37}{18}, ...\right)$

Similarly $f_3(1) = 2 + \frac{1}{3} = \frac{7}{3},\ f_3(2) = 2 + \frac{1}{12} = \frac{25}{12}$

$f_3(3) = \frac{55}{27}$

$\left(\frac{7}{3}, \frac{25}{12}, \frac{55}{27}, ...\right)$

E2) i) Here $f_n(x) = \frac{x}{n}$. Then for each $x \in \mathbb{R}, f_n(x) = \frac{x}{n} \to 0$ as $n \to \infty$. Then if we take f such that $f(x) = 0\ \forall\ x \in \mathbb{R}$, we get that $f_n(x) \to f(x)$ pointwise for every $x \in \mathbb{R}$.

ii) Here $f_n(x) = \frac{\sin nx}{n}$. Also for each $x \in \mathbb{R}, |\sin(nx)| \le 1$ and $\frac{1}{n} \to 0$ as $n \to 0$. Therefore $f_n(x) \to 0$ as $n \to \infty$ [Refer Theorem ? Unit 5 Block 2) since $(f_n(x))_{n\in\mathbb{N}}$ is a product of bounded function and a sequence converges to 0.

iii) Here $f_n(x) = \frac{nx}{1 + n^2x^2} = \frac{n}{n^2}\left[\frac{x}{\frac{1}{n^2} + x^2}\right] = \frac{1}{n}\left[\frac{x}{\frac{1}{n^2} + x^2}\right]$

Then if $x \ne 0$, then $f_n(x) \to 0 \cdot \frac{x}{x^2} = 0$ as $n \to \infty$.

If $x = 0$, then $f_n(0) = 0$ for all n. Therefore $f_n(0) \to 0$ as $n \to \infty$.

If we take f such that $f(x) = 0\ \forall\ x \in \mathbb{R}$, then we get that $f_n(x) \to f(x)$ pointwise for all $x \in \mathbb{R}$.

iv) Let $x \in]-1,1[$. We note that $\sin(nx)$ is bounded and that $\frac{x}{n} \to 0$ as $n \to \infty$. Thus, as the product of a bounded sequence and one that converges to 0, we have $\frac{x}{n}\sin(nx) \to 0$. [Refer Theorem ? Unit 5, Block 2]. Therefore

$$f_n(x) = x + \frac{x}{n}\sin(nx) \to x + 0 = x$$

Thus if we take the function f such that $f(x) = x \ \forall \ x \in \mathbb{R}$, then we get that $f_n(x) \to f(x) \ \forall \ x \in]-1,1[$.

E3) Here $f(x) = (\cos \pi x)^n$.

When $x = 1, \cos \pi = (-1)$. Then we know that the sequence $(-1)^n$ is not convergent.

When $0 < x < 1$, then $\cos \pi x$ is positive and $\cos \pi x < 1$. Therefore $(\cos \pi x)^n$ converges.

E4) Here, $f_n(x) = \begin{cases} 1, & \text{if } \ x \in [-n, n] \\ 0, & \text{otherwise} \end{cases}$

Now, as $n \to \infty, [-n, n] \to \mathbb{R}$.

Assume $f(x) = 1$ for all $x \in \mathbb{R}$.

Now, for $f > 0$ and for $x \in \mathbb{R}$ if you choose $N = [|x|] + 1$ (i.e. $x \in [-N, N]$) then $f_n(x) = 1$ for all $n > N$. So, $|f_n(x) - f(x)| = 0 < \varepsilon$ for $n > N$.

Therefore, $f_n(x)$ pointwise converges to $f(x)$ for all $x \in \mathbb{R}$.

E5) Note that for a given $x \in]0, \infty[$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx} = \lim_{n\to\infty} \frac{x}{\frac{1}{n} + x} = 1.$$

Also, $\lim_{n\to\infty} f_n(0) = 0$. Thus $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the function $f : [0, \infty[$ defined by

$$f(x) = \begin{cases} 1, & \text{if } \ x \in]0, \infty[\\ 0, & \text{if } \ x = 0. \end{cases}$$

If $(f_n)_{n\in\mathbb{N}}$ were to converges uniformly to f, then for $\varepsilon = \frac{1}{4}$, we must get some $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{1}{4} \text{ for all } x \in [0, \infty[\text{ and } n \geq n_0.$$

However for $n = n_0$ and $x = \frac{1}{n_0}$, the above inequality becomes

$$\left|\frac{n_0 \cdot \frac{1}{n_0}}{1+n_0 \cdot \frac{1}{n_0}} - 1\right| < \frac{1}{4},\ i.e., \frac{1}{2} < \frac{1}{4},$$

which is false. Therefore, $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly to f on $[0, \infty[$.

Now consider the domain $[a, \infty[$, for $a > 0$. In this case, $f : [a, \infty[\to \mathbb{R}$ defined by $f(x) = 1$ is the pointwise limit of $(f_n)_{n\in\mathbb{N}}$. Let $\varepsilon > 0$ be given. Then

$$\begin{aligned}|f_n(x) - f(x)| = \left|\frac{nx}{1+nx} - 1\right| &= \frac{1}{1+nx} \\ &\le \frac{1}{nx} \\ &\le \frac{1}{n\,a} \qquad (\because x \ge a)\end{aligned}$$

Now if we take $n_0 \in \mathbb{N}$ in such a way that $\frac{1}{n_0 a} < \varepsilon, i.e., n_0 > \frac{1}{a\,\varepsilon}$, then we have $|f_n(x) - f(x)| < \varepsilon$ for all $n \ge n_0$ and $x \in [a, \infty[$. This implies that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a, \infty[$.

E6) Note that for all $x \in]0, \infty[$

$$\left|\frac{\sin nx}{1+nx}\right| \le \frac{1}{1+nx}.$$

We know that $\lim\limits_{n\to\infty} \frac{1}{1+nx} = 0$. Therefore $\lim\limits_{n\to\infty} f_n(x) = 0$, for $x \in]0, \infty[$. Also $\lim\limits_{n\to\infty} f_n(0) = 0$. Thus $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the limit $f : [0, \infty[\to \mathbb{R}$ defined by $f(x) = 0$. Assume, if possible, that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. Then for every $\varepsilon > 0$, we must get some $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \ge n_0 \text{ and } x \in [0, \infty[.$$

Taking $n = n_0$ and $x = \frac{\pi}{2n_0}$ the above inequality reduces

$$\left|\frac{\sin \frac{\pi}{2}}{1+\frac{\pi}{2}}\right| < \varepsilon, \text{i.e., } \frac{2}{3+\pi} < \varepsilon.$$

But ε is any positive real number. Therefore, we have arrived at a contradiction. Thus, $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly on $[0, \infty[$.

Now, let the domain be $[a,\infty[$, for some $a>0.$ Then the pointwise limit is the same. Now take $\varepsilon>0.$ So,

$$\begin{aligned}|f_n(x)-f(x)| = \left|\frac{\sin nx}{1+nx}\right| &\le \frac{1}{1+nx}\\ &\le \frac{1}{nx}\\ &\le \frac{1}{na}\end{aligned}$$

So, if we take $n_0\in\mathbb{N}$ such that $\frac{1}{n_0 a}<\varepsilon,$ i.e., if we choose $n_0\in\mathbb{N}$ such that $n_0>\frac{1}{a\varepsilon}$, then $|f_n(x)-f(x)|<\varepsilon$ for all $n\ge n_0$ and $x\in[a,\infty[.$
Therefore, $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a,\infty[.$

E7) Let $f:\mathbb{R}\to\mathbb{R}$ be define $f(x)=x.$ Then $\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\left(x+\frac{1}{n}\right)=x=f(x).$ Thus f is the pointwise limit of $(f_n)_{n\in\mathbb{N}}$. Now, let $\varepsilon>0$ be arbitrary. Then $|f_n(x)-f(x)|=\left|x+\frac{1}{n}-x\right|=\frac{1}{n}.$

Choose $n_0>\frac{1}{\varepsilon}.$ Then $|f_n(x)-f(x)|<\varepsilon$ for all $n\ge n_0$ and $x\in\mathbb{R}.$
Therefore, $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $\mathbb{R}.$

E8) We have for $x\in]0,1],$ $\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{1}{1+nx}=0,$ and $\lim_{n\to\infty} f_n(0)=0.$

Thus $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}0, & \text{if } \quad x\in]0,1]\\ 1, & \text{if } \quad x=0\end{cases}$$

is the pointwise limit of $(f_n)_{n\in\mathbb{N}}$. Assume, if possible, that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $[0,1].$ Then for every $\varepsilon>0$ we have some $n_0\in\mathbb{N}$ such that

$$|f_n(x)-f(x)|<\varepsilon \text{ for all } n\ge n_0 \text{ and } n\in[0,1].$$

Take $n=n_0$ and $x=\frac{1}{n_0}.$ Then the above inequality reduces to

$$\left|\frac{1}{1+n_0\cdot\frac{1}{n_0}}-0\right|<\varepsilon, \text{ i.e., } \frac{1}{2}<\varepsilon,$$

which does not hold for all positive real numbers $\varepsilon.$ Therefore, $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly on $[0,1].$

E9) From the definition of f_n, we observe that $(f_n)_{n\in\mathbb{N}}$ converges to $f(x)=0$ for all $x\in[0,1]$. Also, it follows that each f_n and f are continuous on $[0,1]$. Also

$$\int_0^1 f_n(x)dx = \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n}(-n^2x+2n)dx + \int_{2/n}^1 0\,dx$$

$$= n^2 x\frac{x^2}{2}\bigg|_0^{1/n} - n^2\frac{x^2}{2}\bigg|_{1/n}^{2/n} + 2nx\big|_{1/n}^{2/n} + 0$$

$$= \frac{1}{2} - \left[2-\frac{1}{2}\right] + 2 = 1$$

But $\int_0^1 f(x)dx = 0$

$$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$$

Hence by Theorem 10, the sequence is not convergent.

UNIT 18

SERIES OF FUNCTIONS

Structure **Page No**

18.1 INTRODUCTION

In the previous unit we have introduced sequences of functions, and discussed the concepts like pointwise and uniform convergence of such sequences. The next step follows naturally now. We take up the study of series of functions. If you take a quick look at the unit on series of real numbers, you will have an idea of how we are going to proceed in this unit.

Here we shall first define in Section 18.2 the concept of a series of functions, and then discuss the convergence or divergence of such series. These concepts are defined with the help of convergence or divergence of sequences. Next, in Section 18, we shall discuss about power series, and shall study some of their properties.

Objectives

After studying this unit, you should be able to

- define partial sums for a given series of functions;
- decide if the given series is pointwise convergent;
- describe the concept of uniform convergence (of a series of functions), and distinguish it from the pointwise convergence;
- show how uniform convergence establishes the continuity of the limit of a series of continuous functions;
- discuss the termwise integration and differentiation of uniformly convergent series of functions;
- define a power series, and discuss its convergence.

18.2 CONVERGENCE OF SERIES OF FUNCTIONS

In this section, we shall introduce the concept of series of functions. You are already familiar with the series of real numbers. Recall that the sum of two real valued functions f and g defined on a subset S of $\mathbb{R}$ is the function $f+g:S\to\mathbb{R}$ defined by $(f+g)\ (x)=f(x)+g(x)$.

This operation naturally extends to any finite number of functions. That is, if $f_1, f_2, f_3, ..., f_n$ are functions from S to $\mathbb{R}$, then $f_1+f_2+f_3+...+f_n : S\to\mathbb{R}$ is defined by

$$(f_1+f_2+f_3+...+f_n)\ (x)=f_1(x)+f_2(x)+f_3(x)+...+f_n(x).$$

Now look at the following definition.

Definition: If $(f_1, f_2, f_3, ...,)$ is a sequence of functions defined on a subset S of $\mathbb{R}$, then the expression $f_1+f_2+f_3+...$, or $\sum_{n=1}^{\infty} f_n$ is called a **series of functions.**

Note that this expression may not be a function. This is because it involves taking a sum of infinitely many functions. We decide about the convergence of such a series by considering the sequence of its partial sums.

Definition: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real valued functions defined on a subset S of $\mathbb{R}$. We form the sequence of **partial sums**, $(s_n)_{n\in\mathbb{N}}$ of the series $\sum_{n=1}^{\infty} f_n$, where, $s_1=f_1, s_2=f_1+f_2, s_3=f_1+f_2+f_3, ...$

i) If $(s_n)_{n\in\mathbb{N}}$, converges to f pointwise on S, we say that $\sum_{n=1}^{\infty} f_n$ is **pointwise convergent** to f on S.

ii) If $(s_n)_{n\in\mathbb{N}}$, converges uniformly to f on S, we say that $\sum_{n=1}^{\infty} f_n$ is **uniformly convergent** to f on S.

iii) If $(s_n)_{n\in\mathbb{N}}$ does not converge, then we say that $\sum_{n=1}^{\infty} f_n$ is **divergent**.

So, you see, the convergence or divergence of a series of functions is defined in terms of the convergence or divergence of the associated sequence of partial sums. As a result, we can easily carry over the results on sequences of functions to series of functions. But before that, here are a few examples.

Example 1: Find if the series of functions, $\sum f_n$, is pointwise convergent, where

i) $f_n(x)=x^n, x\in\mathbb{R}, n\in\mathbb{N}$

ii) $f_n(x)=x(1-x)^n, x\in[0,1], n=0,1,2,\dots.$

Solution: i) Note that $\sum_{n=1}^{\infty} f_n(x)=\sum_{n=1}^{\infty} x_n$ is the geometric series with common ratio, x. We know that this series is convergent for $|x|<1,$ and divergent for $|x|\geq 1.$ And if $|x|<1,$ the sum of the series is $\frac{x}{1-x}.$

So, the given series of functions converges pointwise on $]-1,1[$ to the function, $f:]-1,1[\to\mathbb{R}, f(x)=\frac{x}{1-x}.$

ii) Here $\sum_{n=0}^{\infty} f_n(x)=\sum_{n=0}^{\infty} x(1-x)^n = x\sum_{n=0}^{\infty}(1-x)^n.$

Now, $f_n(0)=0,$ and $f_n(1)=0, \forall n.$ Therefore, both the series $\sum_{n=0}^{\infty} f_n(0),$ and $\sum_{n=0}^{\infty} f_n(0)$ converge to 0. When $0<x<1,$ then we have $0<1-x<1$.

Now, $\sum_{n=0}^{\infty}(1-x)^n$ is a geometric series, with common ratio $1-x$. Threfore, it converges to $\frac{1}{1-(1-x)}=\frac{1}{x}$. Consequently, $\sum_{n=0}^{\infty} f_n(x)$ converges to 1 for all $0<x<1$.

Thus, the given series of functions converges pointwise to the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}0, \text{ if } x=0\\ 1, \text{ if } 0<x<1.\\ 0, \text{ if } x=1\end{cases}$$

***

In both the series in the example above, we have used our knowledge of the sum of a geometric series to decide the pointwise convergence of the given series, rather than the definition.

Now are these series uniformly convergent?

To find out, we make use of the theorems we have proved in Unit 17. Consider part (i), first. Each of the functions, $f_n(x)=x^n, x\in]-1,1[$ is bounded. So, the partial sums, which are finite sums of these functions, are also bounded. But the pointwise limit of the sequence of partial sums, that is, $f:]-1,1[\to\mathbb{R}, f(x)=\frac{1}{1-x},$ is not bounded on $]-1,1[$. So, applying Theorem 4 of Unit 17, we find that the convergence cannot be uniform.

In part (ii) the given functions are all continuous on the given domain. So, the sequence of partial sums also consists of continuous functions. But as you can see, the pointwise limit is not continuous. So, applying Theorem 5 of Unit 17, we conclude that the convergence is not uniform.

Using similar arguments we prove the next two theorems.

Theorem 1: If for every $n\in\mathbb{N}$, f_n is a real-valued, bounded function defined on a subset S of $\mathbb{R}$, and if the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f:S\to\mathbb{R}$, then f is bounded on S.

Proof: Since f_n is bounded for every $n \in \mathbb{N}$, the partial sum, $s_n = f_1 + f_2 + f_3 + ... + f_n$ is also bounded on S. Since $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on S, the sequence of partial sums $(s_n)_{n\in\mathbb{N}}$ also converges uniformly to f on S. Therefore, by Theorem 3 of Unit 17, f is bounded on S. ■

Theorem 2: If f_n is a real-valued, continuous function defined on a subset S of $\mathbb{R}$, for every $n \in \mathbb{N}$, and if the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f : S \to \mathbb{R}$, then f is continuous on S.

Proof: Since f_n is continuous for every $n \in \mathbb{N}$, the partial sum, $s_n = f_1 + f_2 + f_3 + ... + f_n$ is also continuous on S. Since $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on S, the sequence of partial sums $(s_n)_{n\in\mathbb{N}}$ also converges uniformly to f on S. Therefore, by Theorem 4 of Unit 17, f is continuous on S. ■

Remark 1: If $(s_n)_{n\in\mathbb{N}}$, the sequence of partial sums of a series $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f : S \to \mathbb{R}$, then using Remark 2 of Unit 17, we can say that $\lim_{x\to p} \lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} s_n(p) = \lim_{n\to\infty} \lim_{x\to p} s_n(x)$.

So, if a series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f, then we can write,

$$\lim_{x\to p} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n(p) = \sum_{n=1}^{\infty} \lim_{x\to p} f_n(x).$$

Thus, if the given series is uniformly convergent, we can take the limit term by term.

We now state two theorems dealing with differentiation and integration of a uniformly convergent series. As in the case of Theorems 1 and 2, the proofs of these theorems depend upon the corresponding theorems in Unit 17.

Theorem 3 (Termwise differentiation): Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined and differentiable on a closed and bounded interval, I, of $\mathbb{R}$. Suppose the series $\sum f_n(x_0)$ converges for some $x_0 \in I$. Suppose further, that the series $\sum f_n'$ converges uniformly on I. Then $\sum f_n$ converges uniformly to a function f on I such that f is differentiable on I, and $f' = \sum f_n'$.

Theorem 4 (Termwise integration): Suppose the real-valued functions, $f_n, n \in \mathbb{N}$, are integrable on an interval, $[a,b]$, and suppose the series $\sum f_n$ converges uniformly to a function, f, on $[a,b]$. Then f is integrable on $[a,b]$, and $\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx.$

We now discuss a simple test, which helps us decide whether a given series of functions is uniformly convergent or not. This test was devised by Weierstrass, and is called Weierstrass M-test. It is based on the knowledge of convergent series of numbers, which you have studied in Block 3.

Theorem 5 (Weierstrass' M-Test): Suppose $\sum f_n$ is a series of functions defined on a set $S \subseteq \mathbb{R}$. If $\sum u_n$ is a convergent series of positive real numbers, such that $| f_n (x) | \le u_n$ for all $x \in S, n \in \mathbb{N}$, then $\sum f_n$ is uniformly convergent on S.

Proof: Since $\sum u_n$ is convergent, its sequence of partial sums, $(s_n)_{n\in\mathbb{N}}$, where $s_n = u_1 + u_2 + ... + u_n$, is convergent. Hence, $(s_n)_{n\in\mathbb{N}}$ is cauchy. Therefore, for a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$n > m \ge n_0 \Rightarrow |s_n - s_m| < \varepsilon$$
$$\Rightarrow |u_{m+1} + u_{m+2} + ... + u_n| < \varepsilon.$$

Now, since $|f_n(x)| \le u_n$, we have

$$|f_{m+1}(x) + f_{m+2}(x) + ... + f_n(x)| \le |u_{m+1} + u_{m+2} + ... + u_n|, \forall x \in S, n, m \in \mathbb{N}.$$

Therefore,

$$n > m \ge n_0 \Rightarrow |f_{m+1}(x) + f_{m+2}(x) + ... + f_n(x)| < \varepsilon \text{ for all } x \in S.$$

This means that the sequence of partial sums of $\sum f_n$ is uniformly Cauchy and hence uniformly convergent. (See Theorem 2 of Unit 17.) This means that $\sum f_n$ is uniformly convergent. ■

Here are a few examples to illustrate how these theorems can be applied.

Example 2: Let $f_n(x) = \dfrac{\sin nx}{n!}$ for $0 \le x \le \pi$. Show that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[0, \pi]$. If $f(x) = \sum_{n=1}^{\infty} f_n(x)$, evaluate $\int_0^{\pi} f(x) dx$.

Solution: Since $|\sin nx| \le 1$, we have $\left|\dfrac{\sin nx}{n!}\right| \le \dfrac{1}{n!}$ for all $x \in [0, \pi]$ and $n \in \mathbb{N}$.

Also, recall that $\sum_{n=1}^{\infty} \dfrac{1}{n!}$ converges. Therefore, by Weierstrass' M-test, we conclude that $\sum_{n=1}^{\infty} f_n$ converges uniformly to a limit f. Uniform continuity allows the use of term by term integration by Theorem 4.

Therefore,

$$\int_0^{\pi} f(x)\, dx = \sum_{n=1}^{\infty} \int_0^{\pi} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_0^{\pi} \frac{\sin nx}{n!} dx = \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\pi} \sin nx dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!n}[1-(-1)^n]$$

$$= 2\left(1 + \frac{1}{3!3} + \frac{1}{5!5} + \ldots\right).$$

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Example 3: If $f_n(x) = \frac{x^2}{(1+x^2)^n}, 0 \le x \le 1$, show that $\sum f_n$ is pointwise convergent on $[0,1]$. Is it uniformly convergent on $[0,1]$?

Solution: First, let us consider the case $x = 0$. We have $f_n(0) = 0$ for all $n \in \mathbb{N}$. Therefore, $\sum_{n=1}^{\infty} f_n(0)$ converges to 0. Now let $0 < x \le 1$. Then $0 < \frac{1}{1+x^2} < 1$. Therefore, $(f_n(x))^{\frac{1}{n}} = \frac{x^{2/n}}{1+x^2} < 1$ for all $n \in \mathbb{N}$.

Consequently, by the Root Test $\sum_{n=1}^{\infty} f_n(x)$ converges for each x in $]0,1]$. That is, $\sum_{n=1}^{\infty} f_n$ is pointwise convergent.

The sequence of partial sums of the series is $(s_n)_{n\in\mathbb{N}}$, where

$$s_n(x) = \sum_{k=1}^{n} f_k(x)$$

$$= x^2\left[\frac{1}{(1+x^2)} + \frac{1}{(1+x^2)^2} + \cdots + \frac{1}{(1+x^2)^n}\right]$$

$$= x^2 \frac{\frac{1}{(1+x^2)}\left[1 - \frac{1}{(1+x^2)^n}\right]}{1 - \frac{1}{(1+x^2)}}$$

$$= 1 - \frac{1}{(1+x^2)^n}, \text{ for } x \in]0,1].$$

Now $\lim_{n\to\infty} s_n(x) = 1$ for all $x \in]0,1]$. Therefore, the pointwise limit of the series is the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } \quad x = 0 \\ 1, & \text{if } \quad x \ne 0. \end{cases}$$

The series is not uniformly convergent, because each f_n is continuous on $[0,1]$, but f is not.

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Example 4: Show that the series $\sum \frac{\sin(n^4 x)}{n^2}$ is uniformly convergent on $[0, \infty[$.

Solution: Here $|f_n(x)| = \left|\frac{\sin(n^4 x)}{n^2}\right| \le \frac{1}{n^2}$, for all $x \in [0, \infty[$ and all $n \in \mathbb{N}$. We know that the series of real numbers $\sum \frac{1}{n^2}$ is convergent. Therefore, by Weierstrass' M-test, we conclude that the given series of functions is uniformly convergent on $[0,\infty[$.

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Example 5: Find the derivative of $\sum_{n=1}^{\infty} \frac{1}{n^3+x^2}, x \in [0,b] \subseteq \mathbb{R}$, if possible.

Solution: Here, $|f_n(x)| = \left|\frac{1}{n^3+x^2}\right| \leq \frac{1}{n^3}$, and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent series of real numbers. Therefore, the given series is uniformly convergent on $[0,b]$.

Now, $f_n'(x) = \frac{-2x}{(n^3+x^2)^2}$, and $\sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{-2x}{(n^3+x^2)^2}$. Note that

$|f_n'(x)| = \left|\frac{-2x}{(n^3+x^2)^2}\right| \leq \frac{2b}{n^6}$, for all $x \in [0,b]$ and $n \in \mathbb{N}$.

We know that the series $2b\sum_{n=1}^{\infty} \frac{1}{n^6}$ is convergent. So, by the Weierstrass' M-test we can say that $\sum_{n=1}^{\infty} f_n'(x)$ is uniformly convergent on $[0,b]$.

Now, applying Theorem 3 of Unit 17, we know that this series can be differentiated term by term. So, if

$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^3+x^2}$, then $f'(x) = -2x\sum_{n=1}^{\infty} \frac{1}{(n^3+x^2)^2}$.

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You should be able to try these exercises now.

E1) Let $f_n(x) = \frac{\cos nx}{n^3}$, for $x \in [0,1]$ and $n \in \mathbb{N}$. Let $f(x) = \sum_{n=1}^{\infty} f_n(x), x \in [0,1]$. Show that $\sum_{n=1}^{\infty} f_n$, and $\sum_{n=1}^{\infty} f_n'$ are uniformly convergent on $[0,1]$. Hence, find $f'(x)$.

E2) Show that $\sum_{n=1}^{\infty} \frac{x}{n(x+n)}, x \in [0,10]$ is pointwise convergent on its domain. If $f(x) = \sum_{n=1}^{\infty} \frac{x}{n(x+n)}$, then show that f is differentiable on $[0,10]$. Also find $f'(x)$.

E3) Show that $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ is uniformly convergent on $[0,a]$ for any $a > 0$.

E4) Show that $\sum_{n=1}^{\infty} \frac{1}{1+x^n}$ converges uniformly on $[2,\infty[$. Also show that the series does not converge in $[0,1[$.

We now turn our attention to a special type of series of functions, namely, the power series.

18.3 POWER SERIES

Power series form an important class of series of functions. They can be used to define and study the properties of many well known functions such as sine, conine or natural logarithm. So, let us start with the definition.

Definition: A series of functions, $\sum_{n=0}^{\infty} f_n$ is called a **power series**, if $f_n(x) = a_n (x-c)^n, x \in \mathbb{R}, n = 0,1,2,3,....,$ where c is any real number.

Note that though the functions, $f_n, n = 0,1,2,3,...$ are defined on $\mathbb{R}$ the series may not converge for all $x \in \mathbb{R}$. For example, take the power series, $\sum_{n=0}^{\infty} x^n$. We know that this is a geometric series, and converges if and only if $|x| < 1$. On the other hand, the power series, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$. You can easily check this by using the Ratio Test that you have learnt in Unit 7.

Using Cauchy's nth Root Test, we can say that the series $\sum_{n=0}^{\infty} a_n x^n$ converges, if $\lim\sup|a_n|^{\frac{1}{n}}|x| < 1$, and diverges if $\lim\sup|a_n|^{\frac{1}{n}}|x| > 1$.

So, we have the following definition.

Definition: For the power series $\sum_{n=0}^{\infty} a_n x^n$, we set $\rho = \lim\sup|a_n|^{\frac{1}{n}}$. Note that ρ may be finite or infinite. The **radius of convergence**, R, of the given series is defined as

$$R = \begin{cases} 0, \text{if } \ \rho = \infty \\ \dfrac{1}{\rho}, \text{if } \ 0 < \rho < \infty. \\ \infty, \text{if } \ \rho = 0 \end{cases}$$

The interval, $]-R, R[$ is called the **interval of convergence** of the series.

This definition is justified by the following theorem.

Theorem 6 (Cauchy-Hadamard Theorem): If R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ then it is absolutely convergent if $|x| < R$, and is divergent if $|x| > R$.

Proof: Recall that using the Cauchy's Root Test the series $\sum_{n=0}^{\infty} a_n x^n$ converges if $\lim\sup|a_n|^{\frac{1}{n}}|x| < 1$, and diverges if $\lim\sup|a_n|^{\frac{1}{n}}|x| > 1$. That is, it converges absolutely for $\rho|x| < 1$, and diverges for $\rho|x| > 1$.

If $0 < R < \infty$, then we can say that it converges absolutely for $|x| < R$, and diverges for $|x| < R$.

If $R = 0$, then $\rho = \infty$, and $\rho|x| > 1$ for all x. Hence, in this case, the series diverges for all x. Finally if $R = \infty$, then $\rho = 0$, and $\rho|x| = 0 < 1$ for all x. So, in this case, the series converges absolutely for all x. ■

Remark 2: i) Cauchy-Hadamard Theorem tells us that a power series is convergent if $|x| < R$, and is divergent if $|x| < R$. What happens at $|x| = R$?

In the examples and exercises in this unit we shall consider only those power series, where the limit of $|a_n|^{\frac{1}{n}}$ exists. So, you don't have to worry about the calculation of limsup.

There is no unique answer to it.
For example, the radius of convergence of the series, $\sum x^n$ is 1. This series is divergent at $|x| = 1$. We also have $R = 1$ for the series $\sum_{n=1}^{\infty} \frac{1}{n} x^n$, since $\limsup (n^{\frac{1}{n}}) = 1$. This series converges for $x = -1$, but diverges for $x = 1$.

The series $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ also has the same radius of convergence, 1. This series converges for both $x = 1$, and $x = -1$.

So, the behavior of a power series at the **end points** of its interval of convergence varies from one series to another.

ii) The radius of convergence R of a power series $\sum a_n x^n$ is also given by $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, provided the limit exists. You may have come across this while studying infinite series in Unit 7. You will soon see that sometimes it is easier to calculate R using this formula.

Now we shall see why a power series is uniformly convergent on any closed interval contained in its interval of convergence.

Theorem 7: If $R > 0$ is the radius of convergence of the power series, $\sum_{n=1}^{\infty} a_n x^n$, and if $0 < c < R$, then $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly in $[-c, c]$.

Proof: Let $c < b < R$. By Cauchy-Hadamard Theorem we know that $\sum a_n x^n$ is absolutely convergent for all x such that $|x| < R$. Therefore, $\sum a_n b^n$ converges absolutely. This implies $|a_n b^n| \to 0$. So, for $\varepsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow |a_n b^n| < 1$.

Now, if we take $x \in [-c, c]$, and $n \geq n_0$, then

$$|a_n x^n| = |a_n b^n| \left|\frac{x}{b}\right|^n < \left|\frac{x}{b}\right|^n \leq \left|\frac{c}{b}\right|^n.$$

We have $\left|\frac{c}{b}\right| < 1$. Therefore, $\sum\left(\frac{c}{b}\right)^n$ converges.

Now, using Weierstrass' M-test, we conclude that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$. ■

If we have a power series whose radius of convergence is zero, then the series converges only for $x = 0$, and hence there is no question of testing for uniform convergence. So, the statement of Theorem 7 is trivially true for $R = 0$.

Remark 3: A power series is uniformly convergent on any closed and bounded interval contained in its interval of convergence. Because, if $[p, q]$ is a closed and bounded interval within the interval of convergence of a given power series with the radius of converges R, then we can find $c \in]0, R[$ such that

$$[p, q] \subseteq [-c, c] \subset]-R, R[.$$

Then Theorem 7 tells us that the power series is uniformly convergent on $[-c, c]$. So, it is also uniformly convergent on $[p, q]$.

We are now going to see how the Theorems 2, 3, and 4 can be interpreted for a power series. Let us take these one by one.

Theorem 8: The limit of a power series is continuous on its interval of convergence. That is, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all $x \in]-R, R[$, then f is continuous on $]-R, R[$.

Proof: Suppose $x_0 \in]-R, R[$. Then there is some $\varepsilon > 0$ such that $[x_0 - \varepsilon, x_0 + \varepsilon] \subseteq]-R, R[$. Then by Remark 3, the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[x_0 - \varepsilon, x_0 + \varepsilon]$. Since x^n is continuous on $\mathbb{R}$ for all $n \in \mathbb{N}$, x^n is continuous on $[x_0 - \varepsilon, x_0 + \varepsilon]$ for all $n \in \mathbb{N}$. Therefore, by Theorem 2, f is continuous on $[x_0 - \varepsilon, x_0 + \varepsilon]$, and hence on x_0. Since x_0 is arbitrary, f is continuous on $]-R, R[$. ■

Theorem 9: A power series can be integrated term by term on any closed and bounded interval contained in its interval of convergence.

Proof: By Remark 3, a power series is uniformly convergent on any closed and bounded interval in its interval of convergence. Then by Theorem 4 it follows that the series can be integrated term by term. ■

Next we are going to show that a power series can be differentiated term by term in its interval of convergence. You know that to enable term by term differentiation of a series of functions, the series has to follow some additional criterion, apart from uniform convergence. And, we will show that a power series satisfies that additional criterion also.

Before we state the theorem, recall that $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1.$

Theorem 10: A power series can be differentiated term by term in its interval of convergence.

Proof: Consider the series, $\sum\limits_{n=0}^{\infty} f_n(x) = \sum\limits_{n=0}^{\infty} a_n x^n$. Suppose the radius of convergence of this series is R. Now, the series, $\sum\limits_{n=1}^{\infty} f_n'(x) = \sum\limits_{n=1}^{\infty} n a_n x^{n-1}$ also has the radius of convergence R. This is because $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$ and so,

$$\limsup\left(| na_n |^{\frac{1}{n}}\right) = \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)\left(\limsup | a_n |^{\frac{1}{n}}\right) = \limsup\left(|a_n|^{\frac{1}{n}}\right).$$

Pick any real number c such that $0 < c < R$. Then $\sum\limits_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$, and hence $\sum\limits_{n=0}^{\infty} a_n x_0^n$ converges for some $x_0 \in [-c, c]$. Now the series $\sum\limits_{n=0}^{\infty} f_n'(x) = \sum\limits_{n=0}^{\infty} n a_n x^{n-1}$ converges uniformly in $[-c, c]$. Hence, by Theorem 3, it follows that $\sum\limits_{n=0}^{\infty} a_n x^n$ can be differentiated term by term in $[-c, c]$. Since c is arbitrary, $\sum\limits_{n=0}^{\infty} a_n x^n$ can be differentiated term by term in $]-R, R[$. ■

If $\sum\limits_{n=0}^{\infty} a_n x^n = f(x)$ in its interval of convergence, then by applying Theorem 10 again and again, we can get successive derivatives of $f(x)$. These will be given by $f^{(k)}(x) = \sum\limits_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}$.

All these series have the same radius of convergence, R.

For $x = 0$, we get $f^{(k)}(0) = k! a_k$. This means

$a_k = \frac{f^{(k)}(0)}{k!}$, and $\sum a_n x^n = \sum \frac{f^{(n)}(0)}{n!} x^n$ in $]-R, R[$.

But this is what we call the Maclaurin's series of f. Isn't it?

Instead of starting with a power series, if we are given a real function, f, which has derivatives of all orders at $x = 0$, we can write the Maclaurin's series for f as $\sum \frac{f^{(n)}(0)}{n!} x^n$. This is a power series. If this series converges to f in some interval, $]-R, R[$, we say that f is **analytic at** $x = 0$. So, the property,

"f is **analytic** at $x=0$" is stronger than the property, "f is differentiable at $x=0$".

We now give some examples, which will illustrate the concepts discussed in this section.

Example 6: Find the radius of convergence of the power series, $\sum_{n=0}^{\infty} a_n x^n$, where

i) $a_n = n^{-\sqrt{n}}$, ii) $a_n = \frac{(n!)^2}{(2n)!}$.

Solution: i) We need to find $\rho = \lim\sup|a_n|^{\frac{1}{n}}$, where

$|a_n|^{\frac{1}{n}} = \left(n^{-\sqrt{n}}\right)^{\frac{1}{n}} = n^{\frac{-1}{\sqrt{n}}}$.

Let us find the limit of this as $n \to \infty$, if it exists. We can do this by using logarithmic differentiation. Let $b = n^{\frac{-1}{\sqrt{n}}}$. Then $\ln b = \frac{-1}{\sqrt{n}} \ln n$. This has an $\frac{\infty}{\infty}$ form as $n \to \infty$. So, by L'Hospital's Rule we can write

$$\lim_{n\to\infty} \ln b = \lim_{n\to\infty} \frac{-\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\left(\frac{-1}{n}\right)}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{-1}{2\sqrt{n}} = 0.$$

Therefore,

$\lim_{n\to\infty} b = e^0 = 1$. Hence, the radius of convergence is $R = \frac{1}{\rho} = 1$.

ii) Here the formula for a_n has factorials. In such cases, it is better to use the formula given in Remark 2 ii) to calculate R. Thus,

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

$$= \lim_{n\to\infty} \frac{(n!)^2}{(2n)!} \frac{(2(n+1))!}{((n+1)!)^2} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2} = 4.$$

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Example 7: By integrating the series for $\frac{1}{1+x}, |x|<1,$ find the series for $\ln(1+x)$.

Solution: We write the given function as a geometric series.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n .$$

This power series converges uniformly in $]-1,1[$. So, by Theorem 9, we can integrate it term by term. Thus we get

$$\ln(1+x)=\int\frac{1}{1+x}dx=\int\sum_{n=0}^{\infty}(-1)^n x^n\,dx=\sum_{n=0}^{\infty}\int(-1)^n x^n\,dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}.$$

So, $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...$

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Example 8: By integrating the power series, $\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$ term by term over $[0,1]$, show that $\sum_{n=1}^{\infty}\frac{1}{n!(n+2)}=\frac{1}{2}$.

Solution: The radius of convergence of this power series is ∞. You can easily check this by using the formula given in Remark 2 ii).

Therefore, the series can be integrated term by term over any interval. Now,

$$\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}=x\sum_{n=0}^{\infty}\frac{x^n}{n!}=xe^x.$$

So,

$$\int_0^1\sum\frac{x^{n+1}}{n!}dx=\int_0^1 xe^x dx=1.$$

By Theorem 9,

$$\int_0^1\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}\,dx=\sum_{n=0}^{\infty}\int_0^1\frac{x^{n+1}}{n!}dx$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!(n+2)}=\frac{1}{2}+\frac{1}{3}+\frac{1}{2!4}+\frac{1}{3!5}+....$$

Therefore, $\sum_{n=0}^{\infty}\frac{1}{n!(n+2)}=1$. After subtracting $\frac{1}{2}$ from both the sides, we get

$$\sum_{n=1}^{\infty}\frac{1}{n!(n+2)}=\frac{1}{2}.$$

***

If you have carefully gone through the examples here, you should be able to solve the following exercises now.

E5) Find the radius of convergence of the series, $\sum a_n x^n$, where a_n is as given below.

i) n^n ii) $\frac{1}{n!}$ iii) 1 iv) $\frac{1}{n^2}$ v) $\frac{n^n}{n!}$

E6) Show that the power series, $E(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ is convergent for each $x\in\mathbb{R}$, and uniformly convergent on $[-A,A]$ for any $A\in\mathbb{R}^+$. Further show that $E'(x)=E(x)$. (Note that E6 is the exponential function e^x.)

E7) Find the power series expression for $\tan^{-1} x$. What is the radius of convergence of this series?

E8) Given that the power series, $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence, 2, find the radius of convergence of

i) $\sum_{n=0}^{\infty} a_n^k x^n$ ii) $\sum a_n x^{kn}$.

E9) If $|x| < 1$, show that $\sin^{-1} x = \sum_{0}^{\infty} \frac{1.3.....(2n-1)}{2.4.....2n} \cdot \frac{x^{2n+1}}{2n+1}$.

That brings us to the end of this unit.

18.4 SUMMARY

In this unit we have covered the following points.

1. The concept of pointwise and uniform convergence of series of functions are discussed.
2. Some results such as term by term differentiation and integration of series o functions were discussed.
3. We have discussed the concept of a power series, and of the radius of convergence of a power series.
4. We have seen how to compute the radius of convergence of some power series.

18.5 SOLUTION/ ANSWERS

E1) We know that $|f_n(x)| = \frac{|\cos nx|}{n^3} \le \frac{1}{n^3}$ for all $x \in [0,1]$ and $n \in \mathbb{N}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ has all positive terms, and is convergent. Therefore, by Weierstrass' M-test, $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on $[0,1]$. We have $f_n'(x) = \frac{-\sin n x}{n^2}$ for all $x \in [0,1]$, and for all $n \in \mathbb{N}$. Now $|f_n'(x)| = \left|\frac{\sin n x}{n^2}\right| \le \frac{1}{n^2}$. Again, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series of positive terms. Therefore, by Weierstrass' M-Test, $\sum_{n=1}^{\infty} f_n'$ is also uniformly convergent. Now

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) = -\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

$$= -\sin x - \frac{\sin 2x}{4} - \frac{\sin 3x}{9} - \cdots$$

E2) Since $x \geq 0, x+n \geq n$ which implies $\frac{1}{x+n} \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore,

$$\frac{x}{n(x+n)} \leq \frac{10}{n^2} \text{ for all } n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series, with $p=2$, it is convergent. So $\sum_{n=1}^{\infty} \frac{10}{n^2}$ is also convergent. Therefore, by Weierstrass' M-Test, $\sum_{n=1}^{\infty} \frac{x}{n(x+n)}$ is uniformly convergent, and hence pointwise convergent.

Now, let $f_n(x) = \frac{x}{n(x+n)}$, for $n \in \mathbb{N}$ and $n \in [0,10]$. Clearly, f_n is differentiable and $f_n'(x) = \frac{1}{(x+n)^2}$. Again we can see that $\sum_{n=1}^{\infty} f_n'$ is uniformaly convergent. Thus $\sum_{n=1}^{\infty} f_n$ satisfies all the hypotheses of Theorem 3. Consequently, $\sum_{n=1}^{\infty} f_n$ converges uniformly to a function f such that f is differentiable and for all $x \in [0,10]$, and

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2}.$$

E3) For all $x \in [0,a]$ and for all $n \in \mathbb{N}$ we have $n^3 + x^3 \geq n^3$, which implies $\frac{1}{n^3+x^3} \leq \frac{1}{n^3}$. Also for the same values of x and n, we have $nx^2 \leq na^2$. This gives us

$$\frac{nx^2}{n^3+x^3} \leq \frac{na^2}{n^3} \Rightarrow \left|\frac{nx^2}{n^3+x^3}\right| \leq \frac{a^2}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty} \frac{a^2}{n^2}$ is also convergent. Therefore, by the Weierstrass' M-Test, $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ is uniformly convergent on $[0,a]$.

E4) For all $x \in [2,\infty]$ and for all $n \in \mathbb{N}$

$$x \geq 2 \Rightarrow x^n \geq 2^n \Rightarrow 1+x^n \geq 2^n \Rightarrow \frac{1}{1+x^n} \leq \frac{1}{2^n}$$

$$\Rightarrow \left|\frac{1}{1+x^n}\right| \leq \frac{1}{2^n}.$$

Therefore, by the Weierstrass' M-Test, $\sum_{n=1}^{\infty} \frac{1}{1+x^n}$ is uniformly convergent on $[2,\infty[$.

For $x \in [0,1[$, $\lim_{n\to\infty} x^n = 0$, and hence $\lim_{n\to\infty} \frac{1}{1+x^n} = 1$. This implies $\sum_{n=1}^{\infty} \frac{1}{1+x^n}$ is not convergent in $[0,1[$.

E5) i) Hrere $a_n = n^n$ and so $a_{n+1} = (n+1)^{n+1}$. Now

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n+1} \right) . \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= 0.e^{-1} = 0 \qquad \left(\because \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e. \right)$$

ii) Here $a_n = \frac{1}{n!}$ and $a_{n+1} = \frac{1}{(n+1)!}$. So

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$$

iii) $R = \lim_{n\to\infty} \frac{1}{1} = 1.$

iv) $R = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 = 1.$

v) $R = \lim_{n\to\infty} \frac{n^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = e^{-1}.$

E6) From E5 (ii) we know that the radius of convergance of this series is ∞. Therefore, the series converges for all $x \in \mathbb{R}$, and is uniformly convergent on $[-A, A]$ for every $A \in \mathbb{R}^+$.

By Theroem 10, we can differentiate $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ term by term. That is, $E'(x) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= E(x).$$

E7) To find a power series expression for $\tan^{-1} x$, we shall use the fact that

$$\int_0^x \frac{1}{1+x^2} dx = \tan^{-1} x.$$

We know that

$$\frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_n = \begin{cases} 0, \text{ if } n \text{ is odd} \\ (-1)^{\frac{1}{n}}, \text{ if } n \text{ is even} \end{cases}$

The radius of convergence of this series is

$$R = \limsup |a_n|^{\frac{1}{n}} = \limsup |(-1)|^{\frac{1}{n}} = \limsup 1 = 1.$$

Thus the series converges uniformly in any closed and bounded interval conained in $]-1,1[$. Now, we can integrate the series from 0 to x, for $|x| < 1$. Thus,

$$\int_0^x \frac{1}{1+x^2}\, dx = \int_0^x (1 - x^2 + x^4 - x^6 + \cdots)\, dx$$

$$\Rightarrow \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

E8) i) We are given that $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 2.$ Therefore, the radius of convergence of the series $\sum_{n=0}^{\infty} a_n^k x^n$ is

$$R = \lim_{n\to\infty}\left|\frac{a_n^k}{a_{n+1}^k}\right| = \left(\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|\right)^k = 2^k.$$

ii) The series $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in \mathbb{R}$ such that $|x| < 2.$

Therefore, $\sum_{n=0}^{\infty} a_n x^{kn}$ converges for all $x \in \mathbb{R}$ such that $|x|^k < 2.$

This gives $|x|^k < 2,$ i.e., $|x| < 2^{\frac{1}{k}}.$ Consequently, the radius of convergence of $\sum_{n=0}^{\infty} a_n x^{kn}$ is $2^{\frac{1}{k}}.$

E9) We know that $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$ Binomial Theorem tells us that

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots,$$

where $x, \alpha \in \mathbb{R}.$ Replacing x by $-x^2$ and α by $-\frac{1}{2}$ in this expression we get

$$(1-x^2)^{\frac{-1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2^2 \cdot 2!}x^4 + \frac{1.3.5}{2^3 \cdot 3!}x^6 + \ldots \qquad \ldots (1)$$

$$= \sum_{n=0}^{\infty} \frac{1.3.5\ldots(2n-1)}{2^n \cdot n!} x^{2n}$$

Let $a_n = \dfrac{1.3.5\ldots(2n-1)}{2^n \cdot n!}$. Then, using the formula, $R = \lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right|$, we get $R = 1$. This means that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges in $]-1,1[$.

Hence, the power series in Eqn. (1) converges in $]-1,1[$.

Now integrating both sides in Eq. (1) from 0 to $x, |x| < 1,$ we get

$$\int_0^x \frac{1}{\sqrt{1-x^2}}\,dx = \int_0^x \left(1 + \frac{1}{2!}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots\right)dx$$

$$= \int_0^x 1\,dx + \frac{1}{2!}\int_0^x x^2\,dx + \frac{1\cdot 3}{2\cdot 4}\int_0^x x^4\,dx + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\int_0^x x^6\,dx + \cdots$$

Thus,

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots.$$